A CHARACTERIZATION OF OPEN MAPPING IN TERMS OF CONVERGENT SEQUENCES

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It is certainly well known that a mapping between metric spaces is continuous if and only if it preserves convergent sequences. Does there exist a comparable characterization for the mapping to be open? Of course, the inverse mapping is set-valued, in general. In this research/expository note, we show that a mapping is open if and only if the set-valued inverse mapping preserves convergent sequences in an appropriate set-theoretic sense.

1. Introduction

Let X and Y be topological spaces and $f : X \to Y$ a general mapping. In particular, f need not be onto. For convenience, we will assume that X and Y are metric spaces, so that sequences suffice when considering convergence. (We leave the extension of our main result to nets for the interested reader to pursue.) By definition, f is continuous if $f^{-1}(U)$ is open in X, whenever U is open in Y. It is certainly well known that f is continuous if and only if $\lim_{n \to \infty} f(x_n) = f(x)$ in Y, whenever $\lim_{n \to \infty} x_n = x$ in X (see [1, Theorem 1, page 34]). Also by definition, f is open if $f(U)$ is open in Y, whenever U is open in X. Does there exist a characterization of openness for f in terms of convergent sequences analogous to that for continuity? It would seem that a positive response to this question should be well known. However, we have not been able to find such a result in the literature. Of course, this question is complicated by the fact that f^{-1} need not be one-to-one in general, in which case we obtain a set-valued mapping from Y into X given by $y \to f^{-1}(y)$, for all $y \in Y$, where $f^{-1}(y)$ might be empty, for some $y \in Y$. Nevertheless, the answer to the previous question is yes. Specifically, in this research/expository note, we will show that a mapping is open if and only if its set-valued inverse mapping satisfies a familiar subsequence convergence property (see Theorem 2.2). We have found this characterization to be very useful in our research (i.e., see [9]).

Although we have not been able to find our main result in the literature, there do exist comparable results which require additional assumptions. In [3], the authors establish

such a result (see Theorem 2.4) for which no proof is given. Moreover, their result is stated in terms of ultranets. However, of greatest importance is the fact that they assume the mapping is continuous. The same is true in [4]. It is shown in [4, Proposition 1, Part 4] that our subsequence convergence condition is equivalent to openness, but under the assumption that the mapping in question is once again continuous and onto. (Moreover, the work [4] is not easy to obtain.) Neither one of the two previous references states the relevant results in the context of a set-valued inverse mapping which is, as we have tried to motivate, a more appropriate way to view openness.

It has been suggested that by combining (in order) [6, Proposition 7.2.9], [1, Lemma 6.2.2] and the fact that limit points are cluster points, one may conclude that the subsequence convergence condition is equivalent to openness of the mapping. However, this argument shows only that the subsequence convergence condition is necessary for openness. Thus, this argument does not establish our main result either. Moreover, we feel that the sufficiency is the more important half of the main theorem since it is more likely to be used to show that a given mapping is open (see [9], e.g.).

Finally, our main result is verified in a practical and familiar context. Our proof is direct and easily accessible to students in first-semester real analysis and point-set topology. The most advanced concept we use is that of set-valued mapping, which easily can be circumvented (as in [3, 4]).

As motivation for what follows, let $\{x_n\}_{n=1}^\infty$ be a sequence in X. Let $\limsup_n f(x_n)$ denote the *set* of cluster points of the sequence $\{f(x_n)\}_{n=1}^\infty$ in Y, that is, the set of points y for which there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that the corresponding subsequence $\{f(x_{n_k})\}_{k=1}^\infty$ of $\{f(x_n)\}_{n=1}^\infty$ converges to y. This is common notation for denoting the set of cluster points (see [8, page 337]). Moreover, it is worth noting that $\liminf_n f(x_n)$ is used to denote the set of limit points [8, page 335].

We then have the following alternate ways of looking at continuity.

