# Stay at school or start working? - Optimal timing of leaving school under uncertainty and irreversibility

Draft

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**Abstract**

At any moment a student may decide to leave school and enter the labor market, or stay in the education system. The time of departure from school determines their level of academic achievement and formal qualification. Therefore, the major purpose of this paper is to derive a timing rule for leaving school and thus answer the question: How long should I go to school? To solve this problem we apply the real option approach. Real option theory offers a different perspective of the human capital investment decision under uncertainty and irreversibility. As future income is uncertain, we model future earnings as a continuous stochastic process. We use dynamic programming techniques to derive an income threshold at which a student should leave school irreversibly, and we determine the expected optimal duration of education. Unlike other approaches using real option theory we are able to provide a full analytical discussion of various determinants affecting the timing of the decision to start work. Among other things, we find that a rising income risk increases the duration of education. With a faster growth of expected individual income during working life the duration of schooling will decrease leading to less education. An increase in the no-education wage level will reduce human capital investments. Rising marginal rewards for a year of schooling (in terms of a rising differential in income level) will encourage more investment in human capital. Increasing education costs may also increase human capital investment as long as the marginal reward for a year of schooling is sufficiently high. However, allowing for discontinuities due to various cost and income profiles of formal qualification levels, high costs of schooling may lead to an achievable maximum net wealth of human capital even for lower qualification.

*JEL classifications:* J24, I2, D8

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# 1 Introduction

Education is obtained during a long process of personal maturation and the accumulation of knowledge and abilities. Hence, formal schooling is a learning and investment process that often lasts into one's mid twenties. When a young person makes plans for the future one of the biggest problems is uncertainty. The success of a long education is as uncertain as the process of earning income during a long working life. As time goes on, students repeatedly consider whether to continue their education or enter the labor market. During this process of decision making each moment's conditions determine the eventual attainment level.

Recent literature shows that real option theory can be applied to take into account uncertain time processes and irreversibility in schooling and human capital accumulation decisions. Hogan and Walker (2007) and Jacobs (2007) pick up these ideas in different ways and show that introducing real option theory can generate new insights in the idea of education as an investment decision in human capital.

The first analysis of investment in human capital under uncertainty was conducted by Levhari and Weiss (1974). Levhari and Weiss model risky future earnings in a two-period approach of human capital formation, concluding that an increase in uncertainty regarding the return on human capital investment decreases the level of investment under given assumptions about risk preferences and risk-return technology. This paper remains a major benchmark. Later, e.g. Eaton and Rosen (1980) extended this framework of Levhari and Weiss (1974) to analyse tax effects. Dynamic techniques combined with portfolio theory have also gained considerable attention. Williams (1978) examines risky investment in education using a two-period, mean-variance portfolio model. Considering several sources of risk, Williams can derive more precise hypotheses about observable variables than Levhari and Weiss (1974). Groot and Oosterbeek (1992) discuss the effects of uncertain future earnings and the probability of unemployment on the optimal duration of schooling. Hanchane, Lioui and Touahri (2006) develop a continuous time dynamic programming model which accounts for several sources of uncertainty with regard to earnings and labor market conditions. They show that the global effect of uncertainty is negative, except when a sufficiently high risk premium exists.

The application of real options theory - as established by Dixit and Pindyck (1994) - to the human capital investment decision is a relatively recent development. Hogan and Walker (2007) provide an example of the transfer of real option theory to human capital theory. In their model, at any time a student has the option to leave school to work for wages that reflect the years spent in school. The decision to leave school is irreversible, so once the student has finished education they cannot return. They conclude that high returns on education and increasing risk will cause students to stay in school longer. They also analyze how progressive taxation and education subsidies affect schooling decisions and show that progressive taxes tend to reduce education attainment.

Jacobs (2007) uses the real option approach as well. However, in contrast to Hogan (2007) he uses a discrete time approach and states that the decision to start learning is irreversible. The option value stems from the fact that an individual could wait to enroll and would only do so once the returns are sufficiently large to compensate for the lost option value. The sunk cost of the investment consists of forgone labor earnings and tuition costs.

Like Hogan and Walker (2007) we discuss how uncertain time processes (stochastic processes) determine the duration of schooling, and - with the timing decision to leave school - the accumulation of human capital. As education is a continuous process, a year of schooling also means a year-long deferral of the entry into the labor market. The deferral includes an option to extend schooling for another year and to rise to a higher level of academic achievement. This option for further education is not only a chance to obtain a better expected income track, it also has an implicit value because the uncertainty of working life is postponed and the irreversibility[1] to leave the school system is not realized. However, even if we are in some respect close to Hogan and Walker (2007), we depart from their analysis in various ways. In our approach education implies investment costs accumulate over time with additional years of schooling. These costs may differ depending on the level of formal qualification. That is, we can distinguish between formal qualifications and the related costs. Even more, in addition to a pure Brownian motion, we look at the complete earning profile (the stochastic level of the income path given by the initial wage when entering the labor market, and the stochastic dynamics of income) that is linked to education attainment. In particular we model the stochastic marginal market reward on initial income generated by marginal time of education. Further, not only do we look at threshold reactions like Hogan and Walker, we also explicitly determine the expected timing of market entry. We include stochastic changes in the market evaluation resulting from more years of schooling or even higher levels of formal qualification. Explicitly determining the timing of market entry also enables us to analytically derive how the duration of schooling responds to all relevant variables like risk, income growth, the no-education (minimum) wage level, costs of schooling, etc. Further, we can analyse the optimal choice of formal qualification as well as the optimal timing of dropping out of a formal education program. As will be seen in the analysis, our results are widely consistent with the empirical results recently obtained by Heckman (2008).

[1] There could be also a practical reversability using instruments like exit options build in the model. However, as we would like to analyze a simple case we postpone this discussion to future research.

## 2 A model of optimal timing to leave school

The education decision is an investment in human capital under uncertainty. Education is a choice of various time tracks of opportunities in an uncertain future. Investments in schooling and formal qualifications open up these various time tracks of opportunities and generate the respective uncertain income streams. How many years of life should a person invest in schooling in order to obtain a higher academic achievement or even a higher formal qualification? What is the optimal duration of additional education and skill development outside of formal schooling programs? When is the best time to leave school and to start working?

Modeling the optimal timing to leave the education system we can describe the uncertain future development using various pattern of income streams depicted by different stochastic processes. As soon as working life starts, the individual will enter an uncertain labor market not knowing the income stream and the future success of their professional activity. However, there are expectations about the income track linked to the formal academic qualification. The expected path of a college graduate will differ from that of someone with a high school diploma.

In this model towards an optimal timing decision to leave school we suggest real option theory in terms of a dynamic programming model. Similar to Hogan and Walker (2007) the individual can defer their entry into the labor market and obtain more schooling. A student maximizes their present discounted value of lifetime earnings by deriving an optimal individual income threshold at which it is favourable to enter the labor market. Knowing that threshold, we can determine the expected duration of schooling and the corresponding level of formal academic qualification. This optimal timing decision has two elements: 1) accumulated *investment costs of schooling*, 2) *benefits of schooling* with three components: a) schooling is a determinant of the level of the earning path depicted by the *initial income* when working life starts, b) schooling affects the *dynamic development of the income stream* and the resulting value of *earnings* during working life, and c) postponing working life through longer education, potentially achieving a better income track, or not yet being tied to a specific uncertain earning stream has an own value, i.e. the *option value of education*. While a) and b) are the components of the expected earning profile, c) evaluates the advantage of remaining flexible.

At any point in time a student may decide to stay in school or to leave the education system and start working. From the expected net earning stream (including costs) and the option value of additional education, the student determines a threshold that triggers the decision to leave school and start working. That threshold is the initial income level a student needs to realize in order to have a positive evaluation of the complete education project. Hence they try to navigate their education process towards a situation in which the realized initial market income level matches the required income suggested by the threshold. At any moment the student compares these two values and decides whether it is beneficial to stay at school, or start working and collect the expected income linked to schooling investment. At all given moment the student reassesses their expected further education process. As the decision is repeated we observe a sequence of "staying in school decisions" that adds up to the entire duration of schooling and the eventual level of academic achievement. However, even if not expected before, at any moment sudden (random) changes in conditions can also lead to an unexpected start of working life resulting in an unexpected lower level of education.

## 2.1 Model Components

**Investment costs of schooling:** If we define the time at which an individual decides about schooling as $t = 0$, total schooling costs for an individual student are the sum of the costs of each year until the end of their education.[2] In this model $C$ is defined as *individual cost* of a *successfully completed year of schooling*. Hence, a student with low capabilities would have to spend more to successfully complete a year of schooling. It can be assumed that these expenditures consist of tuition, extra private lessons, purchase of books, computers, materials and other related costs. Total investment expenditure for schooling $I(T)$ is dynamic and increases over time with each additional year of schooling. At time $T$, the end of the schooling phase, the *current* value of total schooling costs is

$$I(T) = \int_0^T C(T)e^{r(T-t)}dt + \bar{C}, \tag{1}$$

where $r$ is the risk-free interest rate and $\bar{C}$ are given cost to successfully graduate, search for an adequate job and realize the market entry. To focus on the major mechanics, in this most simple model taxes are not included. However, this could be easily done by correcting the effective interest rates $r$, the costs of schooling[3] and the income streams for taxes.

However, schooling not only generates costs. Three components define the benefits of schooling. a) Schooling generates a differential in the *initial income level* when entering the labor market[4] (move from A to B along the dotted line in figure 1), b) schooling may lead to a change in the *dynamic development* and *risk* of the income stream during working life (dashed line in figure 1), and c) education time has a *value* of the *option to wait* with market entry and not to be tied to a lifetime earning profile with the corresponding risk and irreversibility.

**Initial income level:** The *initial income* when working life starts and hence the level of income path is the first of two elements of the earning profile linked

[2] Recent empirical studies suggest that education costs are an important ingredient of the education decision see e.g. Heckman (2008). By including cost of schooling we depart from Hogan and Walker(2007) who neglect education costs.

[3] If schooling costs are tax deductible.

[4] In this respect our approach again is different from Hogan and Walker (2007) who do not consider this kind of reward of schooling.

to the education achievement. With each additional (successful) year of schooling the *initial income* of a labor market entrant increases, and hence the level of the earning stream rises (see the doted line in figure 1).[5]. However, even if another year of schooling can be expected to generate a marginal increase in initial income by the rate $\delta$, many random elements determine the initial income when entering the labor market. For the present simple case we describe the development of initial income levels during schooling time as a Brownian motion[6]

$$d\tilde{Y} = \delta\tilde{Y} + \sigma\tilde{Y}dW \quad \text{for} \quad 0 < T < T^*. \tag{2}$$

where $\sigma$ denotes a constant volatility, $dW$ denotes the increments of a standard Wiener process, and $\delta > 0$ is the expected marginal differential in income level with respect to marginal schooling time and educational improvement (expected rate of market reward). This change in the level of the income path is part of the total income reward generated by the schooling process. When a student plans their education at $t = 0$ they expect from their market observation that one year of schooling will give them an initial income when entering the market of $E\tilde{Y}(T = 1)$. For a market entry after two years of schooling they would expect an increase of initial income to the level $E\tilde{Y}(T = 2)$. As we will see later for a given dynamics of the income stream, the expected marginal market reward $\delta$ and hence the differential in the level of the income path must be large[7] enough to compensate sufficiently for the additional investment costs in human capital $C$.

