The restrained geodetic number of a graph

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Abstract

A geodetic set $S \subseteq V(G)$ of a graph $G = (V, E)$ is a *restrained geodetic set* if the subgraph $G[V \setminus S]$ has no isolated vertex. The minimum cardinality of a restrained geodetic set is the *restrained geodetic number*. In this paper, we initiate the study of the restrained geodetic number.

Key Words: geodetic set, restrained geodetic set, restrained geodetic number

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1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ and size $|E|$ of G are denoted $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degrees* of a graph G are denoted

$\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The *complement* $\overline{G}$ of G is the simple graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G. We write K_n for the *complete graph* of order n, C_n for a *cycle* of order n and P_n for a *path* of length $n-1$. For terminology and notation on graph theory not given here the reader is referred to [**?**].

Let $d(u, v)$ denote the minimum length of a path from vertex u to vertex v. An x-y path of length $d(x, y)$ is called an *x-y geodesic*. A vertex v is said to *lie* on an x-y geodesic P if v is an internal vertex of P. The *closed interval* $I[x, y]$ consists of x, y and all vertices lying in some x-y geodesic of G, while for $S \subseteq V(G)$, $I[S] = \cup_{x,y \in S} I[x, y]$.

A set S of vertices is a *geodetic set* if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the *geodetic number* of G, and is denoted $g(G)$. A geodetic set of cardinality $g(G)$ is called a $g(G)$-set. The geodetic number was introduced in [**?**] and has been studied by several authors (see for example [**?**, **?**, **?**, **?**, **?**]).

A set of vertices S in a graph G is a *restrained geodetic set* (RGS) if S is a geodetic set and the subgraph $G[V \setminus S]$ induced by $V \setminus S$ has no isolated vertex. The minimum cardinality of a restrained geodetic set, denoted $g_r(G)$, is called the *restrained geodetic number* of G. A $g_r(G)$-set is a restrained geodetic set of cardinality $g_r(G)$. As the assumption $\delta(G) \geq 1$ is necessary, we always assume that when we discuss $g_r(G)$, all graphs involved satisfy $\delta(G) \geq 1$. Since each restrained geodetic set is a geodetic set, and since the complement of each restrained geodetic set has cardinality different from 1, we have

$$2 \leq g(G) \leq g_r(G) \leq n, \tag{1}$$

$$g_r(G) \neq n - 1. \tag{2}$$

A vertex of G is *simplicial* if the subgraph induced by its neighbors is a complete graph. Note that every end-vertex is simplicial. We make use of the following results in this paper.

Proposition A. ([**?**]) If G is a connected nontrivial graph, then every simplicial vertex belongs to every geodetic set.

Observation 1. Let G be a connected graph of order $n \geq 3$. Then $g_r(G) = 2$ if and only if $g(G) = 2$ and $\operatorname{diam}(G) \geq 3$ or $G = \overline{K}_2 \vee H$, where $\delta(H) \geq 1$.

Proof. First assume $g(G) = 2$. If $\operatorname{diam}(G) \geq 3$, then obviously every $g(G)$-set is a restrained geodetic set and hence $g_r(G) = 2$. If $G = \overline{K}_2 \vee H$ and $\delta(H) \geq 1$, then clearly $V(\overline{K_2})$ is a restrained geodetic set of size 2 and it follows from (**??**) that $g_r(G) = 2$.

Now assume $g_r(G) = 2$. Then $g(G) = 2$ by (**??**). If $\operatorname{diam}(G) \geq 3$, then we are done. Suppose $\operatorname{diam}(G) = 2$ and let $\{u, v\}$ be a $g_r(G)$-set and $H = G[V(G) \setminus \{u, v\}]$. Clearly, $d(u, v) = 2$ and $\delta(H) \geq 1$. Since every vertex $x \in V(G) \setminus \{u, v\}$ lies on a u-v geodetic path, we must have $ux, vx \in E(G)$ for each $x \in V(G) \setminus \{u, v\}$. Thus $G = \overline{K}_2 \vee H$. This completes the proof. □

It is well known that [**?**]:

(a) $g(K_m) = m$, $m \geq 2$,

(b) for $r \geq 2$, $g(C_{2r}) = 2$ and $g(C_{2r+1}) = 3$,

(c) if $m \geq n \geq 2$, then $g(K_{m,n}) = \min\{4, n\}$,

(d) $g(Q_k) = 2$ for $k \geq 2$, where Q_k is the k-dimensional cube, and

(e) $g(W_{n+1}) = \lfloor \frac{n+1}{2} \rfloor$ for $n \geq 4$, where $W_{n+1} = C_n \vee K_1$.

Observation 2. (i) For any tree T different from star, $g_r(T) = |L(T)|$, where $L(T)$ is the set of all leaves of T. Moreover, $g_r(K_{1,r}) = r + 1$.

(ii) For $m \geq 2$, $g_r(K_m) = m$.

(iii) For $n \geq 3$, $g_r(C_{2n}) = 2$ and $g_r(C_{2n+1}) = 3$, and $g_r(C_m) = m$ for $m \in \{3, 4, 5\}$.

(iv) Let $m \geq n \geq 2$ be integers. Then $g_r(K_{2,m}) = m + 2$ and $g_r(K_{m,n}) = 4$ if $n \geq 3$.

(v) For $k \geq 3$, $g_r(Q_k) = 2$.

(vi) For $n \geq 4$, $g_r(W_{n+1}) = \lfloor \frac{n+1}{2} \rfloor$.

(vii) $g_r(K_1 \vee (K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_r})) = 1 + m_1 + \cdots + m_r$ where $r \geq 1$ and $m_i \geq 1$ for $1 \leq i \leq r$.

