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Research Article

Note on the Stability Property of a Cooperative System Incorporating Harvesting

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The stability of a kind of cooperative model incorporating harvesting is revisited in this paper. By using an iterative method, the global attractivity of the interior equilibrium point of the system is investigated. We show that the condition which ensures the existence of a unique positive equilibrium is enough to ensure the global attractivity of the positive equilibrium. Our results significantly improve the corresponding results of Wei and Li (2013).

1. Introduction

In [1], Wei and Li proposed and studied the following cooperative system incorporating harvesting:

$$\begin{aligned} \dot{x} &= x\left(r_1 - b_1 x - \frac{a_1 x}{y + k_1}\right) - Eqx, \\ \dot{y} &= y\left(r_2 - b_2 y - \frac{a_2 y}{x + k_2}\right), \end{aligned} \tag{1}$$

where x and y denote the densities of two populations at time t. The parameters $r_1, r_2, a_1, a_2, b_1, b_2, k_1, k_2, E, q$ are all positive constants. Assume that $r_1 > Eq$; then, the equilibria of (1) are

$$\begin{aligned} &H_0(0,0), \qquad H_1\left(0, \frac{r_2 k_2}{a_2 + k_2 b}\right), \\ &H_3\left(\frac{(r_1 - Eq)k_1}{a_1 + k_1 b_1}, 0\right), \qquad H_3(x^*, y^*), \end{aligned} \tag{2}$$

where

$$\begin{aligned} x^* &= \frac{-(k_2 P - F) + \sqrt{(k_2 P - F)^2 + 4PQM}}{2P}, \\ y^* &= \frac{r_2 (x^* + k_2)}{b_2 x^* + a_2 + k_2 b_2}, \qquad P = r_2 b_1 + k_1 b_1 b_2 + a_1 b_2, \\ Q &= r_1 - Eq, \qquad F = r_2 Q + b_2 k_1 Q - k_1 a_2 b_1 - a_1 a_2, \\ M &= r_2 k_2 + k_1 k_2 b_2 + a_2 k_1. \end{aligned} \tag{3}$$

Wei and Li had showed that H_0, H_1, H_2 are unstable and concerned with the persistence and stability property of the system; by applying the comparison theorem of differential equations and constructing a suitable Lyapunov function, they obtained the following results.

Theorem A. *If $r_1 > Eq$, $k_1 b_1 > a_1$, $k_2 b_2 > a_2$, then the system (1) is persistent. More precisely,*

$$\begin{aligned} C &\le \liminf_{t \to +\infty} x(t) \le \limsup_{t \to +\infty} x(t) \le A, \\ D &\le \liminf_{t \to +\infty} x(t) \le \limsup_{t \to +\infty} y(t) \le B, \end{aligned} \tag{4}$$