**Dynamics and value of the income stream:** The second element of an individual earning profile is the dynamic development of the life-earning stream. The dynamics of an individual income track are also connected to education. From the stylized facts we know that the dynamics of income during working life differs with respect to years of schooling and formal qualification levels. We assume that the lifetime earning path has systematic and random elements. Therefore, we model the life-time earning path as a random process. Upon entry into the market ($t > T$) the student faces a stochastic revenue stream which is characterized by an expected average growth rate $\alpha$ and elements of uncertainty depicted by a constant volatility $\sigma$. In general, individual income dynamics are driven by a stochastic earning process described by a geometric Brownian motion

$$dY = \alpha Y dt + \sigma Y dW, \tag{3}$$

with $dW$ denoting the increments of a standard Wiener process. While in the real world an earning profiles would not be linear and decrease at the end

[5] In figure 1 we consider the logarithm of income in order to draw income streams as linear curves.

[6] Hogan and Walker(2007) consider only one Brownian motion for the dynamics of the income stream. Education does not effect the income profile or give any marginal reward.

[7] In this most simple case $\delta$ is fixed and the process is linear. However, a more realistic case would be $\delta(T)$ with a non linear maybe even an s-shaped marginal reward of additional schooling.This would imply to use more general Ito-processes to describe income development.

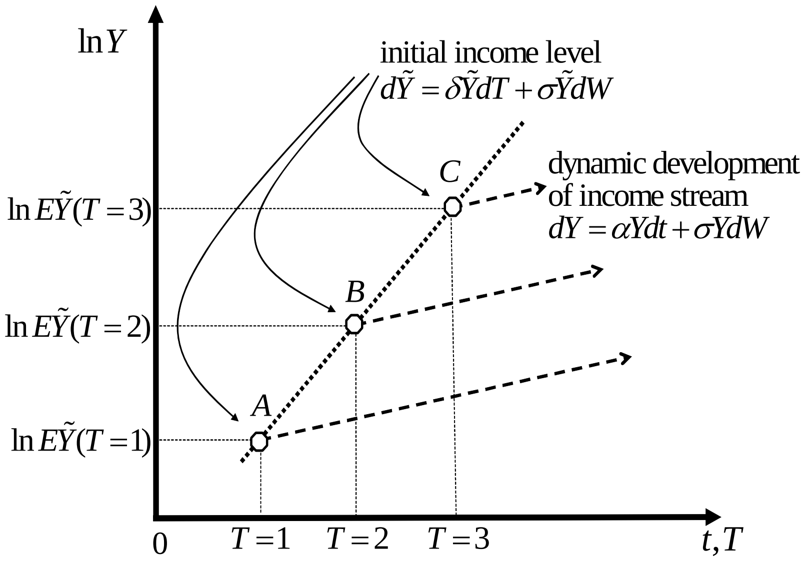


Figure 1: Earning profile: initial income level and dynamics of income

of a working life or even become negative, we try to keep things simple and assume $\alpha$ to be constant. For simplicity we also assume identical developments no matter how many years of schooling were completed. At this point we also do not distinguish between different years of schooling or achievment of formal educational levels like primary, secondary or tertiary education. Hence in figure 1 earning tracks are characterized as parallel processes. In a more realistic setting we need to distinguish between different earning dynamics according to the years of schooling or academic attainment. This will be examined later in this paper.

Once working life begins, the earning profile is fixed within the limits of the random process. Hence other opportunities are excluded and the economic value of the achieved education consists solely of its future income stream. For a risk neutral individual the gross value of *human capital (education wealth)* $V_i^{gross}$ is given by the expected present value of the *wage-income earning stream* $\{Y(t)\}$

$$V^{gross} = E(\int_T^\infty Ye^{-r(t-T)}dt) = \frac{Y}{r-\alpha}; \quad r > \alpha, \tag{4}$$

with $r$ being the risk free interest rate. For simplicity the individual has an infinite lifetime.

**Option value of the waiting and the decision problem:**
The option not to start working and not to irreversibly take the risk of embarking on a particular earning track has its own value - in correspondence to a firm's investment decision (Dixit (1989), and Dixit and Pindyck (1994)). Waiting may open up additional opportunities which could not have been foreseen and realized otherwise. Waiting also protects individuals from an "irreversible" departure from the education track. Once a student has left school they cannot return and are tied to the income and opportunity track they have chosen. In reality this is surely not as strict as suggested by the expression "irreversible".[8] However, the end of schooling often also marks the end of a period of a personal life cycle characterized by a particular measure of independence and flexibility. The entry into working life marks the beginning of a new phase in life often connected with the start of a family or responsibilities that are borne not only to the indiviudal themselfes. A return to an education program is not impossible, but often has rather high costs. Hence, leaving education is a bet that the present education achievement pays off sufficiently. If this bet is not successful the student's investment costs are lost as sunk costs. Therefore, waiting is a value because it offers the student flexibility. Further, once the student has decided to incur the sunk costs and enter the market, they could generally decide to exit if realized revenues are below expectation. In the current model, however, we assume that the student cannot exit the market voluntarily after entering - we thus preclude the exit option and leave it for a future extension to the present model. Accounting for the option value $F$ for the Brownian motion (2), the Hamilton-Jacobi-Bellman equation holds:

$$rFdt = E(dF). \tag{5}$$

This equation indicates that for a time interval dt, the total expected return on the investment opportunity is equal to the expected rate of capital appreciation.

## 2.2 Decision problem

For a student the education decision is a timing problem concerning whether to stay in school or enter the market. In order to clearly state the decision problem for this market entry we need to determine the *net value of education* $V$ (for any education achievement) and compare this value with the *option value* $F$.

To determine the expected *net value of human capital* the expected gross value (4) has to be adjusted for individual education costs $I(T)$ accumulated during the time of schooling (1). Hence, the net value of the earning stream of education investment is

$$V = V^{gross} - I(T) \tag{6}$$

In addition to the expected *net value of human capital (net wealth of education)* the third element of the decision problem, the *option value* of remaining

[8] Exit options could be included.

in the education system, has to be considered. Further, as long as the student delays market entry they retain the option of market entry without the risk of failure and having chosen the wrong earning stream.

Given the expected *net value of the earning stream* (6) the *option value* $F$ of postponing market entry and obtaining a better qualification by adding another year of schooling can be determined by applying dynamic programming.[9] Once the option value of waiting has been determined, the question of whether or not to wait for another period will be solved by the solution to:

$$\max\{V(T), F(T)\} \tag{7}$$

At any time during their education the student will compare the expected *net value of human capital* with the *option value* of remaining at school and not realizing an uncertain earning stream by starting work. As long as the option value of postponing the switch into working life and continuing one's education is higher than the value of realizing the uncertain income stream, the student will opt for another year of schooling. Solving this continuous decision problem determines the time of entry into the labor market and hence the optimal duration of schooling including the decision about the level of formal qualification.

## 2.3 Solving for the optimal time to leave school

Solving for the optimal time of market entry as described above has two steps. First, for each duration of schooling we need to determine the income value ($Y^*(T)$ *threshold*) that would be needed to start working after a certain duration of schooling. This threshold is the required initial income and hence the required level of the income path that would make one's education profitable. When the threshold is reached, the value of the earning stream becomes higher than the option value and hence market entry becomes more profitable than waiting and obtaining more education. Second, as the threshold would trigger the start of working life the student simultaneously observes the *development* of the *relevant initial income* level $\tilde{Y}(T)$. The student compares the threshold for their academic achievement with the corresponding current initial income and verifies if the threshold has already been reached. Third, if they decide to stay at school they will make a prediction about the *expected timing to leave school* and hence the expected duration of schooling. We will model these aspects in the following section.

**Determining the Entry Threshold:**

In order to determine the income value that triggers the switch we need to consider the standard conditions concerning a stochastic dynamic programming problem. In addition to the *Hamilton-Jacobi-Bellman equation* for the option value $F$ and applying Ito's lemma to $dF$ we have to use the well known boundary

[9] See the next section.

conditions, namely (8), the *value matching condition* (*9*), and *smooth pasting* condition (10)

$$\begin{aligned} F_{\tau}(0) &= 0, && (8) \\ F(Y^*) &= V^{gross}(Y^*) - I(T) \quad \textit{value matching condition,} && (9) \\ \frac{dF(Y^*)}{dY} &= \frac{d(V^{gross}(Y^*) - I(T))}{dY} \quad \textit{smooth pasting condition.} && (10) \end{aligned}$$

to solve for the threshold income $Y^*$. The setting of the decision problem implies that the value of the uncertain earning stream must be worth the switch from school to work. Hence, the wage level given by the Brownian motion must be high enough. Reaching this threshold triggers the change in strategy from more education towards entering the labor market. Therefore, determining this optimal threshold is the first part of a solution to the optimal expected timing of market entry problem.

**Proposition 1** *a) For a constant accumulation of costs per year of successful schooling (1), a sequence of increasing earning levels through schooling described by* (2)*, and an earning dynamics after market entry following* (3) *we can determine the threshold* $Y^*(T)$ *that would trigger the start of the earning/working process.*

$$\begin{aligned} Y^*(T) &= \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}\right] = \frac{\lambda}{\lambda-1}(r-\alpha)I(T), && (11) \\ \textit{with} \quad \lambda &= \frac{1}{2}-\frac{\delta}{\sigma^2}+\sqrt{(\frac{1}{2}-\frac{\delta}{\sigma^2})^2+\frac{2r}{\sigma^2}}, && (12) \\ \textit{and} \quad r &> \delta && (13) \end{aligned}$$

**Proof.** For a proof see Appendix 1.

Since each additional year of schooling adds to the total costs of education these costs are dynamic. The investment costs increase with years of schooling. Therefore, the threshold changes with the duration of schooling $T$, i.e. the threshold is a continuous function of $T$.

From the threshold function $Y^*(T)$ in Figure 2 we can see that an additional year of schooling drives up the threshold, that is, the student wants to be compensated for the additional year and costs by a higher initial level of income when entering the labor market. In other words, the student would only complete an addtional year if they expect to be compensated by the market.

**Determining the Path of Expected Initial Income Level:**
Once the student knows from the threshold at which initial income level they should start working, when can they expect to obtain this income from the market?