Proof. (i) It is well known [**?**] that $L(T)$ is the unique $g(T)$-set which is also a restrained geodetic set of G. It follows from (**??**) that $g_r(T) = |L(T)|$.

(ii) It is an immediate consequence of (**??**) and (a).

(iii) It is easy to see that $g_r(C_m) = m$ for $m \in \{3, 4, 5\}$. Let $n \geq 3$. Then $\{x_1, x_{n+1}\}$ is a RGS of C_{2n} and $\{x_1, x_{n+1}, x_{n+2}\}$ is a RGS of C_{2n+1}. Thus $g_r(C_{2n}) \geq 2$ and $g_r(C_{2n+1}) \geq 3$. Now the result follows from (b).

(iv) Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be the partite sets of $K_{m,n}$. If $n \geq 3$, then clearly $\{x_1, x_2, y_1, y_2\}$ is a RGS of $K_{m,n}$ and so $g_r(K_{m,n}) \leq 4$. If $n \geq 4$, then it follows from (**??**) and (c) that $g_r(K_{m,n}) = 4$. Let $n = 3$ and suppose to the contrary that $g_r(K_{m,3}) < 4$. It follows from (c) that $g_r(K_{m,3}) = 3$. Assume S be a $g_r(K_{m,3})$-set. Since X is not a RGS of $K_{m,3}$, we must have $X - S \neq \emptyset$. We may assume without loss of generality that $x_3 \in X \setminus S$. This implies that $|S \cap Y| \geq 2$. A similar argument shows that $|S \cap X| \geq 2$, implying that $|S| \geq 4$, a contradiction. Thus $g_r(K_{m,3}) = 4$.

The proof of $g_r(K_{m,2}) = m + 2$ is straightforward and we leave it to the reader.

(v) It is easy to see that $\{(0, \ldots, 0), (1, \ldots, 1)\}$ is a $g_r(Q_k)$-set for $k \geq 3$. Hence, $g_r(Q_k) = 2$.

(vi) Let $C_n = (x_1, x_2, \ldots, x_n)$ and $W_{n+1} = C_n \vee K_1$. Then obviously $\{x_i \mid i \text{ is odd}\}$ is a RGS of W_{n+1} and so $g_r(W_{n+1}) \geq \lfloor \frac{n+1}{2} \rfloor$. Now the result follows from (e).

(vii) Let $V(K_1) = \{x\}, V(K_{m_1}) = \{x_1^1, \ldots, x_{m_1}^1\}, \ldots, V(K_{m_r}) = \{x_1^r, \ldots, x_{m_r}^r\}$. Then for each pair y, z of vertices, $I[y, z] = \{y, z\}$ or $I[y, z] = \{x, y, z\}$. Thus $V(G)$ is the unique RGS of G and hence $g_r(G) = |V(G)|$. $\square$

2 Graphs with large restrained geodetic numbers

If G is a connected graph of order $n \geq 2$, then $g_r(G) \leq n$ by (**??**). In this section, we characterize the graphs achieving this bound.

Lemma 3. If G is a connected graph of order n with $\mathrm{diam}(G) \geq 3$, then $g_r(G) \leq n - \mathrm{diam}(G) + 1$.

Proof. Let $d = \mathrm{diam}(G)$ and let $P = x_0 x_1 \ldots x_d$ be a diametral path in G. Then obviously $V(G) \setminus \{x_1, \ldots, x_{d-1}\}$ is a restrained geodetic set of G, hence $g_r(G) \leq n - \mathrm{diam}(G) + 1$. $\square$

The next corollary is an immediate consequence of Lemma **??**.

Corollary 4. If G is a connected graph of order $n \geq 2$ with $g_r(G) = n$, then $\operatorname{diam}(G) \leq 2$.

Lemma 5. Let G be a connected graph of order n with $\operatorname{diam}(G) \leq 2$. If $K_4 - e$ is an induced subgraph of G, then $g_r(G) \leq n - 2$.

Proof. Let $K_4 - e$ be an induced subgraph of G with $V(K_4 - e) = \{v_1, v_2, v_3, v_4\}$ and $E(K_4 - e) = \{v_1v_2, v_2v_3, v_3v_4, v_1v_4, v_1v_3\}$. Then $d(v_2, v_4) = 2$ and v_i for $i = 1, 3$ lies on the v_2-v_4 geodetic path $v_2v_iv_4$. Thus $V(G) \setminus \{v_1, v_3\}$ is a restrained geodetic set of G, hence $g_r(G) \leq n - 2$. □

Lemma 6. Let G be a connected graph of order n with $\operatorname{diam}(G) \leq 2$ and $g_r(G) = n$. If G has a cut vertex v, then all components of $G - v$ are complete graphs.

Proof. Let $G_1, \ldots, G_k$ be the components of $G - v$. Since $\operatorname{diam}(G) \leq 2$, each vertex of $G - v$ must be adjacent to v. Assume to the contrary that G_1 is not a complete graph. Then there are three vertices v_1, v_2, v_3 in G such that $v_1v_2, v_1v_3 \in E(G)$ and $v_2v_3 \notin E(G)$. Then the set $\{v, v_1, v_2, v_3\}$ induces the subgraph $K_4 - e$ in G, which leads to a contradiction by Lemma **??**. □

Theorem 7. Let G be a connected graph of order $n \geq 2$. Then $g_r(G) = n$ if and only if one of the following holds:

(i) $G = K_n$;

(ii) $G = K_{2,n-2}$, $n \geq 3$;

(iii) $G = K_1 \vee (K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_r})$, where $r \geq 1$, $m_i \geq 1$ for $1 \leq i \leq r$ and $1 + m_1 + \cdots + m_r = n$;

(iv) G is the graph obtained from $K_{2,n-3}$, $n \geq 4$, by subdividing an edge once.