Lemma 1.1. *Let X and Y be metric spaces and $f : X \to Y$ a map. Then the following are equivalent.*

   (i) *f is continuous.*
   (ii) *$f(x) \in \limsup_n f(x_n)$, for every convergent sequence $\lim_n x_n = x$ in X.*
   (iii) *$\{f(x)\} = \limsup_n f(x_n)$, for every convergent sequence $\lim_n x_n = x$ in X.*

Proof. (ii)⇒(i). Suppose f is not continuous. Then there exists a convergent sequence $\lim_{n\to\infty} x_n = x$ in X for which $\lim_{n\to\infty} y_n \neq y$, where $f(x_n) = y_n$, for all n, and $f(x) = y$. Thus, there exists a convergent subsequence $\lim_{k\to\infty} x_{n_k} = x$ of $\{x_n\}_{n=1}^\infty$ and an open neighborhood U of y in Y such that $y_{n_k} \notin U$, for all k. But, by hypothesis, $f(x) \in \limsup_k f(x_{n_k})$, that is, $y \in \limsup_k y_{n_k}$. Hence, there exists a convergent subsequence $\lim_{j\to\infty} y_{n_{k_j}} = y$ of $\{y_{n_k}\}_{k=1}^\infty$. Hence, $y_{n_{k_j}} \in U$ eventually, which is a contradiction, since $y_{n_k} \notin U$, for all k.

   (iii)⇒(ii). The proof is obvious.
   (i)⇒(iii). Let $\lim_{n\to\infty} x_n = x$ be a convergent sequence in X. Since f is continuous, we have that $\{f(x)\} \subseteq \limsup_n f(x_n)$. Let $y \in \limsup_n f(x_n)$. Then there exists a convergent subsequence $\lim_{k\to\infty} x_k = x$ of $\{x_n\}_{n=1}^\infty$ for which $\lim_{k\to\infty} f(x_{n_k}) = y$. Since f is continuous, $\lim_{k\to\infty} f(x_{n_k}) = f(x)$. Therefore, $f(x) = y$, since Y is a Hausdorff space, that

is, the sequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ cannot converge to two distinct points in Y. Consequently, $\{f(x)\} \supseteq \limsup_n f(x_n)$. $\qquad\square$

2. The main result

For our purposes here, we require the notion of the set of cluster points of a sequence of arbitrary sets. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of nonempty subsets of X. Then $x \in X$ is a cluster point of this sequence of sets if there exists a subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ of $\{A_n\}_{n=1}^{\infty}$ and a corresponding convergent sequence $\lim_{k\to\infty} x_k = x$ in X such that $x_k \in A_{n_k}$, for all k. Let $\limsup_n A_n$ denote the set of all such points (see [8, page 337]). Similarly, $\liminf_n A_n$ denotes the set of limit points of the A_n, that is, those $x \in X$ such that $x = \lim_n x_n$, for $x_n \in A_n$, for all n (see [8, page 335]). We then have the following property (amongst others). We emphasize that the sets A_n are not assumed to be closed.

LEMMA 2.1. *In general,* $\limsup_n A_n$ *is a closed subset of* X.

Proof. It can be shown that $\limsup_n A_n = \cap_{n=1}^{\infty} \overline{\cup_{k\geq n} A_k}$ (see [8, page 337]). See also [2, page 121] and [5, page 169]. $\qquad\square$

The following is our main result.

THEOREM 2.2. *Let X and Y be metric spaces and let $f : X \to Y$ be a map. Then the following are equivalent.*
  (i) *f is open.*
  (ii) *$f^{-1}(y) \subseteq \limsup_n f^{-1}(y_n)$, for every convergent sequence $\lim_n y_n = y$ in Y.*

Proof. (i)$\Rightarrow$(ii). Suppose f is open and $\{y_n\}_{n=1}^{\infty}$ is a sequence in Y which converges to $y \in Y$. Since f is open, the range $f(X)$ of f is an open subset of Y. Either $y \in f(X)$ or not. If $y \notin f(X)$, then $f^{-1}(y) = \varnothing$, so that part (ii) holds. If $y \in f(X)$, then y_n is eventually in $f(X)$, since it is open. Thus, without loss of generality, we may assume $y_n \in f(X)$, for all n. The subset $\limsup_n f^{-1}(y_n)$ of X is clearly closed (see Lemma 2.1). Suppose the subset $f^{-1}(y)$ of X is *not* contained in $\limsup_n f^{-1}(y_n)$. Then there exists $x \in f^{-1}(y)$, that is, $f(x) = y$, such that $x \notin \limsup_n f^{-1}(y_n)$. Let U denote the complement of $\limsup_n f^{-1}(y_n)$ in X, an open neighborhood of x. Let $\{U_n\}_{n=1}^{\infty}$ be a descending open base for the neighborhood system of $\{x\}$, which is contained in U, that is, a sequence of neighborhoods U_n of x in X satisfying

$$\begin{aligned} x \in U_{n+1} \subseteq U_n \subseteq U \quad \forall n, \\ \cap_{n=1}^{\infty} U_n = \{x\}. \end{aligned} \tag{2.1}$$

