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Research Article

Fixed Point Theorems for Multivalued Mappings Involving α-Function

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We obtain some fixed point theorems with error estimates for multivalued mappings satisfying a new α-ψ-contractive type condition. Our theorems generalize many existing fixed point theorems, including some fixed point theorems proved for α-ψ-contractive type conditions.

1. Introduction

Samet et al. [1] introduced and studied the notions of α-ψ-contractive and α-admissible self-mappings and obtained some well-known fixed point and coupled fixed point theorems in complete metric spaces as consequences. Karapınar and Samet [2] generalized these notions and obtained some results as an extension of the results of Samet et al. [1] and those contained therein. Asl et al. [3] extended these notions to multifunctions by introducing the notions of α_*-ψ-contractive and α_*-admissible mappings and obtained some fixed point theorems. Ali and Kamran [4] further generalized the notion of α_*-ψ-contractive mappings and obtained some fixed point theorems for multivalued mappings. Related results in this direction are also given in [5–9]. In addition to that, Ali et al. [10] introduced the notion of (α, ψ, ξ)-contractive multivalued mappings to generalize and extend the notion of α-ψ-contractive mappings to closed valued multifunctions and proved some fixed point theorems for such mappings in complete metric spaces. For details on fixed point theory for multivalued mappings, we refer to [11–16]. The purpose of this paper is to establish some fixed point theorems for a new type of α-ψ-contractive condition for multivalued mappings that also provides convergence rate and error estimates.

We recall the following definitions and results, for the sake of completeness. Let (X, d) be a metric space. For each $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $CL(X)$ the class of all nonempty closed subsets of X. For every $A, B \in CL(X)$, let

$$H(A, B) = \begin{cases} \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases} \quad (1)$$

Such a map H is called a generalized Hausdorff metric induced by d. A point $x \in X$ is said to be a fixed point of $T : X \to CL(X)$ if $x \in Tx$. If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, then $O(T, x_0) = \{x_0, x_1, x_2, \ldots\}$ is said to be an orbit of $T : X \to CL(X)$. A mapping $f : X \to \mathbb{R}$ is said to be T-orbitally lower semicontinuous if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \xi$ implies $f(\xi) \le \liminf_n f(x_n)$. Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is, an interval of the form $[0, A]$, $[0, A)$, or $[0, \infty)$ and $S_n(t)$ denotes the polynomial $S_n(t) = 1 + t + \cdots + t^{n-1}$. We use the abbreviation ψ^n for the nth iterate of a function $\psi : J \to J$.

Definition 1 (see [17]). Let $r \geq 1$. A function $\psi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(i) $\psi(\lambda t) \leq \lambda^r \psi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;

(ii) $\psi(t) < t$ for all $t \in J - \{0\}$.

It is easy to see that the first condition of Definition 1 is equivalent to the following: $\psi(0) = 0$ and $\psi(t)/t^r$ is nondecreasing on $J - \{0\}$.

Definition 2 (see [17]). A nondecreasing function $\psi : J \to J$ is said to be a Bianchini-Grandolfi gauge function [18] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \psi^n(t) < \infty, \quad \forall t \in J. \tag{2}$$

Remark 3. A function $\psi : J \to J$ satisfying (2) can be used as a rate of convergence [19] on J. Also note that ψ satisfies the following functional equation:

$$\sigma(t) = \sigma(\psi(t)) + t. \tag{3}$$

Remark 4 (see [17]). Every gauge function of order $r \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.

Lemma 5. *Let (X, d) be a metric space. Let $B \in CL(X)$ and $x \in X$. Then, for each $\epsilon > 0$, there exists $b \in B$ such that $d(x, b) \leq d(x, B) + \epsilon$.*

Lemma 6 (see [17]). *Let ψ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying*

$$\psi(t) = t\phi(t) \quad \forall t \in J, \tag{4}$$

then it has the following properties:

(i) *$0 \leq \phi(t) < 1$ for all $t \in J$;*

(ii) *$\phi(\lambda t) \leq \lambda^{r-1}\phi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$.*

Moreover, for each $n \geq 0$, we have

(iii) *$\psi^n(t) \leq t\phi(t)^{S_n(r)}$ for all $t \in J$,*

(iv) *$\phi(\psi^n(t)) \leq \phi(t)^{r^n}$ for all $t \in J$.*

Definition 7 (see [3]). Let (X, d) be a metric space and let $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : X \to CL(X)$ is said to be an α_*-admissible if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha_*(Tx, Ty) \geq 1, \tag{5}$$

where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$.

