Evaluating VaR with the ARCH/GARCH Family

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Abstract

The aim of the thesis is to identify an appropriate model in forecasting Value-at-Risk on a more volatile period than that one from which the model is estimated. We estimate 1-day-ahead and 10-days-ahead Value-at-Risk on a number of exchange rates. The Value-at-Risk estimates are based on three models combined with three distributional assumptions of the innovations, and the evaluations are made with Kupiec's (1995) test for unconditional coverage. The data ranges from January 1st 2006 through June 30th 2011. The results suggest that the GARCH(1,1) and GJR-GARCH(1,1) with normally distributed innovations are models adequately capturing the conditional variance in the series.

KEYWORDS: Value-at-Risk, ARCH, GARCH, GJR-GARCH, Exchange rates, Conditional Variance, Volatility Forecasting.

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1 Introduction

Econometricians are often interested in fitting models to various kinds of data, both cross-sectional data and time series data. Historically, they have focused their attention on modeling conditional first moments, see e.g. Bollerslev, Engle and Nelson (1994). An underlying assumption for these analysis is that the error variance is *homoscedastic*, i.e. constant over time. In many instances, however, this assumption is invalid. The error variance might be larger for some ranges in the data, and smaller for others. The error terms is then said to suffer from *heteroscedasticity*.

In financial econometrics this is called volatility clustering and is a common phenomenon. For the investor, volatility means risk, and it is therefore of interest to model and forecast. In a seminal paper, Engle (1982) introduced the Autoregressive Conditional Heteroscedasticity (ARCH) model, a model aimed to capture time varying conditional variance. Later on, Bollerslev (1986) and Taylor (1986) proposed, independently of each other, a generalization of this model, called Generalized ARCH (GARCH). Today, many have designed modifications of the GARCH model, which has given rise to the expression of an ARCH/GARCH family of models, see e.g. Bollerslev, Engle and Nelson (1994) and Teräsvirta (2006).

The purpose of this paper is to identify the best model in modeling financial time series data or, more specifically, exchange rates. The financial crisis and the ongoing debt crisis has created a demand of models that are valid even when a change in regime occurs, from a period of lower volatility to a period of higher volatility. We use data from less volatile periods to explore how accurate the models are in predicting the variation in more volatile periods. The models employed in this thesis are the GARCH(1,1), EGARCH(1,1) and GJR-GARCH(1,1) model.

Capital requirements are imposed on financial institutions by The Basel Committee on Banking Supervision (2010). They base their requirements on Value-at-Risk (VaR) measures, which is commonly used in risk management. In this thesis, we give 1-day-ahead and 10-days-ahead VaR forecasts and evaluate them with Kupiec's (1995) unconditional coverage test. Furthermore, Root Mean Square Error (RMSE) and Mean Square Error (MSE) are also presented as complementary measures.

We base our thesis on the work that two other studies carried out. Angelidos, Benos and Degiannakis (2004) evaluate a family of ARCH models in modeling 1-day-ahead VaR of perfectly diversified portfolios in five stock indices. They conclude that models with leptokurtic distributions, i.e. distributions with excess kurtosis and fat tails, produce better VaR forecasts than models utilizing the normal distribution. They also find strong evidence that the EGARCH produces the most satisfactory results for a majority of the markets. Moreover, the choice of sample size seems to matter for the accuracy of the forecast.

Jánský and Rippel (2011) evaluate ARCH models on six world stock indices, also in modeling 1-day-ahead VaR. With data ranging from 2004 through 2009 they fit models to data from less volatile periods and make forecasts on more volatile ones. They find the GARCH process to most adequately capture the volatility in the indices.

The outline of the paper is as follows. The next section establishes the theoretical framework where the models are presented along with the VaR concept and our methods of evaluation. We also give a description of the data. Section three provides our main results and an analysis of them. Section four gives our main conclusions.

2 Theoretical Framework

The ARCH/GARCH Family

To use financial notation, let the times series sequence of returns $\{r_t\}$ be defined as

$$r_t = \ln p_t - \ln p_{t-1}, \tag{1}$$

where p_t is the exchange rate at time t and, hence, r_t the rate of change in the exchange rate from time $t-1$ to t. A regression of r_t might look like

$$r_t = E\left[r_t|\psi_{t-1}\right] + \varepsilon_t, \tag{2}$$

where $E\left[\cdot|\cdot\right]$ is some mean function, $\psi_{t-1}$the information set at time $t-1$ and ε_t the error term. Earlier studies - see e.g Angelidos, Benos and Degiannakis (2004) and Jánský and Rippel (2011) - conclude that a mean function does not improve the forecast accuracy of the model. Therefore, we drop this part, as it would add more complexity than improvent of the predictions.

When we have heteroscedasticity, we can define the error sequence as

$$\varepsilon_t = z_t \sigma_t, \tag{3}$$

where $\{z_t\}$ is an independently and identically distributed sequence with zero expectation and unit variance and σ_t is the conditional standard deviation at time t. Engle (1982) proposed the Auto Regressive Conditional Heteroscedasticity (ARCH) model

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2, \tag{4}$$

to model the conditional variance. For the variance to be strictly positive we need that $\alpha_0 > 0$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, q$. If we add the possibility for σ_t^2 to be a function of its own lags as well, we get the Generalized ARCH (GARCH) model by Bollerslev (1986) and Taylor (1986). This model can be expressed as

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \tag{5}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, 2, \ldots, q$ and $\beta_j \geq 0$, $j = 1, 2, \ldots, p$ for positivity. If we let $q = p = 1$, we get the GARCH(1,1), the first model that we use in this thesis. For covariance stationarity we need that $\alpha_1 + \beta_1 < 1$. The GARCH(1,1) model's forecast expressions of conditional variance for 1-day-ahead and 10-days-ahead are given by

$${\sigma^2}_{t+1} = \alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2, \tag{6}$$

and

$$\sigma_{t+10}^2 = \sum_{j=0}^{8} \alpha_0 \left[\alpha_1 + \beta_1\right]^j + \left[\alpha_1 + \beta_1\right]^9 \sigma_{t+1}^2, \tag{7}$$

respectively. As h, i.e. the forecast horizon, grows larger, the conditional variance converges to the unconditional variance,

$$\sigma^2 = \frac{\alpha_0}{1-(\alpha_1+\beta_1)}. \tag{8}$$

There is not much argue about the GARCH model's importance in financial and macroeconomic applications. However, Nelson (1991) stresses that the GARCH model have some drawbacks. Firstly, the model treats positive and negative changes in a series in the same way. This is not always an appeasing assumption. Good news and bad news affect people's behavior in different ways. Furthermore, the GARCH model has parameter restrictions that are often violated. Nelson (1991) therefore introduced the Exponential GARCH (EGARCH) model to address these problems. The EGARCH(p,q) can be expressed as

$$\ln\left(\sigma_t^2\right) = \alpha_0 + \sum_{i=1}^{q}\alpha_i|z_{t-i}| + \sum_{k=1}^{r}\gamma_k z_{t-k} + \sum_{j=1}^{p}\beta_j \ln\left(\sigma_{t-j}^2\right), \tag{9}$$

where $\gamma_k \leq 0$ and $z_{t-i} = {}^{\varepsilon_{t-i}}/_{\sigma_{t-i}}$. For the EGARCH model, no parameter restrictions are needed; the log-tranformation assures positivity of the conditional variance by construction.

