Parametric and Nonparametric Tests of Limited Domain and Ordered Hypotheses in Economics

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Abstract.

Technical and conceptual advances in testing multivariate linear and non-linear inequality hypotheses in econometrics are summarized. This is done in the context of substantive empirical settings in economics in which either the null, or the alternative, or both hypotheses define more limited domains than the two-sided alternatives typically tested in the classical testing procedures. The desired goal is increased power which is laudable given the endemic power problems of most of the classical asymptotic tests. The impediments are a lack of familiarity with implementation procedures, and characterization problems of distributions under some composite hypotheses.

several empirically important cases are identified in which practical "one-sided" tests can be conducted by either the $\overline{\chi}^2 - distribution$, or the union intersection mechanisms based on the Gaussian variate, or the increasingly feasible and popular resampling/simulation techniques. Point optimal testing and its derivatives find a natural medium here whenever unique characterization of the null distributions for the "least favorable" cases is not possible.

Most of the recent econometric literature in this area is parametric deriving from the multivariate extensions of the classical Gaussian means test with ordered alternatives. Tests for variance components, random coefficients, over dispersion, heteroskedasticity, regime change, ARCH effects, curvature regularity conditions on flexible supply, demand, and other economic functions, are examples But nonparametric tests for ordered relations between distributions, or their quantiles, or curvature regularity conditions on nonparametric economic relations, have witnessed rapid development and applications in economics and finance. We detail tests for Stochastic Dominance which indicate a major departure in the practice of empirical decision making in, so far, the areas of welfare and optimal financial strategy.

Introduction

The additional information available when hypotheses can restrict the maintained space to subspaces of the usual two-sided (unrestricted) hypotheses, can enhance the power of tests. Since good power is a rare commodity the interest in inequality restricted hypothesis tests has increased dramatically. In addition, the two sided formulation is occasionally too vague to be of help when more sharply ordered alternatives are of interest. An example is the test of order relations (e.g., stochastic dominance) amongst investment strategies, or among income/welfare distributions. The two-sided formulation fails to distinguish between "equivalent" and "unrankable" cases.

In statistics, D.J. Bartholomew, H. Chernoff, V.J. Chacko, A. Kudo, and P.E. Nuesch are among the first to refine and extend the Neyman-Pearson testing procedure for one sided alternatives, first in the one and then in the multivariate settings. See, *inter alia*, Bartholomew (1959a, 1959b), Chernoff (1954), Chacko (1963), and Kudo (1963). Later refinements and advances were obtained by Nuesch (1966), Feder (1968), Perlman (1969), and Shorack (1967). At least in low dimensional cases, the power gains over the two sided counterparts have been shown to be substantial, see Bartholomew (1959b), and Barlow et al (1972). While Chernoff and Feder clarified the local nature of tests and gave some solutions when the true parameter value is "near" the boundaries of the hypotheses regions, Kudo, Nuesch, Perlman and Shorack were among the first to develop the elements of the $\overline{\chi}^2 - distribution$ theory for the likelihood ratio and other classical tests. See the text by Barlow et al (1972).

In econometrics, Gourieroux, Holly, and Monfort (1980, 1982), heretofore GHM, are seminal contributions which introduced and extended this literature to linear and non-linear econometric/regression models (see the important contributions in Judge and Takayama (1966) and Liew (1976)). The focus in GHM (1982) is on the following testing situation:

$$y = X\beta + u \qquad (1)$$

$$R\beta \geq r,\ \ R \sim q \times K,\ \ q \leq K, \text{ the dimension of } \beta$$

We wish to test

$$H_0 : R\beta = r, ...vs....H_1 : R\beta \geq r \qquad (2)$$

Gourieroux et al (1982) derive the Lagrange Multiplier (LM)/Kuhn-Tucker (KT) test, as well as the Likelihood Ratio (LR) and the Wald (W) tests with known and unknown covariance matrix, Ω, of the regression errors. With known covariance all three tests are identical and distributed exactly as a $\bar{\chi}^2 - distribution$. They note that the problem considered here is essentially equivalent to the following in the earlier statistical literature:

Let there be T independent observations from a p-dimensional N(μ,Σ). Test

$$H_0 : \mu = 0 \text{, against the alternative } H_1 : \mu_i \geq 0, \text{ all } i, \text{ with at least one strict inequality} \quad (3)$$

The LR test of this hypothesis has the $\bar{\chi}^2 - distribution$ which is a mixture of chi-squared distributions given by:

$$\sum_{j=0}^{p} w(p,j)\chi^2_{(j)} \quad (4)$$

with $\chi^2_{(0)}$=1 at the origin. The weights $w(.)$ are probabilities to be computed in a multivariate setting over the space of alternatives. This is one of the practical impediments in this area, inviting a variety of solutions which we shall touch upon. These include obtaining bounds, exact tests for low dimension cases, and resampling/Monte Carlo techniques. When Ω is unknown but depends on a finite set of parameters, GHM (1982) and others have shown that the same distribution theory applies asymptotically. In fact GHM show that all three tests are asymptotically equivalent and satisfy the usual inequality : $W \geq LR \geq LM(KT)$. We give the detailed form of these test statistics. In particular the LM version may be desirable as it can avoid the quadratic programming (QP) routine needed to obtain estimators under the inequality restrictions. We also point to routines that are readily available in FORTRAN and GAUSS (but alas not yet in the standard econometric software packages). Kodde and Palm (1986), Rogers (1986), Farebrother (1986), and Hillier (1986) deal with these issues in the linear regression setting.

It should be noted that this simultaneous procedure competes with another approach based on the union intersection technique. In the latter, each univariate hypothesis is tested, with the decision being a rejection of the joint null if the least significant statistic is greater than the α −critical level of a standard Gaussian variate. Consistency of such tests has been established. We will discuss examples of these alternatives. Also, the non-existence generally of an optimal test in the multivariable case has led to consideration of point optimal testing, and tests that attempt to maximize power in the least favorable case, or on suitable "averages". This is similar to recent attempts to deal with power computation when alternatives depend on nuisance parameters. See, inter alia, King and Wu (1997), Lee and King (1993), and Linton and Steigerwald (1999) who test for the existence of ARCH effects in financial processes taking inequality information into account.