Proof. Sufficiency: Obvious.
Necessity: Let $g_r(G) = n$. By Corollary **??**, $\operatorname{diam}(G) \leq 2$ and hence the girth of G is at most 5. If $\operatorname{diam}(G) = 1$, then $G = K_n$ and we are done. Let $\operatorname{diam}(G) = 2$. If G is a tree, then it follows from Observation **??**(i) that G is a star (hence satisfies (iii)) and we are done. So let G have a cycle. We consider three cases.
Case 1 The girth of G is 5.
Let $C = (v_1, v_2, v_3, v_4, v_5)$ be a cycle of order 5 in G. If $n = 5$, then $G = C_5$ and satisfies (iv). Let $n \geq 6$. Clearly, each vertex in $V(G) \setminus V(C)$ has at most one neighbor in $V(C)$. Since G is connected, there is a vertex such as x_1 that is adjacent to some vertex in $V(C)$. Assume without loss of generality that $v_1x_1 \in E(G)$. Since $\operatorname{diam}(G) = 2$, x_1 must have a common neighbor with v_3 and v_4. Suppose $x_2 \in N(x_1) \cap N(v_3)$ and $x_3 \in N(x_1) \cap N(v_4)$. Note that v_1 lies on the v_2-v_5 geodetic path $v_2v_1v_5$ and x_1 lies on the x_2-x_3 geodetic path $x_2x_1x_3$. Hence, $V(G) \setminus \{x_1, v_1\}$ is a restrained geodetic set of G, a contradiction.
Case 2 The girth of G is 4.
We claim that there are not two adjacent vertices each of degree at least 3. Assume to the contrary that u and v are two adjacent vertices each of degree at least 3 and let $u_1, u_2 \in N(u) \setminus \{v\}$ and $v_1, v_2 \in N(v) \setminus \{u\}$. Since G is triangle-free, $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$. Then u lies on the u_1-u_2 geodetic path u_1uu_2 and v lies on the v_1-v_2 geodetic path v_1vv_2, which implies that $V(G) \setminus \{u, v\}$ is a restrained geodetic set of G, a contradiction.

Let now $C = (v_1, v_2, v_3, v_4)$ be a cycle of order 4 in G. If $n = 4$, then $G = K_{2,2}$ and satisfies (ii). Let $n \geq 5$. Since G is connected, there is a vertex, say x_1, adjacent to some vertex in $V(C)$. Assume without loss of generality that $v_1x_1 \in E(G)$. Since $d(x_1, v_3) \leq 2$, x_1 must be adjacent to v_3 or has a common neighbor with v_3, say y_1. Since G has no two adjacent vertices each of degree at least 3, if $x_1v_3 \in E(G)$, then $\deg(v_2) = \deg(v_4) = \deg(x_1) = 2$ and if $y_1 \in N(x_1) \cap N(v_3)$, then $\deg(v_2) = \deg(v_4) = \deg(x_1) = \deg(y_1) = 2$. Hence, either $G = K_{2,n-2}$ and satisfies (ii), or G is obtained from $K_{2,p}$, $3 \leq p \leq n-2$, with partite sets $\{v_1, v_3\}$ and $V(G) \setminus \{v_1, v_3\}$, by subdividing $n-p-2$ edges incident to (without loss of generality) v_3. If y_1 and y_2 are subdivision vertices, then $V(G) \setminus \{y_1, v_3\}$ is a restrained geodetic set of G, a contradiction. Hence, only one edge of G is subdivided and G satisfies (iv).

Case 3 The girth of G is 3.
Let H be the largest clique of G and let $V(H) = \{v_1, \ldots, v_r\}$. Clearly, $r \geq 3$. Since G is not a complete graph, $V(G) \setminus V(H) \neq \emptyset$. First let there exist a vertex $x \in V(G) \setminus V(H)$ such that $|N(x) \cap V(H)| \geq 2$. Assume without loss of generality that $v_1, v_2 \in N(x) \cap V(H)$. By the choice of H, $V(H) \not\subseteq N(x)$. Let $v_3 \notin N(x)$. Then obviously $V(G) \setminus \{v_1, v_2\}$ is a restrained geodetic set of G. This contradicts the fact that $g_r(G) = n$ by assumption.

Thus each vertex in $V(G) \setminus V(H)$ has at most one neighbor in $V(H)$. If there exist two vertices $x, y \in V(G) \setminus V(H)$ with distinct neighbors $v_i, v_j \in V(H)$, then clearly $V(G) \setminus \{v_i, v_j\}$ is a restrained geodetic set of G, a contradiction. If there is a vertex $z \in V(G) \setminus V(H)$ at distance two from $V(H)$, then we must have $N(z) \cap N(v_1) \neq \emptyset$ and $N(z) \cap N(v_2) \neq \emptyset$ because $\mathrm{diam}(G) = 2$ and this leads to a contradiction as above. Therefore all vertices in $V(G) \setminus V(H)$ are adjacent to one vertex of $V(H)$, say v_1. Then v_1 is a cut vertex of G. Let $G_1, \ldots, G_k$ be the components of $G - v_1$. Suppose G_r is not a complete graph for some $1 \leq r \leq k$. Then G_r has a path $z_1z_2z_3$ and $V(G) \setminus \{v_1, z_2\}$ is a restrained geodetic set of G, a contradiction. Thus $G = K_1 \vee (K_{|V(G_1)|} \cup K_{|V(G_2)|} \cup \cdots \cup K_{|V(G_k)|})$ and the result follows. This completes the proof. □

Theorem 8. An ordered triple (a, b, c) of positive integers is realizable as the geodetic number, the restrained geodetic number and the order of some nontrivial connected graph, respectively, if and only if one of the following holds:

1. $(a, b, c) = (3, 5, 5)$;
2. $(a, b, c) \in A = \cup_{i \geq 2}\{(i, i, i), (2, i+1, i+1), (i, i+1, i+1), (4, i+4, i+4)\}$;
3. $2 \leq a \leq b \leq c - 2$.