Again, letting $q = p = 1$, we get the EGARCH(1,1) model. For covariance stationarity we need that $\beta_1 < 1$. The forecasts, 1-day-ahead and 10-days-ahead, are computed as

$$\sigma_{t+1}^2 = \exp\left\{\alpha_0 + \alpha_1|z_t| + \psi z_t + \beta \ln \sigma_t^2\right\}, \tag{10}$$

and

$$\sigma_{t+10}^2 = \exp\left\{\sum_{j=0}^{8}\alpha_o\beta^j + \sum_{j=0}^{8}\alpha_1 E\left[|z|\right]\beta^j + \beta^9 \ln \sigma_{t+1}^2\right\}, \tag{11}$$

respectively. If $z_t \sim N(0,1)$, then $E\left[|z_t|\right] = \sqrt{{}^2/_\pi}$.[1]

The last model we use in this thesis is the GJR-GARCH model developed by Glosten, Jagannathan and Runkle (1993). Formally, it can be expressed as

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2 + \sum_{k=1}^{r}\gamma_k\varepsilon_{t-k}^2 I_{t-k} + \sum_{j=1}^{p}\beta_j\sigma_{t-j}^2, \tag{12}$$

[1] For proof, see Appendix.

where I_{t-k} is an indicator variable taking the value one if the innovation is smaller than zero and zero otherwise,

$$I_t = \begin{cases} 1, \; if \, \varepsilon_t < 0 \\ 0, \; otherwise. \end{cases} \tag{13}$$

As we can see, this model also allows for asymmetric impacts by allowing the sign of the lagged errors to effect the model differently. For the GJR-GARCH(1,1) model, the 1-day-ahead and 10-days-ahead forecasts are given by

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 \varepsilon_t^2 + \gamma_1 \varepsilon_t^2 I_t + \beta_1 \sigma_t^2 \tag{14}$$

and

$$\sigma_{t+10}^2 = \sum_{j=0}^{8} \alpha_0 \delta^j + \delta^9 \sigma_{t+1}^2, \tag{15}$$

respectively. $\delta = \alpha_1 + \beta_1 + \gamma/2$ and I_t is defined as above.

Thus far, we have not discussed the distributional properties of ε_t. Engle (1982) assumed they follow a normal distribution. Others have suggested, see e.g. Bollerslev (1987), that they are t distributed. Later on, Nelson (1991) assumed Generalized Error Distributed (GED) innovations. In this thesis we consider these three distributions of the innovations. The probability density function of the normal distribution is given by

$$f(\varepsilon) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2} \left(\frac{\varepsilon - \mu}{\sigma} \right)^2 \right\}, \tag{16}$$

where μ is the mean and σ^2 the variance. The probability density function of the t distribution is given by

$$f(\varepsilon) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{\varepsilon^2}{\nu} \right)^{-\left(\frac{\nu+1}{2}\right)}, \tag{17}$$

where ν is the degrees of freedom and $\Gamma(\cdot)$ the gamma function,

$$\Gamma(\varepsilon) = \int_0^\infty t^{\varepsilon-1} e^{-t} dt. \tag{18}$$

Finally, the probability density function of the GED can be expressed as

$$f(\varepsilon) = \frac{\beta}{2\alpha\Gamma(1/\beta)} \exp\left\{ -\left(|\varepsilon - \mu| / \alpha \right)^\beta \right\}. \tag{19}$$

These density functions are closely related. For example, if we let $\nu \to \infty$, the t distribution converges to the normal. The GED is the normal if we let $\beta = 2$.

Value at Risk

Value at Risk (VaR) is a commonly used framework to deal with financial risk, see e.g. Engle (2001). VaR can be defined as the percentage loss which will not be exceeded given a certain confidence level. Lets say that we have calculated a 5%VaR of 2%. This means that, in the long run, the loss of an investment of, say, \$1,000,000 will not exceed \$20,000 (2%×\$1,000,000) in 95% of the cases. Explicitly, it can be expressed as

$$VaR\left(\alpha\right) = -\sigma_t \phi_\alpha, \tag{20}$$

where α is the level, σ_t the conditional standard deviation at time t and ϕ_α the quantile which satisfies $Pr(r_t < \phi_\alpha) = \alpha$. In this thesis, we give 1%VaR and 5%VaR. The distribution function will be the normal, the Student's t and the Generalized Error Distribution. If we consider the normal distribution, for example, the quantile is -2.327 and -1,645 for 1% and 5%, respectively. For the t and the GED, the value depends on the degrees of freedom and the GED parameter, respectively. The negative sign makes the VaR a positive value and the interpretation is then in losses. To evaluate the accuracy of the VaR forecast we introduce an indicator variable which satisfies

$$I_t = \begin{cases} 1, \; if \; r_t < -VaR\left(\alpha\right) \\ 0, \; otherwise. \end{cases} \tag{21}$$

We can compute the unconditional coverage as $\frac{1}{T}\sum_{t=1}^{T} I_t$, where T is the number of out-of-sample forecasts. This is called the empirical level. If the model is good, this estimate should lie close to the nominal level α. To get a measure of how accurate this estimate is we employ the framework introduced by Kupiec (1995). It is a likelihood ratio test and can be expressed as

$$LR_{uc} = -2\ln\left[L\left(\alpha; I_1, I_2, \ldots, I_T\right)/L\left(\pi; I_1, I_2, \ldots, I_T\right)\right] \sim \chi^2_{(1)}, \tag{22}$$

$$L\left(\alpha; I_1, I_2, \ldots, I_T\right) = \alpha^{n_1}\left(1-\alpha\right)^{n_0}, \; L\left(\pi; I_1, I_2, \ldots, I_T\right) = \pi^{n_1}\left(1-\pi\right)^{n_0},$$

where n_1 is the number of exceptions, n_0 the number of realizations not exceeding the VaR and π the Maximum Likelihood estimate of the probability of exception. If the number of exceptions deviates significantly from the nominal level α, the model cannot be considered sufficiently capturing the volatility in the index. If the number of exceptions is *too large* in relation to the nominal level, the model *underestimates* the risk; if it is *too low*, the model *overestimates* the risk. Both outcomes are evidence of a bad model, and both have negative implications for the investor. In both cases, the model fails to reveal future risk.