As described above, the path of the initial income level is another random process. For a year of schooling and the investment costs $C$ the student may

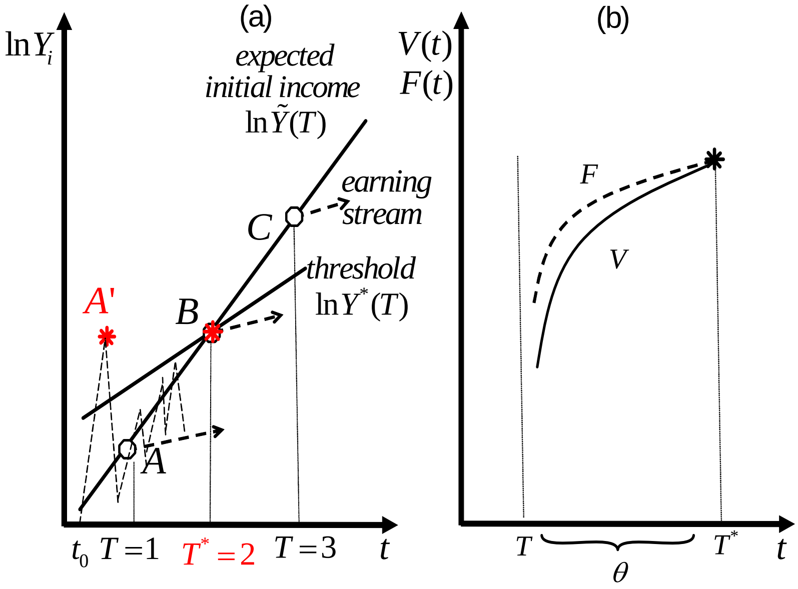


Figure 2: Threshold and income level

expect not only a different earnings dynamic during working life (described by (3)). The first income when entering the labor market is also expected to increase systematically by the rate $\delta$ in (2) when time of schooling $T$ increases. As decribed in figure 1 both income dynamics and the initial level define the complete earning profile. That is, whenever a student considers their education attainment they have expectations about the time path of the initial income level. Hence, the expected *initial income level curve* $\tilde{Y}(T)$ can be also drawn in in figure 2.[10]

**Proposition 2** *a) From the Brownian motion (3) we can derive the path of initial income levels* $\tilde{Y}(T) = \tilde{Y}(0)e^{((\delta-\frac{1}{2}\sigma^2)T+\sigma W(T))}$*, and hence determine the expected value of initial income for each duration of schooling* $T$*.*

$$E\tilde{Y}(T) = \tilde{Y}(0)e^{\delta T}. \tag{14}$$

**Proof.** For a proof see Appendix 2.

**Expected optimal time to leave school:**

The expected time of market entry can now be determined by comparing the current expected *initial income level* with the the *threshold* in figure 2a. Correspondingly, as long as the threshold has not been reached the option value

[10] More precisely, the log of the earning $\tilde{Y}_i(T)$. See also appendix 2.

is larger than the net present value of current human capital (2b)[11]. Schooling is expected to terminate when the expected initial income level reaches the threshold.

**Proposition 3** *a) With the threshold $Y^*(T)$ (see (11)), the expected development of the initial income level $E\tilde{Y}(T)$ (see (2)), and condition (16) and (17) there exist an the expected optimal time to leave school and start working life $T^* = E(T) > 0$. b) For each vector $(\alpha, r, \sigma, T^*, C, \tilde{Y}(0), \delta, \bar{C})$ that fulfils a) there is a marginal environment, such that $T^*$ is an implicit function of $\alpha, \sigma, C, \tilde{Y}(0), \delta, \bar{C}$ and $r$.*

$$T^* = T^*(\alpha, \sigma, C, \tilde{Y}(0), \delta\ r) \tag{15}$$

$$\frac{\lambda}{\lambda - 1}(r - \alpha)\bar{C} \quad > \quad \tilde{Y}(0), \tag{16}$$

$$\bar{C}r \quad > \quad \bar{C}\delta > C0) \tag{17}$$

**Proof.** For a proof see Appendix 3.

In figure 2, a higher *threshold* compared to the expected *initial income level* reflects that learning costs during the education attainment phase (before $T^*$) are not yet sufficiently compensated by the present *initial earning level.* Hence the student will not yet enter the market. Even more, conditions (16) is important to understanding the logic of the decison problem.[12] 1. The decision for any education will only be positive if the minimum wage (no-education income $\tilde{Y}(0)$) is sufficiently small compared to education costs (16).[13]

Further, as randomness is part of reality and random elements are modeled by a random process, the expected time of market entry ($T^*$) is just an indicator of what may actually happen in future. As the future is partly random, an unexpected departure from the education process for the labor market and market entry can easily happen at any time. In figure 2 we draw the time path of the expected initial income for the present state of information at time $t_0$ by $\tilde{Y}(T)$. A moment later, even if the individual expects to enter the market at $T^*$, a randomly occurring incident in the labor market (e.g a business cycle boom) can push initial earnings such that the threshold is reached and the student decides to start working immediately. In figure 2 this is displayed by the randomly upward shift of the realized market value at point $A'$. The observed and hence realized market earnings exceeds the threshold at $t_1$ and hence education is terminated at $T = 1$ and not - as expected before - at $T^*$. It is easy to find

[11] As in Dixit and Pindyck (1994) page160 the curves F,V-I have an upward slope. However, under certain conditions they can also decrease because in t his model costs are accumulated. For details see Appendix/Annotation 3.

[12] Both conditions are required for the existence of a solution of the problem.

[13] This condition is needed for the *threshold curve in figure 2* to start above the *initial income level* curve.

illustrative examples, like an unexpected offer of an extraordinarily well-paid job. In this model the student will take this randomly occurring opportunity at $T = 1$ - no matter what they planned and expected before.

These simple examples also clearly illustrate that education decisions are timing decisions. Leaving education means *taking the opportunity to realize the returns to education at the right time*, even if the opportunity occurs accidentally.

## 2.4 Determinants of the expected time of leaving school

In the previous chapter we determined the optimal expected timing of learning and discussed the implications for academic attainment. In particular we were able to show that the dynamic structure of the problem with special regard to risk and irreversibility is an important ingredient of the decision problem. In this section we examine the most important and most frequently discussed determinants of the expected optimal duration of schooling. In particular we look at the effects of risk, income dynamics, non-education wage level, costs of schooling and the effect of changes in the marginal initial income reward of schooling and interest rate changes.

**Effect of income uncertainty** ($\sigma$)**:**

**Proposition 4** *With an increase in risk $\sigma$ expected duration of schooling $T^*$ will increase,*

$$\frac{dT^*}{d\sigma} = \frac{\left(e^{rT} - 1 + \frac{\bar{C}r}{C}\right)}{\underbrace{\left(e^{rT} - \frac{\delta\left(\lambda-1\right)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)}_{<0}} \frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{\lambda\left(\lambda-1\right)r} > 0 \tag{18}$$

**Proof.** For a proof see Appendix 4.

The effect of rising risk - measured by the volatility of revenues - on the time of market entry can be expected and is consistent with Hogan and Walker (2007), but deviates from the results of Groot and Oosterbeek (1992) and Hanchane et al.(2006). An increasing risk of income will devalue the earning stream and hence will require higher compensation reflected by an increased threshold. As long as additional net rewards of longer education can compensate for the increase in the threshold schooling will be continued. In figure 3a the increasing risk will shift the threshold curve upwards, and hence makes a later market entry more attractive. Interestingly enough, this result is obtained even if we have no explicit evaluation of risk by a utility function (and hence implicitly risk neutral agents). We do not need to make any assumption about the utility function and risk aversion. The pure option value and the irreversibility includes the effects of $\sigma_i$ in a different way.

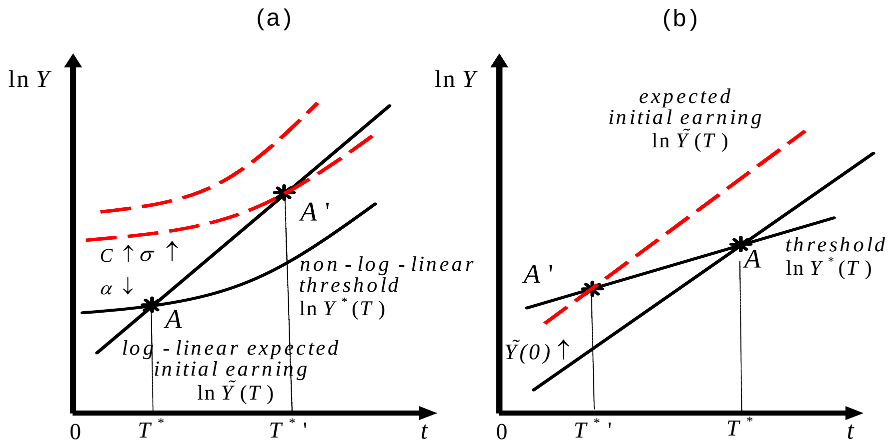


Figure 3: Changes in costs, earning dynamics and risk

**Effects of income growth** $(\alpha)$

**Proposition 5** *With an increase in the growth rate of the future earnings, the expected duration of schooling* $T^*$ *will decrease:*

$$\frac{dT^*}{d\alpha} = \frac{[e^{rT} - 1 + \frac{\bar{C}r}{C}]}{\underbrace{\left(e^{rT} - \frac{\delta\,(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)}_{<0}}\frac{1}{(r-\alpha)r} < 0 \tag{19}$$

**Proof.** For a proof see Appendix 5.

Declining general income growth affects the benefits of education. A lower earnings growth will decrease the expected present value of schooling. Lower growth and hence a less attractive dynamics of the earning track will only pay off if the level of initial earnings increases. This shifts the threshold curve in figure 3a upwards. With a sufficient marginal reward $\delta$ the required threshold can still be reached after more years of schooling. This new earning profile, characterized by a higher initial level of earnings to compensate for a less rapid income growth, still justifies an even longer education.

Effects of an increasing $\alpha$ can also be described by another intuitively plausible story. If $\alpha$ increases, the expected net value of human capital would increase as well. As it is now easier to obtain the same value with lower investments, investments can be reduced.

**Effects of the non-education (minimum) wage level ($\tilde{Y}(0)$)**

**Proposition 6** *A rising level of the non-education (minimum) wage level $\tilde{Y}(0)$ will decrease the duration of education $T^*$*

$$\frac{dT^*}{d\tilde{Y}(0)} = \frac{1}{\underbrace{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{(r-\delta)T} - \delta\tilde{Y}(0)}_{<0}} < 0. \tag{20}$$

**Proof.** For a proof see Appendix 6.

The *minimum wage* level $\tilde{Y}(0)$ represents the *no education wage level* when the schooling decision is made at $t = 0$. If the agent did not obtain any schooling they could start working with this initial wage $\tilde{Y}(0)$ (figure 3b). In case of an increasing *no education wage level* educational attainment will decrease. This finding is intuitively expected. A rise of $\tilde{Y}(0)$ indicates (all else being equal) that no education achieves a higher level in income. The higher the no education wage path the less attractive a long education, and the more attractive a quick market entry.

**Costs of schooling ($C$)**

**Proposition 7** *With increasing education costs the expected duration of schooling $T^*$ and hence academic achievement will rise*

$$\frac{dT^*}{dC} = \frac{-[e^{rT}-1]}{\underbrace{\left(e^{rT} - \frac{\delta\,(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)rC}_{<0}} > 0. \tag{21}$$

**Proof.** For a proof see Appendix 7.

As $C$ denotes the flow of investment costs for schooling, the reaction $\frac{dT^*}{dC} > 0$ is not the intuitively expected reaction. In the standard approach higher investment expenditure would increase the opportunity costs of education and hence would make education less profitable. As a result educational attainment would be reduced. Therefore, this result can be regarded as a *"tuition paradox"*.

In this approach the decision problem is different. With increasing costs of schooling the student needs compensation from the market to stay in the system. Therefore, with increasing $C$, the required threshold shifts upwards in figure 3a. As long as the market rewards the outcome of the additional schooling sufficiently ($\delta$ is sufficiently high), both curves would still intersect at a later time. Even if the project becomes less profitable overall, the response to increasing costs is choosing a higher income path (starting from a higher initial income level) generated by even more education. In other words, the new earning profile promises a sufficiently higher level of the earning path to compensate for the increase in costs and justify even more education. According to empirical results provided by Heckman et al. (2008) increasing costs could be partially compensated by higher investment in schooling and a corresponding rise in initial earnings. Higher costs may lead to longer education as long as the rewards are sufficient.

