EINDHOVEN UNIVERSITY OF TECHNOLOGY

Acyclic and frugal colourings of graphs

Acyclic and frugal colourings of graphs

Ross J. Kang [a,1,*] Tobias Müller [b]

[a] *School of Computer Science, McGill University, 3480 University Street, Montréal, Québec, H2A 2A7, Canada.*

[b] *EURANDOM, Technische Universiteit Eindhoven, PO Box 513, 5600 MB Eindhoven, The Netherlands.*

Abstract

Given a graph $G = (V, E)$, a proper vertex colouring of V is *t-frugal* if no colour appears more than t times in any neighbourhood and is *acyclic* if each of the bipartite graphs consisting of the edges between any two colour classes is acyclic. For graphs of bounded maximum degree, Hind, Molloy and Reed [14] studied proper t-frugal colourings and Yuster [19] studied acyclic proper 2-frugal colourings. In this paper, we expand and generalise this study. In particular, we consider vertex colourings that are not necessarily proper, and in this case, we find qualitative connections with colourings that are *t-improper* — colourings in which the colour classes induce subgraphs of maximum degree at most t — for choices of t near to d.

Key words: graph colouring; frugal colouring; acyclic colouring; linear colouring; improper colouring.

1 Introduction

In this paper, a *(vertex) colouring* of a graph $G = (V, E)$ is any map $f : V \to \mathbb{Z}^+$. The *colour classes* of a colouring f are the preimages $f^{-1}(i), i \in \mathbb{Z}^+$. A colouring of a graph is *proper* if adjacent vertices receive distinct colours; however, in this paper, we will devote a lot of attention to colourings that are not necessarily proper, but that satisfy another condition.

* Corresponding author.
Email addresses: `ross.kang@cs.mcgill.ca` (Ross J. Kang), `t.muller@tue.nl` (Tobias Müller).

1 The research in this paper was carried out while this author was a doctoral research student at the University of Oxford. He was partially supported by NSERC of Canada and the Commonwealth Scholarship Commission (UK).

Given a colouring f of G, the *frugality* of a vertex v under f is the size of the largest monochromatic set of neighbours of v in G, and the *frugality* of f is the value of the largest frugality among all vertices of G. For $t \geq 1$, a colouring of G is *t-frugal* if its frugality is at most t. Alternatively, a colouring of G is t-frugal if no colour appears more than t times in any neighbourhood.

The notion of frugal colouring was introduced a decade ago by Hind, Molloy and Reed [14]. They considered proper t-frugal colourings as a way to improve bounds related to the Total Colouring Conjecture (cf. [15]). In Section 2, we study t-frugal colourings for graphs of bounded maximum degree.

In Section 3, we impose an additional condition that is well-studied in the graph colouring literature (see, e.g. [3, 4, 8, 12]). A colouring of V is *acyclic* if each of the bipartite graphs consisting of the edges between any two colour classes is acyclic. In other words, a colouring of G is acyclic if G contains no *alternating cycle* (that is, an even cycle that alternates between two distinct colours).

For graphs of bounded maximum degree, the study of acyclic proper colourings was instigated by Erdős (cf. [3]) and more or less settled asymptotically by Alon, McDiarmid and Reed [4]. Extending the work of Alon *et al.*, Yuster [19] has investigated acyclic proper 2-frugal colourings (but called them linear colourings) of bounded degree graphs. In Section 3, we expand this study to different values of t and colourings that are not necessarily proper.

In both Sections 2 and 3, we reveal in certain situations a qualitative similarity between the behaviour of t-frugal colourings and another type of "improper" colouring. For $t \geq 0$, a colouring is *t-improper* if each colour class induces a subgraph of G with maximum degree at most t. Notice that t-frugality is a stronger requirement than t-impropriety, i.e. any colouring that is t-frugal is necessarily t-improper, but not conversely. We shall elaborate further upon the connection between improper colouring and frugal colouring later in the paper.

In Sections 2 and 3, we concentrate on using probabilistic techniques to obtain bounds that are asymptotically good. In Section 4, though, we turn our attention away from the probabilistic approach to some deterministic questions on frugal colouring.

Before we proceed, let us outline the notation used throughout the paper, as well as some straightforward observations. As usual, the *(proper) chromatic number* $\chi(G)$ denotes the least number of colours needed in a proper colouring and the *acyclic (proper) chromatic number* $\chi_a(G)$ denotes the least number of colours needed in an acyclic proper colouring. For $t \geq 1$, the *t-frugal chromatic number* $\varphi^t(G)$ denotes the least number of colours needed in a t-frugal colouring; analogously, for $t \geq 1$, we define the *proper t-frugal chromatic* *number* $\chi^t_\varphi(G)$, the *acyclic t-frugal chromatic number* $\varphi^t_a(G)$ and the *acyclic proper t-frugal chromatic number* $\chi^t_{\varphi,a}(G)$. For $t \geq 0$, we analogously define the *t-improper chromatic number* $\chi^t(G)$ and the *acyclic t-improper chromatic number* $\chi^t_a(G)$. In our notation, we have chosen φ to serve as a mnemonic for frugal.

Since we are interested in studying these parameters for graphs G of maximum degree $\Delta(G) = d$, we let $\chi(d)$ denote the maximum possible value of $\chi(G)$ over all graphs G with $\Delta(G) = d$. We analogously define $\chi_a(d)$; $\varphi^t(d)$, $\chi^t_\varphi(d)$, $\varphi^t_a(d)$ and $\chi^t_{\varphi,a}(d)$ for $t \geq 1$; and $\chi^t(d)$ and $\chi^t_a(d)$ for $t \geq 0$. We frequently use the monoticity of these parameters with respect to d: $\chi(d-1) \leq \chi(d)$, $\chi_a(d-1) \leq \chi_a(d)$, $\varphi^t(d-1) \leq \varphi^t(d)$, and so on.

We denote G^2 to be the *square* of a graph G, i.e. the graph formed from G by adding the edges between any two vertices at distance two from each other.

Proposition 1 *For any graph G and any $t \geq 1$, the following hold:*

(i) $\chi^1_\varphi(G) = \chi^1_{\varphi,a}(G) = \chi(G^2)$;
(ii) $\chi^t(G) \leq \varphi^t(G) \leq \chi^t_\varphi(G)$ *and* $\chi^t_a(G) \leq \varphi^t_a(G) \leq \chi^t_{\varphi,a}(G)$;
(iii) $\varphi^t(G) \leq \varphi^t_a(G)$ *and* $\chi^t_\varphi(G) \leq \chi^t_{\varphi,a}(G)$;
(iv) $\varphi^{t+1}(G) \leq \varphi^t(G)$, $\chi^{t+1}_\varphi(G) \leq \chi^t_\varphi(G)$, $\varphi^{t+1}_a(G) \leq \varphi^t_a(G)$, $\chi^{t+1}_{\varphi,a}(G) \leq \chi^t_{\varphi,a}(G)$; *and*
(v) $\varphi^t(G) \geq \Delta(G)/t$.

Proof. Part (i) holds because 1-frugality ensures that vertices at distance two have distinct colours; therefore, a 1-frugal proper colouring is precisely a colouring of the square of the graph (and such a colouring forbids alternating cycles). Part (ii) holds since any proper t-frugal colouring is t-frugal and any t-frugal colouring is t-improper. Part (iii) holds since forbidding alternating cycles can only make the corresponding chromatic number larger. Part (iv) holds as any t-frugal colouring is $(t+1)$-frugal. Part (v) is simply the observation that, in any t-frugal colouring, the $\deg(v)$ vertices in the neighbourhood of any vertex v require at least $\deg(v)/t$ colours. □

For the reader who desires a quick overview of this work, please consult Tables 1, 4 and 5.

1.1 Probabilistic and asymptotic preliminaries

We make use of two versions of the Lovász Local Lemma [9].