In the case of non-linear models and/or non linear inequality restrictions, GHM (1980) and Wolak (1989, 1991) discuss the distribution of the same $\overline{\chi^2}$tests, while Dufour considers modified classical tests. In this setting, however, there is another problem, as pointed out by Wolak (1989, 1991). When q$\leq K$ there is generally no unique solution for the "true β" from $R\beta = r$ (or its nonlinear counterpart). But convention dictates that in this-type case of composite hypotheses, power be computed for the "least favorable" case which arises at the boundary $R\beta = r$. It then follows that the asymptotic distribution (when Ω is consistently estimated in customary ways) cannot, in general, be uniquely characterized for the least favorable case. Sufficient conditions for a unique distribution are given in Wolak (1991) and will be discussed here. In the absence of these conditions, a "localized" version of the hypothesis is testable with the same $\bar{\chi}^2 - distribution$. In this non-linear setting, even one of the least onerous of these conditions, i.e., R ~K$\times K$, is often not of practical interest. Thus the local nature of these tests must be appreciated and properly interpreted in such non-linear applications as economic curvature constraints on flexible functional forms, see Lau (1978), Gallant and Golub (1984), and Diewert and Wales (1987).

All of the above developments are parametric. There is at least an old tradition for the non parametric "two sample" testing of homogeneity between two distributions, often assumed to belong to the same family. Pearson type and Kolmogorov-Smirnov (KS) tests are prominent, as well as the Wilcoxan rank test. In the case of inequality or ordered hypotheses regarding relations between two unknown distributions, Anderson (1986) is an example of the modified Pearson tests based on relative cell frequencies, and Xu, Fisher, and Wilson (1995), Dardanoni and Forcina (1996), and Xu (1995) are examples of quantile-based tests which incorporate the inequality information in the hypotheses and, hence lead to the use of $\bar{\chi}^2 - distribution$ theory. The multivariate versions of the KS test have been studied by McFadden (1989), Klecan, McFadden, and McFadden (1991), Kaur, Rao, and Singh (1994), Maasoumi et al (1997), and Maasoumi and Heshmati (1999). The union intersection alternative is also fully discussed in Davidson and Duclos (1998), representing a culmination of this line of development in, for example, Beach and Davidson (1983), Bishop, Chacraborti, and Thistle (1989), Howes (1993), and Davidson and Duclos (1997). Again, the union intersection techniques do not exploit the inequality information and are expected to be less powerful. We discuss the main features of these alternatives.

The plan of the paper is as follows. In section 2 we introduce the classical multivariate means problem and a general variant of it that makes it amenable to immediate application to very general econometric models in which an asymptotically normal estimator can be obtained. At this level of generality, one can treat very wide classes of processes, as well as linear and non-linear models, as described in Potscher and Prucha (1991a-b). The linear model is given as an example, and the asymptotic distribution of the classical tests is described.

The next section describes the non-linear models and the local nature of the hypothesis that can be tested. Section 4 is devoted to non-parametric setting. Examples from economics and finance are cited throughout the paper. Section 5 concludes.

The general multivariate parametric problem.

Consider the setting in (3) when $\hat{\mu} = \mu + v$, and $v \sim N(0,\Omega)$, is an available unrestricted estimator. Consider the restricted estimator $\tilde{\mu}$ as the solution to the following quadratic programming (QP) problem:

$$\max_{\mu} \hat{\mu}\Sigma^{-1}\hat{\mu} - (\hat{\mu}-\mu)'\Sigma^{-1}(\hat{\mu}-\mu), \text{ subject to } \mu \geq 0 \qquad (5)$$

Then the likelihood ratio (LR) test of the hypothesis in (3) is:

$$LR = \tilde{\mu}'\Sigma^{-1}\tilde{\mu} \qquad (6)$$

Several researchers, for instance Kudo (1963) and Perlman (1969) established the distribution of the LR statistic under the null as:

$$Sup_{\mu \geq 0}.\, pr_{\mu,\Omega}(LR \geq c_\alpha) = pr_{0,\Omega}(LR \geq c_\alpha) = \sum_{i=0}^{p} w(i,p,\Omega) \times pr(\chi^2_{(i)} \geq c_\alpha) \qquad (7)$$

a weighted sum of chi-squared variates for an exact test of size α. The weights $w(i,.)$ sum to unity and each is the probability of $\tilde{\mu}$ having i positive elements.

If the *null* hypothesis is one of *inequality* restrictions, a similar distribution theory applies. To see this, consider:

$$H_0 : \mu \geq 0 \text{ vs. } H_1 : \mu \in R^p \qquad (8)$$

where $\hat{\mu} = \mu + v$, and $v \sim N(0,\Sigma)$. Let $\tilde{\mu}$ be the restricted estimator from the following QP problem:

$$D = \min_{\mu}(\hat{\mu}-\mu)\Sigma^{-1}(\hat{\mu}-\mu) \quad \text{subject to } \mu \geq 0 \quad (9)$$

D is the LR statistic for (8). Perlman (1969) showed that the power function is monotonic in this case. In view of this result, taking C_α as the critical level of a test of size α, we may use the same distribution theory as in (7) above except that the weight $w(i.)$ will be the probability of $\tilde{\mu}$ having exactly $p-i$ positive elements.