VaR is a very intuitive measure of risk. However, the evaluation only tells us whether the ratio of exceptions are in an acceptable range. This makes it difficult to compare models giving the same ratio of exceptions, but where their respective exceptions vary in magnitude.[2] Therefore, we include measures to address this problem. We compute Mean Square Error (MSE) and Root Mean Square Error (RMSE) for all series. They are used to measure how close the forecast of the conditional variance lies to the squared returns. Due to its construction, it cannot be used as an absolute measure, but only to compare models to each other. The MSE and RMSE are computed as

$$MSE = \frac{1}{T}\sum_{t=1}^{T}\left(\hat{\sigma}_t^2 - \sigma_t^2\right)^2 \tag{23}$$

and

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\sigma}_t^2 - \sigma_t^2\right)^2}, \tag{24}$$

respectively, where $\hat{\sigma}_t^2$ is approximated with r_t^2 and T is the number of out-of-sample forecasts.

Data

The data we are working with are exchange rates. The currencies to be evaluated are the U.S. Dollar, the Euro, the Great British Pound, the Japanese Yen and the Swedish Krona.[3] The data is daily and ranges from January 1st 2006 through June 30th 2011 and is the average between the bid and ask prices.

[2] One other thing that is important for an adequate model is that the exceptions are independent of past events. Christoffersen (1995) constructed a test for this purpose. However, this test has bad small sample properties and with confidence levels set as high as 95% or 99% it tends to reject the model too often.

[3] The data is available upon request. For details about the currencies, consult the References.

All in all, this gives us a total of 2,007 observations. A description of the distributional properties of the exchange rates are given in Table 1. Figure 1 illustrates the daily returns for SEK/USD.[4]

In this thesis we do 1-day-ahead and 10-days-ahead forecasts with a dynamic window of 1000 observations. This means that a forecast is made for time $t+h$ at time t, h being 1 or 10. When we stand on day $t+1$, the sample window encloses that observations, which has now been realized, and drops the last observation, $t-999$. This ensures that the window size remains the same. With this new window, a forecast is made for time $t+h+1$. This procedure is then repeated until the end of the forecasting period.

	Mean	St.D	Skewness	Kurtosis	Jarque-Bera
SEK/USD	-0.0001	0.0065	-0.0556	6.6206	1097.28
SEK/EUR	-1.04E-5	0.0033	0.3420	9.4591	3528.06
JPY/SEK	-7.31E-05	0.0082	-0.2558	8.9081	2940.86
GBP/USD	3.56E-05	0.0047	0.6968	11.9396	6845.47
GBP/EUR	0.0001	0.0038	0.2089	10.0386	4157.62
JPY/USD	-0.0001	0.0049	-0.2295	8.3602	2420.39
JPY/EUR	-8.99E-05	0.0062	-0.3844	12.1767	7091.80
USD/EUR	9.71E-05	0.0046	0.1212	8.0346	2124.61

Table 1: Descriptive statistics for the exchange rates.

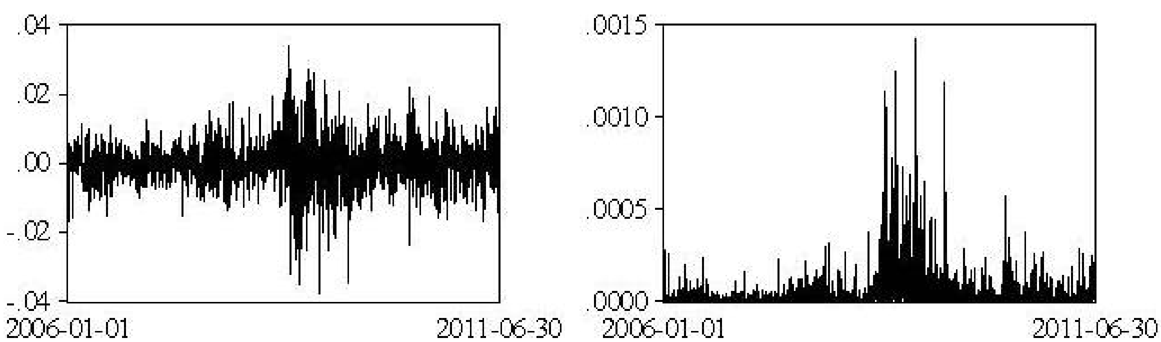

Figure 1: Daily and squared returns for SEK/USD

3 Results

Here we present the results from our estimations. Table 2 shows an extract of the best model for each level, 1% and 5%, and horizon, $h=1$ and $h=10$. Note that the table does not reveal *how* good the

[4]Graphical illustrations of the return series and squared returns can be found in the Appendix.

models are. Neither does it provide any information about how much better the chosen models are than the rest. However, it might give an indication about how the models perform.

	h=1		h=10	
	1%	5%	1%	5%
SEK/USD	EGARCH-GED	EGARCH-GED	GJR-GARCH-GED	EGARCH-GED
GBP/EUR	EGARCH-N	GJR-GARCH-N	EGARCH-N	GJR-GARCH-N
GBP/USD	GJR-GARCH-N	GJR-GARCH-N	GJR-GARCH-N	GJR-GARCH-N
JPY/USD	EGARCH-GED	GARCH-N	EGARCH-N	EGARCH-GED
SEK/EUR	GARCH-N	GJR-GARCH-GED	GARCH-N	GJR-GARCH-GED
SEK/JPY	GARCH-GED	GARCH-N	GARCH-t	GARCH-GED
USD/EUR	GJR-GARCH-N	GARCH-GED	GARCH-GED	GJR-GARCH-GED
JPY/EUR	EGARCH-GED	EGARCH-GED	GARCH-GED	GJR-GARCH-GED

Table 2: The best model for each exchange rate, level and horizon.

The GARCH, EGARCH and GJR-GARCH model all produce similar RMSE and MSE measures, and the small differences among the estimates do not provide much further information about the models' relative accuracy in forecasting conditional variance.

It is difficult to mark out one model that outperforms all other models. The GJR-GARCH and GARCH model often provide good estimates. Almost without exceptions, the three models with Student's t distributed innovations estimate the true risk poorly for both 1-day-ahead and 10-days-ahead VaR. In our estimations, the degrees of freedom turns out low, which generates larger terms to be multiplied with the standard deviations. This we think serves as a possible explanation for the bad estimation. A majority of the times, the estimates are rejected on the 1%-level.

Both GED and normally distributed errors works better with the models than those with t. These two seem to work good interchangebly. However, it seems as if GED errors more often produce better forecasts than the normal does, although in some cases, the opposite is true.

For the 1% level and 1-day-ahead, the EGARCH model generates an empirical coverage ratio closest to the nominal level. Out of those, three are with GED errors. However, GED qualifies just as often as the normal. Not once does the t distribution come out as the best one. That the EGARCH with GED works good is not much of a surprise. When Nelson (1992) introduced the model, he stressed that it follows the GED distribution.

When looking at the 5% level, still with $h = 1$, the results are more dubious. Both the GJR-GARCH and the GARCH are the best one three times each. Twice, they come with the normal distribution.

However, equally many times does the GED appear. For the EGARCH, it does so in both cases.

The GARCH model is the best in a majority of the times on the 1% level when $h = 10$. It is so with all distributions which makes it hard to distinguish one distribution that is more plausible than the others. All in all, the normal distribution appear most frequently.