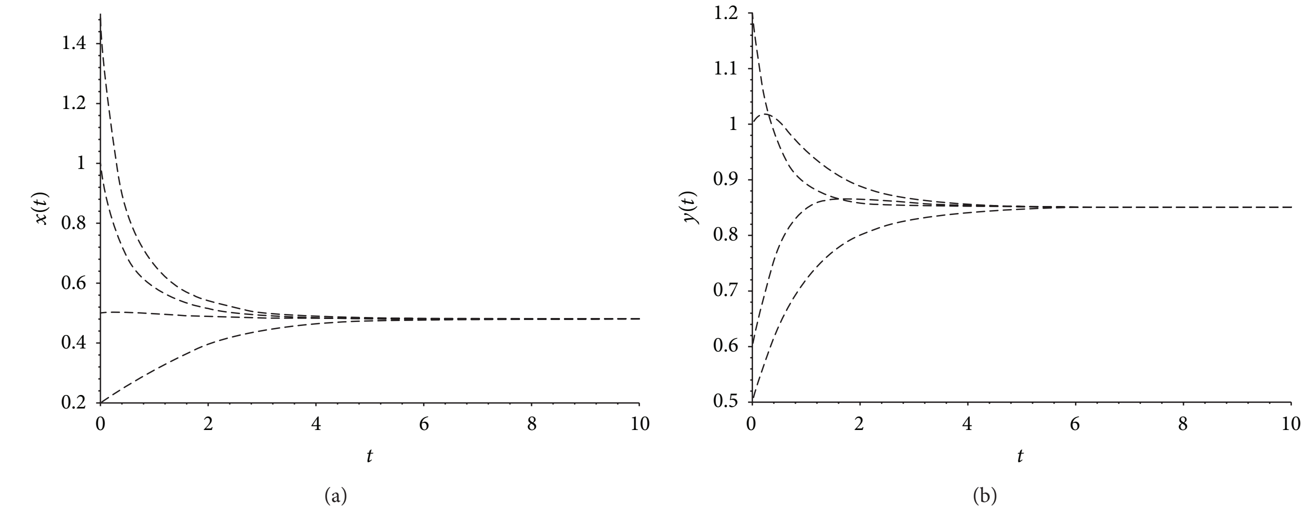

Figure 1: Dynamics behaviors of system (7). Here, we take the initial conditions $(x_1(0), x_2(0)) = (0.5, 1.2)$, $(1.5, 1)$, $(0.2, 0.5)$ and $(1, 0.6)$, respectively.

where

$$A = \frac{r_1 - Eq}{b_1}, \qquad B = \frac{r_2}{b_2},$$
$$C = \frac{(r_1 - Eq)(k_1 b_1 - a_1)}{b_1^2 k_1}, \qquad D = \frac{r_2 (k_2 b_2 - a_2)}{b_2^2 k_2}. \tag{5}$$

Theorem B. *If $r_1 > Eq$, $k_1 b_1 > a_1$, $k_2 b_2 > a_2$,*

$$b_1 + \frac{a_1}{y^* + k_1} > \frac{Ba_2}{2(C + k_2)(x^* + k_2)} + \frac{Aa_1}{2(D + k_1)(y^* + k_1)},$$
$$b_2 + \frac{a_2}{x^* + k_2} > \frac{Ba_2}{2(C + k_2)(x^* + k_2)} + \frac{Aa_1}{2(D + k_1)(y^* + k_1)}, \tag{6}$$

where A, B, C, D are defined by Theorem A, then the positive equilibrium point H_3 of system (1) *is globally asymptotically stable.*

Now let us consider the following example.

Example 1. We have

$$\dot{x} = x\left(2 - x - \frac{2x}{y + 1}\right) - x,$$
$$\dot{y} = y\left(2 - y - \frac{2y}{x + 1}\right). \tag{7}$$

Here we choose $r_1 = r_2 = 2$, $b_1 = b_2 = 1$, $k_1 = k_2 = 1$, $a_1 = a_2 = 2$, $E = q = 1$, and the parameters $r_1, r_2, a_1, a_2, b_1, b_2, k_1, k_2, E, q$ are all positive constants. Obviously, $r_1 = 2 > 1 = Eq$, $k_1 b_1 = 1 < 2 = a_1$, $k_2 b_2 = 1 < 2 = a_2$. Hence, the conditions of Theorems A and B are not all satisfied; however, numeric simulations (Figure 1) show that the unique positive equilibrium (0.4806248475, 0.8507810594) is globally attractive.

The above example shows that it is possible to obtain some weaker conditions than those of Theorems A and B to ensure the persistent and stability of the system. The aim of this paper is to prove the following result.

Theorem 2. *Assume that $r_1 > Eq$ holds; then, the unique positive equilibrium $E^*(x^*, y^*)$ is globally attractive; that is,*

$$\lim_{t \to +\infty} x(t) = x^*, \qquad \lim_{t \to +\infty} y(t) = y^*. \tag{8}$$

Concerned with the persistent property of the system, as a direct corollary of Theorem 2, we have the following.