2. Main Results

Theorem 8. *Let (X, d) be a complete metric space and let $T : X \to CL(X)$ be an α_*-admissible mapping such that*

$$\alpha(x, y) H(Tx, Ty) \leq \psi(d(x, y)), \tag{6}$$

for all $x \in X$ and $y \in Tx$, with $d(x, y) \in J$, where ψ is a Bianchini-Grandolfi gauge function on J. Moreover, the strict inequality holds when $d(x, y) \neq 0$. Suppose that there exists $x_0 \in X$ such that $d(x_0, z) \in J$ and $\alpha(x_0, z) \geq 1$, for some $z \in Tx_0$. Then,

(i) *there exists an orbit $\{x_n\}$ of T in X and $\xi \in X$ such that $\lim_n x_n = \xi$;*

(ii) *ξ is a fixed point of T if and only if the function $f(x) := d(x, Tx)$ is T-orbitally lower semicontinuous at ξ.*

Proof. Consider $x_1 = z \in Tx_0$. We assume that $d(x_0, x_1) \neq 0$, for otherwise x_0 is a fixed point of T. Define $\rho_0 = \sigma(d(x_0, x_1))$, where σ is defined by (2). Since, from (3), $\sigma(t) \geq t$, we have

$$d(x_0, x_1) \leq \rho_0. \tag{7}$$

Notice that $x_1 \in \overline{S}(x_0, \rho_0)$. It follows from (6) that $\alpha(x_0, x_1)H(Tx_0, Tx_1) < \psi(d(x_0, x_1))$. By hypothesis, we have $\alpha(x_0, x_1) \geq 1$. We can choose an $\epsilon_1 > 0$ such that

$$\alpha(x_0, x_1) H(Tx_0, Tx_1) + \epsilon_1 \leq \psi(d(x_0, x_1)). \tag{8}$$

Thus, we have

$$\begin{aligned} d(x_1, Tx_1) + \epsilon_1 &\leq H(Tx_0, Tx_1) + \epsilon_1 \\ &\leq \alpha(x_0, x_1) H(Tx_0, Tx_1) + \epsilon_1 \\ &\leq \psi(d(x_0, x_1)). \end{aligned} \tag{9}$$

It follows from Lemma 5 that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq d(x_1, Tx_1) + \epsilon_1. \tag{10}$$

We assume that $d(x_1, x_2) \neq 0$, for otherwise x_1 is a fixed point of T. From inequalities (9) and (10), we have

$$d(x_1, x_2) \leq \psi(d(x_0, x_1)). \tag{11}$$

Note that $d(x_1, x_2) \in J$. Also, we have $x_2 \in \overline{S}(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \psi(d(x_0, x_1)) \\ &\leq d(x_0, x_1) + \sigma(\psi(d(x_0, x_1))) \\ &= \sigma(d(x_0, x_1)) \quad \text{(by using (3))} \\ &= \rho_0. \end{aligned} \tag{12}$$

Since T is an α_*-admissible, then we have $\alpha(x_1, x_2) \geq 1$. Now choose $\epsilon_2 > 0$ such that

$$\alpha(x_1, x_2) H(Tx_1, Tx_2) + \epsilon_2 \leq \psi(d(x_1, x_2)). \tag{13}$$

Thus, we have

$$\begin{aligned} d(x_2, Tx_2) + \epsilon_2 &\leq H(Tx_1, Tx_2) + \epsilon_2 \\ &\leq \alpha(x_1, x_2) H(Tx_1, Tx_2) + \epsilon_2 \\ &\leq \psi(d(x_1, x_2)). \end{aligned} \tag{14}$$

