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A Numerical Method for Solving Nonlinear Integral Equations in the Urysohn Form

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Abstract

Integral equations occur naturally in many field of mechanics and mathematical physics. In this paper, an iterative scheme based on the homotopy analysis method (HAM) has been used for solving one of the most important cases in nonlinear integral equations which is called urysohn form. For this scope, we applied some examples with known exact solutions and the numerical solutions obtained confirm that the method is very effective and simple. Also, show the advantage of this method, the results obtained by (HAM) are compared with Newton-Kantorovich-quadrature method. Convergence is also observed.

Keywords: Homotopy analysis; Nonlinear integral equation; Newton-Kantorovich-quadrature method; Urysohn integral equation

1 Introduction

As a matter of fact, it might be said that many phenomena of almost all practical engineering and applied science problems like physical applications, potential theory and electrostatics are reduced to solve integral equations. Since these equations usually can not be solved explicitly, so it is required to obtain approximate solutions. Therefore, many different methods are used to obtain the solution of the linear and nonlinear integral equations. The solution of nonlinear integral equations is a complicated problem of computational mathematics, which is related to difficulties of both a principal and computational character. With the advent of computers, the use numerical methods for solving these equations always been important in scientific investigations. Of numerical methods has been popularized and more importantly, people are now able to attack those problems. Which are fundamental to our understanding of scientific phenomenon, but were so much more difficult to study in the past. Several numerical methods for approximating the solution of nonlinear integral equations are known. Several numerical methods for approximating the solution of nonlinear integral equations are known. Variational iteration method in the [6], also Homotopy perturbation method and Adomian decomposition method are effective and convenient for solving integral equations. The homotopy analysis method (HAM) in the [2-5] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-difference equation. More importantly, different from all perturbation and traditional non-perturbation methods, the HAM provides us a simple way to ensure the convergence of solution series, and therefore, the (HAM) is valid even for strongly nonlinear problems. The HAM is based on homotopy, a fundamental concept in topology and differential geometry. Briefly speaking, by means of the HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations and the method enjoys great freedom in choosing initial approximations and auxiliary linear operators [1]. Liao and Tan in [5] shown that by means of this kind of freedom, a complicated nonlinear problem can be transferred into an infinite number of simpler, linear sub-problems. Until recently, the application of the homotopy analysis method in nonlinear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem easy to solve into the difficult problem under study. When solving nonlinear equations, the applicability domain of the method of simple iterations is smaller, and if the process is still convergent. Then in many cases, the rate of convergence can be very low. An effective method that makes it possible to overcome the indicated complication is the Newton-kantorovich method. In [7] Saber-Nadjafi and Heidari with this method solved nonlinear (IE) of the urysohn form in systematic procedure. In the article, we use the (HAM) for in the urysohn form such that:

$$u(x) = f(x) + \int_a^b k(x,t,u(t))dt, \quad a \leq x \leq b.$$

where $k(x,t,u(t))$ is the kernel of the integral equation and $u(x)$ is the unknown function. some example are tested, and the obtained results suggest that newly improvement technique introduces a promising tool and compare with Newton-Kantorovich-quadrature method. This paper is arranged in the

following manner:
After an introduction to the present work, the homotopy analysis method (HAM) is described in section 2. In section 3, Computational procedure method is explained. we explain convergence analysis in section 4. In the section 5, some numerical examples is presented which show efficiency and accuracy of the proposed method and comparison of the obtained results between (HAM) an Newton-Kantorovich-quadrature method are discussed. finally conclusion is drown in section 6.

2 Description Of The Method

The general form of nonlinear Fredholm integral equations of the urysohn form is as follows:

$$u(x) = f(x) + \int_a^b k(x,t,u(t))dt, \qquad a \leq x \leq b,$$

and

$$N[u] = u(x) - f(x) - \int_a^b k(x,t,u(t))dt = 0. \tag{1}$$

Where N is an operator, $u(x)$ is unknown function and x the independent variable. An auxiliary linear operator is chosen to construct such kind of continuous mapping, and an auxiliary parameter is used to ensure the convergence of solution series [1]. Let $u_0(x)$ denote an initial guess of the exact solution $u(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x)] = 0$ when $r(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct such a homotopy

$$(1-q)L[\phi(x;q) - u_0(x)] - qhH(x)N[\phi(x;q)] = \hat{H}[\phi(x;q); u_0(x), H(x), h, q]. \tag{2}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x)$, the auxiliary linear operator L , the non-zero auxiliary parameter h, and the auxiliary function $H(x)$.
Enforcing the homotopy (2) to be zero, i.e;

$$\hat{H}[\phi(x;q); u_0(x), H(x), h, q] = 0$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q) - u_0(x)] = qhH(x)N[\phi(x;q)]. \tag{3}$$