Corollary 3. *Assume that $r_1 > Eq$ holds; then, system* (1) *is permanent.*

Remark 4. A comparison of Theorems A, B, and 2 and Corollary 3 shows that $k_1 b_1 > a_1$, $k_2 a_2 > b_2$, and inequalities (6) are redundant. Therefore, our results significantly improve the corresponding main results of Wei and Li [1].

We will prove Theorem 2 in the next section. For more works on mutualism system, one could refer to [2–10] and the references cited therein.

2. Proof of the Main Results

As a direct corollary of Lemma 2.2 of Chen [11], we have the following.

Lemma 5. *If $a > 0$, $b > 0$ and $\dot{x} \geq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t \to +\infty} x(t) \geq \frac{b}{a}. \tag{9}$$

If $a>0$, $b>0$ and $\dot{x}\le x(b-ax)$, when $t\ge 0$ and $x(0)>0$, we have

$$\limsup_{t\to+\infty} x(t)\le \frac{b}{a}. \quad (10)$$

Proof of Theorem 2. By the first equation of system (1), we have

$$\dot{x}(t)\le x(t)\left(r_1-Eq-b_1x(t)\right). \quad (11)$$

From Lemma 5, it follows that

$$\limsup_{t\to+\infty} x(t)\le \frac{r_1-Eq}{b_1}. \quad (12)$$

Hence, for enough small $\varepsilon>0$ ($\varepsilon<\min\{(r_1-Eq)k_1/(k_1b_1+a_1), r_2k_2/(k_2b_2+a_2)\}$) , it follows from (12) that there exists a $T_1'>0$ such that

$$x(t)<\frac{r_1-Eq}{b_1}+\varepsilon\overset{\text{def}}{=}M_1^{(1)}\quad \forall t>T_1'. \quad (13)$$

Similarly, for the above $\varepsilon>0$, it follows from the second equation of system (1) that there exists a $T_1>T_1'$ such that

$$y(t)<\frac{r_2}{b_2}+\varepsilon\overset{\text{def}}{=}M_2^{(1)}\quad \forall t>T_1. \quad (14)$$

(14) together with the first equation of system (1) implies that

$$\begin{aligned}\dot{x}&=x\left(r_1-b_1x-\frac{a_1x}{y+k_1}\right)-Eqx\\ &\le x\left(r_1-Eq-b_1x-\frac{a_1x}{M_2^{(1)}+k_1}\right)\quad \forall t>T_1.\end{aligned} \quad (15)$$

Therefore, by Lemma 5, we have

$$\limsup_{t\to+\infty} x_1(t)\le \frac{r_1-Eq}{b_1+\left(a_1/\left(M_2^{(1)}+k_1\right)\right)}. \quad (16)$$

That is, for $\varepsilon>0$ to be defined by (12) and (13), there exists a $T_2'>T_1$ such that

$$x(t)<\frac{r_1-Eq}{b_1+\left(a_1/\left(M_2^{(1)}+k_1\right)\right)}+\frac{\varepsilon}{2}\overset{\text{def}}{=}M_1^{(2)}>0\quad \forall t>T_2'. \quad (17)$$

It follows from (13) and the second equation of system (1) that

$$\begin{aligned}\dot{y}&=y\left(r_2-b_2y-\frac{a_2y}{x+k_2}\right)\\ &\le y\left(r_2-b_2y-\frac{a_2y}{M_1^{(1)}+k_2}\right).\end{aligned} \quad (18)$$

Therefore, by Lemma 5, we have

$$\limsup_{t\to+\infty} y(t)\le \frac{r_2}{b_2+\left(a_2/\left(M_1^{(1)}+k_2\right)\right)}. \quad (19)$$