However, this is the simplest case discussed in the proposition. The intuitively expected outcome of reducing education when costs increase can be also obtained as soon as the non-linear expansion of the threshold does not allow for an intersection. If a non-linear threshold is pushed upward as drawn by the dashed line in figure 3a increasing schooling costs cause the student to leave the education system. The shift in the threshold cannot be matched by a sufficient market reward and we find no intersect. For the individual conditions (costs etc.) of this student, there is no inner solution of the timing problem - the student will leave school as soon as possible. This is the simplest solution to the "tuition paradox".

**Effects of marginal initial earning rewards of schooling $\delta$:**

**Proposition 8** *An increase in the marginal initial earning reward for a year of schooling is generally ambiguous. However, an increase in $\delta$ will tend to postpone $T^*$. It encourages more education if $\frac{\tilde{Y}(0)}{C}$ becomes sufficiently large within the limits of conditions (16) and (25) hold,*

$$\frac{dT^*}{d\delta} = \frac{\overbrace{\ln\left(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{Y}(0)}{\bar{C}}\right)}^{<0 \text{ see (16)}}}{(\beta-\delta)^2} + \frac{1}{\beta-\delta}\frac{\overset{(-)}{\frac{\partial\lambda}{\partial\delta}}}{(\lambda-1)\,\lambda} > 0 \tag{22}$$

**Proof.** For a proof see Appendix 8.

Looking at condition (22) the sign of the reaction depends on the relative importance of the two terms, i.e., we can identify two different effects. On the one hand, as education generates an increasing market reward, initial earning approaches the threshold more quickly [first term of (22)].

On the other hand, the threshold itself will be affected. Education time generates a higher reward and hence schooling duration becomes more valueable [second term of (22)]. Depending on these two relative effects we obtain a positive or a negative total effect. In proposition (22) we suggest that costs are relatively high compared to the earning level $\tilde{Y}(0)$. Hence schooling will be extended. Further, as the threshold is - among others - determined by the evaluation of the education time, we can see how the real option approach affects the decision.

**Effects of the interest rate:**

**Proposition 9** *An increase in the interest rate is generally ambiguous. However, an increase in the interest rate will reduce/increase education (decrease/increase $T^*$) if (26)/(27) holds*

$$\frac{dT^*}{dr} = \underbrace{\frac{1}{\beta - \delta}}_{(-)} \left[ \underbrace{\frac{\overset{(+)}{\frac{\partial \lambda}{\partial r}}}{\lambda(\lambda - 1)} - \frac{1}{(r - \alpha)}}_{=:X \gtreqless 0} \right] \gtreqless 0$$

**Proof.** For proof see Appendix 9

As formally discussed in the appendix the reaction is generally ambiguous. However, we can determine conditions to obtain one or the other reaction.

# 3 Education attainment with different levels of formal qualification

The purpose of this section is to account for discontinuities and non-linearities in the decision problem. After the general discussion of an optimal timing decision to leave school we now extend the model to include different levels of formal qualification. Wages and costs of schooling may not only increase with years of schooling but also may jump after an attainment of a certain education level to a higher income stream. The so-called "sheepskin effect" seems of increasing importance in the recent empirical discussion. Although one branch of research argues that there is a linear relationship between wages and years of schooling, recent findings support the hypothesis of non-linearities in incomes, which occur especially with college and high school completition.[14] Further, it seems that both the years of schooling and the achieved level of formal qualification determine the two elements of the earning profile, namely the initial income when entering the labor market, and the dynamics of earnings during working life. Therefore, considering years of schooling and completion of different levels of formal qualification simultaneously is the natural next step in extending the model.

In this extension optimal timing is a choice of a sequence of investments with varying earning profiles connected with the completion of different levels of formal qualification. In general the above model can be regarded as a model for only one level of formal qualification. In order to consider more levels we simply symmetrically add other earning profiles to decide on the full set of opportunities. Specifically, for each level of formal qualification $i$ (e.g. secondary

[14] see Heckman (1995); Denny, Harmon (2001); Skalli, A (2007); Ferrer, Ana (2008); Silles, Mary (2008).

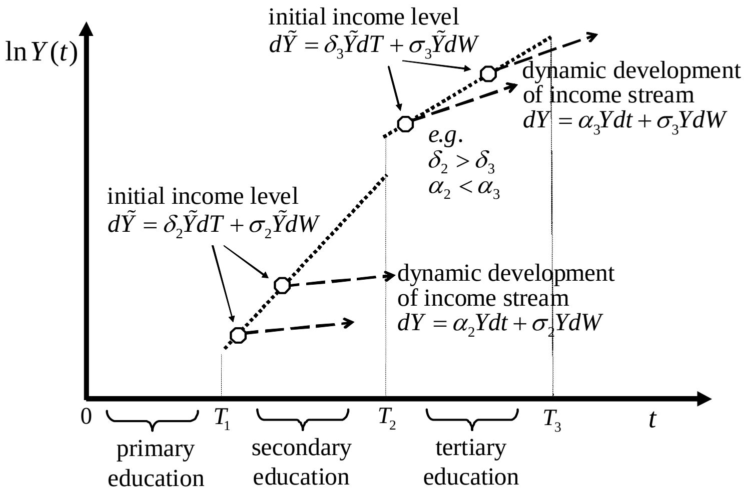


Figure 4: Changes in the minimum wage level and the initial earning rewards of schooling

education) we assume specific costs $C_i$ for a successful year of schooling, and we expect a specific earning profile connected to this formal qualification level. Again, a student observes that for each formal qualification level $i$ a year of additional schooling will increase their *initial income level* $\tilde{Y}_i$ according to the Brownian motion[15]

$$d\tilde{Y}_i = \delta_i \tilde{Y}_i + \sigma_i \tilde{Y}_i dW \quad \text{for} \quad t < T_i.$$

where $T_i$ denotes the years of schooling required to complete the formal qualification level $i$ (e.g in many secondary education programs students have to study four years, and hence $T_i = 4$). Further, the *dynamic development of income* $Y_i$ during working life for each level of formal qualification $i$ is once again described by[16]

$$dY_i = \alpha_i Y_i dt + \sigma_i Y_i dW.$$

It now becomes apparent that the individual earning profiles associated with different levels of formal qualification may be characterized by different marginal rewards in initial income level ($\delta_i$), different growth patterns ($\alpha_i$) and different risks ($\sigma_i$).

In figure 4 we describe an example for three levels of formal qualification (primary, secondary and tertiary education, $i = 1, 2, 3$) to illustrate the additional effects of completing of various qualification levels. Figure 4 describes

[15]Each level of formal qualification is modeled symmetrically to the reference model above. Therefore see also (2) for only one level of formal qualification.

[16]see again (3).

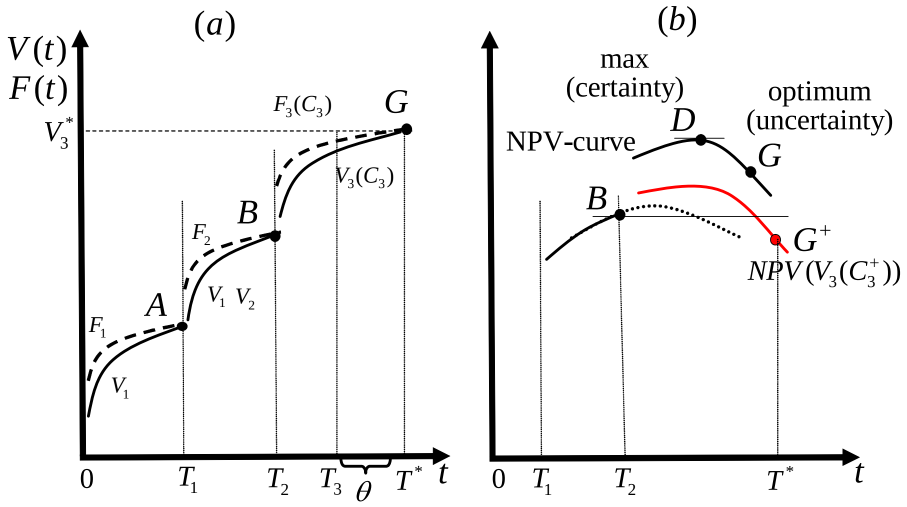


Figure 5: Menue of choices with high education costs

various profiles for various years of schooling and levels of formal qualification. When a specific number of years $T_i$ is reached the student attains a formal qualification $i$ (e.g. secondary education). In this diagram for higher level of formal qualification we assume a smaller $\delta_i$ and an larger $\alpha_i$. As a result, students studying at e.g. secondary level are rewarded for additional schooling with relatively strong boost in initial wages, but will also expect relatively low income dynamics. However, other patterns of income profiles are possible.

As in the previous chapter we can now take the initial income curve $\tilde{Y}_i(T)$, determine the threshold curve $Y_i^*(T)$[17] and derive the optimal time of schooling for each level of qualifcation $T_i^*$. For our example involving three levels of formal qualification figure 5a gives the results in the $V, F - T$-plane. When education starts, a student will make a decision concerning the anticipated duration of schooling and the corresponding formal academic qualification. Their plan uses available information and takes into account potential irreversibility.

Figure 5a exhibits three profiles of net value of human capital *(net wealth of education)* $V_i$ ($i = 1, 2, 3$ see (6)) and the corresponding option value, one for each level of formal qualification. At each moment students will compare the *net value of human capital* when starting to work now $V_i(T)$, with the *value of the option* of staying in school and obtaining more education $F_i(T)$ (see (7)) As long

[17] To keep the discussion simple, we do not explicitly consider the option value of a total sequence, such that the completion of one level education is required to start the next education phase. This extension would increase the option value.

as the option value is higher than the net value of human capital the student will defer their decision to enter the labor market. Waiting and completing more years of schooling is the dominant strategy. In figure 5a we can follow this consideration from the start. If $V_1 < F_1$ the student will defer entering the labor market. They will do so until point $A$ is reached. At point $A$ the value matching condition (9) and smooth pasting condition (10) simultaneously hold and an optimum has been reached. If there are no other alternatives, $A$ would be the optimal duration of schooling at primary level. However, there are alternatives. The student could finish primary education and then opt for additional schooling at secondary level. As the value of a completed primary education and some secondary-level schooling ($F_2$) is higher than $V_1$ at point $A$ the student will complete primary education and enter secondary education. For this higher, secondary education level they go through the same considerations as before. Comparing the value of entering the labor market with the option values of additional education and the attendant opportunities, they will opt for additional years of schooling in the example of figure 5a. The expected optimal time of schooling $T^*$ is reached at the formal qualification of tertiary education plus some additional schooling time. Since for each formal qualification the optimal timing to leave school and entering the labor market can be determined, it is easy to pick the optimal point according to the decision rule. $G$ is the stable optimal solution of this dynamic decision problem which includes irreversibility and uncertainties.

