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Endomorphism Rings of Small Pseudo Projective Modules

Ritu Jaiswal

Department Of Mathematics
Banaras Hindu University
Varanasi-221005, India
ritu11_bhu@yahoo.com

P. C. Bharadwaj

Department Of Mathematics
Banaras Hindu University
Varanasi-221005, India
drpcbharadwaj@yahoo.co.in

Abstract

In this paper I have tried to find some of the results on endomorphism rings of small pseudo projective modules.

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1 Introduction

Throughout this paper the basic ring R is supposed to be ring with unity and all modules are unitary left R-modules.
Let M be an R-module, a submodule K of M is said to be small in M if $K + L = M \Rightarrow L = M$ for any submodule $L \subseteq M$. An R-module M is said to be hollow if all proper submodules of M are small in M. An R-module M is said to be small quasi projective if for any module A, with small epimorphism $g : M \to A$ and homomorphism $f : M \to A$ there exists an $h \in End(M)$ such that $f = goh$. An R-module M is said to be small pseudo projective if for any module A, with small epimorphism $g : M \to A$ and epimorphism $f : M \to A$ there exists an $h \in End(M)$ such that $f = goh$. A ring R is called regular

(in the sense of Von- Neumann) if for each $r \in R$ there exists $x \in R$ such that $r = rxr$. The jacobson radical $J(M)$, of a module M, is the intersection of all maximal submodules of M. An R-module M is called local if it has a unique maximal submodule which contains every proper submodules of M. The socle of an R module M denoted by $Soc(M)$ is defined as intersection of essential submodules of M. Two module epimorphisms $f, g : P \to M$ are right equivalent if $f = goh$ for some automorphism h of P. An R module M is called π-projective if for all submodules U and V of M with $U + V = M$, there exists $f \in S$ with $Imf \subseteq U$ and $Im(1 - f) \subseteq V$. A submodule N of an R-module M is said to be small pseudo stable if for any epimorphism $f : M \to A$ and any small epimorphism $g : M \to A$ with $N \subseteq Kerg \cap Kerf$, there exists $h \in End(M)$ such that $f = goh$ then, $h(N) \subseteq N$. A module M is called a duo module if every submodule of M is fully invariant.

2 Main Results

Proposition 1. *Let M be any small pseudo projective hollow module. Then every epimorphism in $End(M)$ is an automorphism.*

Proof: Let $g : M \to M$ be any epimorphism then we have $Kerg \neq M$. So, $Kerg$ is a proper submodule of M. As M is hollow, g is a small epimorphism, by small pseudo projectivity of M, I_M can be lifted to a homomorphism $h : M \to M$ such that $goh = I_M$.
$\Rightarrow h$ is one-one.
Let $m \in M$ then as g is onto there exists an element $n \in M$ such that $m = g(n) \Rightarrow g(n - h(m) = 0 \Rightarrow n - h(m) \in Kerg \Rightarrow n \in Kerg + h(m)$ $\Rightarrow M \subseteq Kerg + Imh$. So, we have $M = Kerg + Imh \Rightarrow M = Imh$, since M is hollow. Thus h is onto and so h is an automorphism $\Rightarrow h^{-1} = g$ is an automorphism.

Proposition 2. ***(a)****If S is the endomorphism ring of a small quasi projective hollow module M then S is local.*
(b)*If S is the endomorphism ring of a small pseudo projective hollow module M then S is local.*

Proof: Follows from [1, Theorem 1.14]

Proposition 3. *Let M be any pseudo projective module and $End(M)$ denotes the endomorphism ring of M. Then if $\alpha(M) \subseteq^{\oplus} M$ for every $\alpha \in End(M)$ then $ker\alpha \subseteq^{\oplus} M$.*

Proof: Follows from [7, Proposition 8].

Proposition 4. *Let M be any pseudo projective module and $End(M)$ denotes the endomorphism ring of M. Then if $\alpha(M) \subseteq^{\oplus} M$ for every $\alpha \in End(M)$ then $End(M)$ is regular.*

Proof: Follows from [7, Proposition 10].