That is, for $\varepsilon>0$ to be defined by (13) and (14), there exists a $T_2>T_2'$ such that

$$y(t)<\frac{r_2}{b_2+\left(a_2/\left(M_1^{(1)}+k_2\right)\right)}+\frac{\varepsilon}{2}\overset{\text{def}}{=}M_2^{(2)}>0\quad \forall t>T_2. \quad (20)$$

From the first equation of system (1) and the positivity of $y(t)$, we have

$$\begin{aligned}\dot{x}&=x\left(r_1-b_1x-\frac{a_1x}{y+k_1}\right)-Eqx\\ &\ge x\left(r_1-Eq-b_1x-\frac{a_1x}{k_1}\right)\quad \forall t>T_2.\end{aligned} \quad (21)$$

Therefore, by Lemma 5, we have

$$\liminf_{t\to+\infty} x(t)\ge \frac{r_1-Eq}{b_1+\left(a_1/k_1\right)}. \quad (22)$$

Hence, for $\varepsilon>0$ to be defined by (12) and (13), there exists a $T_3'>T_2$ such that

$$x(t)>\frac{r_1-Eq}{b_1+\left(a_1/k_1\right)}-\varepsilon\overset{\text{def}}{=}m_1^{(1)},\quad \forall t>T_3'. \quad (23)$$

Similarly, it follows from the second equation of system (1) that there exists a $T_3>T_3'$ such that

$$y(t)>\frac{r_2}{b_2+\left(a_2/k_2\right)}-\varepsilon\overset{\text{def}}{=}m_2^{(1)},\quad \forall t>T_3. \quad (24)$$

(24) together with the first equation of system (1) implies that

$$\begin{aligned}\dot{x}&=x\left(r_1-b_1x-\frac{a_1x}{y+k_1}\right)-Eqx\\ &\ge x\left(r_1-Eq-b_1x-\frac{a_1x}{m_2^{(1)}+k_1}\right)\quad \forall t>T_3.\end{aligned} \quad (25)$$

Therefore, by Lemma 5, we have

$$\liminf_{t\to+\infty} x(t)\ge \frac{r_1-Eq}{b_1+\left(a_1/\left(m_2^{(1)}+k_1\right)\right)}. \quad (26)$$

That is, for $\varepsilon>0$ to be defined by (12) and (13), there exists a $T_4'>T_3$ such that

$$x(t)>\frac{r_1-Eq}{b_1+\left(a_1/\left(m_2^{(1)}+k_1\right)\right)}-\frac{\varepsilon}{2}\overset{\text{def}}{=}m_1^{(2)}>0,\quad \forall t>T_4'. \quad (27)$$

Similarly, by (23) and the second equation of system (1), for $\varepsilon>0$ to be defined by (12) and (13), there exists a $T_4>T_4'$ such that

$$y(t)>\frac{r_2}{b_2+\left(a_2/\left(m_1^{(1)}+k_2\right)\right)}-\frac{\varepsilon}{2}\overset{\text{def}}{=}m_2^{(2)}>0,\quad \forall t>T_4. \quad (28)$$

Noting that $a_1/(M_2^{(1)} + k_1) > 0$, $a_2/(M_1^{(1)} + k_2) > 0$, it immediately follows that

$$M_1^{(2)} = \frac{r_1 - Eq}{b_1 + \left(a_1/\left(M_2^{(1)} + k_1\right)\right)} + \frac{\varepsilon}{2} < \frac{r_1 - Eq}{b_1} + \varepsilon = M_1^{(1)};$$
$$M_2^{(2)} = \frac{r_2}{b_2 + \left(a_2/\left(M_1^{(1)} + k_2\right)\right)} + \frac{\varepsilon}{2} < \frac{r_2}{b_2} + \varepsilon = M_2^{(1)}. \quad (29)$$