Proof. First let (a, b, c) be realizable as the geodetic number, the restrained geodetic number and the order of some nontrivial connected graph G. Then we must have $2 \leq a \leq b \leq c$ and $c \neq b + 1$ by (**??**). If $b = c$, then it follows from Theorem **??** that (a, b, c) satisfies Condition 1 or Condition 2. If $b \leq c - 2$, then (a, b, c) satisfies Condition 3.

Conversely, let the ordered triple (a, b, c) of positive integers satisfies one of the Conditions 1, 2 or 3. We consider five cases.

Case 1. $2 \leq a \leq b = c$. A graph G of order n has $g_r(G) = n$ if and only if G is one of the graphs stated in Theorem **??**. Since $g(K_n) = n$, (n, n, n) is realizable for $n \geq 2$. Since $g(K_{2,n-2}) = 2$, $(2, n, n)$ is realizable when $n \geq 3$. It is easy to see that $g(K_1 \vee (K_{m_1} \cup \cdots \cup K_{m_r})) = \sum_{i=1}^{r} m_i$ if $r \geq 2$. Hence, $(n-1, n, n)$ is also realizable for $n \geq 3$. Now consider the graph $K_{2,n-3}$ whose partite sets are $\{x_1, x_2\}$ and $\{y_1, y_2, \ldots, y_{n-3}\}$, where $n \geq 4$. Let

G be the graph obtained from $K_{2,n-3}$ by subdividing the edge x_1y_1 with a new vertex z. (This is item (iv) of Theorem **??**.) If $n = 5$, then $G = C_5$ and $(3, 5, 5)$ is realizable. If $n \geq 6$, then $\{x_1, x_2, y_1, z\}$ is a minimum geodetic set of G. Thus $(4, n, n)$, $n \geq 6$, is realizable.

Case 2. $2 \leq a = b \leq c - 2$. Let G be the graph obtained by identifying a vertex in K_a with an end-vertex of P_{c-a+1}. Assume S is the set of all simplicial vertices in G. Clearly, S is a restrained geodetic set of G. Now the result follows by Proposition **??**.

Case 3. $2 = a < b \leq c - 2$. Let $G = \overline{K}_2 \vee (\overline{K}_{b-2} \cup K_{c-b})$. Clearly, $V(\overline{K}_2)$ is the unique $g(G)$-set and $V(\overline{K}_{b-2}) \cup V(\overline{K}_2)$ is the unique $g_r(G)$-set and the result follows.

Case 4. $3 \leq a < b \leq c - 3$. Let $H_1 = K_{2,b-a+1}$ be the complete bipartite graph with bipartite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, \ldots, y_{b-a+1}\}$, and let $H_2 = K_{a-1}$. Let $z \in V(H_2)$ and let G be the graph obtained from $H_1 \cup H_2$ by adding a path $u_1, \ldots, u_k$, $k \geq 1$, and the edges x_1u_1, x_2u_1, u_kz. Note that $S = V(H_2) \setminus \{z\}$ is the set of all simplicial vertices of G. Clearly, $S \cup X$ is the unique $g(G)$-set and $S \cup Y$ is the unique $g_r(G)$-set. Thus $g(G) = a$, $g_r(G) = b$, and $|V(G)| = b + 2 + k$, $k \geq 1$.

Case 5. $3 \leq a < b = c - 2$.

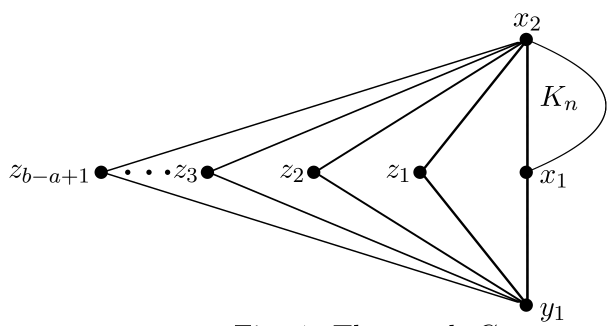

Fig. 1: The graph $G_{a,b}$.

Let $H_1 = K_a$ with vertex set $\{x_1, \ldots, x_a\}$ and let $H_2 = K_{1,b-a+1}$ with $V(H_2) = \{y_1\} \cup \{z_1, \ldots, z_{b-a+1}\}$, where y_1 is its central vertex. Let $G_{a,b}$ be the graph obtained from $H_1 \cup H_2$ by adding the edges x_1y_1 and x_2z_i for $1 \leq i \leq b - a + 1$ (see Figure 1). Clearly, $S = V(H_1) \setminus \{x_1, x_2\}$ is the set of all simplicial vertices of $G_{a,b}$. It is easy to see that $S \cup \{x_2, y_1\}$ is the unique $g(G_{a,b})$-set and $V(G_{a,b}) \setminus \{x_1, x_2\}$ is the unique $g_r(G_{a,b})$-set. □

Theorem 9. Let n, d and k be integers such that $3 \leq d < n$, $2 \leq k < n$, and $n - d - k + 1 \geq 0$. Then there exists a graph G of order n, diameter d and $g_r(G) = k$.