Symmetric Lovász Local Lemma 2 ([9]**, cf.** [18]**, page 40)** *Let $\mathcal{E}$ be a set* *of (typically bad) events such that for each* $A \in \mathcal{E}$

(i) $\Pr(A) \le p < 1$, *and*
(ii) A *is mutually independent of a set of all but at most* δ *of the other events.*

If $ep(\delta+1) < 1$, *then with positive probability none of the events in* $\mathcal{E}$ *occur.*

General Lovász Local Lemma 3 ([9]**, cf.** [18]**, page 222)** *Let* $\mathcal{E}$ *be a set* $\{A_1, \ldots, A_n\}$ *of (typically bad) events such that for each* A_i *there exists a set* $\mathcal{D}_i \subseteq \mathcal{E}$ *such that* A_i *is mutually independent of all* A_j *not in* $\mathcal{D}_i$. *If there are real weights* $0 \le x_i < 1$ *such that for all* i

$$\Pr(A_i) \le x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j),$$

then with positive probability none of the events in $\mathcal{E}$ *occur.*

We will also use of the following bound on the upper tail of the binomial distribution $\mathrm{BIN}(n, p)$.

A Chernoff Bound 4 (cf. Equation (2.5) of [17]**)** *If* $t \ge 0$, *then*

$$\Pr(\mathrm{BIN}(n,p) \ge np + t) \le \exp\left(-\frac{t^2}{2(np + t/3)}\right).$$

We will frequently make use of standard notation to compare the relative asymptotic behaviour of two real sequences $f, g : \mathbb{Z}^+ \to \mathbb{R}$ that depend on a parameter $n \to \infty$. We write that (as $n \to \infty$)

- $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ if there exist constants $C > 0$ and n_0 such that $|f(n)| < C|g(n)|$ for all $n \ge n_0$;
- $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$;
- $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$;
- $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.

2 Frugal colourings

As a tool to improve bounds for total colouring (cf. [15]), Hind *et al.* [14], using probabilistic techniques (including use of the Local Lemma), developed the following result for proper frugal colouring.

Theorem 5 (Hind *et al.* [14]**)** *For sufficiently large* d,

$$\chi_\varphi^{(\ln d)^5}(d) \le d + 1.$$

Since $\chi^t_\varphi(K_{d+1}) \geq d+1$, it follows that $\chi^t_\varphi(d) = d+1$ for $t = t(d) \geq (\ln d)^5$. For smaller frugalities, they also showed the following.

Theorem 6 (Hind *et al.* [14]**)** *For any $t \geq 1$ and sufficiently large d,*

$$\chi^t_\varphi(d) \leq \max\left\{(t+1)d, \left\lceil e^3 \frac{d^{1+1/t}}{t} \right\rceil\right\}.$$

Note that $\chi^1_\varphi(d) \sim d^2$ by Proposition 1(i). By Proposition 1(ii), the following example essentially due to Alon (cf. [14]) shows that Theorem 6 is asymptotically correct up to a constant multiple when $t = o(\ln d/\ln\ln d)$.

Proposition 7 *For any $t \geq 1$ and any prime power n,*

$$\varphi^t(n^t + \cdots + 1) \geq \frac{n^{t+1} + \cdots + 1}{t}.$$

Proof. Set $d = n^t + \cdots + 1$ and $m = n^{t+1} + \cdots + 1$. Let P be a $(t+2)$-dimensional projective geometry with m points. We form a bipartite graph G with parts A and B, where A is the set of points in P and B is the set of $(t+1)$-flats (hyperplanes), and an edge between two vertices $a \in A, b \in B$ if the point a lies in the hyperplane b.

Every hyperplane contains exactly d points in P, so G has maximum degree d by projective geometry duality. Since every set of $t+1$ points lies in a $(t+1)$-flat, no colour can appear more than t times on A in any t-frugal colouring of G (*whether proper or not*); thus, at least m/t colours are required. □

Corollary 8 *Suppose that $t = t(d) \geq 2$ and $t = o(\ln d/\ln\ln d)$. Then, for any $\epsilon > 0$, it holds that*

$$\varphi^t(d) \geq (1-\epsilon)\frac{d^{1+1/t}}{t}$$

for sufficiently large d.

Proof. Let x solve $d = x^{t(d)} + \cdots + 1$ where d is chosen large enough to satisfy certain inequalities specified below. Set $m = x^{t(d)+1} + \cdots + 1$. Note that, since $t = o(\ln d)$, $x \to \infty$ as $d \to \infty$. It follows that $d = (1+o(1))x^{t(d)}$ and $x = (1+o(1))d^{1/t(d)}$.

Due to a classical result of Ingham [16] on the gaps between primes, there is a prime n between $x - Cx^{5/8}$ and x, for some absolute constant C. Let $d' = n^{t(d)} + \cdots + 1$ and $m' = n^{t(d)+1} + \cdots + 1$. We have, using Proposition 7,

$$\varphi^{t(d)}(d) \geq \varphi^{t(d)}(d') \geq \frac{m'}{t(d)} \geq \left(1 - \frac{C}{x^{3/8}}\right)^{t(d)+1} \frac{m}{t(d)} \tag{1}$$

Since $x = (1 + o(1))d^{1/t(d)}$, we have

$$\left(1 - \frac{C}{x^{3/8}}\right)^{t(d)+1} \geq 1 - \frac{C(t(d)+1)}{x^{3/8}} \geq 1 - \frac{2C(t(d)+1)}{d^{0.375/t(d)}} \geq 1 - \epsilon/2$$

for d sufficiently large, with this last inequality due to $t(d) = o(\ln d/ \ln\ln d)$. Also, using $d = (1 + o(1))x^{t(d)}$ and $x = (1 + o(1))d^{1/t(d)}$, we have that

$$m = (x^{t(d)+2} - 1)/(x - 1) = (1 + o(1))x^{t(d)+1} = (1 + o(1))d^{1+1/t(d)}.$$

Substituting these last two estimates into Inequality (1), we obtain that

$$\varphi^{t(d)}(d) \geq (1 - \epsilon/2)^2 \frac{d^{1+1/t(d)}}{t(d)} \geq (1 - \epsilon)\frac{d^{1+1/t(d)}}{t(d)}$$

for large enough d, as claimed. $\square$

Theorems 5 and 6 determine the behaviour of $\chi^t_\varphi(d)$ up to a constant multiple for all t except for the range such that $t = \Omega(\ln d/ \ln\ln d)$ and $t = O((\ln d)^5)$. Complete closure of this logarithmic gap would be likely to require significant effort and ingenuity. In the next result, we numerically refine the upper bound of Theorem 6.

Theorem 9 *For positive integers t and d with $t \leq d$, let γ be such that*

$$\gamma > \left(\frac{5(t+1)}{\sqrt{2\pi t}e^{1/(12t+1)}}\right)^{1/t} e.$$

Then

$$\chi^t_\varphi(d) \leq \max\left\{12(t+1)d, \left\lceil \gamma\frac{d^{1+1/t}}{t}\right\rceil\right\}.$$

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \max\left\{12(t+1)d, \left\lceil\gamma d^{1+1/t}/t\right\rceil\right\}$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$.

For vertices $v, v_1, \ldots, v_{t+1}$ with $\{v_1, \ldots, v_{t+1}\} \subseteq N(v)$, let $A_{\{v,v_1,\ldots,v_{t+1}\}}$ be the event that $f(v_1) = \cdots = f(v_{t+1})$. Call these events of Type I and observe that, if no Type I events hold, then f is t-frugal. For vertices $v_1, v_2 \in E$, let $B_{\{v_1,v_2\}}$ be the event that $f(v_1) = f(v_2)$. Call these events of Type II and observe that, if no Type II events hold, then f is proper.

Clearly, $\Pr\left(A_{\{v,v_1,\ldots,v_{t+1}\}}\right) = 1/x^t$ and $\Pr\left(B_{\{v_1,v_2\}}\right) = 1/x$. Furthermore, each vertex participates in at most $d\binom{d-1}{t}$ events of Type I and at most d events of Type II. It follows that each Type I event is independent of all but at most

$(t+1)d\binom{d-1}{t}$ other Type I events and at most $(t+1)d$ Type II events, and each Type II event is independent of all but at most $2d\binom{d-1}{t}$ other Type I events and at most $2d$ other Type II events.