When $h = 10$, on the 5% level the GJR-GARCH seems to be the best model. It appears five times as the best one and utilizes the GED three times and the normal twice. For the rest of the models, GED appears exclusively.

If we look at one exchange rate at a time, we can see that for GBP/USD the best model is the GJR-GARCH-N. For SEK/JPY, the best model is the GARCH, but it has different distributions. For the rest of the exchange rates, the results are more dubious. For SEK/USD the GED is the only distribution that seems appropriate, and three times it is so with the EGARCH model. Also for JPY/USD, the EGARCH appear three times, but here the normal distribution comes up twice.

The results in Table 2 are not conclusive. They do not mark out one specific model that outperforms all other. Rather, they recommend different ones for every combination of level and horizon. Moreover, which one that is the best model also depends on which exchange rate we look at. However, what can be said is that GED and Normal seem to be good distributional assumptions. When looking at the complete results in the tables presented in the Appendix, it can be seen that the GJR-GARCH-N model is rejected the least number of times, closely followed by the GARCH-N. This might be an indication that these models more often provide even and good results.

4 Conclusion

Our objective have been to identify a suitable model to forecast risk in a more volatile future than the data from which the model is estimated. The data is exchange rates constructed from various combinations of the U.S. Dollar, the Euro, the Great British Pound, the Japanese Yen and the Swedish Krona. The data ranges from January 1st 2006 through June 30th 2011, and thus covers the peak of the 2008 financial crisis, as well as the anxiety that striked markets in the aftermatch. We have analyzed three principal members belonging to the ARCH/GARCH class of models, and by means of these models, we computed VaR estimates. In our thesis, we find support for the GARCH and

GJR-GARCH model as adequate captures of the dynamics in the exchange rate series. Additionally, these models combined with the assumption of normally distributed innovations generates empirical coverage ratios of VaR estimates close to the nominal level. However, there are more to do. To apply the VaR concept on other models from the ARCH/GARCH family or from the field of high frequency MIDAS models, both with further distributional assumptions about the innovations, are examples of possible extensions of our analysis that future research can adopt.

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Data collected at http://www.oanda.com/lang/sv/currency/historical-rates/

A Appendix

Results

The computed estimates of VaR in the tables in this section show the ratio of number of exceptions to the total forecasting period. A model is considered good whenever this ratio is consistent with the nominal level, 1% or 5%, and the Kupiec test rejects the VaR if the realized and theoretical ratio differ too much from each other. Also presented are RMSE and MSE for all series.

		SEK/USD								
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0451	0.0180	0.0441	0.0531	0.0240	0.0511	0.0461	0.0251	0.0471
			***			***			***	
	1%VaR	0.0150	0.0010	0.0050	0.0160	0.0030	0.0100	0.0130	0.0020	0.0090
			***	*	*	***			***	
	RMSE	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
	MSE	1.69E-08	1.69E-08	1.71E-08	1.70E-08	1.79E-08	1.75E-08	1.64E-08	1.65E-08	1.64E-08
$h=10$	5%VaR	0.0461	0.0160	0.0421	0.0471	0.0481	0.0511	0.0481	0.0240	0.0521
			***						***	
	1%VaR	0.0130	0.0030	0.0160	0.0280	0.0170	0.0060	0.0130	0.0030	0.0090
			***		*	***	*		***	
	RMSE	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
	MSE	1.72E-08	1.74E-08	1.79E-08	1.74E-08	1.96E-08	1.81E-08	1.66E-08	1.67E-08	1.66E-08

Table 3: 5% and 1% VaR for SEK/USD. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

		GBP/EUR								
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0441	0.0170	0.0401	0.0291	0.0090	0.0371	0.0471	0.0220	0.0421
			***		***	***	**		***	
	1%VaR	0.0150	0.0010	0.0030	0.0090	0.0010	0.0040	0.0160	0.0010	0.0040
			***	***		***	**	*	***	**
	RMSE	5.97E-05	5.97E-05	6.02E-05	6.00E-05	6.08E-05	6.02E-05	6.02E-05	6.09E-05	6.14E-05
	MSE	3.56E-09	3.56E-09	3.62E-09	3.60E-09	3.70E-09	3.62E-09	3.63E-09	3.71E-09	3.77E-09
$h=10$	5%VaR	0.0351	0.0160	0.0351	0.0331	0.0421	0.0591	0.0481	0.0220	0.0301
		**	***	**	***				***	***
	1%VaR	0.0140	0.0000	0.0030	0.0090	0.0360	0.0340	0.0140	0.0010	0.0040
			***	***		***	***		***	**
	RMSE	5.94E-05	5.96E-05	6.09E-05	6.02E-05	6.53E-05	6.45E-05	6.02E-05	6.13E-05	6.26E-05
	MSE	3.52E-09	3.55E-09	3.71E-09	3.62E-09	4.27E-09	4.16E-09	3.63E-09	3.76E-09	3.92E-09

Table 4: 5% and 1% VaR for GBP/EUR. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

GBP/USD										
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0431	0.0150	0.0391	0.0467	0.0100	0.0291	0.0501	0.0220	0.0361
			***			***	***		***	**
	1%VaR	0.0080	0.0010	0.0020	0.0110	0.0020	0.0040	0.0100	0.0030	0.0030
			***	***		***	**		***	***
	RMSE	9.85E-05	9.87E-05	9.86E-05	0.0001	0.0001	0.0001	9.94E-05	0.0001	0.0001
	MSE	9.73E-09	9.74E-09	9.95E-09	1.00E-08	1.03E-08	1.04E-08	9.88E-09	1.00E-08	1.01E-08
$h=10$	5%VaR	0.0351	0.0120	0.0341	0.0417	0.1042	0.1242	0.0441	0.0261	0.0251
		**	***	**		***	***		***	***
	1%VaR	0.0110	0.0010	0.0020	0.0110	0.0020	0.0040	0.0110	0.0040	0.0040
			***	***		***	**		**	**
	RMSE	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
	MSE	1.05E-08	1.05E-08	1.12E-08	1.04E-08	1.04E-08	1.07E-08	1.01E-08	1.01E-08	1.05E-08

Table 5: 5% and 1% VaR for GBP/USD. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

JPY/USD										
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0501	0.0170	0.0471	0.0331	0.0140	0.0501	0.0501	0.0220	0.0421
			***		***	***			***	
	1%VaR	0.0150	0.0010	0.0030	0.0130	0.0020	0.0080	0.0150	0.0010	0.0020
			***	***		***			***	***
	RMSE	5.97E-05	5.97E-05	6.02E-05	8.18E-05	8.91E-05	8.17E-05	7.59E-05	7.66E-05	7.72E-05
	MSE	3.56E-09	3.56E-09	3.62E-09	6.69E-09	7.94E-09	6.68E-09	5.76E-09	5.86E-09	5.95E-09
$h=10$	5%VaR	0.0461	0.0110	0.0441	0.0341	0.1102	0.0501	0.0401	0.0220	0.0361
			***		**	***			***	**
	1%VaR	0.0140	0.0000	0.0030	0.0120	0.1032	0.0120	0.0210	0.0020	0.0050
			***	***		***		***	***	*
	RMSE	5.94E-05	5.96E-05	6.09E-05	8.37E-05	8.81E-05	8.50E-05	7.66E-05	7.64E-05	7.74E-05
	MSE	3.52E-09	3.55E-09	3.71E-09	7.01E-09	7.76E-09	7.22E-09	5.87E-09	5.84E-09	6.00E-09