It again follows from Lemma 5 that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \le d(x_2, Tx_2) + \epsilon_2. \quad (15)$$

We assume that $d(x_2, x_3) \neq 0$, for otherwise x_2 is a fixed point of T. From (11), (14), and (15), we have

$$d(x_2, x_3) \le \psi^2(d(x_0, x_1)). \quad (16)$$

Note that $d(x_2, x_3) \in J$. Also, we have $x_3 \in \overline{S}(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_3) &\le d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\le d(x_0, x_1) + \psi(d(x_0, x_1)) + \psi^2(d(x_0, x_1)) \\ &\le \sum_{j=0}^{\infty} \psi^j(d(x_0, x_1)) \\ &= \sigma(d(x_0, x_1)) = \rho_0. \end{aligned} \quad (17)$$

Repeating the above argument, inductively we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n \in Tx_{n-1}, \quad (18)$$

$$\alpha(x_{n-1}, x_n) \ge 1, \quad (19)$$

$$d(x_n, x_{n+1}) \le \psi^n(d(x_0, x_1)), \quad (20)$$

$$d(x_{n-1}, x_n) \in J, \quad x_n \in \overline{S}(x_0, \rho_0). \quad (21)$$

We claim that $\{x_n\}$ is a Cauchy sequence. For $n, p \in \mathbb{N}$, from (20) we have

$$\begin{aligned} d(x_n, x_{n+p}) &\le d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\le \psi^n(d(x_0, x_1)) + \cdots + \psi^{n+p-1}(d(x_0, x_1)) \\ &\le \sum_{j=n}^{\infty} \psi^j(d(x_0, x_1)). \end{aligned} \quad (22)$$

By using (2), it follows from (22) that $\{x_n\}$ is a Cauchy sequence. Thus, there exists $\xi \in \overline{S}(x_0, \rho_0)$ with $x_n \to \xi$ as $n \to \infty$. Since $x_n \in Tx_{n-1}$, from (6), (19), and (20), we have

$$\begin{aligned} d(x_n, Tx_n) &\le \alpha(x_{n-1}, x_n) H(Tx_{n-1}, Tx_n) \\ &\le \psi(d(x_{n-1}, x_n)) \\ &\le \psi^n(d(x_0, x_1)). \end{aligned} \quad (23)$$

Letting $n \to \infty$, from (23), we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \quad (24)$$

Suppose $f(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at ξ; then,

$$d(\xi, T\xi) = f(\xi) \le \liminf_n f(x_n) = \liminf_n d(x_n, Tx_n) = 0. \quad (25)$$

Hence, $\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is fixed point of T, then $f(\xi) = 0 \le \lim_n \inf f(x_n)$. □

Example 9. Let $X = [-100, \infty)$ be endowed with the usual metric d and let $J = [0, \infty)$. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} \left[0, \dfrac{x}{3}\right] & \text{if } x \ge 0 \\ [x+1, 0] & \text{otherwise,} \end{cases} \quad (26)$$

and define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, \infty) \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

Take $\psi(t) = t/2$ for each $t \ge 0$. Let $x_0 = 1$; then, we have $z = 1/3 \in Tx_0$ such that $d(x_0, z) \in J$ and $\alpha(x_0, z) = 1$. As we know, $\alpha(x, y) = 1$ for $x, y \in [0, \infty)$. Then, we have $\alpha_*(Tx, Ty) = 1$ whenever $\alpha(x, y) = 1$. Thus, T is an α_*-admissible mapping. For $x \ge 0$ and $y \in Tx$, from (6), we have

$$\alpha(x, y) H(Tx, Ty) = \frac{1}{3}|x - y| \le \frac{1}{2}|x - y| = \psi(d(x, y)); \quad (28)$$

for $x < 0$ and $y \in Tx$, we have

$$\alpha(x, y) H(Tx, Ty) = 0 \le \frac{1}{2}|x - y| = \psi(d(x, y)). \quad (29)$$

Hence, (6) holds for each $x \in X$ and $y \in Tx$ with $d(x, y) \in J$. Therefore, all the conditions of Theorem 8 hold and hence T has a fixed point.