We define the weight x_i of each event i to be twice its probability and, in order to apply the General Local Lemma, we want to show that each of the following inequalities hold:

$$\frac{1}{x^t} \le \frac{2}{x^t}\left(1-\frac{2}{x^t}\right)^{(t+1)d^{t+1}/t!}\left(1-\frac{2}{x}\right)^{(t+1)d} \quad \text{and} \tag{2}$$

$$\frac{1}{x} \le \frac{2}{x}\left(1-\frac{2}{x^t}\right)^{2d^{t+1}/t!}\left(1-\frac{2}{x}\right)^{2d}. \tag{3}$$

Clearly, the validity of Inequality (3) follows from that of Inequality (2). First, observe that

$$\left(1-\frac{2}{x}\right)^{(t+1)d} \ge 1-\frac{2(t+1)d}{x} \ge \frac{5}{6} \text{ and}$$
$$\left(1-\frac{2}{x^t}\right)^{(t+1)d^{t+1}/t!} \ge 1-\frac{2(t+1)d^{t+1}}{x^t t!}.$$

Thus, substituting these into the left-hand side of Inequality (2) and then rearranging, we see that it suffices to show that $5(t+1)d^{t+1}/(x^t t!) \le 1$. By a precise form of Stirling's formula (cf. (1.4) of [7]),

$$t! \ge (t/e)^t\sqrt{2\pi t}e^{1/(12t+1)}; \tag{4}$$

therefore,

$$\frac{5(t+1)d^{t+1}}{x^t t!} \le \frac{5(t+1)d^{t+1}t^t}{\gamma^t d^{t+1} t!} \le \frac{5(t+1)e^t}{\gamma^t\sqrt{2\pi t}e^{1/(12t+1)}}.$$

By the choice of γ, this last expression is less than one; therefore, by the General Lovász Local Lemma, f is a proper t-frugal colouring with positive probability. $\square$

To get a more concrete feeling of this bound, note that, for example,

- when $t=2$, we need $\gamma > \left(\frac{15}{\sqrt{4\pi}e^{1/25}}\right)^{1/2} e \approx 2.016e$,
- when $t=3$, we need $\gamma > \left(\frac{20}{\sqrt{6\pi}e^{1/37}}\right)^{1/3} e \approx 1.649e$,
- when $t=4$, we need $\gamma > \left(\frac{25}{\sqrt{8\pi}e^{1/49}}\right)^{1/4} e \approx 1.487e$, and
- when $t=1000$, we need $\gamma > \left(\frac{5005}{\sqrt{2000\pi}e^{1/12001}}\right)^{1/1000} e \approx 1.004e$.

By adjusting the constant 12 before $(t+1)d$, it is possible to improve upon the constant 5 within the constraint on γ; however, this would realise a modest improvement for γ only for the smallest values of t. Note that, in the range of t such that $t \to \infty$ as $d \to \infty$ and $t = o(\ln d / \ln \ln d)$, Theorem 9 provides an improvement upon Theorem 6 of a factor $e^2 \approx 7.389$. This is because we can choose $\gamma = e + \epsilon$ for any fixed ϵ. In this case, the bound is asymptotically correct up to a multiplicative factor of e due to Corollary 8.

Next, for colourings that are not necessarily proper, we can obtain a further improvement upon the upper bound of the last theorem.

Theorem 10 *For positive integers t and d with $t \leq d$, let γ be such that*

$$\gamma > \left(\frac{e(t+1)}{\sqrt{2\pi t} e^{1/(12t+1)}} \right)^{1/t} e.$$

Then

$$\varphi^t(d) \leq \left\lceil \gamma \frac{d^{1+1/t}}{t} \right\rceil.$$

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \left\lceil \gamma d^{1+1/t}/t \right\rceil$. Let $f : V \to \{1, \dots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \dots, x\}$.

For vertices $v, v_1, \dots, v_{t+1}$ with $\{v_1, \dots, v_{t+1}\} \subseteq N(v)$, let $A_{\{v,v_1,\dots,v_{t+1}\}}$ be the event that $f(v_1) = \cdots = f(v_{t+1})$. If none of these events hold, then f is t-frugal.

Clearly, $\Pr\left(A_{\{v,v_1,\dots,v_{t+1}\}}\right) = 1/x^t$. Furthermore, each vertex participates in at most $d\binom{d-1}{t}$ of these events; thus, each event is independent of all but at most $(t+1)d\binom{d-1}{t}$ other events. We have that

$$e \Pr\left(A_{\{v,v_1,\dots,v_{t+1}\}}\right) \left((t+1)d\binom{d-1}{t} + 1 \right) < e \frac{t^t}{\gamma^t d^{t+1}} (t+1) \frac{d^{t+1}}{t!} = \frac{e(t+1)t^t}{\gamma^t t!}.$$

By Inequality (4),

$$\frac{e(t+1)t^t}{\gamma^t t!} \leq \frac{e(t+1)}{\gamma^t \sqrt{2\pi t} e^{1/(12t+1)}}.$$

It follows that $e \Pr\left(A_{\{v,v_1,\dots,v_{t+1}\}}\right) \left((t+1)d\binom{d-1}{t} + 1 \right) < 1$; thus, by the Symmetric Lovász Local Lemma, f is a t-frugal colouring with positive probability. $\square$

To get a more concrete feeling of this bound, note that, for example,

- when $t = 2$, we need $\gamma > \left(\frac{3}{\sqrt{4\pi}} e^{24/25}\right)^{1/2} e \approx 1.487e$,
- when $t = 3$, we need $\gamma > \left(\frac{4}{\sqrt{6\pi}} e^{36/37}\right)^{1/3} e \approx 1.346e$,
- when $t = 4$, we need $\gamma > \left(\frac{5}{\sqrt{8\pi}} e^{48/49}\right)^{1/4} e \approx 1.277e$, and
- when $t = 1000$, we need $\gamma > \left(\frac{1001}{\sqrt{2000\pi}} e^{12000/12001}\right)^{1/1000} e \approx 1.004e$.

If $t \to \infty$ as $d \to \infty$, then we can choose $\gamma = e + \epsilon$ for any fixed ϵ. By Corollary 8, this result is asymptotically correct up to a constant multiple when $t = o(\ln d / \ln \ln d)$. When $t \to \infty$ as $d \to \infty$ and $t = o(\ln d / \ln \ln d)$, this result is asymptotically correct up to a multiplicative factor of e.

Recall from the Proposition 1(v) that $\varphi^t(d) \geq d/t$. Now, for the case $t = \omega(\ln d)$, we give an essentially optimal upper bound for $\varphi^t(d)$. By contrast, necessarily $\chi^t_\varphi(d) = \Omega(d)$ in this range, and we see that the value of $\varphi^t(d)$ is asymptotically lower.

Theorem 11 *Suppose $t = \omega(\ln d)$. For any $\epsilon > 0$, it holds that*

$$\varphi^t(d) \leq \left\lceil (1+\epsilon)\frac{d}{t} \right\rceil$$

for sufficiently large d.

Proof. We first remark that if G is a subgraph of G', then $\varphi^t(G) \leq \varphi^t(G')$. As any graph of maximum degree d is contained in a d-regular graph, it therefore suffices to show the theorem holds for d-regular graphs. Let $G = (V, E)$ be any d-regular graph and let $x = \lceil (1+\epsilon)d/t \rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$.

For a vertex v and a colour $i \in \{1, \ldots, x\}$, let $A_{v,i}$ be the event that v has more than t neighbours with colour i. If none of these events hold, then f is t-frugal. Each event is independent of all but at most $d^2 x \ll d^3$ other events.