Moreover, as completing formal qualification $i = 3$ (tertiary education) is reached at $T = T_{i=3}$ and $T^* > T_{i=3}$ the student decides to have more years of schooling than the minimum years required to complete tertiary education. If $\theta = T^* - T_i > 0$ are the additional years of schooling beyond the minimum formal requirement, why these extra years of education? In real life we find many education activities which are not directly connected to a formal qualification. Internships, a language course in a language school abroad, time spent on searching for a job and collecting information and even time spent on developing ones personal profile can extend one's duration of education beyond the minimum required duration for a given formal qualification. Therefore, these additional years are part of the value of the waiting option after formal education has been attained. The additional years $\theta$ do not necessarily take place after a formal education has been reached; they can, and in fact normally do, take place at any time during the formal education program.
We can also think of other examples especially at tertiary level when an extra year may be completed during the official term of study. At universities many students take a sabbatical from their home program. In this case the extra year is taken before formal graduation. The extra year will extend the time until the formal qualification is completed. In this case working life still starts right after graduation, which seems to be the normal pattern. Therefore, equation (15) is the number of school years including those elements that are not formally required for formal qualification at secondary level.

Further, to have a reference system the NPV curve under certainty is drawn in figure 5b. First, under certainty, the standard problem is to pick the optimal duration of schooling that maximizes the net present value (point D in figure 5b). Under uncertainty the optimal strategy leads to a later market entry leading to a lower net present value than under certainty.[18] Sustaining flexible is compensated by a reduction of the NPV (point G in 5b). Second, The diagram also illustrates how a person can choose the optimal duration of schooling at the tertiary level, even if secondary education would lead to a higher expected net value of human capital. In order to illustrate this decision we assume high tuition costs at tertiary level of qualification. The high cost of a college education ($C_3 < C_3^+$) would shift the $V_3$ and $F_3$ curve.[19] dwonward. Hence, after secondary education ($T \geqq T_2$), the expected optimal duration of schooling at tertiary level is indicated by point $G^+$. $G^+$ is the only stable point at the tertiary level.

The optimal plan suggests that education is expected to finish at tertiary level. Therefore, sufficiently high education costs compared to the level of the earnings path and the resulting effects on the net value of human capital would drive a student to extend their education, even if less education has as higher expected net value. The dynamic decision problem suggests more education. The expectation to start at a higher initial income level sufficiently compensates for the risk and inflexibility involved in being tied to a certain income track. Hence market entry is postponed even if the expected net wealth decreases as a result. Heckman et al.(2008) estimate marginal internal rates of return for several schooling levels taking into account tuition costs (consisting of tuition and non-pecuniary costs), income taxes and non-linearities in the earnings-schooling-experience relationship. The internal rate of return to schooling is used to rate whether educational expenditure should be expanded or contracted. The authors find that there are relatively greater returns to graduating from high school than to graduating from college. These findings seem consistent with the results in this approach.

However, as discussed in the subsequent section (see figure 2), this menue of choices reflects long term expectations regarding on years of schooling and completing formal education, while the future is uncertain. Therefore, this optimal plan that is valid only under the present conditions of period $t = 0$. It is a preliminary optimal plan which needs to be revised permanently. One period later the optimal plan and the marginal decision to wait or start working is adjusted to account for new information and conditions. The sequence of marginal decisions terminates when the decision to switch is made. Then the earnings track has been chosen irreversibly. In other words, even if the plan suggests that schooling should be expected to be terminated in 3 years (e.g. after tertiary education is completed), a sudden change in conditions can terminate schooling overnight, leading to an unexpected entry into working life.

[18] The differences of the outcome of optimal decisions under certainty and uncertainty using this approach are also described for a simple example by Dixit and Pindyck (1994) chapter 5.1.

[19] See appendix 3d.

# 4 Summary

The major purpose of this paper is to derive a timing rule for leaving school and entering the labor market. Thus, we answer the question: How long should one go to school? Considering uncertainty of future developments we use real option theory in terms of a dynamic programming model. Schooling expenditure, earning streams and the option values of remaining in education determine the optimal timing of departure from school and one's academic achievement. In addition to the recent literature our approach includes schooling costs depending on individual abilities and qualification programs. Future income level rewards for additional schooling are uncertain and described by a Brownian motion. A second Brownian motion describes the dynamics of the income earning profile once working life has started. At all times an individual can decide to remain in school or leave the education system and start working. However, once the decision to leave the education system is made, it is not possible to reenter. The decision is irreversible. As the individual knows the expected path of earning streams and the optimal threshold, the expected optimal duration of schooling can be identified. With the optimal duration of schooling we also determine the optimal academic achievement and the level of formal qualification. We solve the model purely analytically and find: 1) An increasing income risk increases the duration of education. 2) With an increase in the generally expected growth of individual income the expected time of schooling will decrease leading to less education. 3) An increase in the no-education wage level will reduce human capital investments. 4) A rising marginal reward for a year of schooling (in terms of a rising differential in the income levels) will under certain conditions favour investments in human capital. 5) Increasing education costs may also increase human capital investments as long as the marginal reward for a year of schooling is sufficiently high and hence additional costs have a sufficient income compensation. However, if we allow for discontinuities due to various cost and income profiles of formal qualification levels, very high (individual) costs of schooling may lead to an achievable maximum net wealth of human capital at the lower levels of formal qualification.

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# 6 Annotations (to help the referee follow the calculations easily)

## 6.1 Annotation 1: Proof of Proposition 1 and Derivatives of $\lambda$

The value of the revenue stream is determined by

$$\begin{aligned} V^{gross} &= E\int_T^{\infty} e^{-r(t-T)} Y\, dt \\ &= \int_T^{\infty} Y e^{-r(t-T)} e^{\alpha(t-T)} \\ &= \left[\frac{1}{\alpha - r} e^{(\alpha-r)(t-T)} Y\right]_T^{\infty} \\ &= \frac{Y}{r-\alpha} \end{aligned}$$

For the option values $F_i$ the Hamilton-Jacobi-Bellman equation for the Brownian motion of 2 holds:

$$rF = \frac{1}{dt} E(dF)$$

From Ito's Lemma we know:

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial t} + \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma^2\tilde{Y}^2\frac{\partial F}{\partial \tilde{Y}^2}\right)dt + \sigma\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} dW \\ \Rightarrow E(dF) &= \left(\frac{\partial F}{\partial t} + \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma^2\tilde{Y}^2\frac{\partial F}{\partial \tilde{Y}^2}\right)dt \end{aligned}$$

because $E(dW) = 0$.

From the last two equations we obtain the following differential equation:

$$\underbrace{\frac{\partial F}{\partial t}}_{=0} + \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma^2\tilde{Y}^2\frac{\partial F}{\partial \tilde{Y}^2} - rF = 0$$

$$\Leftrightarrow \delta\tilde{Y}\frac{\partial F}{\partial \tilde{Y}} + \frac{1}{2}\sigma^2\tilde{Y}^2\frac{\partial F}{\partial \tilde{Y}^2} - rF = 0$$

This is a second-order homogenous ordinary differential equation with a free boundary.

A general solution to this differential equation will be

$$F = B\tilde{Y}^{\lambda}.$$

$B\tilde{Y}^{\lambda}$ solves the homogenous differential equation.

$$\begin{aligned}
\delta \tilde{Y} B\lambda \tilde{Y}^{\lambda-1} + \frac{1}{2}\sigma^2 B \tilde{Y}^2 \lambda(\lambda-1)\tilde{Y}^{\lambda-2} - rB\tilde{Y}^{\lambda} &= 0 \\
\delta B\lambda \tilde{Y}^{\lambda} + \frac{1}{2}\sigma^2 B\lambda(\lambda-1)\tilde{Y}^{\lambda} - rB\tilde{Y}^{\lambda} &= 0 \\
\delta\lambda + \frac{1}{2}\sigma^2\lambda(\lambda-1) - r &= 0
\end{aligned}$$

$$\Leftrightarrow \quad \lambda = \frac{1}{2} - \frac{\delta}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} > 1 \qquad \text{see (12)}$$

with $\quad \delta \quad < \quad r \quad$ see *(13)*

As $\tilde{Y}$ goes to zero, $F$ tends to 0. This implies that the negative root of the characteristic polynomial should have no influence on $F$ as $\tilde{Y}$ tends to zero.

Besides $\lambda > 1 \Leftrightarrow r > \delta$ :

$$\begin{aligned}
\frac{1}{2} - \frac{\delta}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} &> 1 \\
\sqrt{(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} &> \frac{1}{2} + \frac{\delta}{\sigma^2} \\
(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2} &> \left(\frac{1}{2} + \frac{\delta}{\sigma^2}\right)^2 \\
-2\frac{\delta}{\sigma^2}\frac{1}{2} + \frac{2r}{\sigma^2} &> 2\frac{\delta}{\sigma^2}\frac{1}{2} \\
-\frac{\delta}{\sigma^2} + \frac{2r}{\sigma^2} &> \frac{\delta}{\sigma^2} \\
r &> \delta
\end{aligned}$$

For the derivatives of $\lambda$ we get:

$$\begin{aligned}
\frac{d\lambda}{d\delta} &= -\frac{1}{\sigma^2} - \frac{2}{2}\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}-1}\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)\frac{1}{\sigma^2} \\
&= -\frac{1}{\sigma^2}\left[1 + \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)\right] < 0 \\
&= -\frac{\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}}{\sigma^2}\left[\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}} + \left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)\right] < 0 \\
&= -\frac{\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}}{\sigma^2}\lambda < 0
\end{aligned}$$

$$\frac{d\lambda}{dr} = \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}} \frac{1}{\sigma^2} > 0$$

$$\begin{aligned}
\frac{d\lambda}{d\sigma} &= \frac{2\delta}{\sigma^3} + \frac{1}{2}\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}} (2(\frac{1}{2} - \frac{\delta}{\sigma^2}) \cdot \frac{2\delta}{\sigma^3} - \frac{4r}{\sigma^3}) \\
&= \frac{2\delta}{\sigma^3} + \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}} ((\frac{1}{2} - \frac{\delta}{\sigma^2}) \cdot \frac{2\delta}{\sigma^3} - \frac{2r}{\sigma^3}) \\
&= \frac{2\delta}{\sigma^3}\left[1 + \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}} \left((\frac{1}{2} - \frac{\delta}{\sigma^2}) - \frac{r}{\delta}\right)\right] \\
&= \frac{2\delta\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}}{\sigma^3}\left[\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}} + \frac{1}{2} - \frac{\delta}{\sigma^2} - \frac{r}{\delta}\right] < 0
\end{aligned}$$

At the investment trigger point $Y^*$ the value of the option must equal the net value obtained by exercising it (value of the active project minus sunk cost of the investment). Hence the following must hold:

$$\begin{aligned}
F(Y^*) &= V^{gross}(Y^*) - I(T). \\
&= \int_T^\infty Y^* e^{-r(t-T)} e^{\alpha(t-T)} - \left[\int_0^T e^{r(T-t)} C dt + \bar{C}\right] \\
&= \left[\frac{1}{\alpha - r} e^{(\alpha-r)(t-T)} Y^*\right]_T^\infty - \left[\left[-\frac{C}{r} e^{r(T-t)}\right]_0^T + \bar{C}\right] \\
&= 0 - \frac{Y^*}{\alpha - r} e^{(\alpha-r)(T-T)} - (-\frac{C}{r} + \frac{C}{r} e^{rT} + \bar{C}) \\
&= \frac{Y^*}{r - \alpha} - \frac{C}{r}\left(e^{rT} - 1\right) - \bar{C}
\end{aligned}$$

$$B(Y^*)^\lambda = \frac{Y^*}{r - \alpha} - \frac{C}{r}\left(e^{rT} - 1\right) - \bar{C}$$

Besides for $I(T) > 0$ we have to assume that $\bar{C} > \frac{C}{r}$.