Example 10. Let $X = [-1, \infty)$ be endowed with the usual metric d and let $J = [0, \infty)$. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} \left[-1, \dfrac{x}{3}\right] & \text{if } x \in [-1, 0) \\ [0, x^2] & \text{if } x \in \left[0, \dfrac{3}{5}\right] \\ [x, e^x] & \text{if } x \in \left(\dfrac{3}{5}, \infty\right), \end{cases} \quad (30)$$

and define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \left[0, \dfrac{3}{5}\right] \\ 0 & \text{otherwise.} \end{cases} \quad (31)$$

Take $\psi(t) = (24/25)t$ for each $t \ge 0$. Let $x_0 = 3/5$; then, we have $z = 9/25 \in Tx_0$ such that $d(x_0, z) \in J$ and $\alpha(x_0, z) = 1$. As we know, $\alpha(x, y) = 1$ for $x, y \in [0, 3/5]$. Then, we have $\alpha_*(Tx, Ty) = 1$ whenever $\alpha(x, y) = 1$. Thus, T is an α_*-admissible mapping. For $x \in [0, 3/5]$ and $y \in Tx$, from (6), we have

$$\begin{aligned} &\alpha(x, y) H(Tx, Ty) \\ &\quad \le \left(\frac{3}{5} + \frac{9}{25}\right)|x - y| = \frac{24}{25}|x - y| = \psi(d(x, y)), \end{aligned} \quad (32)$$

for otherwise we have

$$\alpha(x,y)H(Tx,Ty)=0\leq\frac{24}{25}|x-y|=\psi(d(x,y)). \quad (33)$$

Hence, (6) holds for each $x\in X$ and $y\in Tx$ with $d(x,y)\in J$. Therefore, all the conditions of Theorem 8 hold and hence T has a fixed point.

Theorem 11. *Let (X,d) be a complete metric space and let $T: X\to CL(X)$ be an α_*-admissible mapping such that*

$$\alpha(x,y)H(Tx,Ty)\leq\psi(d(x,y)), \quad (34)$$

for all $x\in X$ and $y\in Tx$, with $d(x,y)\in J$, where ψ is a gauge function of order $r\geq 1$ on J and $\phi: J\to\mathbb{R}^+$ is a nondecreasing function defined by (4). Moreover, the strict inequality holds when $d(x,y)\neq 0$. Suppose that there exists $x_0\in X$ such that $d(x_0,z)\in J$ and $\alpha(x_0,z)\geq 1$, for some $z\in Tx_0$. Then,

(i) *there exists an orbit $\{x_n\}$ of T in $\overline{S}(x_0,\rho_0)$ that converges with rate of convergence at least r to a point $\xi\in\overline{S}(x_0,\rho_0)$, where $\rho_0=\sigma(d(x_0,z))$ and σ is defined by (2);*

(ii) *for all $n\geq 0$, we have the following a priori estimate:*

$$d(x_n,\xi)\leq\frac{\lambda^{S_n(r)}d(x_0,x_1)}{1-\lambda^{r^n}}, \quad (35)$$

where $\lambda=\phi(d(x_0,x_1))$;

(iii) *for all $n\geq 1$, we have the following a posteriori estimate:*

$$d(x_n,\xi)\leq\frac{\psi(d(x_n,x_{n-1}))}{1-[\phi(d(x_n,x_{n-1}))]^r}; \quad (36)$$

(iv) *for all $n\geq 1$, we have*

$$d(x_n,x_{n+1})\leq\lambda^{S_n(r)}d(x_0,x_1), \quad (37)$$

where $\lambda=\phi(d(x_0,x_1))$;

(v) *ξ is a fixed point of T if and only if the function $f(x):=d(x,Tx)$ is T-orbitally lower semicontinuous at ξ.*

Proof. (i) Following the proof of Theorem 8, we have an orbit $\{x_n\}$ of T at x_0 in $\overline{S}(x_0,\rho_0)$ such that $\lim_{n\to\infty}x_n=\xi$ and $\xi\in\overline{S}(x_0,\rho_0)$.