By a Chernoff bound, we have that

$$\begin{aligned}\Pr(A_{v,i}) &= \Pr(\mathrm{BIN}(d, 1/x) > t) \leq \Pr(\mathrm{BIN}(d, 1/x) > d/x + ct) \\ &\leq \exp\left(-c^2 t^2 / (2d/x + 2ct/3)\right)\end{aligned}$$

where $c = \epsilon/(1+\epsilon)$. Thus, $e \Pr(A_{v,i})(d^3 + 1) = \exp(-\Omega(t)) d^3 < 1$ for large enough d, and by the Symmetric Lovász Local Lemma, f is t-frugal with positive probability for large enough d. $\Box$

We see from this last theorem that when $t = \omega(\ln d)$, the behaviour of $\varphi^t(d)$ is closely tied to that of $\chi^t(d)$. Because cliques satisfy $\chi^t(K_{d+1}) = \lceil (d+1)/(t+1) \rceil$, it follows that $\varphi^t(d) \sim \chi^t(d) \sim \lceil d/t \rceil$ if $t = \omega(\ln d)$ and $t = o(d)$. If $d/t \to x$ for some $0 < x < \infty$, then $\chi^t(d) = \lceil x \rceil$ while Theorem 11 implies that $\varphi^t(d) = \lceil x \rceil$ if x is not integral and $\varphi^t(d) \in \{x, x+1\}$ otherwise.

When $t = \Theta(\ln d)$, our bounds are weaker. For convenience, let $t \sim \tau \ln d$. Theorem 10 implies that $\varphi^t(d) \le \left\lceil (e+\epsilon)e^{\frac{1}{\tau}} d/t \right\rceil$. If $\tau > 2$, then, by following the proof of Theorem 11 and performing some straightforward calculations, we obtain that

$$\varphi^t(d) \le \left\lceil \left(1 + (4 + \sqrt{4 + 6\tau})/(\tau - 2)\right) d/t \right\rceil .$$

For purely aesthetic reasons, we suspect the behaviour of $\varphi^t(d)$ to be closer to $d^{1+1/t}/t = e^{\frac{1}{\tau}} d/t$ (see Conjecture 29 below).

To summarise our results in this section, we first observed in Proposition 7 that the example of Alon can be easily adapted to frugal colourings that are not necessarily proper. Next, in Theorems 9 and 10, in the case of smaller frugalities, i.e. $t = O(\ln d)$, we refined an upper bound by Hind *et al.* [14] for (proper) t-frugal colourings. Lastly, we showed that for $t = \omega(\ln d)$, the behaviour of $\varphi^t(d)$ is asymptotically the same as that of $\chi^t(d)$. Table 1 gives a rough overview of the behaviour we have outlined in this section.

Table 1
Bounds for $\varphi^t(d)$ and $\chi^t_\varphi(d)$.

<table>
<tr><th rowspan="2">t</th><th colspan="2">$\varphi^t(d)$</th><th colspan="2">$\chi^t_\varphi(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td>$O(\ln d)$</td><td>$\Omega\left(\frac{d^{1+1/t}}{t}\right)$</td><td rowspan="2">$O\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$\Omega\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$O\left(\frac{d^{1+1/t}}{t}\right)$</td></tr>
<tr><td>$\Omega\left(\frac{\ln d}{\ln\ln d}\right)$
$O((\ln d)^5)$</td><td rowspan="3">$\left\lceil \frac{d}{t} \right\rceil$</td><td rowspan="3">$d+1$</td><td rowspan="2">$(t+1)d$</td></tr>
<tr><td rowspan="2">$\omega(\ln d)$</td><td rowspan="2">$\left\lceil (1+\epsilon)\frac{d}{t} \right\rceil$</td></tr>
<tr><td>$d+1$</td></tr>
</table>

3 Acyclic frugal colourings

Alon *et al.* [4] tackled the question of the asymptotic behaviour of $\chi_a(d)$. They found a nearly optimal upper bound for the acyclic chromatic number of graphs of maximum degree d, answering a long-standing question of Erdős (cf. [3]). Using the Lovász Local Lemma, they showed the following.

Theorem 12 (Alon *et al.* [4]) $\chi_a(d) \le \lceil 50d^{4/3} \rceil$.

The above statement still holds if 50 is replaced by any constant C satisfying

$$\left(1 - \frac{20}{C}\right)\left(1 - \frac{30}{C^3}\right)\left(1 - \frac{10}{C^2}\right) > \frac{1}{2},$$

e.g. $C = 40.27$ suffices. Using a probabilistic construction, they showed this upper bound to be asymptotically correct up to a logarithmic multiple.

Theorem 13 (Alon *et al.* [4]) $\chi_a(d) = \Omega\left(d^{4/3}/(\ln d)^{1/3}\right)$.

Yuster [19] considered acyclic proper 2-frugal colourings of graphs and showed $\chi^2_{\varphi,a}(d) = \Theta(d^{3/2})$. In particular, by an adaptation of Theorem 12, he showed that $\chi^2_{\varphi,a}(d) \le \lceil \max\{50d^{4/3}, 10d^{3/2}\} \rceil$. We note that a more precise analysis of Yuster's proof gives the following.

Theorem 14 (Yuster [19]) *For sufficiently large d,*

$$\chi^2_{\varphi,a}(d) \le \lceil 5.478 d^{3/2} \rceil.$$

The above statement still holds if 5.478 is replaced by any fixed constant greater than $\sqrt{30}$.

For acyclic frugal colourings, we start by considering the smallest cases $t = 1, 2, 3$ and establish upper bounds for acyclic proper frugal colourings. Later in the section, we consider larger values of t and concentrate our attention upon acyclic frugal colourings that are not necessarily proper.

For $t = 1, 2, 3$, notice that Corollary 8 implies the bounds $\varphi^1_a(d) \ge (1-\epsilon)d^2$, $\varphi^2_a(d) \ge (1/2 - \epsilon)d^{3/2}$ and $\varphi^3_a(d) \ge (1/3 - \epsilon)d^{4/3}$, for fixed $\epsilon > 0$ and large enough d. By Proposition 1(i), it follows that $\varphi^1_a(d) \sim \chi^1_{\varphi,a}(d) \sim d^2$.

Theorem 14 implies that $\varphi^2_a(d) = \Theta(d^{3/2})$ and $\chi^2_{\varphi,a}(d) = \Theta(d^{3/2})$. We give two extensions to Theorem 14, one for the case $t = 2$ with a slightly stronger notion of acyclic colouring, and the other for the case $t = 3$. In both cases we employ the General Lovász Local Lemma.

For $t = 2$, the slightly stronger notion we use is that of *star colouring*. A *star colouring* of G is a colouring such that no path of length three (i.e. with four vertices) is alternating; in other words, each bipartite subgraph consisting of the edges between two colour classes is a disjoint union of stars. Clearly, every star colouring is acyclic. The *star chromatic number* $\chi_s(G)$ of a graph G is the least number of colours needed in a proper star colouring. It was shown by Fertin, Raspaud and Reed [11] that the star chromatic number satisfies $\chi_s(d) = O(d^{3/2})$ and $\chi_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$.

We define the *t-frugal star chromatic number* $\varphi_s^t(G)$ of a graph G to be the least number of colours needed in a t-frugal star colouring and the *proper t-frugal star chromatic number* $\chi_{\varphi,s}^t(G)$ to be the least number of colours needed in a proper t-frugal star colouring of G. We define $\varphi_s^t(d)$ and $\chi_{\varphi,s}^t(d)$ similarly as before.

For the following result, we give an upper bound for the proper 2-frugal star chromatic number. With this bound, we provide a slightly simpler proof of the fact that $\chi_{\varphi,a}^2(d) = O(d^{3/2})$. Our proof is an extension of the proof of Theorem 8.1 in Fertin *et al.* [11].

Theorem 15 *For sufficiently large d,*

$$\chi_{\varphi,s}^2(d) \leq \lceil 6.325 d^{3/2} \rceil.$$

The above statement still holds if 6.325 is replaced by any fixed constant greater than $2\sqrt{10}$.

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \lceil 6.325 d^{3/2} \rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$. We define three types of events, the first two of which are from Fertin *et al.* [11].

I For adjacent vertices u, v, let $A_{\{u,v\}}$ be the event that $f(u) = f(v)$.

II For a path of length three $v_1v_2v_3v_4$, let $B_{\{v_1,\ldots,v_4\}}$ be the event that $f(v_1) = f(v_3)$ and $f(v_2) = f(v_4)$.