The smooth-pasting condition requires that the two value functions meet tangentially:

$$(F(Y^*))' = (V^{gross}(Y^*))'$$

$$\Leftrightarrow B\lambda(Y^*)^{\lambda-1} = \frac{1}{r - \alpha}$$

This implies

$$B(Y^*)^\lambda = \frac{Y^*}{(r-\alpha)\lambda}$$

Now we compute the threshold $Y^*$:

$$\frac{Y^*}{r-\alpha} - \frac{C}{r}\left(e^{rT}-1\right) - \bar{C} = \frac{Y^*}{(r-\alpha)\lambda}$$

$$\Leftrightarrow \frac{Y^*\lambda - Y^*}{(r-\alpha)\lambda} = \frac{C}{r}\left(e^{rT}-1\right) + \bar{C}$$

$$\Leftrightarrow Y^*(\lambda-1) = (r-\alpha)\lambda\left[\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}\right]$$

$$\Leftrightarrow Y^*(T) = \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}\right] = \frac{\lambda}{\lambda-1}(r-\alpha)I(T)$$

with $\ln I(T)$ being convex and hence $\ln Y^*(T)$ being a convex function in $T$.:

$$\frac{\partial \ln I}{\partial T} = \frac{Ce^{rT}}{\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}} > 0,$$

$$\begin{aligned}
\frac{\partial^2 \ln I}{\partial T^2} &= \frac{Cre^{rT}(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}) - C^2e^{2rT}}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} > 0 \\
&= \frac{Cre^{rT}\frac{C}{r}e^{rT} - \frac{C}{r}Cre^{rT} + Cre^{rT}\bar{C}) - C^2e^{2rT}}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} \\
&= \frac{-\frac{C}{r}Cre^{rT} + Cre^{rT}\bar{C}}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} \\
&= \frac{Ce^{rT}\left(r\bar{C}-C\right)}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} > 0 \text{ (convex) as we assume condition (17)} \qquad (23)
\end{aligned}$$

$$\begin{aligned}
\lim_{T\to\infty}\frac{\partial \ln I}{\partial T} &= \lim_{T\to\infty}\frac{Ce^{rT}}{\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}} = \lim_{T\to\infty}\frac{re^{rT}}{\left[e^{rT}-1+\frac{r\bar{C}}{C}\right]} \\
&= \lim_{T\to\infty}\frac{re^{rT}}{\left[1+\frac{-1+\frac{r\bar{C}}{C}}{e^{rT}}\right]e^{rT}} = r
\end{aligned}$$

## 6.2 Annotation 2: Deriving T and Proof of Proposition 2

**a) Development of the initial income level value:** The development of the pre-start-up market value is determined by

$$d\tilde{Y} = \delta\tilde{Y} + \sigma\tilde{Y}dW$$

We put $g(x) = \log x$ to get the Ito formula for $\log\tilde{Y}(t)$:

$$\begin{aligned} d(\log\tilde{Y}(t)) &= (\frac{1}{\tilde{Y}(t)}\delta\tilde{Y}(t) + \frac{1}{2}(-\frac{1}{\tilde{Y}(t)^2})\tilde{Y}^2\sigma^2)dt + \frac{1}{\tilde{Y}(t)}\sigma\tilde{Y}(t)dW \\ &= (\delta - \frac{1}{2}\sigma^2)dt + \sigma dW \end{aligned}$$

We obtain after integration

$$\begin{aligned} \log\tilde{Y}(T) - \log\tilde{Y}(0) &= \int_0^T(\delta - \frac{1}{2}\sigma^2)dt + \int_0^T\sigma dW \\ &\Leftrightarrow \log\tilde{Y}(T) = \log\tilde{Y}(0) + (\delta - \frac{1}{2}\sigma^2)T + \sigma W(T), \text{ and hence} \\ \tilde{Y}(T) &= \tilde{Y}(0)e^{((\delta-\frac{1}{2}\sigma^2)T+\sigma W(T))} \quad \text{and hence} \\ E\tilde{Y}(T) &= \tilde{Y}(0)e^{\delta T}. \end{aligned}$$

$$\frac{\partial E\tilde{Y}(T)}{\partial T} = \delta\tilde{Y}(0)e^{\delta T}$$

and $\ln E\tilde{Y}(T)$ is a linear function in $T$:

$$\ln E\tilde{Y}(T) = \ln\tilde{Y}(0) + \delta T$$

## 6.3 Annotation 3: Existence of a solution for the expected time $T^*$ of market entry, and determination of $T^*$ as an implicit function/Proof of Proposition 3:

In general we look for conditions described in figure 6. The threshold starts above the initial income curve. For positive $T$ the threshold will have an unique intersection with the initial value curve from below at $A$. Hence at the time of expected market entry denoted by $T^*$ $G = Y^*(T) - E\tilde{Y}(T) = 0$ and the $G-$curve has a negative slope $\frac{dG}{dT} < 0$.

Further, at $T^*$ the threshold $Y^*(T = 0)$ must start above the expected initial income $E\tilde{Y}(T = 0)$, and $G > 0$ during the pre-market entry period $(0 < t < T^*)$. Otherwise the market entry would have been taken place.

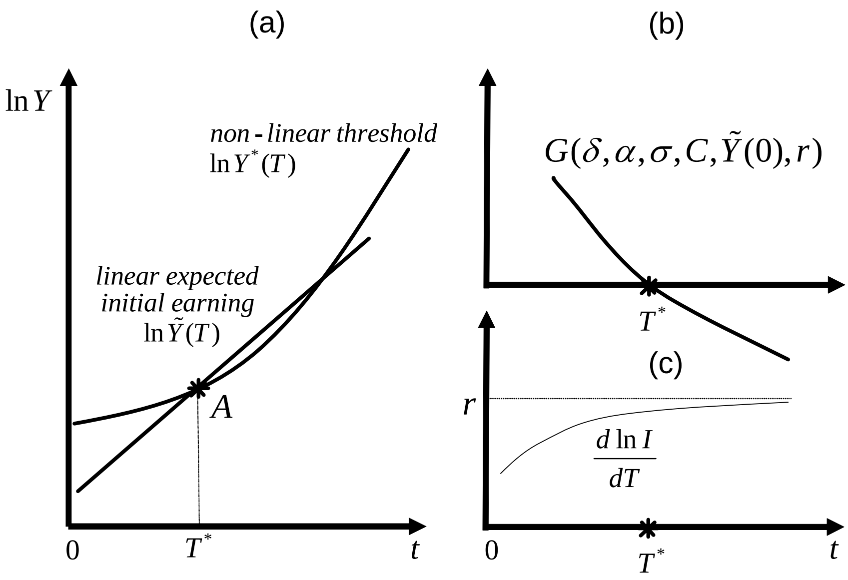


Figure 6: Intersection of the threshold and initial income function, and distance function G.

#### 6.3.1 Negative slope of $G$

$$\begin{aligned}
\frac{\partial G}{\partial T^*} &= \frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT^*} - \delta\tilde{Y}(0)e^{\delta T^*} < 0 \qquad (24)\\
&\Leftrightarrow \tilde{Y}(0) > \frac{\lambda}{\lambda-1}\frac{(r-\alpha)}{\delta}Ce^{(r-\delta)T^*}\\
&\Leftrightarrow \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}e^{(r-\delta)T^*} - (\frac{C}{r}-\bar{C})e^{-\delta T^*}\right] > \frac{\lambda}{\lambda-1}\frac{(r-\alpha)}{\delta}Ce^{(r-\delta)T^*}
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \frac{C}{r}e^{(r-\delta)T^*} - (\frac{C}{r}-\bar{C})e^{-\delta T^*} > \frac{C}{\delta}e^{(r-\delta)T^*}\\
&\Leftrightarrow \frac{C}{r} + (-\frac{C}{r}+\bar{C})e^{-rT^*} > \frac{C}{\delta}\\
&\Leftrightarrow (-\frac{C}{r}+\bar{C})e^{-rT^*} > \frac{C}{\delta} - \frac{C}{r}\\
&\Leftrightarrow e^{-rT^*} > \frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}
\end{aligned}$$

$$\Leftrightarrow \quad -rT^* > \ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right) \Leftrightarrow T^* < \frac{-1}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)$$

$$\text{and } \frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}} \quad < \quad 1 \Leftrightarrow \frac{C}{\delta}-\frac{C}{r} < -\frac{C}{r}+\bar{C}$$

$$\Leftrightarrow \quad \frac{C}{\delta} < \bar{C} \Leftrightarrow C < \delta\bar{C} < r\bar{C} \quad \textit{see} \quad 17$$

Before market entry the initial income curve must grow faster than the threshold curve. Only for a negative slope $G$ can approach and eventually reach zero. $\frac{\partial G}{\partial T} < 0$ is fulfilled if condition $\bar{C} > \frac{C}{r}$ (condition 17)

### 6.3.2 Existence of an intersect of $Y^*(T^*)$ and $E\tilde{Y}(T^*)$ for positive $T^*$

a) As the function $\ln Y^*(T)$ is convex if condition (17) holds (see 23) and the function $\ln E\tilde{Y}(T)$ is linear, there are at most two intersections. We are interested only in intersections at $T > 0$. An intersection for positive values of both functions exsits if condition (16) and (17) holds and $G = 0$ for positive values of $T^*$.

$$\begin{aligned}
G &= Y^*(T^*) - E\tilde{Y}(T^*) = 0 \\
&= \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right] - \tilde{Y}(0)e^{\delta T^*} = 0 \\
\Leftrightarrow &\quad \ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right] = \ln\tilde{Y}(0) + \delta T^* \\
\Leftrightarrow &\quad \ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right] - \ln\tilde{Y}(0) = \delta T^* \\
\Leftrightarrow &\quad \frac{1}{\delta}\ln\left[\frac{\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]}{\tilde{Y}(0)}\right] = T^* \overset{!}{>} 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow &\quad \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right] > \tilde{Y}(0) \\
\Leftrightarrow &\quad \frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^{rT^*} > \tilde{Y}(0) + \frac{\lambda}{\lambda-1}(r-\alpha)\left(\frac{C}{r}-\bar{C}\right) \\
\Leftrightarrow &\quad e^{rT^*} > \frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)}\frac{r}{C}\tilde{Y}(0) + 1 - \bar{C}\frac{r}{C} \\
\Leftrightarrow &\quad T^* > \frac{1}{r}\ln\Bigg[\underbrace{\frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)}\frac{r}{C}\tilde{Y}(0) + 1 - \frac{r}{C}\bar{C}}_{<1}\Bigg] \text{see (16)}
\end{aligned}$$

$$\Leftrightarrow \tfrac{r}{C}\bar{C} > \tfrac{\lambda-1}{\lambda}\tfrac{1}{(r-\alpha)}\tfrac{r}{C}\tilde{Y}(0)$$

$$\Leftrightarrow \bar{C} > \tfrac{\lambda-1}{\lambda}\tfrac{1}{(r-\alpha)}\tilde{Y}(0)$$

The last inequality is a condition for the axis intercepts of $Y^*$and $E\tilde{Y}$. It guarantees that $E\tilde{Y}$ has a lower value in $T = 0$ than $Y^*$.