There is a relatively extensive literature dealing with the computation of the weights $w(.)$. Their computation requires evaluation of multivariate integrals which become tedious for $p \geq 8$. Kudo (1963) provides exact expressions for $p \leq 4$. Shapiro (1985) gives expressions for $p = 4$, and Bohrer and Chow (1978) provide computational algorithms for $p \leq 10$. But these can be slow for large p. Kodde and Palm(1986) suggest an attractive bounds test solution which requires obtaining lower and upper bounds, c_l and c_u, to the critical value, as follows:

$$\alpha_l = \frac{1}{2}pr(\chi^2_{(1)} \geq c_l), \text{ and}$$

$$\alpha_u = \frac{1}{2}pr(\chi^2_{(p-1)} \geq c_u) + \frac{1}{2}pr(\chi^2_{(p)} \geq c_u) \qquad (10)$$

The null in (8) is rejected if $D \geq c_u$, but is inconclusive when $c_l \leq D \leq c_u$.

Advances on Monte Carlo integration suggest resampling techniques may be used for large p, especially if the bounds test is inconclusive.

In the case of a single hypothesis $(\mu_1 = 0)$, the above test is the one sided UMP test. In this situation:

$$pr(LR \geq c_\alpha) = pr(\frac{1}{2}\chi^2_{(0)} + \frac{1}{2}\chi^2_{(1)}) \geq c_\alpha) = \alpha \quad (11)$$

The standard two sided test would be based on the critical values c'_α from a $\chi^2_{(1)}$ distribution. But $pr(\chi^2_{(1)} \geq c'_\alpha) = \alpha$ makes clear that $c'_\alpha \geq c_\alpha$, indicating the substantial power loss which was demonstrated by Bartholomew (1959a-b) and Nuesch (1966).

In the two dimension $(p = 2)$ case, under the null we have:

$$pr(LR \geq c_\alpha) = w(2,0)\chi^2_{(0)} + w(2,1)\chi^2_{(1)} + w(2,2)\chi^2_{(2)} \quad (12)$$

where $w(2,0) = pr[LR = 0] = pr[\hat{\mu}_1 \leq 0, \hat{\mu}_2 \leq 0]$, $w(2,1) = \frac{1}{2}$, $w(2,2) = \frac{1}{2} - w(2,0)$. While difficult to establish analytically, the power gains over the standard case can be substantial in higher dimensions where UMP tests do not generally exist. See Kudo (1963) and GHM (1982).

The equivalence of LR, W and LM tests.

We give an account of the three classical tests in the context of the general linear regression model introduced in (1) above. We take R to be a $(p \times K)$ known matrix of rank $p \leq K$. Consider three estimators of β under the exact linear restrictions, under inequality restrictions, and when $\beta \in R^p$ (no restrictions). Denote these by $\bar{\beta}$, $\tilde{\beta}$, and $\hat{\beta}$, respectively. We note that $\hat{\beta} = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}y)$ is the ML (GLS) estimator here. Let $(X'\Omega^{-1}X) = G$, and consider the following optimization programs:

$$\max -(y - X\beta)'\Omega^{-1}(y - X\beta), \quad \text{subject to } R\beta \geq r \qquad (13)$$

and the same objective function but with equality restrictions. Denote by $\tilde{\lambda}$ and $\bar{\lambda}$ the Lagrange multipliers, respectively, of these two programs (conventionally, $\lambda = 0$ for $\hat{\beta}$). Then:

$$\tilde{\beta} = \hat{\beta} + G^{-1}R'\tilde{\lambda}/2, \quad \text{and} \quad \bar{\beta} = \hat{\beta} + G^{-1}R'\bar{\lambda}/2 \qquad (14)$$

See GHM (1982). Employing these relations it is straightforward to show that the following three classical tests are identical:

$$\xi_{LR} = -2\log LR = 2(\tilde{L} - \bar{L}) \qquad (15)$$

where $\bar{L}$ and $\tilde{L}$ are the logarithms of the maxima of the respective likelihood functions;

$$\xi_{LM} = \min(\lambda - \bar{\lambda})'RG^{-1}R'(\lambda - \bar{\lambda})/4 \text{ , subject to } \lambda \leq 0 \qquad (16)$$

is the Kuhn-Tucker/Lagrangian Multiplier (LM) test. computed at $\tilde{\lambda}$, and,

$$\xi_W = (R\tilde{\beta} - r)'[RG^{-1}R']^{-1}(R\tilde{\beta} - r) \qquad (17)$$

is the Wald test. In order to utilize the classical results stated above for problems in (3) , or (8), it is customary to note that the LR test in (15) above is identical to the LR test of the following problem:

$$\hat{\beta} = \beta + v$$

$$R\beta \geq r$$

$$v \sim N(0, G^{-1}) \qquad (18)$$

For this problem ξ_{LR} is the optimum of the following QP problem:

$$\max -(\beta - \hat{\beta})'G(\beta - \hat{\beta}) + (\bar{\beta} - \hat{\beta})'G(\bar{\beta} - \hat{\beta})$$

$$\text{subject to } \quad R\beta \geq r \qquad (19)$$

This is identical to the one-sided multivariate problem in (3). It also suggests that the context for applications can be very general indeed. All that is needed is normally distributed estimators, $\hat{\beta}$, which are then projected onto the cone defined by the inequality restrictions in order to obtain the restricted estimator $\tilde{\beta}$.

Asymtotically normal estimators.

The assumption of normality can be relaxed for the situations that allow asymptotically normal estimators for β. This is because the inference theory developed for problems (3) or (8) is asymptotically valid for much broader classes of models and hypotheses. In fact, when consistent estimators of Ω are available and, in (18), v has the stated distribution *asymptotically*, an asymptotically exact test of size α is based on the same $\bar{\chi}^2 -$ *distribution* given above. To obtain this result one needs to replace Ω in the optimization problems with its corresponding consistent estimator . This is routinely possible when Ω is a continuous function of a finite set of parameters other than β.