III For vertices v, v_1, v_2, v_3 with $\{v_1, v_2, v_3\} \subseteq N(v)$, let $A_{\{v_1,v_2,v_3\}}$ be the event that $f(v_1) = f(v_2) = f(v_3)$.

It is clear that if none of these events occur, then f is a proper 2-frugal star colouring. Furthermore, $\Pr(A) = 1/x$ and $\Pr(B) = \Pr(C) = 1/x^2$, where A, B, C are events of Types I, II, III, respectively. Also, since G has maximum degree d, each vertex participates in at most $d \cdot \binom{d-1}{2} < d^3/2$ events of Type III. It is routine to check that in Table 2, the (i, j) entry is an upper bound on the number of nodes corresponding to events of Type j which are adjacent in the dependency graph to a node corresponding to an event of Type i.

Table 2
Upper bounds in the dependency graph for Theorem 15.

	I	II	III
I	$2d$	$4d^3$	d^3
II	$4d$	$8d^3$	$2d^3$
III	$3d$	$6d^3$	$3d^3/2$

We define the weight x_i of each event i to be twice its probability and we want to show that each of the following inequalities hold:

$$\frac{1}{x} \leq \frac{2}{x}\left(1-\frac{2}{x}\right)^{2d}\left(1-\frac{2}{x^2}\right)^{5d^3}, \tag{5}$$

$$\frac{1}{x^2} \leq \frac{2}{x^2}\left(1-\frac{2}{x}\right)^{4d}\left(1-\frac{2}{x^2}\right)^{10d^3}, \text{ and} \tag{6}$$

$$\frac{1}{x^2} \leq \frac{2}{x^2}\left(1-\frac{2}{x}\right)^{3d}\left(1-\frac{2}{x^2}\right)^{15d^3/2}. \tag{7}$$

Inequalities (5), (6) and (7) correspond to events of Type I, II and III, respectively. Inequality (6) implies the other two and it is valid for sufficiently large d since

$$\left(1-\frac{2}{x}\right)^{4d}\left(1-\frac{2}{x^2}\right)^{10d^3} \geq \left(1-\frac{8}{6.325\sqrt{d}}\right)\left(1-\frac{20}{6.325^2}\right) > \frac{1}{2}$$

if $d \geq 9 \cdot 10^7$. Therefore, by the General Lovász Local Lemma, f is an acyclic proper 3-frugal colouring with positive probability. $\square$

Corollary 16 *For any $t \geq 2$ and sufficiently large d,*

$$\chi^t_{\varphi,s}(d) \leq \lceil 6.325 d^{3/2} \rceil.$$

Since $\chi^t_{\varphi,s}(d) \geq \chi_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$ for any $t \geq 1$, this corollary is correct up to a logarithmic multiple. Note that $\chi^1_{\varphi,s}(G) = \chi(G^2)$ for any graph G, so that $\chi^1_{\varphi,s}(d) \sim d^2$.

Next, for the case $t = 3$, we show that $\chi^3_{\varphi,a}(d) = O(d^{4/3})$, giving a bound that is within a constant multiple of the asymptotic lower bound for $\varphi^3_a(d)$. Our proof is an extension of the proof of Theorem 1.1 in Alon *et al.* [4].

Theorem 17 $\chi^3_{\varphi,a}(d) \leq \lceil 40.27 d^{4/3} \rceil$.

The above statement still holds if 40.27 is replaced by any constant C satisfying

$$\left(1-\frac{20}{C}\right)\left(1-\frac{95/3}{C^3}\right)\left(1-\frac{10}{C^2}\right) > \frac{1}{2}.$$

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \lceil 40.27 d^{4/3} \rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$. We define five types of events, the first four of which are from [4]:

I For adjacent vertices u, v, let $A_{\{u,v\}}$ be the event that $f(u) = f(v)$.
II For an induced path of length four $v_1v_2v_3v_4v_5$, let $B_{\{v_1,\ldots,v_5\}}$ be the event that $f(v_1) = f(v_3) = f(v_5)$ and $f(v_2) = f(v_4)$.
III For an induced 4-cycle $v_1v_2v_3v_4$ such that v_1, v_3 share at most $d^{2/3}$ common neighbours and v_2, v_4 share at most $d^{2/3}$ common neighbours, let $C_{\{v_1,v_2,v_3,v_4\}}$ be the event that $f(v_1) = f(v_3)$ and $f(v_2) = f(v_4)$.
IV For non-adjacent vertices u, v that share more than $d^{2/3}$ common neighbours, let $D_{\{u,v\}}$ be the event that $f(u) = f(v)$.
V For vertices v, v_1, v_2, v_3, v_4 with $\{v_1, v_2, v_3, v_4\} \subseteq N(v)$, let $E_{\{v_1,\ldots,v_4\}}$ be the event that $f(v_1) = f(v_2) = f(v_3) = f(v_4)$.

In [4], it was shown that if none of the events of Type I–IV occur, then f is an acyclic proper colouring. If no event of Type V occurs, then f is 3-frugal. Thus, if none of these events occur, then f is an acyclic proper 3-frugal colouring.

Clearly, $\Pr(A) = \Pr(D) = 1/x$, $\Pr(B) = \Pr(E) = 1/x^3$ and $\Pr(C) = 1/x^2$, where A, B, C, D and E are events of Types I, II, III, IV and V, respectively. Also, since G has maximum degree d, each vertex participates in at most $d \cdot \binom{d-1}{3} < d^4/6$ events of Type V. It is easy to check that in Table 3, the (i, j) entry is an upper bound on the number of nodes corresponding to events of Type j which are adjacent in the dependency graph to a node corresponding to an event of Type i.

Table 3
Upper bounds in the dependency graph for Theorem 17.

	I	II	III	IV	V
I	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$	$d^4/3$
II	$5d$	$15d^4$	$5d^{8/3}$	$5d^{4/3}$	$5d^4/6$
III	$4d$	$12d^4$	$4d^{8/3}$	$4d^{4/3}$	$2d^4/3$
IV	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$	$d^4/3$
V	$4d$	$12d^4$	$4d^{8/3}$	$4d^{4/3}$	$2d^4/3$

We define the weight x_i of each event i to be twice its probability and we want to show that each of the following inequalities hold:

$$\frac{1}{x} \leq \frac{2}{x}\left(1-\frac{2}{x}\right)^{2d+2d^{4/3}}\left(1-\frac{2}{x^3}\right)^{19d^4/3}\left(1-\frac{2}{x^2}\right)^{2d^{8/3}}, \tag{8}$$

$$\frac{1}{x^3} \leq \frac{2}{x^3}\left(1-\frac{2}{x}\right)^{5d+5d^{4/3}}\left(1-\frac{2}{x^3}\right)^{95d^4/6}\left(1-\frac{2}{x^2}\right)^{5d^{8/3}}, \text{ and} \tag{9}$$

$$\frac{1}{x^2} \leq \frac{2}{x^2}\left(1-\frac{2}{x}\right)^{4d+4d^{4/3}}\left(1-\frac{2}{x^3}\right)^{26d^4/3}\left(1-\frac{2}{x^2}\right)^{4d^{8/3}}. \tag{10}$$

Inequality (8) corresponds to events of Types I and IV, inequality (9) to events of Types II and V, and inequality (10) to events of Type III. Inequality (9) implies the other two and it is valid since

$$\left(1-\frac{2}{x}\right)^{5d+5d^{\frac{4}{3}}}\left(1-\frac{2}{x^3}\right)^{95d^4/6}\left(1-\frac{2}{x^2}\right)^{5d^{\frac{8}{3}}}$$
$$\geq\left(1-\frac{20}{40.27}\right)\left(1-\frac{95/3}{40.27^3}\right)\left(1-\frac{10}{40.27^2}\right)>\frac{1}{2}$$

and therefore, by the General Lovász Local Lemma, f is an acyclic proper 3-frugal colouring with positive probability. □

Corollary 18 *For any $t \geq 3$,*

$$\chi^t_{\varphi,a}(d) \leq \lceil 40.27 d^{4/3} \rceil.$$

Since $\chi^t_{\varphi,a}(d) \geq \chi_a(d) = \Omega\left(d^{4/3}/(\ln d)^{1/3}\right)$ for any $t \geq 1$, this corollary is correct up to a logarithmic multiple. This partially answers a question by Esperet, Montassier and Raspaud [10].