Proof. Let G be the graph obtained from a path $P_d = u_0u_1 \ldots u_d$ of length d by adding $k - 2$ pendant edges $u_1v_1, \ldots, u_1v_{k-2}$ and $n - d - k + 1$ new vertices $w_1, \ldots, w_{n-k-d+1}$ and joining w_i to u_0, u_2 for each $1 \leq i \leq n - k - d + 1$. Clearly, G has order n and diameter d. By Proposition **??**, the set $\{u_0, u_d, v_1, \ldots, v_{k-2}\}$ is the unique minimum restrained geodetic set of G and so G has the desired property. □

It is well-known that for every connected graph G, $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\ \mathrm{rad}(G)$. Chartrand et al. [**?**] showed that every three positive integers r, d, and $k \geq 2$ with $r \leq d \leq 2r$ are realizable as the radius, the diameter and the geodetic number, respectively. Their theorem can be extended so that restrained geodetic number instead of the geodetic number, can be prescribed as well.

Theorem 10. For positive integers r, d, and $p \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $\mathrm{rad}(G) = r$, $\mathrm{diam}(G) = d$ and $g_r(G) = p$.

Proof. If $r = 1$, then we let $G = K_p$ or $G = K_2 \vee (K_{p-1} \cup K_1)$ according to whether $d = 1$ or 2, respectively. Let $r \geq 2$. If $r = d = 2$, then we let $G = \overline{K}_2 \vee (K_2 \cup K_2), G = (K_1 \cup K_2) \vee (K_2 \cup K_2)$ or $G = K_{2,p-2}$ according to $p = 2, 3$ or $p \geq 4$, respectively. If $r = d \geq 3$ and $p = 2$, then we let $G = C_{2r}$ ($r \geq 3$).

Now assume $r = d \geq 3$ and $p \geq 3$. Let G be the graph obtained from disjoint union of a cycle $C_{2r} = (x_1, x_2, \ldots, x_{2r})$ of order $2r$ and the complete graph K_{p-2} by joining x_1 and x_{2r} to all vertices of K_{p-2}. Clearly, $\mathrm{rad}(G) = \mathrm{diam}(G) = r$. Since the vertices of K_{p-2} lie on no geodetic path, each restrained geodetic set of G must contain $V(K_{p-2})$. Since also $I[V(K_{p-2}) \cup \{x_i\}] \neq V(G)$ for each $1 \leq i \leq 2r$, we have $g_r(G) \geq p$. On the other hand, the set $V(K_{p-2}) \cup \{x_r, x_{r+1}\}$ is a restrained geodetic set of G and hence $g_r(G) = p$.

Finally, let $2 \leq r < d$ and $p \geq 2$. Let $C_{2r} = (x_1, x_2, \ldots, x_{2r})$ be a cycle of order $2r$ and let $P_{d-r+1} = u_0u_1 \ldots u_{d-r}$ be a path of order $d - r + 1$ (and length $d - r$). Let G be the graph obtained from C_{2r} and P_{d-r+1} by identifying x_1 and u_0. Add $p - 2$ new pendant edges $u_{d-r-1}w_1, \ldots, u_{d-r-1}w_{p-2}$. Then $\mathrm{rad}(G) = r$ and $\mathrm{diam}(G) = d$. The graph G has $p - 1$ end-vertices, that is, $L = \{u_{d-r}, w_1, \ldots, w_{p-2}\}$. By Proposition **??**, L is contained in each restrained geodetic set of G and that $I[L] \neq V(G)$. Hence, $g_r(G) \geq p$. On the other hand, we have $I[L \cup \{x_{r+1}\}] = V(G)$, implying that $g_r(G) = p$. This completes the proof. □

Next we study the effect of adding an edge on the restrained geodetic number of a graph.

Lemma 11. For any connected graph G and any two nonadjacent vertices x and y of G, $g_r(G) \leq g_r(G + xy) + 2$. In addition, if M' is a geodetic set of $G + xy$, then $M' \cup \{x, y\}$ is a geodetic set of G.

Proof. Let M' be a geodetic set of $G + xy$. Assume there exists $z \in V(G) \setminus I_G[M']$. Then there is a u-v geodesic path P: $u \ldots z \ldots v$ in $G + xy$ with $u, v \in M'$ such that x and y are neighbors in P. Moreover, without loss of generality, there exist the following possibilities: (i) P: $u \ldots xy \ldots z \ldots v$, (ii) P: $uy \ldots z \ldots v$, ($u = x$) (iii) P: $uz \ldots v$, ($u = x$ and $y = z$). Hence, $I_G[M' \cup \{x, y\}] = V(G)$, as is required. □

The following corollary is an immediate result of Lemma **??**.

Corollary 12. Let x and y be nonadjacent vertices of a connected graph G and let M' be a $g_r(G + xy)$-set. If $G[V(G) \setminus (M' \cup \{x, y\})]$ has no isolated vertices, then $M' \cup \{x, y\}$ is a restrained geodetic set of G and $g_r(G + xy) \geq g_r(G) - 2$.

Theorem 13. For integers a and b with $a \geq 3$, $b \geq -2$ and $a + b \geq 2$, there exists a connected graph G and an edge $e \in E(\overline{G})$ such that $g_r(G) = a$ and $g_r(G + e) = a + b$.

Proof. We consider four cases.

Case 1. $b = -2$.

Let $H_{a,-2}$ be the graph obtained from a path $P_6 = u_0u_1 \ldots u_6$, by adding $a-3$ pendant edges $u_1v_1, \ldots, u_1v_{a-3}$ and a pendant edge u_4w. By Observation **??**, the set $\{w, u_0, u_6, v_1, \ldots, v_{a-3}\}$ is the unique minimum restrained geodetic set of G and hence $g_r(H_{a,-2}) = a$. Assume $H'_{a,-2} = H_{a,-2} + v_{a-3}w$. It is easy to see that the set $\{u_0, u_6, v_1, \ldots, v_{a-4}\}$ is a minimum restrained geodetic set of $H'_{a,-2}$ and so $g_r(H'_{a,-2}) = a - 2$.