Corollary 4.1: Endomorphism ring of a completely reducible pseudo projective module is regular.

Proposition 5. *Let M be a small pseudo projective hollow module S denotes the endomorphism ring of M, $J(S)$ denotes the jacobson radical of S then*
(a) $J(S) = \{\alpha \in S | Im\alpha \text{ is small in } M\}$
(b)$J(S) \subseteq Hom(M, J(M))$
(C)$S/J(S)$ *is Von-neumann regular ring.*

Proof: Follows from [1, Theorem 1.15]

Proposition 6. *Let M be a small pseudo projective hollow module and K be any small submodule of M then for any automorphism $g \in Aut(M/K)$ there exists an automorphism $h \in Aut(M)$ such that $g(m+K) = h(m)+K \ \forall m \in M$.*

Proof: Let K be any small submodule of M and $\nu : M \to M/K$ be any natural map, and $g : M/K \to M/K$ be any automorphism in $Aut(M/K)$. Then by small pseudo projectivity of M $\exists\ h \in End(M)$ such that $go\nu = \nu oh$ i.e. $go\nu(m) = \nu oh(m) \forall m \in M \Rightarrow g(m + K) = h(m) + K \ \forall m \in M$. Then by [5, Proposition 4] h is an epi-endomorphism. By Proposition 1 we get h is an automorphism.

Proposition 7. *Let M be a small pseudo projective hollow module then for any $\alpha \in End(M)$ and any small submodule K of M with $\alpha(M) + K = M$ and $\alpha^{-1}(K) = K$ there exists $\beta \in End(M)$ such that $\beta(M) \subseteq K$ and $\alpha + \beta \in Aut(M)$.*

Proof: Suppose $\alpha \in End(M)$ and K is any small submodule of M satisfying $\alpha(M) + K = M$ and $\alpha^{-1}(K) = K$. Let $f : M \to M/K$ be the natural map. Now we have $Ker(fo\alpha) = \alpha^{-1}(Kerf) = \alpha^{-1}(K) = K = Kerf$. Thus, $Ker(fo\alpha) = Kerf$. Now, $\alpha(M) + K = M \Rightarrow \alpha(M) = M \Rightarrow \alpha$ is onto and therefore $fo\alpha$ is onto. So by [2, Theorem 3.6] $\exists$ an automorphism $g \in End(M/K) \ni gof = fo\alpha$. So by assumption there exists $h \in Aut(M)$ such that $g(m + K) = h(m) + K \Rightarrow g(M/K) = foh(M) \Rightarrow gof(M) = foh(M)$ $\Rightarrow gof = foh \Rightarrow fo\alpha = foh \Rightarrow f(h - \alpha) = 0$. Let $\beta = h - \alpha$. We have $\beta(M) \subseteq K$. Also $\alpha + \beta = h$ is an automorphism in $Aut(M)$.

Proposition 8. *Let M be a small pseudo projective hollow module then any pair of small epimorphisms from M to any module N are right equivalent if for given any $\alpha \in End(M)$ and any small submodule K of M with $\alpha(M)+K = M$ there exists $\beta \in End(M)$ such that $\beta(M) \subseteq K$ and $\alpha + \beta \in Aut(M)$.*

Proof: Suppose $f, g : M \to N$ are small epimorphism. By small pseudo projective of M there exists $\alpha \in End(M)$ such that $f = go\alpha$. Since f is epimorphism we have $\alpha(M) + Ker(g) = M$ then by assumption there exists $\beta \in End(M)$ such that $\alpha + \beta \in Aut(M)$ and $\beta(M) \subseteq K$. So $g(\alpha + \beta) = go\alpha + go\beta = go\alpha = f$. So, f and g are right equivalent.