Table 6: 5% and 1% VaR for JPY/USD. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

		SEK/EUR								
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0451	0.0120	0.0441	0.0671	0.0240	0.0621	0.0491	0.0180	0.0497
			***		**	***	*		***	
	1%VaR	0.0100	0.0000	0.0000	0.0090	0.0000	0.0050	0.0120	0.0010	0.0020
			***	***		***	*		***	***
	RMSE	4.31E-05	4.31E-05	4.37E-05	4.28E-05	4.42E-05	4.42E-05	4.40E-05	4.44E-05	4.50E-05
	MSE	1.85E-09	1.86E-09	1.91E-09	1.83E-09	1.95E-09	1.95E-09	1.93E-09	1.97E-09	2.03E-09
$h=10$	5%VaR	0.0451	0.0130	0.0431	0.0671	0.0481	0.0872	0.0461	0.0190	0.0506
			***			**	***		***	
	1%VaR	0.0100	0.0000	0.0000	0.0120	0.09120	0.0972	0.0100	0.0020	0.0020
			***	***		***	***		***	***
	RMSE	4.33E-05	4.35E-05	4.50E-05	4.40E-05	4.77E-05	4.75E-05	4.34E-05	4.38E-05	4.52E-05
	MSE	1.87E-09	1.89E-09	2.03E-09	1.94E-09	2.28E-09	2.26E-09	1.89E-09	1.92E-09	2.04E-09

Table 7: 5% and 1% VaR for SEK/EUR. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

		JPY/SEK								
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0551	0.0261	0.0551	0.0681	0.0401	0.0661	0.0661	0.0311	0.0631
			***		**		**	**	***	*
	1%VaR	0.0180	0.0030	0.0090	0.0160	0.0030	0.0080	0.0230	0.0050	0.0130
		**	***		*	***		***	*	
	RMSE	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001	0.0002	0.0002	0.0002
	MSE	5.97E-08	5.97E-08	6.06E-08	1.71E-08	1.74E-08	1.73E-08	5.69E-08	5.68E-08	5.74E-08
$h=10$	5%VaR	0.0531	0.0261	0.0521	0.0601	0.0401	0.0601	0.0671	0.0361	0.0571
			***					**	**	
	1%VaR	0.0200	0.0100	0.0140	0.0120	0.0040	0.0070	0.0250	0.0110	0.0170
		***				**		***		**
	RMSE	0.0003	0.0003	0.0003	0.0001	0.0001	0.0001	0.0003	0.0003	0.0003
	MSE	6.52E-08	6.51E-08	6.89E-08	1.75E-08	1.79E-08	1.76E-08	6.31E-08	6.35E-08	6.45E-08

Table 8: 5% and 1% VaR for JPY/SEK. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

		USD/EUR								
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0521	0.0200 ***	0.0491	0.0541	0.0220 ***	0.0451	0.0571	0.0200 ***	0.0531
	1%VaR	0.0170 **	0.0020 ***	0.0030 ***	0.0160 *	0.0010 ***	0.0040 **	0.0150	0.0010 ***	0.0020 ***
	RMSE	7.50E-05	7.52E-05	7.58E-05	7.50E-05	7.51E-05	7.54E-05	7.59E-05	7.66E-05	7.72E-05
	MSE	5.63E-09	5.65E-09	5.75E-09	5.63E-09	5.64E-09	5.69E-09	5.76E-09	5.86E-09	5.96E-09
$h=10$	5%VaR	0.0551	0.0210 ***	0.0531	0.0541	0.1413 ***	0.1573 ***	0.0581	0.0200 ***	0.0491
	1%VaR	0.0190 **	0.0040 ***	0.0060	0.0160 *	0.1343 ***	0.1333 ***	0.0210 ***	0.0020 ***	0.0050 *
	RMSE	7.78E-05	7.80E-05	7.98E-05	7.72E-05	8.05E-05	7.99E-05	7.66E-05	7.64E-05	7.74E-05
	MSE	6.05E-09	6.08E-09	6.37E-09	5.97E-09	6.48E-09	6.39E-09	5.87E-09	5.84E-09	6.00E-09

Table 9: 5% and 1% VaR for USD/EUR. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

		JPY/EUR								
		GARCH(1,1)			EGARCH(1,1)			GJR-GARCH(1,1)		
		N	t	GED	N	t	GED	N	t	GED
$h=1$	5%VaR	0.0641 **	0.0261 ***	0.0611	0.0661 **	0.0180 ***	0.0471	0.0641 **	0.0311 ***	0.0541
	1%VaR	0.0230 ***	0.0050 *	0.0070	0.0230 ***	0.0060	0.0100	0.0140	0.0010 ***	0.0030 ***
	RMSE	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	6.06E-05	6.13E-05	6.28E-05
	MSE	2.74E-08	2.73E-08	2.79E-08	2.63E-08	2.63E-08	2.62E-08	3.68E-09	3.76E-09	3.94E-09
$h=10$	5%VaR	0.0621 *	0.0291 ***	0.0581	0.0611	0.0240 ***	0.1062 ***	0.0701 ***	0.0351 **	0.0541
	1%VaR	0.0230 ***	0.050 *	0.0080	0.0280 ***	0.0321 ***	0.0411 ***	0.0150	0.0020 ***	0.0030 ***
	RMSE	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	6.12E-05	6.27E-05	6.81E-05
	MSE	3.21E-08	3.12E-08	3.38E-08	2.99E-08	3.01E-08	3.02E-08	3.74E-09	3.93E-09	4.64E-09

Table 10: 5% and 1% VaR for JPY/EUR. ***, ** and * indicates rejection of Kupiec's unconditional coverage test on 1%, 5% and 10%, respectively.

Graphics

Here we give graphs over the return series and squared returns. From the graphs it is clear that there were an increase in volatily in the midst of the period.