Case 2. $b = -1$.
Suppose $C_5 = (u_1, u_2, u_3, u_4, u_5)$ is a cycle on five vertices and $K_{1,a-1}$ is a star with the center x and end-vertices $y_1, \ldots, y_{a-1}$. Let $H_{a,-1} = (C_5 \cup K_{1,a-1}) + y_1u_5$ and $H'_{a,-1} = H_{a,-1} + y_1u_1$. It is easy to verify that $\{y_2, \ldots, y_{a-1}, u_2, u_3\}$ and $\{y_2, \ldots, y_{a-1}, u_3\}$ are $g_r(H_{a,-1})$-set and $g_r(H'_{a,-1})$-set, respectively. Hence, $g_r(H_{a,-1}) = a$ and $g_r(H'_{a,-1}) = a - 1$.
Case 3. $b = 0$.
Let $H_{a,0}$ be the graph obtained from a path $P_5 = u_1u_2u_3u_4u_5$ by adding $a - 1$ pendant edges $u_5v_1, \ldots, u_5v_{a-1}$. Assume $H'_{a,0} = H_{a,0} + u_1u_4$. Clearly, $\{u_1, v_1, \ldots, v_{a-1}\}$ is the unique $g_r(H_{a,0})$-set and $\{u_2, v_1, \ldots, v_{a-1}\}$ is the unique $g_r(H'_{a,0})$-set. Thus $g_r(H_{a,0}) = g_r(H'_{a,0}) = a$.
Case 4. $b \geq 1$.
Let $K_{2,b}$ be a complete bipartite graph with partite sets $\{u_1, u_2\}$ and $\{v_1, \ldots, v_b\}$, and let $K_{1,a-1}$ be a star with the center x and end-vertices $\{y_1, \ldots, y_{a-1}\}$. Assume $H_{a,b} = (K_{2,b} \cup K_{1,a-1}) + u_2x$ and $H'_{a,b} = H_{a,b} + u_1u_2$. It is easy to see that $\{u_1, y_1, \ldots, y_{a-1}\}$ and $\{u_1, y_1, \ldots, y_{a-1},\ v_1, \ldots, v_b\}$ are the unique minimum restrained geodetic set of $H_{a,b}$ and $H'_{a,b}$, respectively. Hence $g_r(H_{a,b}) = a$ and $g_r(H'_{a,b}) = a + b$ and the proof is complete. $\square$

3 Forcing subsets in restrained geodetic sets of a graph

Let G be a connected nontrivial graph and let S be a $g_r(G)$-set. A subset T of S is called a *forcing subset* of S if S is the unique extension of T to a $g_r(G)$-set. The *forcing restrained geodetic number* $f(S, g_r)$ of S is defined by $f(S, g_r) = \min\{|T| \mid T \text{ is a forcing subset of } S\}$. An $f(S, g_r)$-set is a forcing subset of S of size $f(S, g_r)$. The *forcing restrained geodetic number* $f(G, g_r)$ is defined by $f(G, g_r) = \min\{f(S, g_r) \mid S \text{ is a } g_r(G)\text{-set}\}$. Hence, for every connected graph G, $f(G, g_r) \geq 0$.

The concept of forcing numbers has been studied in different areas of combinatorics and graph theory, including the chromatic number [?], the domination number [?, ?, ?] and the geodetic number [?, ?, ?, ?, ?]. The forcing geodetic set and the forcing geodetic number in a graph were introduced by Chartrand et al. in [?].

Observation 14. Let G be a connected graph of order $n \geq 2$.

(i) $0 \leq f(G, g_r) \leq g_r(G) \leq n$;

(ii) $f(G, g_r) = 0$ if and only if G has a unique $g_r(G)$-set;

(iii) $f(G, g_r) = 1$ if and only if G has at least two distinct $g_r(G)$-sets but some vertices of G belongs to exactly one $g_r(G)$-set;

(iv) $f(G, g_r) \geq 2$ if and only if every vertex of each $g_r(G)$-set belongs to at least two $g_r(G)$-sets.

The following corollaries are immediate consequences of Observations **??** and **??**.

Corollary 15. For a tree T, $f(T, g_r) = 0$.

Corollary 16. For $n \geq 1$, $f(K_n, g_r) = 0$.

Theorem 17. Every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 3$, can be realized as the forcing restrained geodetic number and the restrained geodetic number of some connected graph, respectively.

Proof. We have already seen that $f(K_b, g_r) = 0$ and $g_r(K_b) = b$. Thus, we assume that $0 < a \leq b$. We consider the following cases.

Case 1. $a = b - 1$.
Let $D_{a,a+1}$ be the graph with vertex set $V(D_{a,a+1}) = \{x_1, x_2\} \cup \{u_i, v_i \mid 1 \leq i \leq a+1\}$ and edge set $E(D_{a,a+1}) = \{x_1x_2\} \cup \{x_1u_i, x_2v_i, u_iv_i \mid 1 \leq i \leq a+1\}$ (see Figure 2).

Obviously, $\{u_1, \ldots, u_a, v_1\}$ is a restrained geodetic set of $D_{a,a+1}$ and hence $g_r(D_{a,a+1}) \leq a+1$. Let S be an arbitrary $g_r(D_{a,a+1})$-set. If $u_i, v_i \notin S$ for some $1 \leq i \leq a+1$, then we must have a geodesic path P containing the path $x_1u_iv_ix_2$, which is a contradiction because $x_1x_2 \in E(D_{a,a+1})$. Therefore $|S \cap \{u_i, v_i\}| \geq 1$ for each $1 \leq i \leq a+1$ and hence $g_r(D_{a,a+1}) = |S| \geq a+1$. Thus $g_r(D_{a,a+1}) = a+1$.