Let $n = 1$. Then, by hypothesis, $f(U_1)$ is an open neighborhood of $f(x) = y$ in Y. Since $\lim_n y_n = y$, there exists n_1 such that $y_n \in f(U_1)$, for all $n \geq n_1$. Suppose we have chosen $y_{n_1}, \dots, y_{n_k}$ such that $y_{n_j} \in U_j$, for all $j = 1, \dots, k$, and $y_n \in f(U_k)$, for all $n \geq n_k$. For U_{k+1}, there exists $n_{k+1} > n_k$ such that $y_n \in f(U_{k+1})$, for all $n \geq n_{k+1}$. Then $\{y_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{y_n\}_{n=1}^{\infty}$. Let $x_k \in X$ be such that $x_k \in U_k$, and $f(x_k) = y_{n_k}$, that is, $x_k \in f^{-1}(y_{n_k})$, for all k. Then the sequence $\{x_j\}_{j=1}^{\infty}$ is eventually in each U_k. In fact, given

k, we have $x_k \in U_k$, and $x_j \in U_j \subseteq U_k$, for all $j \geq k$. Consequently, $\lim_{k \to \infty} x_k = x$. By definition of $\limsup_n f^{-1}(y_n)$, it follows that $x \in \limsup_n f^{-1}(y_n)$, a contradiction. Thus,

$$f^{-1}(y) \subseteq \limsup_n f^{-1}(y_n). \tag{2.2}$$

(ii)$\Rightarrow$(i). Suppose the condition in (ii) holds and f is not open. Then there exists an open subset U of X such that $f(U)$ is not open in Y, that is, its complement $Y \setminus f(U)$ is not closed in Y. Consequently, there exists a convergent sequence $\lim_{n \to \infty} y_n = y$ in Y such that $y_n \notin f(U)$, for all n, while $y \in f(U)$. Since $y \in f(U)$, there exists $x \in U$ such that $f(x) = y$, that is, $x \in f^{-1}(y)$. By hypothesis,

$$f^{-1}(y) \subseteq \limsup_n f^{-1}(y_n). \tag{2.3}$$

Thus, there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$, and a corresponding sequence $\{x_k\}_{k=1}^{\infty}$ in X such that $x_k \in f^{-1}(y_{n_k})$, for all k, and $\lim_{k \to \infty} x_k = x$. Then $x_k \notin U$, for all k; if not, $x_k \in U$, for some k, which implies

$$y_{n_k} = f(x_k) \in f(U), \tag{2.4}$$

a contradiction. Since $X \setminus U$ is closed, and $x_k \in X \setminus U$, for all k, it follows that $x \in X \setminus U$, that is, $x \notin U$, a contradiction. $\qquad\square$

As expected, we have the following special case.

COROLLARY 2.3. *If f is one-to-one and onto, then f is open if and only if f^{-1} is continuous.*

Proof. This follows from Theorem 2.2 and Lemma 1.1. $\qquad\square$

Remark 2.4. (1) We urge the reader to compare part (ii) of Lemma 1.1 for continuous maps with part (ii) of Theorem 2.2 for open maps. (2) In [2], the author gives three notions of semicontinuity and continuity for set-valued maps, each of which is a generalization of classical continuity for single-valued maps. One of these is part (ii) of our main theorem for set-valued inverse mappings. Thus, our theorem says that a point-valued mapping f is open if and only if the corresponding set-valued inverse mapping f^{-1} is continuous (in the sense of part (ii) of the main theorem). (3) The results contained herein can be reworked for continuity and openness at a point. The details are straightforward and left to the reader.

Acknowledgment

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