Also, since $m_1^{(1)} > 0$, $m_2^{(1)} > 0$, it follows that $a_1/(m_2^{(1)} + k_1) < a_1/k_1$, $a_2/(m_1^{(1)} + k_2) < a_2/k_2$, and so

$$m_1^{(2)} = \frac{r_1 - Eq}{b_1 + \left(a_1/\left(m_2^{(1)} + k_1\right)\right)} - \frac{\varepsilon}{2} > \frac{r_1 - Eq}{b_1 + (a_1/k_1)} - \varepsilon = m_1^{(1)};$$
$$m_2^{(2)} = \frac{r_2}{b_2 + \left(a_2/\left(m_1^{(1)} + k_2\right)\right)} - \frac{\varepsilon}{2} > \frac{r_2}{b_2 + (a_2/k_2)} - \varepsilon = m_2^{(1)}. \quad (30)$$

Repeating the above procedure, we get four sequences $M_i^{(n)}$, $m_i^{(n)}$, $i = 1, 2$, $n = 1, 2, \ldots$, such that for $n \geq 2$

$$M_1^{(n)} = \frac{r_1 - Eq}{b_1 + \left(a_1/\left(M_2^{(n-1)} + k_1\right)\right)} + \frac{\varepsilon}{n};$$
$$M_2^{(n)} = \frac{r_2}{b_2 + \left(a_2/\left(M_1^{(n-1)} + k_2\right)\right)} + \frac{\varepsilon}{n};$$
$$m_1^{(n)} = \frac{r_1 - Eq}{b_1 + \left(a_1/\left(m_2^{(n-1)} + k_1\right)\right)} - \frac{\varepsilon}{n};$$
$$m_2^{(n)} = \frac{r_2}{b_2 + \left(a_2/\left(m_1^{(n-1)} + k_2\right)\right)} - \frac{\varepsilon}{n}. \quad (31)$$

Obviously,

$$m_i^{(n)} < x_i(t) < M_i^{(n)} \quad \forall t \geq T_{2n}, \ i = 1, 2. \quad (32)$$

We claim that sequences $M_i^{(n)}$, $i = 1, 2$ are strictly decreasing, and sequences $m_i^{(n)}$, $i = 1, 2$ are strictly increasing. To proof this claim, we will carry them out by induction. Firstly, from (29) and (30) we have

$$M_i^{(2)} < M_i^{(1)}, \quad m_i^{(2)} > m_i^{(1)}, \quad i = 1, 2. \quad (33)$$

Let us assume now that our claim is true for n; that is,

$$M_i^{(n)} < M_i^{(n-1)}, \quad m_i^{(n)} > m_i^{(n-1)}, \quad i = 1, 2. \quad (34)$$

Then,

$$\frac{a_1}{M_2^{(n)} + k_1} > \frac{a_1}{M_2^{(n-1)} + k_1},$$
$$\frac{r_2}{b_2 + \left(a_2/\left(M_1^{(n)} + k_2\right)\right)} > \frac{r_2}{b_2 + \left(a_2/\left(M_1^{(n-1)} + k_2\right)\right)}. \quad (35)$$

From (34) and the expression of $M_i^{(n)}$, it immediately follows that

$$M_1^{(n+1)} = \frac{r_1 - Eq}{b_1 + \left(a_1/\left(M_2^{(n)} + k_1\right)\right)} + \frac{\varepsilon}{n+1} < \frac{r_1 - Eq}{b_1 + \left(a_1/\left(M_2^{(n-1)} + k_1\right)\right)} + \frac{\varepsilon}{n} = M_1^{(n)},$$
$$M_2^{(n+1)} = \frac{r_2}{b_2 + \left(a_2/\left(M_1^{(n)} + k_2\right)\right)} + \frac{\varepsilon}{n+1} < \frac{r_2}{b_2 + \left(a_2/\left(M_1^{(n-1)} + k_2\right)\right)} + \frac{\varepsilon}{n} = M_2^{(n)}. \quad (36)$$