For acyclic proper t-frugal colourings and proper t-frugal star colourings when t is large, Corollaries 18 and 16, respectively, give fairly reasonable answers, so now we would like to consider what happens when we no longer prescribe that the colourings be proper.

As we shall see, the study of the acyclic t-improper chromatic number is quite important here. We first note that, by a probabilistic construction applying bounds on the t-dependence number of random graphs, Addario-Berry *et al.* [1] showed the following non-trivial extensions of the lower bounds for $\chi_a(d)$ and $\chi_s(d)$ of Alon *et al.* [4] and Fertin *et al.* [11], respectively.

Theorem 19 (Addario-Berry *et al.* [1]) *For any $t = t(d) \leq d - 10\sqrt{d \ln d}$ and sufficiently large d,*

$$\chi^t_a(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$

For any $t = t(d) \leq d - 16\sqrt{d \ln d}$ and sufficiently large d,

$$\chi^t_s(d) \geq \frac{(d-t)^{3/2}}{2^{12}(\ln d)^{1/2}}.$$

By Proposition 1(ii), the following holds.

Corollary 20 *For any $t = t(d) \leq d - 16\sqrt{d \ln d}$,*

$$\varphi^t_a(d) = \Omega\left((d-t)^{4/3}/(\ln d)^{1/3}\right) \text{ and } \varphi^t_s(d) = \Omega\left((d-t)^{3/2}/(\ln d)^{1/2}\right).$$

An important implication of this is that, even if $t = (1-\epsilon)d$ for some fixed $\epsilon > 0$, the bounds of Corollaries 18 and 16 accurately describe the behaviours of $\varphi_a^t(d)$ and $\varphi_s^t(d)$, respectively, up to a logarithmic multiple.

We can obtain an asymptotic improvement upon Corollaries 18 and 16 when $t = t(d)$ is *very* close to d. We adapt an argument of Addario-Berry *et al.* [1]. The following theorem, for example, implies that $\varphi_a^t(d)$ is asymptotically smaller than $\chi_{\varphi,a}^t(d)$ when $d - t(d) = o\left(d^{1/3}/(\ln d)^{1/3}\right)$.

Theorem 21 *For any $t = t(d) \geq 1$ and sufficiently large d,*

$$\varphi_s^t(d) \leq d \cdot \max\{3(d-t), 31 \ln d\} + 2.$$

The proof of this result relies on the following lemma which is essentially from Addario-Berry *et al.* [1]. A *total k-dominating set $\mathcal{D}$* in a graph is a set of vertices such that each vertex has at least k neighbours in $\mathcal{D}$. Given a d-regular graph $G = (V, E)$ and $1 \leq k \leq d$, let $\psi(G, k)$ be the least integer $k \leq k' \leq d$ such that there exists a total k-dominating set $\mathcal{D}$ for which $|N(v) \cap \mathcal{D}| \leq k'$ for all $v \in V$. The quantity $\psi(G, k)$ is well-defined due to the fact that V is a total k-dominating set in G for $1 \leq k \leq d$. Let $\psi(d, k)$ be the maximum over all d-regular graphs G of $\psi(G, k)$.

Lemma 22 *For any $1 \leq k \leq d$ and sufficiently large d,*

$$\psi(d, k) \leq \max\{3k, 31 \ln d\}.$$

Because this is only a slight modification to an analogous lemma in Addario-Berry *et al.* [1], we omit its proof here and mention that it is an application of the Symmetric Lovász Local Lemma.

ProofProof of Theorem 21. We first remark that if G is a subgraph of G', then $\varphi_s^t(G) \leq \varphi_s^t(G')$. As any graph of maximum degree d is contained in a d-regular graph, it therefore suffices to show the theorem holds for d-regular graphs. We hereafter assume $G = (V, E)$ is d-regular and d is large enough to apply Lemma 22. Let $k = d - t$. We will show that $\varphi_s^t(G) \leq d\psi(d, k) + 2$, which proves the theorem.

By the definition of $\psi(d, k)$, if d is sufficiently large, there is a set $\mathcal{D}$ such that $k \leq |N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for any $v \in V$. Fix such a set $\mathcal{D}$ and form the auxiliary graph H as follows: let H have vertex set $\mathcal{D}$ and let uv be an edge of H precisely if u and v have graph distance at most two in G. As $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for any $v \in V$, H has maximum degree at most $d\psi(d, k)$.

To colour G, we first properly colour H by using the greedy algorithm to choose colours from the set $\{1, \ldots, d\psi(d, k) + 1\}$ and then assign each vertex v of $\mathcal{D}$

the colour it received in H. We next assign colour $d\psi(d,k)+2$ to all vertices of $V \setminus \mathcal{D}$. Since $k \leq |N(v) \cap \mathcal{D}|$ for any $v \in V$, colour $d\psi(d,k)+2$ appears at most $d-k=t$ times in any neighbourhood. Since the vertices of H at distance two have distinct colours, each colour other than $d\psi(d,k)+2$ appears at most once in any neighbourhood. So the resulting colouring is t-frugal.

Furthermore, given any path $P = v_1v_2v_3v_4$ of length three in G, either two consecutive vertices v_i, v_{i+1} of P are not in $\mathcal{D}$ (in which case v_i and v_{i+1} have the same colour and P is not alternating), or two vertices v_i, v_{i+2} are in $\mathcal{D}$ (in which case v_i and v_{i+2} have different colours and P is not alternating). Thus, the above colouring is a star colouring of G with frugality at most t and using at most $d\psi(d,k)+2$ colours. $\square$

Corollary 23 *For any $t = t(d) \geq 1$,*

$$\varphi_a^t(d) \leq \varphi_s^t(d) = O(d \ln d + (d-t)d).$$

A deterministic construction by Addario-Berry, Kang and Müller [2] based on a "doubled grid" shows that Corollary 23 is correct up to a logarithmic multiple when $d-t$ itself is logarithmic.

Theorem 24 (Addario-Berry, Kang and Müller [2]**)** $\chi_a^{d-1}(d) = \Omega(d)$.

Corollary 25 $\varphi_s^{d-1}(d) \geq \varphi_a^{d-1}(d) = \Omega(d)$.

Observe that the lower bound for $\varphi_a^t(d)$ (respectively, $\varphi_s^t(d)$) of Corollary 20 is stronger than that of Corollary 25 when $d-t = \omega(d^{3/4}(\ln d)^{1/4})$ (respectively, $d-t = \omega(d^{2/3}(\ln d)^{1/3})$).

Table 4
Asymptotic bounds for $\chi_{\varphi,a}^t(d)$ and $\chi_{\varphi,s}^t(d)$.

<table>
<tr><th rowspan="2">t</th><th colspan="2">$\chi_{\varphi,a}^t(d)$</th><th colspan="2">$\chi_{\varphi,s}^t(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td>1</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td></tr>
<tr><td>2</td><td>$\Omega\left(d^{3/2}\right)$</td><td>$O\left(d^{3/2}\right)$</td><td>$\Omega\left(d^{3/2}\right)$</td><td rowspan="3">$O(d^{3/2})$</td></tr>
<tr><td>3</td><td>$\Omega\left(d^{4/3}\right)$</td><td rowspan="2">$O(d^{4/3})$</td><td rowspan="2">$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$</td></tr>
<tr><td>≥ 4</td><td>$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$</td></tr>
</table>

What we have demonstrated in this section is, first, that the asymptotic behaviour of the acyclic proper t-frugal chromatic number and the proper t-frugal star chromatic number can be determined up to at most a logarithmic multiple. Second, we showed that the asymptotic behaviour of the acyclic t-frugal chromatic number (respectively, t-frugal star chromatic number) of graphs of bounded maximum degree seems closely tied to that of their acyclic t-improper chromatic number (respectively, t-improper star chromatic number) as long as $t \geq 3$ (respectively, $t \geq 2$). Tables 4 and 5 give a summary of the bounds we have obtained.