(ii) For $m>n$, by using (20) and Lemma 6(iii), we have

$$\begin{aligned}
&d(x_n,x_m)\\
&\quad\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)\\
&\quad\leq\psi^n(d(x_0,x_1))+\psi^{n+1}(d(x_0,x_1))\\
&\qquad+\cdots+\psi^{m-1}(d(x_0,x_1))\\
&\quad\leq d(x_0,x_1)\left[\lambda^{S_n(r)}+\lambda^{S_{n+1}(r)}+\cdots+\lambda^{S_{m-1}(r)}\right]\\
&\quad=d(x_0,x_1)\sum_{j=n}^{m-1}\lambda^{S_j(r)}.
\end{aligned} \quad (38)$$

Taking n fixed and letting $m\to\infty$, we get

$$d(x_n,\xi)\leq d(x_0,x_1)\sum_{j=n}^{\infty}\lambda^{S_j(r)}. \quad (39)$$

Note that

$$\begin{aligned}
\sum_{j=n}^{\infty}\lambda^{S_j(r)}&=\lambda^{S_n(r)}+\lambda^{S_{n+1}(r)}+\cdots\\
&=\lambda^{S_n(r)}\left[1+\lambda^{r^n}+\lambda^{r^n+r^{n+1}}+\lambda^{r^n+r^{n+1}+r^{n+2}}+\cdots\right].
\end{aligned} \quad (40)$$

Since $r\geq 1$, therefore

$$\begin{gathered}
r^n+r^{n+1}\geq 2r^n,\qquad r^n+r^{n+1}+r^{n+2}\geq 3r^n\cdots,\\
\lambda^{r^n+r^{n+1}}\leq\lambda^{2r^n},\qquad \lambda^{r^n+r^{n+1}+r^{n+2}}\leq\lambda^{3r^n}\cdots,
\end{gathered} \quad (41)$$

since $0\leq\lambda<1$. Thus, we have

$$\sum_{j=n}^{\infty}\lambda^{S_j(r)}\leq\lambda^{S_n(r)}\left[1+\lambda^{r^n}+\lambda^{2r^n}+\lambda^{3r^n}+\cdots\right]=\frac{\lambda^{S_n(r)}}{1-\lambda^{r^n}}. \quad (42)$$

Substituting this in (39), we get

$$d(x_n,\xi)\leq d(x_0,x_1)\frac{\lambda^{S_n(r)}}{1-\lambda^{r^n}}. \quad (43)$$

(iii) For $n\geq 0$, from (39), we have

$$d(x_n,\xi)\leq d(x_0,x_1)\sum_{j=n}^{\infty}[\phi(d(x_0,x_1))]^{S_j(r)}. \quad (44)$$

Putting $n=0$, $y_0=x_n$, and $y_1=x_1$, we have

$$d(y_0,\xi)\leq d(y_0,y_1)\sum_{j=0}^{\infty}[\phi(d(y_0,y_1))]^{S_j(r)}. \quad (45)$$

Putting $y_0=x_n$ and $y_1=x_{n+1}$, we have

$$\begin{aligned}
d(x_n,\xi)&\leq d(x_n,x_{n+1})\sum_{j=0}^{\infty}[\phi(d(x_n,x_{n+1}))]^{S_j(r)}\\
&\leq\psi(d(x_n,x_{n-1}))\sum_{j=0}^{\infty}[\phi(\psi(d(x_n,x_{n-1})))]^{S_j(r)}\\
&\leq\psi(d(x_n,x_{n-1}))\sum_{j=0}^{\infty}[\phi(\psi(d(x_n,x_{n-1})))]^{j}
\end{aligned} \quad (46)$$

$$=\frac{\psi(d(x_n,x_{n-1}))}{1-\phi(\psi(d(x_n,x_{n-1})))}, \quad (47)$$

since $S_j(r)\geq j$. Now, by Lemma 6(iv), we have

$$\phi(\psi(d(x_n,x_{n-1})))\leq[\phi(d(x_n,x_{n-1}))]^r \quad (48)$$

which means that

$$\frac{1}{1-\phi\left(\psi\left(d\left(x_n, x_{n-1}\right)\right)\right)} \leq \frac{1}{1-\left[\phi\left(d\left(x_n, x_{n-1}\right)\right)\right]^r}. \quad (49)$$