The three tests are not identical in this situation, of course, but have the same asymptotic distribution. Furthermore, the usual inequality , vis. $\xi_W \geq \xi_{LR} \geq \xi_{LM}$ is still valid, see GHM (1982). Often the test which can avoid the QP problem is preferred, which means the LM test for the test of the null of equality of the restrictions, and the Wald test when the null is one of inequality and the alternative is unrestricted. But much recent evidence, as well as invariance arguments, suggest that the LR test be used.

Given this remarkable range, we defer the details of this section to be discussed as special cases of the general nonlinear models and nonlinear restrictions discussed in the next section.

In the general linear regression models with linear and or nonlinear inequality restrictions, other approaches are available. Kodde and Palm (1986, 1987), Dufour (1989), Dufour and Khalaf (1993), Farebrother (1986), King and Smith (1986), Rogers (1986), and Stewart (1997) are examples of theoretical and empirical applications in economics and finance. Dufour (1989) is an alternative "conservative" bounds test for the following type situation:

$$H_0 : R\beta \in \Gamma_0 \text{ against } H_1 : \beta \in \Gamma_1 \qquad (20)$$

where Γ_0 and Γ_1 are non empty subsets, respectively of R^p and R^K. This also allows a consideration of such cases as $h(R\beta) = 0$, or $h(R\beta) \geq 0$. Dofour (1989) suggests a generalization of the well known, two-sided F test in this situation as follows:

$$pr[\frac{SS_0 - SS_1}{SS_1} \geq \frac{p}{T-K}F_\alpha(p, T-K)] \leq \alpha \quad (21)$$

where there are T observations from which $SS_i, i = o, 1,$ are calculated as residual sums of squares under the null and the alternative, respectively. Thus the traditional p-values will be upper-bounds for the true values and offer a conservative bounds testing strategy. Dufour (1989), Dofour and Khalaf (1993) and Stewart (1997), inter alia, consider "liberal bounds", and extensions to multivariate/simultaneous equations models and nonlinear inequality restrictions. Applications to demand functions and negativity constraints on the substitution matrix, as well as tests of non-linear nulls in the CAPM models (e.g.,as discussed in Gibbons (1982)), show size and power improvements over the traditional asymptotic tests. The latter are known for their tendency to over-reject in any case. Stewart (1997) considered the performance of the standard LR, the Kodde and Palm (1986) bounds for the $\bar{\chi}^2$ distribution, and the Dufour-type bound test of negativity of the substitution matrix for the demand data for

Germany and Holland. This was first studied in Barten and Geyskens (1975). He looked at, among other things, the hypothesis of negativity against an unrestricted alternative, and the null of negativity when symmetry and homogeneity are maintained. It appeared that, while the Dufour test did well in most cases, certainly reversing the conclusions of the traditional LR test (which rejects everything!), the Kodde and Palm bounds test did consistently well when the conservative bounds test was not informative (with $\alpha = 1$). Both the lower and upper bounds for the chi-bar squared distribution are available, while the "liberal"/lower bounds for the Dufour adjustment are not in this case.

Nonlinear models and nonlinear inequality restrictions

Wolak (1989, 1991) gives a general account of this topic. He considers the general formulation in (18) with nonlinear restrictions. Specifically, consider the following problem:

$$\hat{\beta} = \beta + v$$

$$h(\beta) \geq 0$$

$$v \underset{a}{\sim} N(0, \Psi) \qquad (22)$$

where $h(.)$ is a smooth vector function of dimension p and a derivative matrix denoted by H(.). We wish to test

$$H_0 : h(\beta) \geq 0\,, \text{ vs } H_1 : \beta \in R^K \quad (23)$$

This is very general since model classes that allow for estimation results given in (22) are very broad indeed. As the results in Potscher and Prucha (1991a-b), and Burguete et al (1982) indicate, many non-linear dynamic processes in econometrics permit consistent and asymptotically normal estimators under regularity conditions.

In general an asymptotically exact size test of the null in (23) is not possible without a localization to some suitable neighborhood of the parameter space. To see this, let $h(\beta^0) = 0$ define β^0, and $H(\beta^0), and\ I(\beta^0)$ the evaluations of $\partial h/\partial\beta = H(\beta)$, and the information matrix, respectively. Let

$$C = \{\beta | h(\beta) \geq 0,\ \beta \in R^K\} \qquad (24)$$

and $N_{\delta_T}(\beta^0)$ as a δ_T-neighborhood with $\delta_T = O(T^{\frac{-1}{2}})$. It is well known that the global hypotheses of the type in (23)-(24) do not generally permit large sample approximations to the power function for fixed alternative hypotheses for nonlinear multivariate equality restrictions. See Chernoff (1954), Feder (1968), and Wolak (1989). If we localize the null in (23) to only $\beta \in N_{\delta_T}(\beta^0)$, then exact size tests are available and are as given by the appropriately defined chi-bar distribution. In fact, we would be testing whether $\beta \in$ {cone of tangents of C at β^0}, where $\Psi = H(\beta^0)I(\beta^0)^{-1}H(\beta^0)$, in (22).

In order to appreciate the issues, we note that the asymptotic distribution of the test depends on Ψ, which in turn depends on $H(.)$ and $I(.)$. But the latter generally vary with β^0. Also, we note that $h(\beta^0) = 0$ does not have a unique solution unless it is linear and of rank $K = p$. Thus the case of $H_0 : \beta \geq 0$ does not have a problem. The case $R\beta \geq 0$, will present a problem in nonlinear models when rank $(R) < K$ since $I(\beta)$ will generally depend on the $K - p$ free parameters in β.

It must be appreciated that this is specially serious since, inequalities define composite hypotheses which force a consideration of power over regions. Optimal tests do not exist in multivariate situations. Thus other conventions must be developed for test selection. One method is to consider power in the least favorable case. Another is to maximize power at given points that are known to be "desirable", leading to point optimal testing. Closely related, and since such points may be difficult to select a priori, are tests that maximize mean power over sets of desirable points. See King and Wu (1996), and Lee and King (1993) for discussion. This is related to the general problem of testing with dependence on nuisance parameters under the alternative. In the instant case, the "least favorable" cases are all those defined by $h(\beta^0) = 0$. Hence the indeterminacy of the asymptotic power function.