$$\begin{aligned} Y^*(0) &> E\tilde{Y}(0) \\ &\Rightarrow \bar{C} > \frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)}\tilde{Y}(0) \end{aligned}$$

b) Further, In figure 6 the condition for an intersection and a negative slope have to hold simultaneously at $T^*$. We need to show that there is a $T^*$ were both $\frac{dG}{dT} < 0$ and $G = 0$ hold. That is, we can find a minimum level for $\tilde{Y}(0)$ in order to ensure an intersection and a negative slope:

$$\underbrace{\frac{1}{\delta}\ln\left[\frac{\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]}{\tilde{Y}(0)}\right]}_{\text{follows from } G=0} = T^* < \underbrace{\frac{-1}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)}_{\text{follows from the slope condition}}$$

$$\frac{1}{\delta}\ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right] - \frac{1}{\delta}\ln\tilde{Y}(0) < \frac{-1}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)$$

$$\ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right] + \frac{\delta}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right) < \ln\tilde{Y}(0)$$

$$\ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right] + \underbrace{\frac{\delta}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)}_{c<0} < \ln\tilde{Y}(0)$$

$$\begin{aligned} \left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right]e^c &< \tilde{Y}(0) \\ \frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^{rT^*}e^c - \frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^c + \frac{\lambda}{\lambda-1}(r-\alpha)\bar{C}e^c &< \tilde{Y}(0) \end{aligned}$$

$$\begin{aligned} \frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^{rT^*}e^c &< \tilde{Y}(0) + \frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^c - \frac{\lambda}{\lambda-1}(r-\alpha)\bar{C}e^c \\ e^{rT^*}e^c &< \tilde{Y}(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda} + e^c - \frac{r\bar{C}}{C}e^c \\ e^{rT^*} &< \tilde{Y}(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda}e^{-c} + 1 - \frac{r\bar{C}}{C} \\ T^* &< \frac{1}{r}\ln\left(\tilde{Y}(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda}e^{-c} + 1 - \frac{r\bar{C}}{C}\right) \end{aligned}$$

$$\begin{aligned} \tilde{Y}(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda}e^{-c} + 1 - \frac{r\bar{C}}{C} &> 1 \\ \tilde{Y}(0) &> \bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)e^c \end{aligned}$$

$$\tilde{Y}(0) > \bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)\underbrace{\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)^{\frac{\delta}{r}}}_{<1}$$

Finally, $\tilde{Y}(0)$ has to lie in the open interval $\left(\bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)\ ,\ \bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)^{\frac{\delta}{r}}\right)$.

### 6.3.3 c) $T^*$ as implicit function of various variables: Proof of propositon 3

**Proof of Proposition 3:** (i) condition , (16) hold,

(ii) the derivative $\frac{\partial G}{\partial T}(\alpha, r, \sigma, T^*, C, \tilde{Y}(0), \delta, \bar{C})$ is negative (see condition (24)) for each vector $(\alpha, r, \sigma, T^*, C, \tilde{Y}(0), \delta, \bar{C})$ and

(iii) the partial derivatives of $G$ by of $\alpha, \sigma, C, \tilde{Y}(0), \delta, \bar{C}$ and $r$ are continuous (vide infra), we can apply the implicit function theorem. Hence for a marginal environment of any vector $(\alpha, r, \sigma, T^*, C, \tilde{Y}(0), \delta, \bar{C})$, $T^*$ is an implicit function of of $\alpha, \sigma, C, \tilde{Y}(0), \delta, \bar{C}$ and $r$. q.e.d.

$$T^* = T^*(\alpha, \sigma, C, \tilde{Y}(0), \delta\ , r, \bar{C})$$

### 6.3.4 d) Curve properties of $V = V^{gross} - I$ (Net Current Value)

$$\begin{aligned} V &= \frac{\tilde{Y}(0)e^{\delta T}}{r-\alpha} - \frac{C}{r}\left(e^{rT}-1\right) - \bar{C} \\ &\Rightarrow \quad (V^{gross} - I)(0) = \frac{\tilde{Y}(0)}{r-\alpha} - \bar{C} \end{aligned}$$

$$\frac{d(V)}{dT} = \frac{\delta\tilde{Y}(0)e^{\delta T}}{r-\alpha} - Ce^{rT}$$

Maximum of the curve:

$$\begin{aligned} 0 &= \frac{d(V)}{dT} = \frac{\delta\tilde{Y}(0)e^{\delta T}}{r-\alpha} - Ce^{rT} \Rightarrow \ln\left[\frac{\delta\tilde{Y}(0)}{r-\alpha}\right] + \delta T = \ln C + rT \\ &\Leftrightarrow \quad T = \frac{1}{r-\delta}\ln\left[\frac{\tilde{Y}(0)}{r-\alpha}\frac{\delta}{C}\right] \end{aligned}$$

$$\begin{aligned}
\frac{d^2(V)}{dT^2} &= \frac{\delta^2\tilde{Y}(0)e^{\delta T}}{r-\alpha} - rCe^{rT} < 0 \\
&\Leftrightarrow rCe^{rT} > \frac{\delta^2\tilde{Y}(0)e^{\delta T}}{r-\alpha} \\
&\Leftrightarrow \ln(rC) + rT > \ln(\frac{\delta^2\tilde{Y}(0)}{r-\alpha}) + \delta T \\
&\Leftrightarrow T > \frac{1}{r-\delta}\ln(\frac{\delta^2\tilde{Y}(0)}{r-\alpha}\frac{1}{rC}) \\
\text{for } T &= \frac{1}{r-\delta}\ln\left[\frac{\tilde{Y}(0)}{r-\alpha}\frac{\delta}{C}\right] \text{ we get} \\
\frac{\delta}{r} &> 1
\end{aligned}$$

$$\frac{\partial V}{\partial C} = -\frac{1}{r}e^{rT} < 0$$

## 6.4 Annotation 4: Proof of Proposition 4

To apply comparative statics for the implicit function $T^* = T^*(\alpha, \sigma, C, \tilde{Y}(0), \delta\ r)$ we need to consider

$$\frac{\partial G}{\partial T} = \frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{Y}(0)e^{\delta T} \gtreqless 0$$

Since we are only interested in values of $T^*$ described by point $A$ in figure 6 conditions (16) and $\frac{\partial G}{\partial T^*} < 0$ (17). Then at $T^*$ we obtain:

$$\begin{aligned}
\frac{dT^*}{d\sigma} &= -\frac{\frac{dG}{d\sigma}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{Y}(0)e^{\delta T}}[\frac{-\frac{\partial\lambda}{\partial\sigma}}{(\lambda-1)^2}\{(r-\alpha)\frac{C}{r}e^{rT} - (r-\alpha)\frac{C}{r} + \bar{C}(r-\alpha)\}] \\
&= \frac{1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{Y}(0)e^{\delta T}}\frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{(\lambda-1)^2}(r-\alpha)(\frac{C}{r}\left(e^{rT}-1\right) + \bar{C}) > 0 \\
&= \frac{1}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)\frac{(r-\alpha)\lambda}{\lambda-1}C}\frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{(\lambda-1)^2}(r-\alpha)\frac{C}{r}(e^{rT} - 1 + \frac{\bar{C}r}{C}) > 0 \\
&= \frac{\left(e^{rT} - 1 + \frac{\bar{C}r}{C}\right)}{\underbrace{\left(e^{rT} - \frac{\delta\,(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)}_{<0 \text{ see (24 and 17)}}}\frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{\lambda\,(\lambda-1)\,r} > 0
\end{aligned}$$

## 6.5 Annotation 5: Proof of Proposition 5

$$
\begin{aligned}
\frac{dT^*}{d\alpha} &= -\frac{\frac{dG}{d\alpha}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{Y}(0)e^{\delta T}}[-\frac{\lambda}{\lambda-1}\frac{C}{r}e^{rT} + \frac{\lambda}{\lambda-1}\frac{C}{r} - \bar{C}\frac{\lambda}{\lambda-1}] \\
&= \frac{1}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)\frac{\lambda(r-\alpha)}{\lambda-1}C}\frac{\lambda}{\lambda-1}\frac{C}{r}[e^{rT} - 1 + \frac{\bar{C}r}{C}] < 0 \\
&= \frac{[e^{rT} - 1 + \frac{\bar{C}r}{C}]}{\underbrace{\left(e^{rT} - \frac{\delta\,(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)}_{<0 \text{ see (24 and 17)}}}\frac{1}{(r-\alpha)r} < 0
\end{aligned}
$$

## 6.6 Annotation 6: Proof of Proposition 6

$$
\begin{aligned}
\frac{dT^*}{d\tilde{Y}(0)} &= -\frac{\frac{dG}{d\tilde{Y}(0)}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{Y}(0)e^{\delta T}}[-e^{\delta T}] \\
&= \frac{1}{\underbrace{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{(r-\delta)T} - \delta\tilde{Y}(0)}_{<0 \text{ see (24 and 17)}}} < 0
\end{aligned}
$$

## 6.7 Annotation 7: Proof of Proposition 7

$$
\begin{aligned}
\frac{dT^*}{dC} &= -\frac{\frac{dG}{dC}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{Y}(0)e^{\delta T}}[\frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}e^{rT} - \frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}] \\
&= \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{Y}(0)e^{\delta T}}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}[e^{rT} - 1] \\
&= \frac{-1}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)\frac{\lambda(r-\alpha)}{\lambda-1}C}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}[e^{rT} - 1] \\
&= \frac{-[e^{rT} - 1]}{\underbrace{\left(e^{rT} - \frac{\delta\,(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{Y}(0)}{C}e^{\delta T}\right)}_{<0 \text{ see (24 and 17)}}rC} > 0
\end{aligned}
$$

## 6.8 Annotation 8: Proof of Proposition 8

For the derivative with respect to $\delta$ and $r$ we need an approximation of $I(T)$ to examine the sign. We approximate $I(T)$ for the time range between 0 and the point $T^*$ by a log linear function with the parameter $\beta$ denoting the average growth rate of total accumulated costs between 0 and $T^*$. Economically this simplification describes a approximation where total costs are payable only at the end of the education period. The non log linear path of cost accumulation is proximated as a continous geometric growth process. Therefore we introduce a parameter $\beta$ which determines the average growth rate of $I(T)$.

$$\bar{C}e^{\beta T} \approx \frac{C}{r}\left(e^{rT}-1\right)+\bar{C}$$

Note that we approximate a non-linear function ($I$) with a non-linear growth rate through a log-linear function which has the same unique positive intersection with the logarithmized income threshold curve. We consider the shape of $\ln I(T)$, which is convex as shown in appendix 1 condition (23).

$$\begin{aligned}
\frac{\partial \ln I}{\partial T} &= \frac{Ce^{rT}}{\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}} > 0, \\
&= \frac{Ce^{rT}\left(r\bar{C}-C\right)}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} > 0 \quad \text{since (17)} \\
&= \lim_{T\to\infty} \frac{re^{rT}}{\left[1+\frac{-1+\frac{r\bar{C}}{C}}{e^{rT}}\right]e^{rT}} = r
\end{aligned}$$

and obviously

$$\ln\left[\bar{C}e^{\beta T}\right]_{T=0} = \ln\bar{C} = \ln I(0).$$

As both curves intersect in $T = T^*$ we can determine a $\beta$ that satisfies the condition $\bar{C}e^{\beta T} \approx \frac{C}{r}\left(e^{rT}-1\right)+\bar{C}$ :

$$\begin{aligned}
\bar{C}e^{\beta T^*} &\approx \frac{C}{r}\left(e^{rT^*}-1\right)+\bar{C} \\
\beta T^* &= \ln\left[\frac{C}{r\bar{C}}\left(e^{rT^*}-1\right)+1\right] \\
\beta &= \frac{\ln\left[\frac{C}{r\bar{C}}\left(e^{rT^*}-1\right)+1\right]}{T^*}
\end{aligned}$$

