Total Variation Reconstruction From Quasi-Random Samples

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***Abstract*— Pseudo-random numbers are often used for generating incoherent uniformly distributed sample distributions. However randomness is a sufficient – not necessary – condition to ensure incoherence. If one wants to reconstruct an image from few samples, choosing a globally optimized set of evenly distributed points could capture the visual content more efficiently. This work compares classical random sampling with a simple construction based on properties of the fractional Golden ratio sequence and the Hilbert space filling curve. Images are then reconstructed using a total variation prior. Results show improvements in terms of peak signal to noise ratio over pseudo-random sampling.**

1 Introduction

In compressed sensing (CS) [1], random linear measurements are used to efficiently recover an unknown but assumed sparse signal. We propose the use of quasi-random samples as a better way of sampling. As an initial illustration of the benefits of non-random sampling methods, we reconstruct the Lena image from quasi-random point samples and compare it with the usual pseudo-random point sample reconstruction. As a regularization of the inversion, we use the total variation (TV) prior, *i.e.*, we impose sparsity of the local image gradient.

2 Quasi-random sampling

The law of large number says that any sequence of randomly distributed numbers converge to a uniform distribution. This assumption is however valid only for very large numbers. When a strictly limited budget of samples should be selected for image reconstruction, it might be advantageous to rely on deterministic constructions ensuring more local uniformity in the point distribution, in contrast to clusters of points generated by true (or computer-generated pseudo-) randomness [2].

Elements of the fractional golden ratio sequence $G_s(i)$ with given seed constant $s \in [0,1)$ are given by the fractional part of the sum between s and an integer multiple of the golden ratio:

$$G_s(i) = \{s + i \cdot \phi\}, \forall\, i \geq 1\ , \quad \text{with} \quad \phi = \frac{1+\sqrt{5}}{2} \tag{1}$$

where $\{t\}$ is the fractional part of the real number t. Note that the conjugate golden section $\tau = \frac{1}{\phi} = \phi - 1$ can be subtitued to ϕ since only fractional parts are retained. Hence, computing $G_s(i+1)$ given $G_s(i)$ only requires an addition and a bit mask.

Such sequences are especially interesting for distributing coordinates on the 1D unit range. A key corollary of the strong irrationality of ϕ (and τ) is that the fractional parts of integer multiples will not align on any regular grid. In previous work [3], the coordinates of this one-dimensional sequence are mapped to higher dimensions, using the invertible Hilbert space filling curve mapping $H : \mathbb{R} \to \mathbb{R}^2$.

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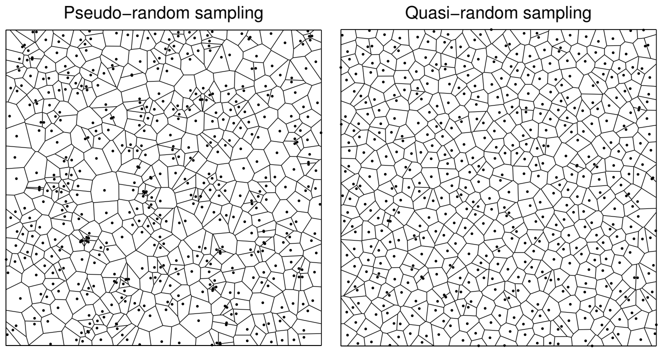

Figure 1: Voronoi diagram for the first 512 points of a pseudo-random and quasi-random sequence. Spurious clusters emerges in the random distribution, while the quasi-random distribution ensures a more evenly uniform coverage.

The Hilbert space filling curve $H\,(t) = (x, y)\ ,\ t \in [0, 1)$ defines a nested recursive grid and a locally-preserving traversal order of grid elements. Plugin-in $G_s(\cdot)$ in $H(\cdot)$, we obtain the following sets of N uniformly distributed points:

$$H(G_s(i)) = H(\{s + i \cdot \tau\}),\ i \in [1 \ldots N]. \tag{2}$$

Figure 1 shows the Voronoi tessellation for two sets of $N = 512$ point samples, using pseudo-random and the proposed quasi-random strategy with $s = 0$. Variability of Voronoi cell's areas indicates increases in local discrepancy.

3 Total variation reconstruction

Let S be the sampling operator $S : \mathbb{R}^{N_1 \times N_2} \to \mathbb{R}^N : u \to y = Su$ that corresponds to choosing a certain N pixels from the $N_1 \times N_2$ image u. Based on samples y of some given image ($y = Su^{\text{orig}}$), the reconstructed image u^{rec} minimizes the total variation and coincides with u^{orig} in the chosen samples, *i.e.*:

$$u^{\text{rec}} = \underset{Su=y}{\arg\min}\ \mathrm{TV}(u). \tag{3}$$

Here the total variation $\mathrm{TV}(u)$ is defined in terms of the local differences (image gradients) $D(u) \equiv (D_{\mathrm{h}}u, D_{\mathrm{v}}u) \in \mathbb{R}^2$:

$$\begin{cases} D_{\mathrm{h}}u_{i,j} = u_{i+1,j} - u_{i,j} & 1 \leq i \leq N_1 - 1, \\ D_{\mathrm{v}}u_{i,j} = u_{i,j+1} - u_{i,j} & 1 \leq j \leq N_2 - 1, \end{cases} \tag{4}$$

with $D_{\mathrm{h}}u_{N_1,j} = D_{\mathrm{v}}u_{i,N_2} = 0$. Thus, the total variation $\mathrm{TV}(u)$ in expression (3) is given explicitly by

$$\mathrm{TV}(u) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sqrt{(D_{\mathrm{h}}u_{i,j})^2 + (D_{\mathrm{v}}u_{i,j})^2}. \tag{5}$$

Figure 2: Original Lena image and reconstructions from 50.000 samples using pseudo-random and the proposed quasi-random algorithm. Visual quality and signal to noise ratio are improved when using the quasi-random method.