From this point on our discussion pertains to the "local" test whenever the estimates of Ψ cannot converge to a unique $H(\beta^0)I(\beta^0)^{-1}H(\beta^0) = \Psi$.

Let $x_t = (x_{t1}, x_{t2}, ...,x_{tn})'$, $t = 1,....T$, be a realization of a random variable in R^n, with a density function $f(x_t, \beta)$ which is continuous in β for all x_t. We assume a compact subspace of R^n contains β, $h(.)$ is continuous with continuous partial derivatives $\partial h_i(\beta)/\partial\beta_j = H_{ij}$, defining the $p \times K$ matrix $H(\beta)$ that is assumed to have full rank $p \leq K$ at an interior point β^0 such that $h(\beta^0) = 0$. Finally, let β_T^0 denote the "true" value under the local hypothesis, then,

$$\beta_T^0 \in C_T = \{\beta | h(\beta) \geq 0, \beta \in N_{\delta_T}(\beta^0)\} \qquad \textit{for all } T$$

and $(\beta_T^0 - \beta^0)$=o(1) and $T^{\frac{1}{2}}(\beta_T^0 - \beta^0) = O(1)$.

Let x represent T random observations from $f(x_t, \beta)$, and the log-likelihood function given below:

$$L(\beta) = L(x, \beta) = \sum_{t=1}^{T} \ln(f(x_t, \beta)) \qquad (25)$$

Following GHM (1982), again we consider the three estimators of (β, λ), obtained under the inequality constraints, equality constraints, and no constraints as $(\tilde{\beta}, \tilde{\lambda}), (\bar{\beta}, \bar{\lambda}), and\ (\hat{\beta}, \hat{\lambda} = 0)$, respectively. It can be verified that (see Wolak (1989)):

The three tests LR, Wald, and LM are asymptotically equivalent and have the distribution given earlier. They are computed as follows:

$$\xi_{LR} = 2[L(\hat{\beta}) - L(\tilde{\beta})] \quad (26)$$
$$\xi_W = T(h(\tilde{\beta}) - h(\hat{\beta}))'[H(\hat{\beta})I(\beta^0)^{-1}H(\hat{\beta})'](h(\tilde{\beta}) - h(\hat{\beta})) \quad (27)$$
$$\xi'_W = T(\tilde{\beta} - \hat{\beta})'I(\beta^0)(\tilde{\beta} - \hat{\beta}) \quad (28)$$
$$\xi_{LM} = T\tilde{\lambda}'H(\tilde{\beta})I(\beta^0)^{-1}H(\tilde{\beta})'\tilde{\lambda} \quad (29)$$

where $I(\beta^0)$ is the value of the information matrix, $\lim_{T\to\infty} T^{-1}E_{\beta^0}[-\partial^2 L/\partial\beta\partial\beta']$, at β^0, and (27)-(28) are two asymptotically equivalent ways of computing the Wald test. This testifies to its lack of invariance which has been widely appreciated in econometrics. The above results also benefit from the well known asymptotic approximations:

$$h(\beta^0) \simeq H(\beta^0)(\beta - \beta^0), \qquad \text{and} \quad H(\beta^0) - H(\beta) \simeq 0$$

which hold for all of the three estimators of β. As Wolak (1989) showed, these statistics are asymptotically equivalent to the generalized distance statistic D introduced in Kodde and Palm (1986) :

$$D = \min_{\beta} T(\hat{\beta} - \beta)'I(\beta^0)(\hat{\beta} - \beta) \,, \quad (30)$$

$$\textit{subject to } H(\beta^0)(\beta - \beta^0) \geq 0, \text{ and } \beta \in N_{\delta_T}(\beta^0)$$

For local β defined above, all these statistics have the same chi-bar squared distribution given earlier. Kodde and Palm (1987) employed this statistic for an empirical test of the negativity of the substitution matrix of demand systems. They found that it outperforms the two-sided asymptotic LR test. Their bounds also appear to deal with the related problem of over rejection when nominal significance levels are used with other classical tests against the two sided alternatives. Gourieroux et al (1982) gave the popular artificial regression method of deriving the LM test.

In the same general context, Wolak (1989) specialized the above tests to a test of joint nonlinear inequality and equality restrictions.

With the advent of cheap computing and Monte Carlo integration in high dimensions, the above tests are quite accessible. Certainly, the critical values from the bounds procedures deserve to be incorporated in standard econometric routines, as well as the exact bounds for low dimensional cases ($p \leq 8$). The power gains justify the extra effort. .

Nonparametric tests of inequality restrictions

All of the above models and hypotheses were concerned with comparing means and/or variance parameters of either known or asymptotically normal distributions. We may not know the distributions, we may be interested in comparing more general characteristics than the first few moments, and the distributions being compared may not be from the same family. All of these situations require a nonparametric development that can also deal with ordered hypotheses.

Order relations between distributions present one of the most important and exciting areas of development in economics and finance. These include stochastic dominance relations which in turn include Lorenz dominance, and such others as Likelihood and uniform orders. See Dardanoni and Forcina (1996) for the latter two relations. Below we focus on the example of Stochastic Dominance (SD) of various orders. An account of the definitions and tests is first given, followed by some applications.