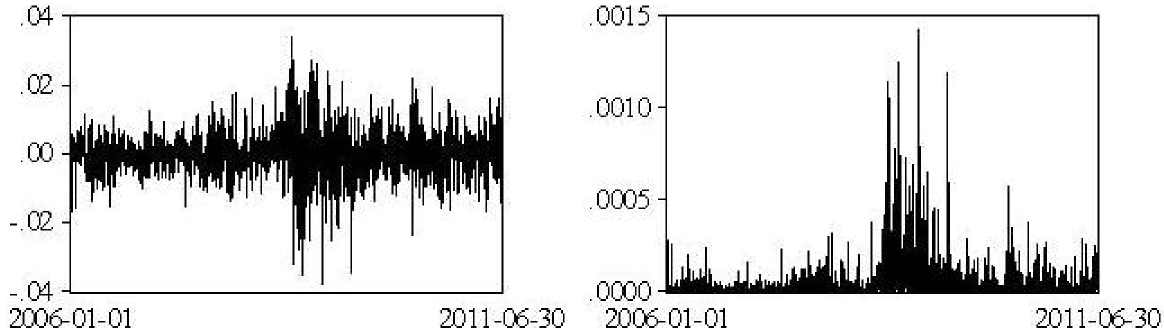

Figure 2: Daily and squared returns for SEK/USD

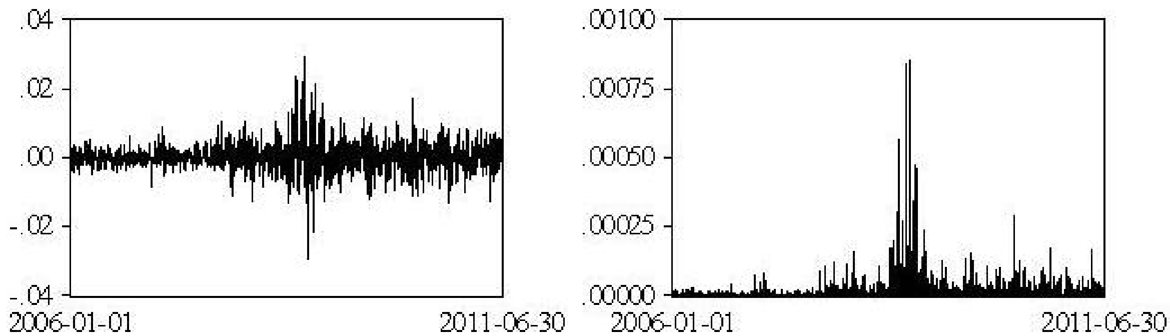

Figure 3: Daily and squared returns for GBP/EUR

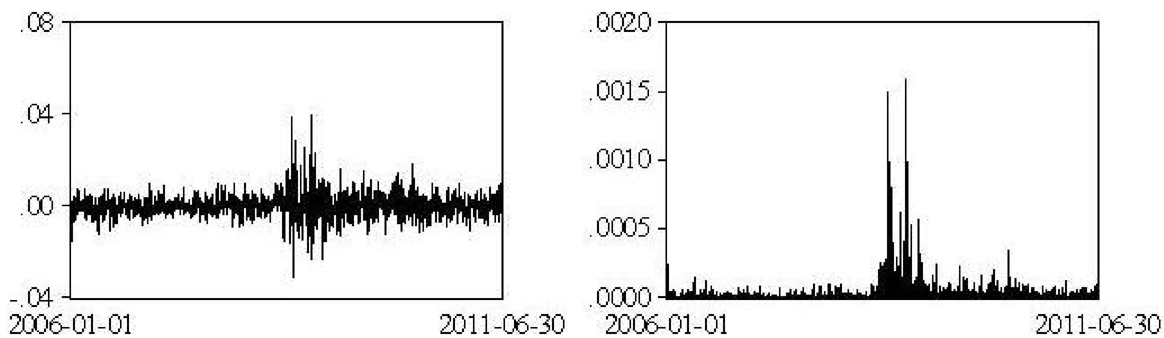

Figure 4: Daily and squared returns for GBP/USD

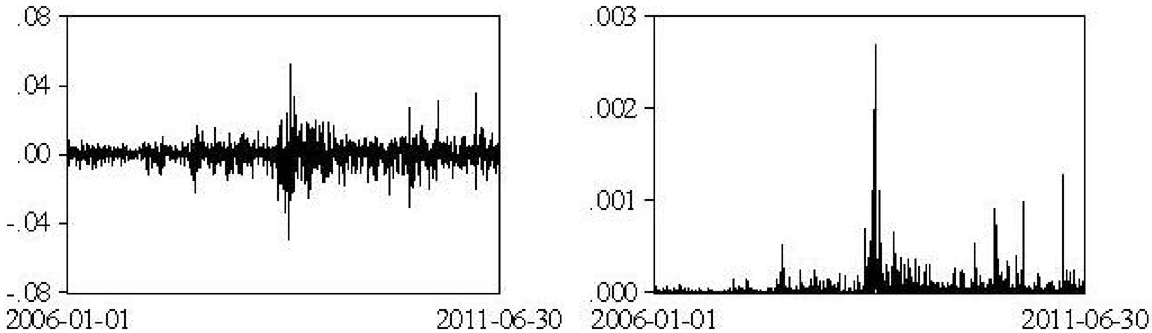

Figure 5: Daily and squared returns for JPY/EUR

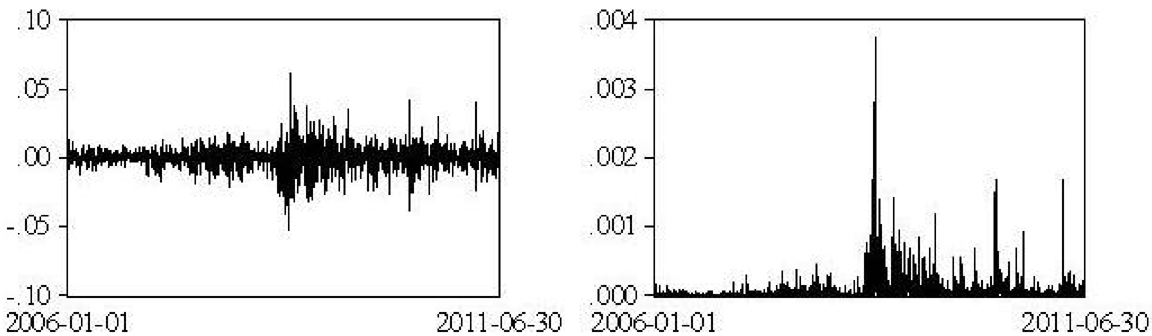

Figure 6: Daily and squared returns for JPY/SEK

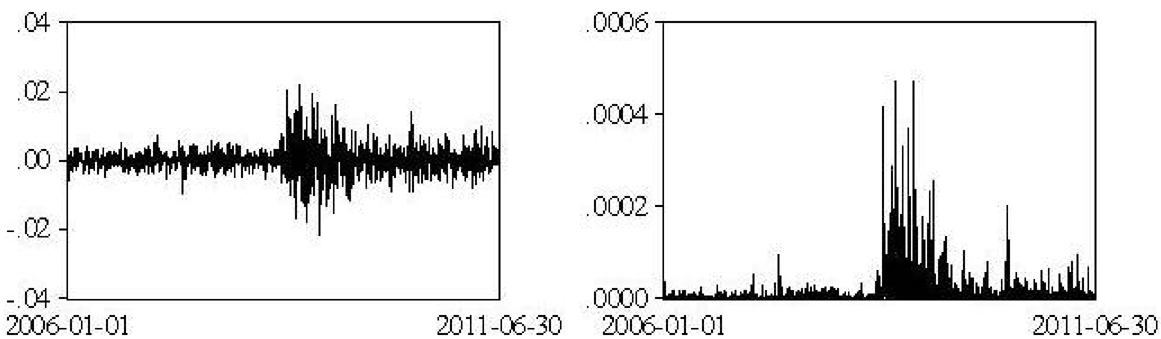

Figure 7: Daily and squared returns for SEK/EUR

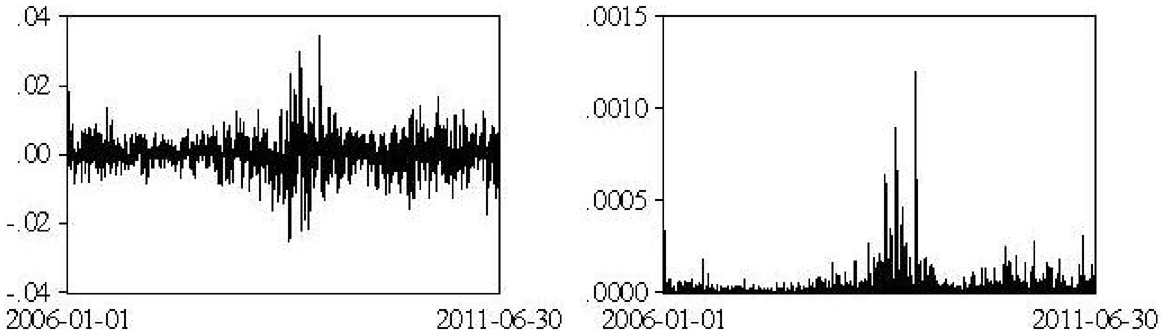

Figure 8: Daily and squared returns for USD/EUR

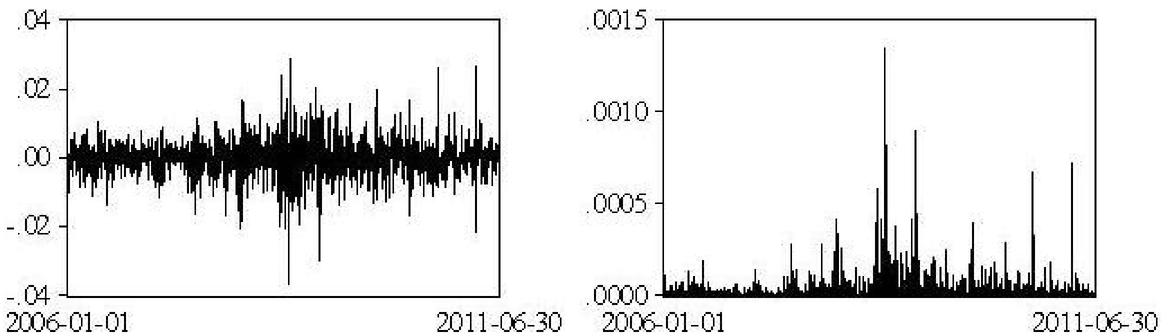

Figure 9: Daily and squared returns for JPY/USD

Proofs

In this subsection we give inductive proofs for the forecast equations with $h \geq 2$, which can be applied to our 10-days-ahead forecast equations.

Proposition 1: The 10-days-ahead forecast equation of the conditional variance for GARCH(1,1) is computed as

$$\sigma_{t+10}^2 = \sum_{j=0}^{8} \alpha_0 \left(\alpha_1 + \beta_1\right)^j + \left(\alpha_1 + \beta_1\right)^9 \sigma_{t+1}^2 .$$

Proof: For $t+2$

$$\begin{aligned} \sigma_{t+2}^2 &= \alpha_0 + \alpha_1 E\left[\varepsilon_{t+1}^2\right] + \beta_1 \sigma_{t+1}^2 \\ &= \alpha_0 + \left(\alpha_1 + \beta_1\right) \sigma_{t+1}^2 , \end{aligned}$$

because $E\left[\varepsilon_{t+1}^2\right] = \sigma_{t+1}^2$. For $t+3$