Total variation was introduced in imaging in 1992[4]. This penalty promotes sparse gradients in the image. In other words, it promotes a piecewise constant reconstruction. In order to compute the solution to the convex minimization problem (3), we use the following iterative scheme:

$$\begin{cases} \bar{u}^{n+1} &= u^n - \tau_1 S^T \left(2v^n - v^{n-1}\right) - \tau_1 D^T w^n \\ w^{n+1} &= P_\lambda \left(w^n + \frac{\tau_2}{\tau_1} D\bar{u}^{n+1}\right) \\ u^{n+1} &= u^n - \tau_1 S^T \left(2v^n - v^{n-1}\right) - \tau_1 D^T w^{n+1} \\ v^{n+1} &= v^n + Su^{n+1} - y \end{cases} \tag{6}$$

where the horizontal (h) and vertical (v) projection P_λ is

$$P_\lambda(w_\mathrm{h}, w_\mathrm{v}) = \begin{cases} \lambda \dfrac{(w_\mathrm{h}, w_\mathrm{v})}{\sqrt{w_\mathrm{h}^2 + w_\mathrm{v}^2}} & \text{if } \sqrt{w_\mathrm{h}^2 + w_\mathrm{v}^2} \geq \lambda \\ (w_\mathrm{h}, w_\mathrm{v}) & \text{otherwise} \end{cases} \tag{7}$$

and it is understood that it is applied in each pixel separately.

The algorithm (6) converges for positive step sizes $\tau_1 < \|S\|_2^{-2}$ and $\tau_2 < \|D\|_2^{-2}$. In this case, the spectral norm of S is $\|S\|_2 = 1$ and the spectral norm of D is $\|D\|_2 = 2\sqrt{2}$. The convergence speed of this algorithm depends on the positive parameter λ, but its choice is not critical. The algorithm depend on three auxiliary variables ($\bar{u}, v$ and w). All variables are initialized ($n = 0$) by zero. A proof of convergence can be found in [6]. Other methods can be found *e.g.* in [5].

4 Results

In order to compare the two sampling methods, we constructed a numerical experiment in which an image is sampled at a limited number of pixels. The whole image is then reconstructed using the TV penalty.

Figure 2, first row, shows the results of an initial comparison of the proposed quasi-random sampling method and standard pseudo-random sampling. Both reconstructions are based on 50.000 distinct point samples of the Lena image. In both cases we used the TV prior to regularize the inversion from 50.000 samples to a dense image of 512×512 pixels. Figure 2, second row, represents a close-up of the the first row images. More visual detail is discernable in the reconstruction from quasi-random samples, than from pseudo-random samples.

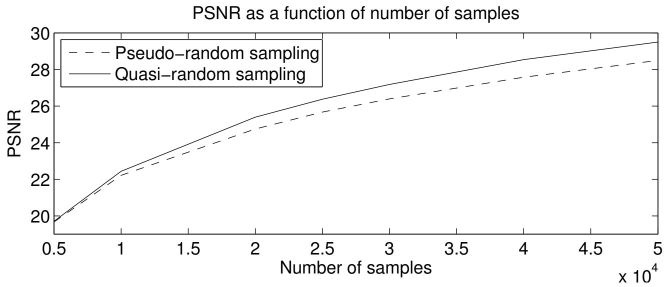

Figure 3: Peak signal to noise ratio (PSNR) is monotonously increasing with the number of images samples available for image reconstruction. The quasi-random sampling strategy yields better recovery of the original image content.

The algorithm was run with up to 1000 iterations. The final peak signal to noise ratio (PSNR) of the reconstruction from quasi-random samples is 29.5dB whereas the final PSNR for the reconstruction from pseudo-random samples is 28.5dB. Furthermore, seven experiments were performed with $5000, 10.000, 20.000, 25.000, 30.000, 40.000$ and 50.000 samples. Figure 3 plots the PSNR for the seven pairs of reconstructions of the Lena image. In both case the PSNR increases monotonically with the number of samples taken, but the quasi-random samples give better reconstruction results in all cases.

5 Conclusion

The law of large numbers says that randomly distributed points are expected to tend towards a uniform distribution. Experiments demonstrate that if the number of point samples is limited, spurious random clustering impairs sensing efficiency since some regions of the sampling domain are more sparsely populated. Fortunately, deterministic quasi-random methods exist for providing evenly distributed but still incoherent points.

Our first results on a natural image were encouraging and future work will quantify the potential of the method on a practical medical imaging setup. Another possibility consists of sampling the wavelet or Fourier domains in quasi-random points, as opposed to the usual pseudo-randomly placed samples, as is often done in applications of the compressed sensing framework.

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