When $q = 0$ and $q = 1$, since $h \neq 0$ and $H(x) \neq 0$, the zero-order deformation equation (3) is equivalent to

$$\phi(x; 0) = u_0(x), \quad (4)$$

$$\phi(x; 1) = u(x). \quad (5)$$

Thus, as q increase from 0 to 1, the solution $\phi(x; q)$ varies from initial guess $u_0(x)$ to the solution $u(x)$. By Taylor's theorem, $\phi(x; q)$ can be expanded in a power series of q as follows

$$\phi(x; q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \quad (6)$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} \mid q = 0. \quad (7)$$

If the initial guess $u_0(x)$, the auxiliary linear parameter L , the nonzero auxiliary function $H(x)$ are properly chosen so that the power series (6) of $\phi(x; q)$ convergence at $q = 1$. Then, we have under these assumptions the solution series

$$u(x) = \phi(x; 1) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \quad (8)$$

For brevity, define the vector

$$\overrightarrow{u}_n(x) = \{u_0(x), u_1(x), u_2(x), u_3(x), \ldots, u_n(x)\}. \quad (9)$$

According to the definition (7), the governing equation of $u_m(x)$ can be derived from the zero-order deformation equation (3). Differentiating the zero-order deformation equation (3) m times with respective to q and then dividing by m! and finally setting $q = 0$, we have the so-called mth-order deformation equation

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x) R_m(\overrightarrow{u}_{m-1}(x)), \quad u_m(0) = 0. \quad (10)$$

where

$$R_m(\overrightarrow{u}_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x; q)]}{\partial q^m}, \quad (11)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation equation (10) is governing by the linear operator L, and the term $R_m(\overrightarrow{u}_{m-1}(x))$ can be expressed simply by (11) for any nonlinear operator N.
According to the definition (11), the right-hand side of equation (10) is only dependent upon $u_{m-1}(x)$. Therefore $u_m(x)$ can be easily gained, especially by means of computational software such as MATLAB. Here, we rigorous definitions and then give some properties of the homotopy derivative. These properties are useful to deduce the high-order deformation equations and provide us with a simple and convenient way to apply the HAM to nonlinear problems. Let ϕ be a function of the homotopy-parameter q , then

$$D_m(\phi) = \frac{1}{(m)!}\frac{\partial^m[\phi(x;q)]}{\partial q^m} \mid q = 0,$$

is called the mth-order homotopy-derivative of ϕ, where $m \geq 0$ is an integer. According to the Leibnitz's rule for derivatives and using the induction, one can show the following properties of the homotopy-derivative. For homotopy series we have:

$$\phi(x;q) = \sum_{m=0}^{\infty} u_m(x) q^m,$$

where m is positive integers, then (1) For $p \geq 0$, a positive integer, it holds

$$D_m(\phi^p) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{p_{ij}-2}=0}^{r_{p_{ij}-3}} u_{r_{p-3}-r_{p-2}} \sum_{r_{p_{ij}-1}=0}^{r_{p_{ij}-2}} u_{r_{p-2}-r_{p-1}} u_{r_{p-1}}.$$

3 Computational procedure

One kind of the nonlinear integral equation is the nonlinear integral equation in the urysohn form. This kind of integral equation is defined in the following general form:

$$u(x) = f(x) + \int_a^b k(x,t,u(t))dt \quad a \leq x \leq b. \tag{12}$$

Where $k(x,t,u(t))$ is the kernel of the integral equation and $u(x)$ is the unknown function. All function in (12) are usually assumed to be continuous. Also, $k(x,t,u(t))$ equal with $k(x,t,Fu(t))$ synonym. Where the unknown function $u(x)$ occurs inside and outside the integral sign. For this type of equations the kernel $k(x,t)$ and function $f(x)$ are given real-valued function, and $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x)$, $sin(u(x))$. Whit hypothesis $F(u(t)) = [u(t)]^p$ where, p is a positive integer. We have:

$$u(x) = f(x) + \int_a^b k(x,t)[u(t)]^p dt.$$

In this section we will use the HAM approach to consider above equation.