Proposition 9. *Let M be a duo and small pseudo projective module. Let S denotes the endomorphism ring of M and $T = \{\alpha \in S | Im\alpha \text{ is small in } M\}$. Then for every $f \in T$, Imf is a small pseudo stable submodule of M.*

Proof: Let $f \in T$ then Imf is a small submodule of M. Let $g : M/Imf \to A$ be a small epimorphism, $\psi : M/Imf \to A$ be an epimorphism and $\nu : M \to M/Imf$ be the natural map. Then $Ker\nu = Imf$ is a small submodule of $M \Rightarrow \nu$ is a small epimorphism. Now, $Imf \subseteq Ker(go\nu) \cap Ker(\psi o\nu)$, since $Ker\nu = Imf \Rightarrow \nu(Imf) = 0 \Rightarrow g(\nu(Imf)) = 0 \Rightarrow Imf \subseteq Ker(go\nu)$. Similarly $Imf \subseteq Ker(\psi o\nu))$. By small pseudo projectivity of M there exists $h \in End(M)$ such that $\psi o\nu = go\nu oh$. We have, $h(Imf) \subseteq Imf$, since M is duo and $Imf \subseteq M$. So, Imf is a small pseudo stable submodule of M.

Proposition 10. *Let M be a duo and small pseudo projective hollow module. Let S denotes the endomorphism ring of M and $J(S)$ denotes the jacobson radical of M. Then for every $f \in J(S)$, Imf is a small pseudo stable submodule of M.*

Proof: By Proposition 5(a), we have $T = J(S)$. Rest of the proof follows from Proposition 9.

Proposition 11. *Let M be a small pseudo projective module if S is local and M is π- projective then M is hollow.*

Proof: Let U and V be submodules of M such that $U + V = M$. As M is π-projective there exists $f \in S$ such that $Imf \subseteq U$ and $Im(1-f) \subseteq V$. Now S is local so, $f \in S \Rightarrow$ either f or $(1-f)$ is invertible. Now f is invertible $\Rightarrow \exists\ g \in S \ni fog = I_M \Rightarrow f$ is onto and so $Imf = M \Rightarrow U = M$. Thus V is small. Similarly we can show that when $(1-f)$ is invertible then $V = M$ $\Rightarrow U$ is small, and therefore M is hollow.

Proposition 12. *Let M be a small pseudo projective $D2$ module. Then M is S.F.*

Proof: Follows from [5, Proposition 3]

Proposition 13. *Let M be a small quasi projective duo module. If $S = End(M)$, is local, then M is not supplemented.*

Proof: Suppose that M is supplemented and A is any submodule of M. Let B be supplement of A in M then we have $M = A + B$ and $A \cap B$ is small in M. Let $0 \neq s(M) = A$ and $0 \neq t(M) = B$, $s, t \in S$. Define the map $f : M = (s+t)(M) \to M/(A \cap B)$ such that $f(s+t)(m) = s(m) + (A \cap B)$. For any $m, m^{'} \in M$, $(s+t)(m) = (s+t)(m^{'})$ implies that $s(m - m^{'}) = t(m^{'} - m) \in A \cap B$. So $s(m) + (A \cap B) = s(m^{'}) + (A \cap B)$. Thus f is well defind and f is also an R-homomorphism. Let $\nu : M \to M/(A \cap B)$ is the natural map. By small quasi projectivity of M, there exist $g \in S$ such that $\nu og = f$. We have $\nu og(s+t))(m) = f(s+t)(m) = s(m) + (A \cap B)$. Then, $g(s+t))(m) + (A \cap B) = s(m) + (A \cap B) \Rightarrow ((1-g)os - got)(M) \subseteq (A \cap B)$. Since S is local, g or $(1-g)$ is invertible. If $(1-g)$ is invertible we have, $(s - (1-g)^{-1}ogot)(M) \subseteq (1-g)^{-1}(A \cap B) \subseteq (A \cap B)$. Now $A \subseteq (s - (1 - g)^{-1}ogot)(M) \subseteq (A \cap B)$. Then $A \subseteq (A \cap B)$, which is a contradiction. Similarly if g is invertible we have $B \subseteq (g^{-1}o(1-g)os - t) \subseteq g^{-1}(A \cap B) \subseteq (A \cap B)$. Then $B \subseteq (A \cap B)$, that is also a contradiction. Hence M is not supplemented.

Corollary 13.1: Let M be a hollow small quasi projective duo module. Then M is not supplemented.
Proof: Follows from Proposition 2(a) and Proposition 13.

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