Tests for Stochastic Dominance

In the area of income distributions and tax analysis, early attempts to test for Lorenz curve comparisons may be exemplified by Beach and Davidson (1983), and Bishop, Formby, and Thistle (1989). In practice, a finite number of ordinates of the desired curves or functions are compared. These ordinates are typically represented by quantiles and/or conditional interval means. Thus, the distribution theory of the proposed tests are typically derived from the existing asymptotic theory for ordered statistics or conditional means and variances. Recently Beach, Davidson, and Slotsve (1995), Davidson and Duclos (1997, 1998) have outlined the asymptotic distribution theory for cumulative/conditional means and variances which are the essential ingredients of Lorenz and GL curves, and in testing for any order of stochastic dominance when these curves cross. To control for the size of a sequence of tests at several points the union intersection (UI) and Studentized Maximum Modulus technique for multiple comparisons is generally favored in this area. In this line of inquiry the inequality nature of the order relations is not explicitly utilized in the manner described above for parametric tests. Therefore procedures that do so may offer power gain. Some alternatives to these multiple comparison techniques have been suggested which are typically based on Wald type joint tests of *equality* of the same ordinates, see Bishop et al (1994) and Anderson (1996). These alternatives are somewhat problematic since their implicit null and alternative hypotheses are typically not a satisfactory representation of the *inequality* (order) relations that need to be tested. Xu et al (1995), and Xu (1995) take proper account of the inequality nature of such hypotheses and adapt econometric tests for inequality restrictions to testing for FSD and SSD, and to GL dominance, respectively. Their tests follow the work in econometrics of Gourieroux et al (1982) Kodde and Palm (1986), and Wolak (1988, 1989), which was outlined in the previous sections.

McFadden (1989) and Klecan, McFadden, and McFadden (1991) have proposed tests of first and second order "maximality" for stochastic dominance which are extensions of the Kolmogorov-Smirnov statistic. McFadden (1989) assumes i.i.d. observations and independent variates, allowing him to derive the asymptotic distribution of his test, in general, and its exact distribution in some cases (see Durbin (1973, 1985). He provides a Fortran and a GAUSS program for computing his tests. Klecan et al generalize this earlier test by allowing for weak dependence in the processes and replace independence with exchangeability. They demonstrate with an application for ranking investment portfolios. The asymptotic distribution of these tests cannot be fully characterized, however, prompting Klecan et al to propose Monte Carlo methods for evaluating critical levels. Similarly, Maasoumi, Mills and Zandvakili (1997) and Maasoumi and Heshmati (1998) propose bootstrap-KS tests with several empirical applications. In the following subsections some definitions and results are summarized which help to describe these tests.

Definitions and Tests

Let X and Y be two income variables at either two different points in time, before and after taxes, or for different regions or countries. Let $X_1, X_2, ..., X_n$ be n not necessarily i.i.d observations on X, and $Y_1, Y_2, ..., Y_m$ be similar observations on Y. Let U_1 denote the class of all utility functions u such that $u' \geq 0$, (increasing). Also, let U_2 denote the class of all utility functions in U_1 for which $u'' \leq 0$ (strict concavity), and U_3 denote a subset of U_2 for which $u''' \geq 0$. Let $X_{(i)}$ and $Y_{(i)}$ denote the i-th order statistics, and assume F(x) and G(x) are continuous and monotonic cumulative distribution functions (cdf,s) of X and Y, respectively.

The quantile functions $X(p)$ and $Y(p)$ are defined by, for example, $Y(p) = \inf\{y : F(y) \geq p\}$. Then:

definition

definition

Weaker versions of these relations drop the requirement of strict inequality at some point. When either Lorenz or Generalized Lorenz Curves of two distributions cross, unambiguous ranking by FSD and SSD may not be possible. Whitmore introduced the concept of third order stochastic dominance (TSD) in finance, see (e.g.) Whitmore and Findley (1978). Shorrocks and Foster (1987) showed that the addition of a "transfer sensitivity" requirement leads to TSD ranking of income distributions. This requirement is stronger than the Pigou-Dalton principle of transfers since it makes regressive transfers less desirable at lower income levels. TSD is defined as follows:

definition

(1) $E[u(X)] \geq E[u(Y)]$ for all $u \in U_3$, with strict inequality for some u.

(2) $\int_{-\infty}^{x}\int_{-\infty}^{v}[F(t) - G(t)]dt\, dv \leq 0$, for all x in the support, with strict inequality for some x,

with the end-point condition:

$\int_{-\infty}^{+\infty}[F(t) - G(t)]dt \leq 0.$

(3) When E[X] = E[Y], X TSD Y iff $\bar{\sigma}_x^2(q_i) \leq \bar{\sigma}_y^2(q_i)$, for all Lorenz curve crossing points $q_i,\ i = 1,2,......., (n+1)$; where $\bar{\sigma}_x^2(q_i)$ denotes the "cumulative variance" for incomes upto the *ith* crossing point. See Davies and Hoy (1996).

When $n = 1$, Shorrocks and Foster (1987) show that X TSD Y if (a) the Lorenz curve of X cuts that of Y from above, and (b) $Var(X) \leq Var(Y)$. This situation seemingly revives the coefficient of variation as a useful statistical index for ranking distributions. But a distinction is needed between the well known (unconditional) coefficient of variation for a distribution, on the one hand, and the sequence of several conditional coefficients of variation involved in the TSD.

The tests of FSD and SSD are based on empirical evaluations of conditions (2) or (3) in the above definitions. Mounting tests on conditions (3) typically relies on the fact that quantiles are consistently estimated by the corresponding order statistics at a finite number of sample points. Mounting tests on conditions (2) requires empirical cdfs and comparisons at a finite number of observed ordinates. Also, from Shorrocks (1983) it is clear that condition (3) of SSD is equivalent to the requirement of Generalized Lorenz (GL) dominance. FSD implies SSD.

The Lorenz and the generalized Lorenz curves are, respectively, defined by:

$L(p) = (1/\mu)\int_0^p Y(u)du$, and $GL(p) = \mu L(p) = \int Y(u)du$, with $GL(0) = 0, and\ GL(1) = \mu$, see Shorrocks (1983).