$$N[u] = u(x) - f(x) - \int_a^b k(x,t)[u(t)]^p dt.$$

The corresponding mth-order deformation equation (10) reads

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)R_{m-1}(\overrightarrow{u}_{m-1}(x)) \quad , \quad u_m(0) = 0 \tag{13}$$

where

$$R_{m-1}(\overrightarrow{u}_{m-1}(x)) = u_{m-1} - (1-\chi_m)f - \int_a^b k(x,t)R_{m-1}(\phi^p)dt$$

and

$$R_m(\phi^p) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{p-2}=0}^{r_{p-3}} u_{r_{p-3}-r_{p-2}} \sum_{r_{p-1}=0}^{r_{p-2}} u_{r_{p-2}-r_{p-1}} u_{r_{p-1}}.$$

To obtain a simple iteration formula for $u_m(x)$, choose $Lu = u$ as an auxiliary linear operator, as a zero-order approximation to the desired function $u(x)$, the solution $u_0(x) = f(x)$, is taken, the nonzero auxiliary parameter h and the auxiliary function $H(x)$, can be taken as $h = -1$ and $H(x) = 1$. This is substituted into (13) to obtain

$$u_0(x) = f(x), \qquad u_m(x) = \int_a^b k(x,t)R_{m-1}(\phi^p)dt, \qquad m = 1,2,3,\ldots$$

The corresponding homotopy-series solution is given by

$$u(x) = \sum_{m=0}^{\infty} u_m(x). \tag{14}$$

4 Convergence Analysis

Theorem 1. The urysohn Integral Equation

$$u(x) = f(x) + \int_a^b k(x,t)F(u(t))dt. \tag{15}$$

With the kernel $k(x,t)$ satisfies $|k(x,t)| < M$ for all $(x,t) \in [a,b] \times [a,b]$, also $f(x)$ is a given continuous function defined on $[a,b]$ and $f(u)$ is Libschitz continuous with $|f(u) - f(w)| \leq L|u - w|$, has a unique solution whenever $0 < \alpha < 1$, where, $\alpha = LM(b-a)$. As the function $f(x) = x^p$ is Lipschitz continuous, the integral equation (12) has a unique solution. If the kernel is separable , i.e. $k(x,t) = g(x)h(t)$, then the following condition

$$|1 - \int_a^b K(t,t)dt| < 1,$$

must be justified for convergence.

Theorem 2. Let $S(x) = \sum_{n=0}^{\infty} u_n(x)$ then for $k \geq 2$, where k is an integer,

$$\sum_{m=0}^{\infty} R_m(\phi^k) = S^k(x).$$

See the proof **Theorem 1 , 2** in [1].

Theorem 3. As long as the series (14) convergence, it must be the exact solution of the integral equation (12).

Proof. If the series (14) convergence, we can write

$$S(x) = \sum_{n=0}^{\infty} u_m(x),$$

and it hold that

$$\lim_{m \to \infty} u_m(x) = 0.$$

We can verify that

$$\sum_{m=1}^{n} [u_m(x) - \chi_m u_{m-1}(x)] = u_1 + (u_2 + u_1) + \ldots + (u_n + u_{n-1}) = u_n(x),$$

which gives us, according to (15),

$$\sum_{m=1}^{\infty} [u_m(x) - \chi_m u_{m-1}(x)] = \lim_{n \to \infty} u_n(x) = 0. \tag{16}$$

Furthermore, using (16) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = L[\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)]] = 0,$$

In this line, we can obtain that

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)\sum_{m=1}^{\infty} R_{m-1}(\overrightarrow{u}_{m-1}(x)) = 0,$$

which gives, since $h \neq 0$and $H \neq 0$, that

$$\sum_{m=1}^{\infty} R_{m-1}(\overrightarrow{u}_{m-1}(x)) = 0. \tag{17}$$

substituting $R_{m-1}(\overrightarrow{u}_{m-1}(x))$ into the above expression, recall Theorem 3, and simplifying it, we have

$$\sum_{m=1}^{\infty} R_{m-1}(\overrightarrow{u}_{m-1}(x)) = \sum_{m=1}^{\infty}[u_{m-1} - (1-\chi_m)f - \int_a^b k(x,t)R_{m-1}(\phi^p)dt]$$

$$= \sum_{m=0}^{\infty} u_m(x) - f(x) - (1-\chi_m)f - \int_a^b k(x,t)\sum_{m=1}^{\infty} R_{m-1}(\phi^p)dt$$