Also, it follows from (34) that $m_i^{(n)} \geq m_i^{(n-1)}$, $i = 1, 2$. Then,

$$\frac{a_1}{m_2^{(n)} + k_1} < \frac{a_1}{m_2^{(n-1)} + k_1}, \qquad \frac{a_2}{m_1^{(n)} + k_2} < \frac{a_2}{m_1^{(n-1)} + k_2}. \quad (37)$$

From (37) and the expression of $m_i^{(n)}$, it immediately follows that

$$m_1^{(n+1)} = \frac{r_1 - Eq}{b_1 + \left(a_1/\left(m_2^{(n)} + k_1\right)\right)} - \frac{\varepsilon}{n+1} > \frac{r_1 - Eq}{b_1 + \left(a_1/\left(m_2^{(n-1)} + k_1\right)\right)} - \frac{\varepsilon}{n} = m_1^{(n)},$$
$$m_2^{(n+1)} = \frac{r_2}{b_2 + \left(a_2/\left(m_1^{(n)} + k_2\right)\right)} - \frac{\varepsilon}{n+1} > \frac{r_2}{b_2 + \left(a_2/\left(m_1^{(n-1)} + k_2\right)\right)} - \frac{\varepsilon}{n} = m_2^{(n)}. \quad (38)$$

Therefore,

$$\lim_{t \to +\infty} M_1^{(n)} = \overline{x}, \qquad \lim_{t \to +\infty} M_2^{(n)} = \overline{y},$$
$$\lim_{t \to +\infty} m_1^{(n)} = \underline{x}, \qquad \lim_{t \to +\infty} m_2^{(n)} = \underline{y}. \quad (39)$$

Letting $n \to +\infty$ in (31), we obtain

$$b_1\overline{x} + \frac{a_1\overline{x}}{\overline{y} + k_1} = r_1 - Eq,$$
$$b_2\overline{y} + \frac{a_2\overline{y}}{\overline{x} + k_2} = r_2;$$
$$b_1\underline{x} + \frac{a_1\underline{x}}{\underline{y} + k_1} = r_1 - Eq,$$
$$b_2\underline{y} + \frac{a_2\underline{y}}{\underline{x} + k_2} = r_2. \quad (40)$$

(40) shows that $(\overline{x}, \overline{y})$ and $(\underline{x}, \underline{y})$ are positive solutions of the equations

$$b_1 x + \frac{a_1 x}{y + k_1} = r_1 - Eq,$$
$$b_2 y + \frac{a_2 y}{x + k_2} = r_2. \quad (41)$$

Wei and Li [1] had already showed that, under the assumption that $r_1 > Eq$ holds, (41) has a unique positive solution $E^*(x^*, y^*)$. Hence, we conclude that

$$\overline{x} = \underline{x} = x^*, \qquad \overline{y} = \underline{y} = y^*; \tag{42}$$

that is,

$$\lim_{t \to +\infty} x(t) = x^*, \qquad \lim_{t \to +\infty} y(t) = y^*. \tag{43}$$

Thus, the unique interior equilibrium $E^*(x^*, y^*)$ is globally attractive. This completes the proof of Theorem 2. □

Proof of Corollary 3. Noting that $M_1^{(1)}$, $M_2^{(1)}$, $m_1^{(1)}$, $m_2^{(1)}$ are only dependent on the coefficients of the system (1) and independent of the solution of system (1), hence, (13), (14), (23), and (24) show that the system is permanent. This ends the proof of Corollary 3. □

3. Discussion

In this paper, we revisited the stability property of a cooperative system incorporating harvesting which was proposed by Wei and Li [1]; by using the iterative method, we show that the condition which ensures the existence of a unique positive equilibrium is enough to ensure the global attractivity of the positive equilibrium. The numeric simulation of Example 1 shows the feasibility of our results. It seems interesting to investigate the stability property of the corresponding discrete type model of system (1); we leave this for future study.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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