For $n \geq 1$, from (46), we have

$$\begin{aligned} d\left(x_n, \xi\right) & \\ & \leq \psi\left(d\left(x_n, x_{n-1}\right)\right) \sum_{j=0}^{\infty}\left[\phi\left(\psi\left(d\left(x_n, x_{n-1}\right)\right)\right)\right]^{S_j(r)} \\ & \leq \frac{\psi\left(d\left(x_n, x_{n-1}\right)\right)}{1-\phi\left(\psi\left(d\left(x_n, x_{n-1}\right)\right)\right)} \\ & \leq \frac{\psi\left(d\left(x_n, x_{n-1}\right)\right)}{1-\left[\phi\left(d\left(x_n, x_{n-1}\right)\right)\right]^r} \quad \text{(by using (49))}. \end{aligned} \quad (50)$$

(iv) For $n \geq 1$, by using (20) and Lemma 6(iii), we have

$$\begin{aligned} d\left(x_{n+1}, x_n\right) & \leq \psi^n\left(d\left(x_1, x_0\right)\right) \\ & \leq d\left(x_0, x_1\right) \phi(d\left(x_0, x_1\right))^{S_n(r)} \\ & = d\left(x_0, x_1\right) \lambda^{S_n(r)}. \end{aligned} \quad (51)$$

(v) The proof follows from the same arguments as in the proof of Theorem 8. □

Corollary 12. *Let (X, d) be complete metric space and let $T: X \rightarrow CL(X)$ be an α_*-admissible mapping such that*

$$\alpha(x, y) H(Tx, Ty) \leq \psi(d(x, y)), \quad (52)$$

for all $x, y \in X$ $(x \neq y)$, with $d(x, y) \in J$, where ψ is a gauge function of order $r \geq 1$ on an interval J. Suppose that there exists x_0 in X such that $d(x_0, z) \in J$ and $\alpha(x_0, z) \geq 1$ for some $z \in Tx_0$. Suppose that for any sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$ with $\alpha(x_{n-1}, x_n) \geq 1$, for each $n \in \mathbb{N}$, implies $\alpha(x_n, x) \geq 1$, for each $n \in \mathbb{N}$. Then, the following statements hold true:

(i) *there exists an orbit $\{x_n\}$ of T in $\bar{S}(x_0, \rho_0)$ that converges to a fixed point $\xi \in \bar{S}(x_0, \rho_0)$, where $\rho_0 = \sigma(d(x_0, z))$ and σ is defined by (2);*

(ii) *the estimates (35)–(37) are valid.*

Theorem 13. *Let (X, d) be a complete metric space and let $T: X \rightarrow CL(X)$ be a continuous and α_*-admissible mapping such that*

$$\alpha(x, y) H(Tx, Ty) \leq \psi(m(x, y)), \quad \forall x \in X, \ y \in Tx, \quad (53)$$

with strict inequality holds if $m(x, y) \neq 0$, where ψ is a gauge function of the first order on $J = [0, \infty)$ and

$$\begin{aligned} m(x, y) = \max \Big\{ & d(x, y), d(x, Tx), d(y, Ty), \\ & \frac{1}{2}[d(x, Ty) + d(y, Tx)] \Big\}. \end{aligned} \quad (54)$$

Suppose that there exists $x_0 \in X$ such that $d(x_0, z) \in J$ and $\alpha(x_0, z) \geq 1$ for some $z \in Tx_0$. Then, the following statements hold true:

(i) *there exists an orbit of T in X that converges to a fixed point ξ of T;*

(ii) *for $n \geq 0$, we have the following a priori estimate:*

$$d\left(x_n, \xi\right) \leq \frac{\lambda^n}{1-\lambda} d\left(x_0, x_1\right), \quad (55)$$

where $\lambda = \phi(d(x_0, x_1))$ and $\phi: J \rightarrow \mathbb{R}^+$ is a nondecreasing function defined by (4);

(iii) *for all $n \geq 1$, we have the following a posteriori estimate:*

$$d\left(x_n, \xi\right) \leq \frac{\psi\left(d\left(x_n, x_{n-1}\right)\right)}{1-\phi\left[\psi\left(d\left(x_n, x_{n-1}\right)\right)\right]}. \quad (56)$$

Proof. Consider $x_1 = z \in Tx_0$. Define $\rho_0 = \sigma(d(x_0, x_1))$, where σ is defined by (2). Since, from (3), $\sigma(t) \geq t$, we have

$$d\left(x_0, x_1\right) \leq \rho_0. \quad (57)$$

Assume that $m(x_0, x_1) \neq 0$, for otherwise $d(x_0, Tx_0) \leq m(x_0, x_1) = 0$ and x_0 is a fixed point of T. From (53), we have $\alpha(x_0, x_1) H(Tx_0, Tx_1) < \psi(m(x_0, x_1))$. By hypothesis, we have $\alpha(x_0, x_1) \geq 1$. We can choose $\epsilon_1 > 0$ such that

$$\alpha\left(x_0, x_1\right) H\left(Tx_0, Tx_1\right) + \epsilon_1 \leq \psi\left(m\left(x_0, x_1\right)\right). \quad (58)$$

Thus, we have

$$\begin{aligned} d\left(x_1, Tx_1\right) + \epsilon_1 & \leq H\left(Tx_0, Tx_1\right) + \epsilon_1 \\ & \leq \alpha\left(x_0, x_1\right) H\left(Tx_0, Tx_1\right) + \epsilon_1 \\ & \leq \psi\left(m\left(x_0, x_1\right)\right). \end{aligned} \quad (59)$$

It follows from Lemma 5 that there exists $x_2 \in Tx_1$ such that

$$d\left(x_1, x_2\right) \leq d\left(x_1, Tx_1\right) + \epsilon_1. \quad (60)$$

From the last two inequalities, we have

$$\begin{aligned} d & \left(x_1, x_2\right) \\ & \leq \psi\left(m\left(x_0, x_1\right)\right) \\ & = \psi\left(\max\left\{d\left(x_0, x_1\right), d\left(x_0, Tx_0\right), d\left(x_1, Tx_1\right), \frac{d\left(x_0, Tx_1\right) + d\left(x_1, Tx_0\right)}{2}\right\}\right) \\ & = \psi\left(\max\left\{d\left(x_0, x_1\right), d\left(x_1, Tx_1\right)\right\}\right), \end{aligned} \quad (61)$$

since $d(x_0, Tx_1)/2 \leq \max\{d(x_0, x_1), d(x_1, Tx_1)\}$. Assume that $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$. From (61), we have

$$d\left(x_1, Tx_1\right) \leq d\left(x_1, x_2\right) \leq \psi\left(d\left(x_1, Tx_1\right)\right), \quad (62)$$

which is not possible. Thus, $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1)$. From (61), we have

$$d(x_1, x_2) \leq \psi(d(x_0, x_1)). \tag{63}$$

Proceeding inductively in a similar way as in Theorem 8, we obtain the sequence $\{x_n\}$ in X such that $x_n \to \xi \in X$ as $n \to \infty$. Since T is continuous, by taking limit as $n \to \infty$, we have $\xi \in T\xi$. Estimates (35) and (36) become (55) and (56) for $r = 1$. □

Remark 14. Note that our results generalize [3, Theorem 2.1]; [5, Theorem 3.4]; [17, Theorems 4.1 and 4.2; and Corollary 4.5]; [20, Theorems 2.1 and 2.8; and Corollary 2.12]; [21, Theorem 2.1]; [22, Theorems 2.11 and 2.15]; and [23, Theorems 2.1 and 2.2].

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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