Table 5
Asymptotic bounds for $\varphi_a^t(d)$ and $\varphi_s^t(d)$.

<table>
<tr><td rowspan="2">$d-t$</td><td colspan="2">$\varphi_a^t(d)$</td><td colspan="2">$\varphi_s^t(d)$</td></tr>
<tr><td>lower</td><td>upper</td><td>lower</td><td>upper</td></tr>
<tr><td>$d-1$</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td></tr>
<tr><td>$d-2$</td><td>$\Omega\left(d^{3/2}\right)$</td><td>$O\left(d^{3/2}\right)$</td><td>$\Omega\left(d^{3/2}\right)$</td><td rowspan="4">$O(d^{3/2})$</td></tr>
<tr><td>$d-3$</td><td>$\Omega\left(d^{4/3}\right)$</td><td rowspan="4">$O(d^{4/3})$</td><td rowspan="3">$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$</td></tr>
<tr><td>$\omega(d^{3/4}(\ln d)^{1/4})$</td><td>$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$</td></tr>
<tr><td>$\omega(d^{2/3}(\ln d)^{1/3})$</td><td rowspan="4">$\Omega\left(d\right)$</td></tr>
<tr><td>$O(d^{1/2})$</td><td rowspan="3">$\Omega\left(d\right)$</td><td rowspan="2">$O((d-t)d)$</td></tr>
<tr><td>$O(d^{1/3})$</td><td>$O((d-t)d)$</td></tr>
<tr><td>$O(\ln d)$</td><td>$O(d\ln d)$</td><td>$O(d\ln d)$</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

4 Deterministic questions

All of the upper bounds obtained so far are probabilistic in nature and rely on the Lovász Local Lemma. Although there are methods to convert applications of the Local Lemma into deterministic algorithms (cf. [6]), it remains an interesting problem to find *elementary* deterministic algorithms that do better than the most obvious bounds.

For instance, a simple way to acyclically $(d-1)$-frugally colour a graph of maximum degree d is to colour the square of the graph, using at most d^2+1 colours; however, Theorem 23 implies that we should be able to use many fewer colours. It is vexing that there does not appear to be an elementary deterministic algorithm giving $\varphi_a^{d-1}(d) = o(d^2)$.

Similarly, we can ask if there is any elementary deterministic algorithm that t-frugally colours a graph of maximum degree d with fewer colours than the following simple greedy algorithm.

Proposition 26 *If $t \geq (d-1)/k$, then $\varphi^t(d) \leq kd+1$.*

Proof. Let $G = (V, E)$ be a graph of maximum degree d. Colour the vertices according to an arbitrary ordering $(u_1, \ldots, u_{|V|})$ of V and, at each step, assign the smallest available colour (i.e. greedily). Suppose we are at step i, for $i \in \{1, \ldots, |V|\}$. For each $v \in N(u_i)$, there are at most k colour classes which have t members in the set $N(v) \setminus \{u_i\}$. As there are at most d such v, at most kd choices of colour are forbidden for u_i. Thus, at most $kd+1$ colours are required by this procedure. $\square$

We remark that, for $t = 1$ (setting $k = d-1$), this algorithm also produces an acyclic colouring; thus, $\varphi^1(d) \leq \varphi^1_a(d) \leq d^2 - d + 1$ and the projective planes show via Proposition 7 that $\varphi^1(d) = \varphi^1_a(d) = d^2 - d + 1$ for infinitely many d. It was shown by Hahn *et al.* [13] that these are the only graphs G which attain $\varphi^1(G) = \Delta(G)^2 - \Delta(G) + 1$.

For $t(d) = d-1$ (setting $k = 1$), Proposition 26 gives that $\varphi^{d-1}(d) \leq d+1$. We can give the following minor improvement which is essentially a backtracking version of the greedy algorithm:

Proposition 27 *If $d \geq 3$, then $\varphi^{d-1}(d) \leq d$.*

Proof. Let $G = (V, E)$ be a graph of maximum degree d. Without loss of generality, we may assume that G is d-regular. As in Proposition 26, let $\sigma = (u_1, \ldots, u_{|V|})$ be an arbitrary ordering of V and start by colouring the vertices greedily according to σ. Let i be the smallest $i \in \{1, \ldots, |V|\}$ such that at stage i we are forced to use colour $d+1$ under the greedy algorithm.

We will now perform a search to find a recolouring that will allow u_i to be coloured from $\{1, \ldots, d\}$. As we search, we will define a sequence of vertex subsets $V_0, V_1, \ldots$. For convenience, let v_{-1} denote u_i.

Stage 0. It must be that for each $v \in N(u_i)$, there exists a colour $c_v \in \{1, \ldots, d\}$ such that each vertex of $N(v) \setminus \{u_i\}$ is coloured c_v; furthermore, the set $\{c_v \mid v \in N(u_i)\}$ is precisely the set of integers $\{1, \ldots, d\}$. Let $V_0 = \bigcup_{v \in N(u_i)} N(v) \setminus \{u_i\}$. For each member $x \in V_0$, there is a unique vertex $p_0(x) \in N(x) \cap N(u_i)$, which we call the *parent* of x.

For any $x \in V_0$, if we can recolour x to another colour without violating the frugality of its neighbours other than $p_0(x)$, then we can safely assign the colour $c_{p_0(x)} \in \{1, \ldots, d\}$ to u_i. So assume that none of the members of V_0

may be recoloured and pick an arbitrary $v_0 \in V_0$.

Stage j, $j \geq 1$. Assuming that V_{j-1}, $p_{j-1}(\cdot)$, and v_{j-1} were defined in previous stages, we form V_j as follows. Since v_{j-1} could not be recoloured at Stage $j-1$, it must be that for each $v \in N(v_{j-1}) \setminus \{p_{j-1}(v_{j-1})\}$, there exists a colour $c_v \in \{1, \ldots, d\} \setminus \{c_{p_{j-1}(v)}\}$ such that each vertex of $N(v) \setminus \{v_{j-1}\}$ is coloured c_v; furthermore, the set $\{c_v \mid v \in N(v_{j-1}) \setminus \{p_{j-1}(v_{j-1})\}\}$ is precisely the set $\{1, \ldots, d\} \setminus \{c_{p_{j-1}(v_{j-1})}\}$. Let $V_j = \bigcup_{v \in N(v_{j-1}) \setminus \{p_{j-1}(v_{j-1})\}} N(v) \setminus \{v_{j-1}\}$. For each member $x \in V_j$, there is a unique vertex $p_j(x) \in N(x) \cap N(v_{j-1})$, which we call the *parent* of x.

For any $x \in V_j$, if we can recolour x to another colour without violating the frugality of its neighbours other than $p_j(x)$, then we can safely recolour v_{j-1} to the colour $c_{p_j(x)} \neq c_{p_{j-1}(v_{j-1})}$, and then recolour v_{j-2}, and so on until we recolour u_i. So assume that none of the members of V_j may be recoloured, pick an arbitrary $v_j \in V_j$, and continue to Stage $j+1$.

We want to show that this sequence terminates at some stage so that u_i is recoloured eventually; for a contradiction, suppose not. It is important to observe that, for any $j \geq 0$, the colour assigned to v_{j-1} is distinct from the colours assigned to the vertices in V_j (otherwise, we could have recoloured v_{j-1} at Stage $j-1$).

If it exists, let j_* be the smallest index such that for some $0 \leq j < j_*$ there exists $v \in V_j \cap V_{j_*}$. It follows that $v_{j-1} \neq v_{j_*-1}$; otherwise, either there is a smaller choice of j_* or $v_{j-1} = v_{j_*-1} = v_{-1}$ which is impossible since V_{j_*-1} contains only vertices with colours in $\{1, \ldots, d\}$. We consider two cases.

(i) Suppose that $p_j(v) = p_{j_*}(v)$. Since v and v_{j_*-1} are in $N(p_j(v)) \setminus \{v_{j-1}\}$, they must both have colour $c_{p_j(v)}$; however, by an observation above, v_{j_*-1} and $v \in V_{j_*}$ cannot have the same colour, a contradiction.