It is customary to consider K points on the L(or GL or the support) curves for empirical evaluation with $0 < p_1 < p_2 < .... < p_K = 1$, and $p_i = i/K$. Denote the corresponding quantiles by $Y(p_i)$, and the conditional moments $\gamma_i = E(Y|Y \leq Y(p_i))$, and $\bar{\sigma}_i^2 = E\{(Y - \gamma_i)^2|Y \leq Y(p_i)\}$.

The vector of GL ordinates is given by $\eta = (p_1\bar{\sigma}_1^2, p_2\bar{\sigma}^2, ...p_K\bar{\sigma}_K^2)'$.

Xu (1995) and Xu, Fisher and Wilson (1995) adopt the $\bar{\chi}^2$ approach described above to test quantile conditions (3) of FSD and SSD. A short description follows:

Consider the random sequence $\{Z_t\} = \{X_t, Y_t\}'$, a stationary $\phi - mixing$ sequence of random vectors on a probability space $(\Omega, \Re, P)$. Similarly, denote the stacked vector of GL ordinates for the two variables as $\eta^Z = (\eta^X, \eta^Y)'$, and the stacked vector of quantiles of the two variables by $q^Z = (q^X, q^Y)'$, where $q^X = (X(p_1), X(p_2), ...X(p_K))'$, and similarly for Y. In order to utilize the general theory given for the chi-bar distribution, three ingredients are required. One is to show that the various hypotheses of interest in this context are representable as in (23) above. This is possible and simple. The second is to verify if and when the unrestricted estimators of the η and q functions satisfy the asymptotic representation given in (22). This is possible under conditions on the processes and their relationships, as we'll summarize shortly. The third is to be able to empirically implement the chi-bar statistics that ensue. In this last step, resampling techniques are and will become even more prominent.

To see that hypotheses of interest are suitably representable, we note that for the case of conditions (2) of FSD and SSD, the testing problem is the following:

$H_0 : h(q^Z) \geq 0$ against $H_1 : h(q^Z) \ngeq 0$,

where $h(q^Z) = [I_K : -I_K]q^Z = I^* q^Z, say$, for FSD, and $h(q^Z) = BI^* \times q^Z$, for the test of SSD, where,

$B = (B_{ij}), B_{ij} = 1,\ i \geq j,\ B_{ij} = 0$, otherwise, is the "summation" matrix which obtains the successive cumulated quantile (Φ) and other functions.

Tests for GL dominance (SSD) which are based on the ordinate vector η^Z are also of the "linear inequality" form and require $h(\eta^Z) = I^* \eta^Z$.

Sen (1972) gives a good account of the conditions under which sample quantiles are asymptotically normally distributed. Davidson and Duclos (1998) provides the most general treatment of the asymptotic normality of the nonparametric sample estimators of the ordinates in η. In both cases the asymptotic variance matrix, Ψ, noted in the general setup (22) is derived. What is needed is to appropriately replace R in the formulations of Kodde and Palm (1986) , or Gourieroux et al (1982), and to implement the procedure with consistent estimates of Ω in $\Psi = R\Omega R'$.

For sample order statistics , $\hat{q}_T^Z$ it is well known that, if X and Y are independent,

$$\sqrt{T}(\hat{q}^Z - q^Z) \underset{\longrightarrow}{d} N(0, \Omega)$$

$$\Omega = G^{-1} V G'^{-1}$$

$$G = diag[f_x(X(p_i)), \ldots\ldots\ldots; f_y(Y(p_i)), \ldots..],\ i = 1, \ldots.., K$$

$$V = \lim_{T \to \infty} E(gg'),\ g = T^{-1}(F_x F_y'),$$

$$F_x = [\{F(X(p_1)) - p_1\}, \ldots, \{F(X(p_K)) - p_K\}], F_y \text{ similarly defined}$$

As is generally appreciated, these density components are notoriously difficult to estimate. Kernel density methods can be used, as can Newey-West type robust estimators. But it is desirable to obtain bootstrap estimates based on block bootstrap and/or iterated bootstrap techniques. These are equally accessible computationally, but may perform much better in smaller samples and for larger numbers of ordinates K. Xu (1995) demonstrates with an application to testing for GL dominance in Canadian wages, and Xu et al (1995) demonstrate with an application to the hypothesis of term premia based on one and two months US Treasury bills. FSD and SSD of two months over one month bills can be inferred at 5% level and not the other way round. This application was based on the Kodde and Palm (1986) critical bounds and encountered some realizations in the inconclusive region. Xu et Al (1995) employed Monte Carlo simulations to obtain the exact critical levels in those cases.

Sample analogues of η and similar functions for testing any stochastic order also have asymptotically normal distributions. Davidson and Duclos (1998) exploit the following interesting result which translates conditions (2) of the FSD and SSD into inequality restrictions among the members of the η functions defined above:

Let $D_X^1(x) = F_X(x),\ D_Y^1(y) = F_Y(y)$; then,

$$D_i^s(x) = \int_0^x D^{s-1}(u)du = \frac{1}{(s-1)!}\int_0^x (x-u)^{s-1} dF(u), \text{for } s \geq 2$$

This last equality clearly shows that tests of any order stochastic dominance can be based on the conditional moments estimated at a suitable finite number of K ordinates as defined above. Indeed , since poverty measures are often defined over lower subsets of the domain such that $x \leq$ *Poverty line*, dominance relations over poverty measures can also be tested in the same fashion. Using empirical distribution functions, Davidson and Duclos (1998) demonstrate with an example from the panels for six countries in the Luxembourg study. It should be appreciated, however, that the latter do not exploit the inequality nature of the alternative hypotheses. The union intersection method determines the critical level of the inference process here. The cases of unrankable distributions include both "equivalence" and crossing (non-dominant) distributions. A usual asymptotic χ^2 test will have power in both directions. In order to improve upon this, therefore, one must employ the chi-bar distribution technique.