As $Y^*(0) > E\tilde{Y}(0)$ and $ln(\bar{C}e^{\beta T^*})$ and $lnI(T)$ start at the same point, the corresponding condition for the approximation to (24) is

$$\delta - \beta > 0 \tag{25}$$

Plugging the above approximation into the threshold we can explicitly determine $T^*$ :

$$\begin{aligned}
\frac{\lambda}{\lambda-1}(r-\alpha)\bar{C}e^{\beta T^*}-\tilde{Y}(0)e^{\delta T^*} &= 0\\
&\Leftrightarrow\ T^*=\ln\Big(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{Y}(0)}{\bar{C}}\Big)\frac{1}{\beta-\delta}\\
&>\ 0,\ \text{for}\ \frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}<\frac{\bar{C}}{\tilde{Y}(0)}.\quad \text{see (16)}
\end{aligned}$$

$$\begin{aligned}
\frac{dT^*}{d\delta} &= \frac{1}{(\beta-\delta)^2}\ln\Big(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{Y}(0)}{\bar{C}}\Big)+\frac{1}{\beta-\delta}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{\bar{C}}{\tilde{Y}(0)}\frac{\frac{\partial\lambda}{\partial\delta}}{\lambda^2}\frac{1}{r-\alpha}\frac{\tilde{Y}(0)}{\bar{C}}\\
&= \frac{1}{(\beta-\delta)^2}\overbrace{\ln\Big(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{Y}(0)}{\bar{C}}\Big)}^{<0\text{ see (16)}}+\frac{1}{\beta-\delta}\frac{1}{\lambda-1}\frac{\overset{(-)}{\frac{\partial\lambda}{\partial\delta}}}{\lambda}>0
\end{aligned}$$

Similar to the derivative of $T^*$ with respect to $r$ we have to examine under which condition which summand prevails. Here we assume that the effect of the option value is dominant.

## 6.9 Annotation 9: Proof of Proposition 9

$$\frac{dT^*}{dr}=\frac{1}{\beta-\delta}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{\bar{C}}{\tilde{Y}(0)}\left[\frac{\frac{\partial\lambda}{\partial r}}{\lambda^2}\frac{1}{r-\alpha}\frac{\tilde{Y}(0)}{\bar{C}}-\frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)^2}\frac{\tilde{Y}(0)}{\bar{C}}\right]$$

$$\begin{aligned}
\frac{dT^*}{dr} &= \frac{1}{\beta-\delta}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{\bar{C}}{\tilde{Y}(0)}\left[\frac{\frac{\partial\lambda}{\partial r}}{\lambda^2}\frac{1}{r-\alpha}\frac{\tilde{Y}(0)}{\bar{C}}-\frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)^2}\frac{\tilde{Y}(0)}{\bar{C}}\right]\\
&= \underbrace{\frac{1}{\beta-\delta}}_{(-)}\left[\underbrace{\frac{\overset{(+)}{\frac{\partial\lambda}{\partial r}}}{\lambda(\lambda-1)}-\frac{1}{(r-\alpha)}}_{=:X}\right]
\end{aligned}$$

To find out if $X \gtreqless 0$ we need to follow three steps:

**1) Assume a sufficient condition for $X>0$ and $X<0$ :** From our knowledge of the system we assume a sufficient condition for an unambigious sign. It is supposed that

$$\tfrac{3}{8}\sigma^2+\tfrac{3}{2}\delta>r \quad \text{implies } X>0. \tag{26}$$
$$\tfrac{3}{8}\sigma^2+\tfrac{3}{2}\delta<r \quad \text{implies } X<0. \tag{27}$$

**2) Show that conditon (26) and (27) hold** We now show that (26) and (27) are sufficient conditions to obtain an unambigious sign for $X$:

**a) From (26) we obtain $\lambda < 3/2$ and from (27) we obtain $\lambda > 3/2$ the latter case will be in brackets:** [<]

$$\begin{aligned}
\frac{3}{4}\sigma^2 + 3\delta &> [<]\, 2r \\
\frac{3}{4} + 3\frac{\delta}{\sigma^2} + \frac{\delta^2}{\sigma^4} &> [<]\, \frac{\delta^2}{\sigma^4} + \frac{2r}{\sigma^2} \\
\frac{3}{4} + 2\frac{\delta}{\sigma^2} + \frac{\delta^2}{\sigma^4} &> [<] - \frac{\delta}{\sigma^2} + \frac{\delta^2}{\sigma^4} + \frac{2r}{\sigma^2} \\
1 + 2\frac{\delta}{\sigma^2} + \left(\frac{\delta}{\sigma^2}\right)^2 &> [<]\, \frac{1}{4} - \frac{\delta}{\sigma^2} + \frac{\delta^2}{\sigma^4} + \frac{2r}{\sigma^2} \\
\left(1 + \frac{\delta}{\sigma^2}\right)\left(1 + \frac{\delta}{\sigma^2}\right) &> [<] \left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2} \\
\left(1 + \frac{\delta}{\sigma^2}\right)^2 &> [<] \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]
\end{aligned}$$

$$\begin{aligned}
1 + \frac{\delta}{\sigma^2} &> [<] \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}} \\
\frac{3}{2} &> [<]\, \frac{1}{2} - \frac{\delta}{\sigma^2} + \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}}
\end{aligned}$$

As $\lambda = \frac{1}{2} - \frac{\delta}{\sigma^2} + \left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}}$ we proved that the value of $\lambda$ depends on the conditions (26) and (27). Therefore we get

$$\frac{3}{2} > \lambda \quad \text{for } \frac{3}{4}\sigma^2 + 3\delta > 2r \tag{28}$$

$$\frac{3}{2} < \lambda \quad \text{for } \frac{3}{4}\sigma^2 + 3\delta < 2r \tag{29}$$

**b) Now we apply this condition for X:**

$$X = \frac{\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}} \frac{1}{\sigma^2}}{\lambda(\lambda - 1)} - \frac{1}{r - \alpha} > [<]\, 0$$

$$\begin{aligned}
\frac{\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}} \frac{1}{\sigma^2}}{\lambda(\lambda - 1)} &> [<]\, \frac{1}{r - \alpha} \\
\frac{(r - \alpha)\left[\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}} \frac{1}{\sigma^2}}{\lambda(\lambda - 1)} &> [<]\, 1
\end{aligned}$$

$$\begin{aligned}
\frac{(r-\alpha)}{\lambda(\lambda-1)}\frac{1}{\sigma^2} &> [<] \left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}} \\
\frac{(r-\alpha)}{\lambda(\lambda-1)\delta}\frac{\delta}{\sigma^2}-\frac{\delta}{\sigma^2}+\frac{1}{2} &> [<] \frac{1}{2}-\frac{\delta}{\sigma^2}+\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}} \\
\left[\frac{(r-\alpha)}{\lambda(\lambda-1)\delta}-1\right]\frac{\delta}{\sigma^2} &> [<] \lambda-\frac{1}{2} \\
\frac{(r-\alpha)}{\lambda(\lambda-1)\delta}-1 &> [<] \left(\lambda-\frac{1}{2}\right)\frac{\sigma^2}{\delta} \\
\frac{(r-\alpha)}{\delta}-\lambda(\lambda-1) &> [<] \left(\lambda-\frac{1}{2}\right)\frac{\sigma^2\lambda(\lambda-1)}{\delta} \\
(r-\alpha) &> [<] \lambda(\lambda-1)\left[\left(\lambda-\frac{1}{2}\right)\sigma^2+\delta\right] \\
r &> [<] \lambda(\lambda-1)\left[\left(\lambda-\frac{1}{2}\right)\sigma^2+\delta\right]+\alpha
\end{aligned}$$

With the conditions (29) and (28) we know $\frac{3}{2} > \lambda$ for $\frac{3}{4}\sigma^2 + 3\delta > 2r$ and $\frac{3}{2} < \lambda$ for $\frac{3}{4}\sigma^2 + 3\delta < 2r$ . Therefore we can check if we find true conditions for $X > 0, X < 0$ using the highest/lowest value of $\lambda$ .

$$\begin{aligned}
\frac{(r-\alpha)}{\delta} &> (<) \frac{3}{2}(\frac{3}{2}-1)\left[\left(\frac{3}{2}-\frac{1}{2}\right)\frac{\sigma^2}{\delta}+1\right] \\
& (<) \frac{3}{2}(\frac{1}{2})\left[\left(\frac{2}{2}\right)\frac{\sigma^2}{\delta}+1\right] \\
&> (<) \frac{3}{4}(1)\left[\frac{2}{2}\frac{\sigma^2}{\delta}+1\right] \\
(r-\alpha) &> (<) \frac{3}{4}\left[\sigma^2+\delta\right]
\end{aligned}$$

(i) conditions for $X > 0$ :

$$\begin{aligned}
r &> \frac{3}{4}\left[\sigma^2+\delta\right]+\alpha \\
\frac{3}{8}\sigma^2+\frac{3}{2}\delta &> r \quad \text{see} \quad \text{above} \\
\frac{3}{8}\sigma^2+\frac{3}{2}\delta &> r > \frac{3}{4}\left[\sigma^2+\delta\right]+\alpha
\end{aligned}$$

$$\begin{aligned}
\Delta r &= \frac{3}{8}\sigma^2+\frac{3}{2}\delta-\frac{3}{4}\sigma^2-\frac{3}{4}\delta-\alpha > 0 \\
\Delta r &= -\frac{3}{8}\sigma^2+\frac{3}{4}\delta-\alpha > 0
\end{aligned}$$

$$\begin{aligned} 3\delta &> \frac{3}{2}\sigma^2 + 4\alpha \\ \delta &> \frac{1}{2}\sigma^2 + \frac{4}{3}\alpha \end{aligned}$$

(i) conditions for $X < 0$ :

$$\begin{aligned} r &< \frac{3}{4}\left[\sigma^2 + \delta\right] + \alpha \\ \frac{3}{8}\sigma^2 + \frac{3}{2}\delta &< r \quad \text{see} \quad (26) \\ \frac{3}{4}\left[\sigma^2 + \delta\right] + \alpha &> r > \frac{3}{8}\sigma^2 + \frac{3}{2}\delta \end{aligned}$$

$$\begin{aligned} \Delta r &= \frac{3}{4}\left[\sigma^2 + \delta\right] + \alpha - \frac{3}{8}\sigma^2 - \frac{3}{2}\delta > 0 \\ &= \frac{6}{8}\sigma^2 + \frac{3}{4}\delta + \alpha - \frac{3}{8}\sigma^2 - \frac{6}{4}\delta > 0 \\ \Delta r &= \frac{3}{8}\sigma^2 - \frac{3}{4}\delta + \alpha > 0 \end{aligned}$$

$$\delta < \frac{1}{2}\sigma^2 + \frac{4}{3}\alpha$$

As we can see there is a feasible combination of $\delta, \sigma^2$, and $\alpha$ satisfying this condition. The assumption that $X > 0$ and $X < 0$ is proven under the derived conditions (26) and (27).