$$= \sum_{m=0}^{\infty} u_m(x) - f(x) - (1-\chi_m)f - \int_a^b k(x,t)[\sum_{m=0}^{\infty} u_m(t)^p]dt$$

$$= S(x) - f(x) - \int_a^b k(x,t)S(t)^p dt \tag{18}$$

and so, $S(x)$ must be the exact solution of Eq.(12).■

5 Numerical Result and Discussion

The HAM provides an analytical solution terms of an infinite power series. However, there is a particle need to evaluate this solution. The consequent series truncation, and the particle procedure conducted to accomplish this task, together transform the analytical results into an exact solution, which is evaluated to a finite degree of accuracy. In order to investigated the accuracy of the HAM solution with a finite number of terms, two examples were solved.

For efficiency of the present method for our problem in comparison with those obtained by the Newton-Kantorovich-quadrature method.

Example 6.1. Consider the following nonlinear Fredholm integral equation urysohn form:

$$u(x) = sin(\pi x) + \frac{1}{5}\int_0^1 cos(\pi x)sin(\pi t)u^3(t)dt, \quad 0 \le x \le 1.$$

The exact solution to this integral equation is $u(x) = sin(\pi x) + \frac{20-\sqrt{391}}{3}cos(\pi x)$. The formulas corresponding to this problem are

$$u_0(x) = sin(\pi x),$$

$$u_m(x) = \frac{1}{5}\int_0^1 [cos(\pi x)sin(\pi t)\sum_{i=0}^{m-1} u_{m-i-1}\sum_{j=0}^{i} u_j u_{i-j}]dt \quad m = 1, 2 \ldots$$

Table 1. shows numerical results calculated according the presented method and Newton-Kantorovich-quadrature.

x_i	u_{exact}	u^{15}_{HAM}	$u_{(Newoton-Kantorovich)}$
0	0.0754266889	0.0754266889	0.0255947861
0.1	0.3807520383	0.3807520383	0.3333590824
0.2	0.6488067254	0.6488067254	0.6084918692
0.3	0.8533516897	0.8533516897	0.8240612321
0.4	0.9743646449	0.9743646449	0.9589657401
0.5	1.0000000000	1.0000000000	1.0000000000
0.6	0.9277483875	0.9277483875	0.9431472924
0.7	0.7646822990	0.7646822990	0.7939727565
0.8	0.5267637791	0.5267637791	0.5670786353
0.9	0.2372819503	0.2372819503	0.2446749062
1	-0.0754266889	-0.0754266889	-0.025597861

Table 1 solution to the example6.1.

Example 6.2. The presented HAM iterative scheme is applied for solving urysohn integral equation, that the exact solution is $u(x) = \ln(x + 1)$.

$$u(x) = \ln(x+1) + 2\ln 2(1 - x\ln 2 + x) - 2x - \frac{5}{4} + \int_0^1 (x-t)u^2(t)dt, \quad 0 \le x \le 1.$$

We begin with

$$u_0(x) = \ln(x+1) + 2\ln 2(1 - x\ln 2 + x - 2x - \frac{5}{4}.$$

Its iteration formulation reads

$$u_m(x) = \int_0^1 [(x-t)\sum_{j=0}^{m-1} u_j(t)u_{m-j-1}(t)]dt, \quad m = 1, 2, \ldots$$

Some numerical results of these solutions are presented and compare results calculated by Newton-Kantorovich method in follow table.

x_i	u_{exact}	u^{15}_{HAM}	$u_{(Newoton-Kantorovich)}$
0	0	0.000000026768	0.000000026768
0.1	0.095310179804	0.095310206285	0.080610206788
0.2	0.182321556793	0.182321579887	0.195465677887
0.3	0.262364264467	0.262364280170	0.223454563210
0.4	0.336472236621	0.336472248418	0.315434533211
0.5	0.405465108108	0.405465120209	0.374545130278
0.6	0.470003629245	0.470003629794	0.421234567300
0.7	0.530628251062	0.530628259053	0.500678789650
0.8	0.587786664902	0.587786677053	0.564455564053
0.9	0.641853886172	0.641853890141	0.621756891141
1	0.693147180559	0.693147181293	0.682314518129

6 Conclusions

The proposed method is a powerful procedure for solving nonlinear urysohn integral equations. The examples analyzed illustrate the ability and reliability of the method presented in this paper and reveals that this one is very simple and effective. The obtained solutions, are compare with Newton-Kantorovich method. Results indicate that the convergence rate is very fast, and lower approximations can achieve high accuracy.

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