Now we show that $f(D_{a,a+1}, g_r) = a$. If S is a $g_r(D_{a,a+1})$-set, then $|S \cap \{u_i, v_i\}| = 1$ for each $1 \leq i \leq a+1$. Since the sets $\{u_1, \ldots, u_{a+1}\}$ and $\{v_1, \ldots, v_{a+1}\}$ are not $g_r(D_{a,a+1})$-set, we must have $S \cap \{u_1, \ldots, u_{a+1}\} \neq \emptyset$ and $S \cap \{v_1, \ldots, v_{a+1}\} \neq \emptyset$. It is easy to see that every set S of vertices with $|S \cap \{u_i, v_i\}| = 1$ for each $1 \leq i \leq a+1$, $S \cap \{u_1, \ldots, u_{a+1}\} \neq \emptyset$ and $S \cap \{v_1, \ldots, v_{a+1}\} \neq \emptyset$ is a $g_r(D_{a,a+1})$-set. It follows that every set S of vertices of order at most $a-1$ with $|S \cap \{u_i, v_i\}| \leq 1$ for each $1 \leq i \leq a+1$, can be extended to at least two $g_r(D_{a,a+1})$-sets. Thus $f(D_{a,a+1}, g_r) \geq a$. On the other hand, clearly the set $\{u_1, \ldots, u_a\}$ is a forcing subset of the $g_r(D_{a,a+1})$-set $\{u_1, \ldots, u_a, v_1\}$. Thus $f(D_{a,a+1}, g_r) \leq a$ and hence $f(D_{a,a+1}, g_r) = a$.

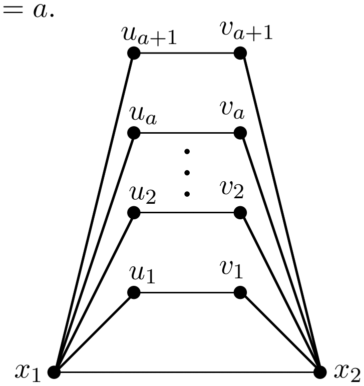

Fig. 2: The graph $D_{a,a+1}$

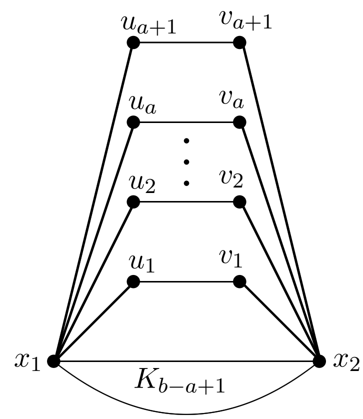

Fig. 3: The graph $D_{a,b}$

Case 2. Assume $1 \leq a \leq b-2$.
Let $D_{a,b}$ be a graph obtained from $K_{b-a-1} \cup D_{a,a+1}$ by joining x_1 and x_2 to all vertices of K_{b-a-1} (see Figure 3). Let S be a $g_r(D_{a,b})$-set. Since the vertices of K_{b-a-1} lie on no geodetic path, we have $V(K_{b-a-1}) \subseteq S$. An argument similar to that described in Case 1 shows that $|S \cap \{u_i, v_i\}| \geq 1$ for each $1 \leq i \leq a+1$. Thus $g_r(D_{a,b}) = |S| \geq b$. On the other hand, it is easy to see that the set $\{u_1, \ldots, u_{a+1}, v_1\}$ is a restrained geodetic set of $D_{a,b}$, implying that $g_r(D_{a,b}) \leq b$. Thus $g_r(D_{a,b}) = b$.

Now let S be a $g_r(D_{a,b})$-set and let F be a forcing set of S. As above, we must have $|F| \geq a$. It is easy to see that the set $\{u_i \mid 1 \leq i \leq a\}$ is a forcing set of the restrained geodetic set $V(K_{b-a-1}) \cup \{u_1, \ldots, u_a, v_1\}$, which implies that $f(D_{a,b}, g_r) = a$.

Case 3. $a = b$.
We consider three subcases.

Subcase 3.1 $a = b = 2k$, $k \geq 2$
For $1 \leq i \leq k$, suppose G_i is a copy of $K_{3,3}$ with bipartite sets $X_i = \{u_1^i, u_2^i, u_3^i\}$ and

$Y_i = \{v_1^i, v_2^i, v_3^i\}$. Let M_{2k} be the graph obtained from $\cup_{i=1}^k G_i$ by adding two new vertices x_1, x_2, adding the edge x_1x_2 and joining x_1 to u_j^i and x_2 to v_j^i for each $1 \le i \le k$ and $j = 1, 2, 3$ (see Figure 4).

It is easy to see that the set $\{u_1^i, v_1^i \mid 1 \le i \le k\}$ is a restrained geodetic set of M_{2k}, implying that $g_r(M_{2k}) \le 2k$. Assume S is a $g_r(M_{2k})$-set. We claim that $|S \cap X_i| \ge 1$ and $|S \cap Y_i| \ge 1$ for each $1 \le i \le k$. Assume to the contrary that $S \cap X_i = \emptyset$ (the case $S \cap Y_i = \emptyset$ is similar) for some i, say $i = 1$. It follows that $Y_1 \subseteq S$. A similar argument shows that $|S \cup V(G_i)| \ge 2$ for each $i \ge 2$, which implies that $|S| \ge 2k + 1$, a contradiction. Thus $|S \cap X_i| \ge 1$ and $|S \cap Y_i| \ge 1$ for each $1 \le i \le k$, implying that $|S| \ge 2k$ and hence $g_r(M_{2k}) = |S| = 2k$. Moreover, we must have $|S \cap X_i| = 1$ and $|S \cap Y_i| = 1$ for each $1 \le i \le k$.