$$\begin{aligned}
\sigma_{t+3}^2 &= \alpha_0 + (\alpha_1 + \beta_1)\,\sigma_{t+2}^2 \\
&= \alpha_0 + (\alpha_1 + \beta_1)\left(\alpha_0 + (\alpha_1 + \beta_1)\,\sigma_{t+1}^2\right) \\
&= \alpha_0 + \alpha_0\,(\alpha_1 + \beta_1) + (\alpha_1 + \beta_1)^2\,\sigma_{t+1}^2,
\end{aligned}$$

for $t+4$

$$\begin{aligned}
\sigma_{t+4}^2 &= \alpha_0 + (\alpha_1 + \beta_1)\,\sigma_{t+3}^2 \\
&= \alpha_0 + (\alpha_1 + \beta_1)\left(\alpha_0 + \alpha_0\,(\alpha_1 + \beta_1) + (\alpha_1 + \beta_1)^2\,\sigma_{t+1}^2\right) \\
&= \alpha_0 + \alpha_0\,(\alpha_1 + \beta_1) + \alpha_0\,(\alpha_1 + \beta_1)^2 + (\alpha_1 + \beta_1)^3\,\sigma_{t+1}^2,
\end{aligned}$$

and now we can see that for any $h \geq 2$ we have

$$\begin{aligned}
\sigma_{t+h}^2 &= \alpha_0\,(\alpha_1 + \beta_1)^0 + \alpha_0\,(\alpha_1 + \beta_1)^1 + \alpha_0\,(\alpha_1 + \beta_1)^2 + \ldots + \\
&\quad + \alpha_0\,(\alpha_1 + \beta_1)^{h-2} + (\alpha_1 + \beta_1)^{h-1}\,\sigma_{t+1}^2 \\
&= \sum_{j=0}^{h-2} \alpha_0\,(\alpha_1 + \beta_1)^j + (\alpha_1 + \beta_1)^{h-1}\,\sigma_{t+1}^2,\ Q.E.D.
\end{aligned}$$

Proposition 2a: The 10-days-ahead forecast equation of the conditional variance for EGARCH(1,1) is computed as

$$\sigma_{t+10}^2 = \exp\left\{\sum_{j=0}^{8} \alpha_0 \beta^j + \sum_{j=0}^{8} \alpha_0 E\left[|z|\right]\beta^j + \beta^9 \ln \sigma_{t+1}^2\right\}.$$