(ii) Suppose that $p_j(v) \neq p_{j_*}(v)$. Note that $p_j(v) \in N(v) \setminus \{p_{j_*}(v)\}$. Since $d \geq 3$, it follows from the definition of V_j that $N(p_j(v)) \setminus \{v\}$ contains v_{j-1} and at least one vertex v' coloured $c_{p_j(v)}$. Now, v_{j-1} and $v' \in V_j$ have different colours (again by the above observation). Now consider Stage j_*. The vertex $v \in V_{j_*}$ has a neighbour w other than $p_{j_*}(v)$ (namely $p_j(v)$) such that $N(w) \setminus \{v\}$ is not monochromatic. This means we could have recoloured v at Stage j_* without violating the frugality of its neighbours other than $p_{j_*}(v)$, a contradiction.

Now no such j_* exists, but this means that the V_j's are pairwise disjoint, and we have a contradiction as G is finite. □

Note that if G is an odd cycle, then $\varphi^1(G) = 3$. It can be checked that the point-line incidence graph for the Fano plane is a 3-regular graph verifying

$\varphi^2(3) = 3$; however, by Theorem 11 we know that $\varphi^{d-1}(d) = 2$ for all but finitely many values of d.

Problem 28 *Are $d = 2, 3$ the only obstructions to $\varphi^{d-1}(d) = 2$?*

5 Concluding remarks and open problems

There are a handful of works that have examined proper t-frugal colourings [5, 10, 19], but, as far as we are aware, t-frugal colourings (i.e. ones that are not necessarily proper) have not been studied much at all. One exception is that the parameter $\varphi^1(G)$, referred to as the *injective chromatic number*, was considered by Hahn *et al.* [13]. There remain several problems in this area of graph colouring which we hope will be of interest to other researchers, and we outline these below.

We believe the following conjecture to be natural in light of the results we obtained in Section 2.

Conjecture 29 $\varphi^t(d) \sim \lceil d^{1+1/t}/t \rceil$ *for any $t = t(d) \geq 1$.*

This conjecture essentially holds for $t = \omega(\ln d)$, but, when $t = O(\ln d)$, the upper and lower bounds that we outlined are separated by at least a constant multiple.

In Sections 2 and 3, by dropping the condition that the colourings be proper, we demonstrated what seems to be a close qualitative link between t-frugal and t-improper colourings for t large enough. In Section 2, we established a threshold for t, namely $t = \Theta(\ln d)$, above which, the t-frugal chromatic number is asymptotically equal to the t-improper chromatic number. For $\varphi^t_a(d)$ (respectively, $\varphi^t_s(d)$), the threshold of convergence (if it exists) with $\chi^t_a(d)$ (respectively, $\chi^t_s(d)$) may possibly be $t = 3$ (respectively, $t = 2$). Indeed, we conjecture the following.

Conjecture 30 $\varphi^t_a(d) = \Theta(\chi^t_a(d))$ *for any $t = t(d) \geq 1$ unless $t \in \{1, 2\}$. Analogously, $\varphi^t_s(d) = \Theta(\chi^t_s(d))$ for any $t = t(d) \geq 1$ unless $t = 1$.*

We point out here that, in the setting of planar graphs, such an asymptotic "convergence" does not occur. The acyclic t-improper chromatic number of planar graphs is bounded by a constant (namely, 5); whereas, the t-frugal chromatic number of a planar graph G is at least $\Delta(G)/t$ and thus can be arbitrarily large. It remains interesting to determine, for fixed t,

$$\varphi^t(P, d) := \max\{\varphi^t(G) \mid G \text{ is planar and } \Delta(G) \leq d\}$$

and, in particular, what is the smallest constant $K \geq 1$ such that $\varphi^t(P, d) \leq Kd/t + o(d)$. That such a constant K exists is implied by work of Amini, Esperet and van den Heuvel [5].

Another interesting line of inquiry is to determine the asymptotically slowest-growing choice of $t = t(d)$ for which $\varphi^t(d)$ is asymptotically smaller than $\chi^t_\varphi(d)$ or for which $\varphi^t_a(d)$ (respectively, $\varphi^t_s(d)$) is asymptotically smaller than $\chi^t_{\varphi,a}(d)$ (respectively, $\chi^t_s(d)$). Theorem 10 implies that the answer in the former case is in the range $t = \Theta(\ln d/\ln\ln d)$. Corollaries 20 and 23 (respectively, Corollary 20 and Theorem 21) suggest that t, for the latter question, is in the range such that $d - t = \Omega(d^{1/3}/(\ln d)^{1/3})$ and $d - t = o(d)$ (respectively, $d - t = \Omega(d^{1/2}/(\ln d)^{1/2})$ and $d - t = o(d)$).

There are also intriguing questions concerning the case $t = d - 1$, for example, Problem 28. Another natural challenge (which is a subproblem of Conjecture 30) is to determine the asymptotic values of $\varphi^{d-1}_a(d)$ and $\varphi^{d-1}_s(d)$, as these parameters lie in the range $\Omega(d)$ and $O(d \ln d)$.

Acknowledgements

We thank Colin McDiarmid and Jean-Sébastien Sereni for helpful discussions. We are grateful to EURANDOM, Technische Universiteit Eindhoven for sponsoring a visit of the first author, during which the bulk of this work was carried out.

References

[1] L. Addario-Berry, L. Esperet, R. J. Kang, C. J. H. McDiarmid, and A. Pinlou, *Acyclic t-improper colourings of graphs with bounded maximum degree*, 2007. Submitted for publication.

[2] L. Addario-Berry, R. J. Kang, and T. Müller, *Acyclic dominating partitions*, 2007. Submitted for publication.

[3] M. O. Albertson and D. M. Berman, *The acyclic chromatic number*, Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing (Louisiana State Univ., Baton Rouge, La., 1976), 1976, pp. 51–69.

[4] N. Alon, C. J. H. McDiarmid, and B. Reed, *Acyclic coloring of graphs*, Random Structures and Algorithms **2** (1991), no. 3, 277–288.

[5] O. Amini, L. Esperet, and J. van den Heuvel, *Frugal colouring of graphs*, Technical Report RR-1428-07, LaBRI, Université Bordeaux 1, 2007.

[6] J. Beck, *An algorithmic approach to the Lovász local lemma. I*, Random Structures Algorithms **2** (1991), no. 4, 343–365. MR **MR1125954 (92i:**05219)

[7] B. Bollobás, *Random Graphs*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001.

[8] O. V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979), no. 3, 211–236.

[9] P. Erdős and L. Lovász, *Problems and results on* 3*-chromatic hypergraphs and some related questions*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, 1975, pp. 609–627.

[10] L. Esperet, M. Montassier, and A. Raspaud, *Linear choosability of graphs*, Technical Report RR-1382-06, LaBRI, Université Bordeaux 1, 2006.

[11] G. Fertin, A. Raspaud, and B. Reed, *Star coloring of graphs*, J. Graph Theory **47** (2004), no. 3, 163–182.

[12] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973), 390–408.

[13] G. Hahn, J. Kratochvíl, J. Širáň, and D. Sotteau, *On the injective chromatic number of graphs*, Discrete Math. **256** (2002), no. 1-2, 179–192.

[14] H. Hind, M. Molloy, and B. Reed, *Colouring a graph frugally*, Combinatorica **17** (1997), no. 4, 469–482.

[15] ———, *Total coloring with* Δ+poly($\log \Delta$) *colors*, SIAM J. Comput. **28** (1999), no. 3, 816–821 (electronic).

[16] A. E. Ingham, *On the difference between consecutive primes*, Quart. J. Math. **8** (1937), no. 1, 255–266.

[17] S. Janson, T. Łuczak, and A. Rucinski, *Random Graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

[18] M. Molloy and B. Reed, *Graph Colouring and the Probabilistic Method*, Algorithms and Combinatorics, vol. 23, Springer-Verlag, Berlin, 2002.

[19] R. Yuster, *Linear coloring of graphs*, Discrete Math. **185** (1998), no. 1-3, 293–297.