Now let S be a $g_r(M_{2k})$-set and let F be a forcing set of S. We claim that $F = S$. Suppose to the contrary that $F \subset S$. We may assume without loss of generality that $u_1^1 \in (S \setminus F)$. It is easy to see that $(S \setminus \{u_1^1\}) \cup \{u_2^1\}$ is a $g_r(M_{2k})$-set containing F, a contradiction. Hence, $f(M_{2k}, g_r) = |F| = |S| = g_r(M_{2k})$.

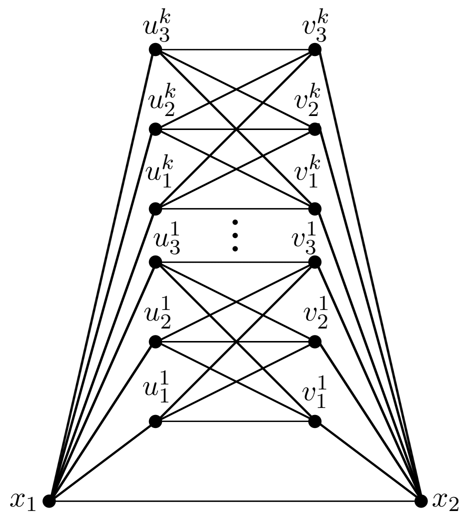

Fig. 4: The graph M_{2k}

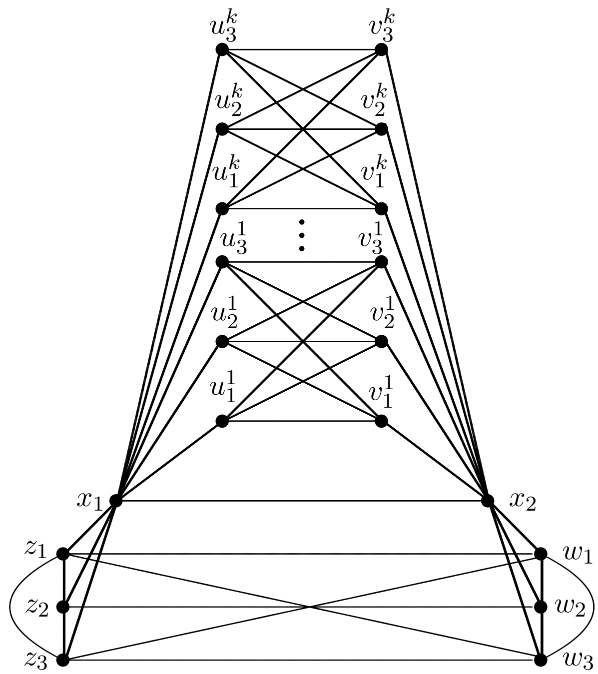

Fig. 5: The graph H_{2k+3}

Subcase 3.2 $a = b = 2k + 3$, $k \ge 1$.
Let H_{2k+3} be the graph obtained from M_{2k} by adding new vertices $z_1, z_2, z_3, w_1, w_2, w_3$ and adding the edges $z_1z_2, z_1z_3, z_2z_3, w_1w_2, w_1w_3, w_2w_3, z_1w_1, z_1w_3, z_2w_2, z_3w_1, z_3w_3$, and x_1z_i and x_2w_i for $i = 1, 2, 3$ (see Figure 5).

It is easy to verify that the set $\{u_1^i, v_1^i \mid 1 \le i \le k\} \cup \{x_1, x_2, w_1\}$ is a restrained geodetic set of H_{2k+3}, this implies $g_r(H_{2k+3}) \le 2k + 3$. If S is a $g_r(H_{2k+3})$-set, then as in Case 2, we can see that $|S \cap X_i| \ge 1$ and $|S \cap Y_i| \ge 1$ for each $1 \le i \le k$, and $|S \cap \{z_1, z_2, z_3, w_1, w_2, w_3\}| \ge 3$. Hence, $g_r(H_{2k+3}) = 2k + 3$. Since every set of vertices $S \subset V(H_{2k+3})$ with $|S \cap \{z_1, z_2, z_3, w_1, w_2, w_3\}| = 3$, $|S \cap X_i| = 1$ and $|S \cap Y_i| = 1$ for each $1 \le i \le k$, is a restrained geodetic set of H_{2k+3} whenever the set $S \cap \{z_1, z_2, z_3, w_1, w_2, w_3\}$ is not one of $\{z_1, z_3, w_3\}$, $\{w_1, z_3, w_3\}$, $\{z_1, w_1, z_3\}$ and $\{z_1, w_1, w_3\}$, we deduce that for every $g_r(H_{2k+3})$-set S, $f(S, g_r) = |S|$. It follows that $f(M_{2k}, g_r) = g_r(M_{2k})$.

Subcase 3.2 $a = b = 3$.

For $1 \leq i \leq 3$, let R_i be the graph obtained from a complete bipartite graph $K_{2,3}$ with partite sets $\{u_i, v_i\}$ and $\{x_1^i, x_2^i, x_3^i\}$ by joining x_1^i to x_2^i and x_2^i to x_3^i. Assume H_3 results from $R_1 \cup R_2 \cup R_3$ by adding the edges v_1u_2, v_2u_3 and v_3u_1. It is easy to see that $f(H_3, g_r) = g_r(H_3) = 3$. □

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