Proof: For $t+2$

$$\begin{aligned}
\ln \sigma_{t+2}^2 &= \alpha_0 + \alpha_1 E\left[|z_{t+1}|\right] + \gamma_1 E\left[z_{t+1}\right] + \beta_1 \ln \sigma_{t+1}^2 \\
&= \alpha_0 + \alpha_1 E\left[|z_{t+1}|\right] + \beta_1 \ln \sigma_{t+1}^2,
\end{aligned}$$

as $E[z_{t+1}] = 0$. For $t+3$

$$\begin{aligned}
\ln \sigma_{t+3}^2 &= \alpha_0 + \alpha_1 E[|z_{t+2}|] + \beta_1 \ln \sigma_{t+2}^2 \\
&= \alpha_0 + \alpha_1 E[|z_{t+2}|] + \beta_1 \left(\alpha_0 + \alpha_1 E[|z_{t+1}|] + \beta_1 \ln \sigma_{t+1}^2\right) \\
&= \alpha_0 + \alpha_0 \beta_1 + \alpha_1 E[|z_{t+2}|] + \alpha_1 \beta_1 E[|z_{t+1}|] + \beta_1^2 \ln \sigma_{t+1}^2,
\end{aligned}$$

for $t+4$

$$\begin{aligned}
\ln \sigma_{t+4}^2 &= \alpha_0 + \alpha_1 E[|z_{t+3}|] + \beta_1 \ln \sigma_{t+3}^2 \\
&= \alpha_0 + \alpha_1 E[|z_{t+3}|] + \beta_1 \left(\alpha_0 + \alpha_0 \beta + \alpha_1 E[|z_{t+2}|] + \alpha_1 \beta_1 E[|z_{t+1}|] + \beta_1^2 \ln \sigma_{t+1}^2\right) \\
&= \alpha_0 + \alpha_0 \beta_1 + \alpha_0 \beta_1^2 + \alpha_1 E[|z_{t+3}|] + \alpha_1 \beta_1 E[|z_{t+2}|] + \alpha_1 \beta_1^2 E[|z_{t+1}|] + \beta_1^3 \ln \sigma_{t+1}^2.
\end{aligned}$$

Now we can draw the conclusion that for any $h \geq 2$

$$\begin{aligned}
\ln \sigma_{t+h}^2 &= \alpha_0 \beta_1^0 + \alpha_0 \beta_1^1 + \ldots + \alpha_0 \beta_1^{h-2} + \alpha_1 \beta_1^0 E[|z_{t+h-1}|] + \alpha_1 \beta_1^1 E[|z_{t+h-2}|] + \ldots + \\
&\quad + \alpha_1 \beta_1^{h-2} E[|z_{t+1}|] + \beta^{h-1} \ln \sigma_{t+1}^2 \\
&= \sum_{j=0}^{h-2} \alpha_0 \beta^j + \sum_{j=0}^{h-2} \alpha_1 E[|z|] \beta^j + \beta^{h-1} \ln \sigma_{t+1}^2.
\end{aligned}$$

That $E[|z_t|] = E[|z|]$ follows from the fact that z_t is independently and identically distributed. We solve the last part by taking the anti-logarithm,

$$\sigma_{t+h}^2 = \exp\left\{\sum_{j=0}^{h-2} \alpha_0 \beta^j + \sum_{j=0}^{h-2} E[|z|] \beta^j + \beta^{h-1} \ln \sigma_{t+1}^2\right\}, \; Q.E.D.$$

Proposition 2b: If $z \sim N(0,1)$, then $E[|z|] = \sqrt{\frac{2}{\pi}}$.

Proof: Calculus yields

$$\begin{aligned} E\left[|z|\right] &= 2\int_0^{\infty} \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\ &= \frac{2}{\sqrt{2\pi}} \left[\exp\left\{-\frac{z^2}{2}\right\}\right]_{\infty}^{0} \\ &= \frac{2}{\sqrt{2\pi}}, \end{aligned}$$

and thus $E\left[|z|\right] = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$, $Q.E.D.$ The first equality follows from the symmetric property of the normal distribution.

Proposition 3: The 10-days-ahead forecast equation of the conditional variance for GJR-GARCH(1,1) is computed as

$$\sigma_{t+10}^2 = \sum_{j=0}^{8} \alpha_0 \delta^j + \delta^9 \sigma_{t+1}^2.$$

Proof: For $t+2$

$$\begin{aligned} \sigma_{t+2}^2 &= \alpha_0 + \alpha_1 E\left[\varepsilon_{t+1}^2\right] + \gamma_1 E\left[\varepsilon_{t+1}^2 I_{t+1}\right] + \beta_1 \sigma_{t+1}^2 \\ &= \alpha_0 + \alpha_1 \sigma_{t+1}^2 + \frac{\gamma_1}{2}\sigma_{t+1}^2 + \beta_1 \sigma_{t+1}^2 \\ &= \alpha_0 + \left(\alpha_1 + \frac{\gamma_1}{2} + \beta_1\right)\sigma_{t+1}^2 \\ &= \alpha_0 + \delta \sigma_{t+1}^2, \end{aligned}$$

where $\delta = \alpha_1 + \frac{\gamma_1}{2} + \beta_1$. That $E\left[\varepsilon_{t+1}^2 I_{t+1}\right] = \sigma_{t+1}^2/2$ follows from the fact that

$$\begin{aligned} E\left[\varepsilon_{t+1}^2 I_{t+1}\right] &= E\left[\varepsilon^2 | I^{(0)}\right] + E\left[\varepsilon^2 | I^{(1)}\right] \\ &= 0 \times \int_0^{\infty} \varepsilon^2 f\left(\varepsilon\right) d\varepsilon + 1 \times \int_{-\infty}^{0} \varepsilon^2 f\left(\varepsilon\right) d\varepsilon \\ &= \int_0^{\infty} \varepsilon^2 f\left(\varepsilon\right) d\varepsilon \\ &= \frac{1}{2} \times \sigma^2, \end{aligned}$$

where the third equality is due to the symmetric property of the Normal, the t and Generalized Error

Distribution and the last equality because they all have expectation zero. For $t+3$

$$\begin{aligned}\sigma_{t+3}^2 &= \alpha_0 + \delta\sigma_{t+2}^2 \\ &= \alpha_0 + \delta\left(\alpha_0 + \delta\sigma_{t+1}^2\right) \\ &= \alpha_0 + \alpha_0\delta + \delta^2\sigma_{t+1}^2,\end{aligned}$$

for $t+4$

$$\begin{aligned}\sigma_{t+4}^2 &= \alpha_0 + \delta\sigma_{t+3}^2 \\ &= \alpha_0 + \delta\left(\alpha_0 + \alpha_0\delta + \delta^2\sigma_{t+1}^2\right) \\ &= \alpha_0 + \alpha_0\delta + \alpha_0\delta^2 + \delta^3\sigma_{t+1}^2.\end{aligned}$$

Now we can see that for any $h \geq 2$

$$\begin{aligned}\sigma_{t+h}^2 &= \alpha_0\delta^0 + \alpha_0\delta^1 + \alpha_0\delta^2 + \ldots + \alpha_0\delta^{h-2} + \delta^{h-1}\sigma_{t+1}^2 \\ &= \sum_{j=0}^{h-2}\alpha_0\delta^j + \delta^{h-1}\sigma_{t+1}^2,\ Q.E.D.\end{aligned}$$