Similarly, Kaur et al (1994) proposed a test for condition (2) of SSD when i.i.d observations are assumed for independent prospects X and Y. Their null hypothesis is condition (2) of SSD *for each x* against the alternative of strict violation of the same condition *for all x*. The test of SSD then requires an appeal to union intersection technique which results in a test procedure with maximum asymptotic size of α if the test statistic at each x is compared with the critical value Z_α of the standard Normal distribution. They showed their test is consistent. One rejects the null of dominance if any negative distances at the K ordinates is significant.

In contrast, McFadden (1982), and Klecan et al (1991) test for dominance jointly *for all x*. McFadden's analysis of the multivariate Kolmogorov-Smirnov type test is developed for a set of variables and requires a definition of "maximal" sets, as follows:

definition

Let $X_{.n} = (x_{1n}, x_{2n}, \ldots.., x_{Kn})$, n = 1, 2,....,N , be the observed data. We assume $X_{.n}$ is strictly stationary and $\alpha - mixing$. As in Klecan et al., we also assume $F_i(X_i)$, i = 1,2,...,K are *exchangeable* random variables, so that our resampling estimates of the test statistics converge appropriately. This is less demanding than the assumption of independence which is not realistic in many applications (as in before and after tax scenarios). We also assume F_k is unknown and estimated by the empirical distribution function $F_{kN}(X_k)$. Finally, we adopt Klecan et al's mathematical regularity conditions pertaining to von Neumann-Morgenstern (VNM) utility functions that generally underlie the expected utility maximization paradigm. The following theorem defines the tests and the hypotheses being tested:

Theorem Given the mathematical regularity conditions;

(a) The variables in Æ are first-order stochastically maximal; i.e.,

(1) $d = \min_{i \neq j} \max_{x} [F_i(x) - F_j(x)] > 0,$

if and only if for each i and j, there exists a continuous increasing function u such that $E\,u(X_i) > E\,u(X_j)$.

(b) The variables in Æ are second order stochastically maximal; i.e.,

(2) $S = \min_{i \neq j} \max_x \int_{-\infty}^{x} [F_i(\mu) - F_j(\mu)] d\mu > 0,$

if and only if for each i and j, there exists a continuous increasing and strictly concave function u such that $Eu(X_i) > Eu(X_j)$.

(c) Assuming the stochastic process $X_{.n}$, n = 1, 2,.... , to be strictly stationary and $\alpha - mixing$ with $\alpha(j) = O(j^{-\delta})$, for some $\delta > 1$, and ,

(d) Assuming the variables in the set are exchangeable (relaxing independence in McFadden (1989)),

Theorem

$d_{2N} \to d$, and $S_{2N} \to S$, where d_{2N} and S_{2N} are the empirical test statistics defined as :

(3) $d_{2N} = \min_{i \neq j} \max_{x} [F_{iN}(x) - F_{jN}(x)]$

and,

(4) $S_{2N} = \min_{i \neq j} \max_{x} \int_{0}^{x} [F_{iN}(\mu) - F_{jN}(\mu)] d\mu$

[Proof] . See Theorems 1. and 5 of Klecan et al (1991).

The null hypothesis tested by these two statistics is that, respectively, Æ is *not* first (second) order maximal— i.e., X_i FSD(SSD) X_j for some i and j. We reject the null when the statistics are positive and large. Since the null hypothesis in each case is composite, power is conventionally determined in the least favorable case of identical marginals $F_i = F_j$. As is shown in Kaur et al (1994) and Klecan et al (1991), when X and Y are independent, tests based on d_{2N} and S_{2N} are consistent. Furthermore, the asymptotic distribution of these statistics are non-degenerate in the least favorable case, being Gaussian (see Klecan et al (1991), Theorems 6-7).

As is pointed out by Klecan et al (1991), for non-independent variables, the statistic S_{2N} has, in general, neither a tractable distribution, nor an asymptotic distribution for which there are convenient computational approximations. The situation for d_{2N} is similar except for some special cases—-see Durbin (1973, 1985), and McFadden (1989) who assume i.i.d. observations (not crucial), and independent variables in Æ (consequential). Unequal sample sizes may be handled as in Kaur et al.(1994).

Klecan et al (1991) suggest Monte Carlo procedures for computing the significance levels of these tests. This forces a dependence on an assumed parametric distribution for generating MC iterations, but is otherwise quite appealing for very large iterations. Maasoumi et al (1997) employ the bootstrap method to obtain the empirical distributions of the test statistics and of p-values. Pilot studies show that their computations obtain similar results to the algorithm proposed in Klecan et al (1991).

In the bootstrap procedure we compute d_{2N} and S_{2N} for a finite number K of the income ordinates. This requires a computation of sample frequencies, cdfs and sums of cdfs, as well as the differences of the last two quantities at all the K points. Bootstrap samples are generated from which empirical distributions of the differences, of the d_{2N} and S_{2N} statistics, and their bootstrap confidence intervals are determined. The bootstrap probability of these statistics being negative and/or falling in these intervals leads to rejection of the hypotheses.

Maasoumi et al (1997) demonstrate by several applications to the US income distributions based on the CPS and the panel data fro the Michigan study. In contrast to the sometimes confusing picture drawn by comparisons based on inequality indices, they find frequent SSD relations, including between population subgroups, that suggest a “welfare” deterioration in the 1980s compared to the previous periods.

Conclusion

Taking the one-sided nature of some linear and non-linear hypotheses is both desirable and practical. It can improve power and to the improved computation of the critical levels. A chi-bar squared and a multivariate KS testing strategy were described and contrasted with some alternatives, either the less powerful two-sided methods, or the union intersection procedures. The latter deserves to be studied further in comparison to the methods that are expected to have better power. Computational issues involve having to solve QP problems to obtain inequality restricted estimators, and numerical techniques for computation of the weights in the chi-bar statistic. Bounds tests for the latter are available and may be sufficient in many cases.

Applications in the parametric/semi-parametric , and the non-parametric testing area have been cited. They tend to occur in substantive attempts at empirical evaluation and incorporation of economic theories.

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