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OSCILLATION CRITERIA FOR SECOND ORDER FUNCTIONAL DYNAMIC EQUATIONS ON TIME-SCALES

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Abstract: Using a Riccati transformation technique, the authors establish some new oscillation criteria for the second-order functional dynamic equation

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta}+F(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))=0,$$

on a time scale $\mathbb{T}$, where $\gamma>0$ is a constant. The cases

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}=\infty \quad \text{and} \quad \int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}<\infty$$

are both considered. Examples are provided to illustrate the relevance of the results.

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1. Introduction

In this paper, we are concerned with the oscillatory behavior of solutions of the

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second-order functional dynamic equation

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta}+F(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))=0, \tag{1}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma>0$ is a constant, r is a positive real-valued rd-continuous function defined on $\mathbb{T}$, $\tau:\mathbb{T}\to\mathbb{T}$ is a positive rd-continuous function such that $\tau(t)\to\infty$ as $t\to\infty$, and $F:\mathbb{T}\times\mathbb{R}^4\to\mathbb{R}$ is a continuous function. We shall consider the two cases

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}=\infty \tag{2}$$

and

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}<\infty. \tag{3}$$

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. By a *solution* of (1) we mean that there exists a $t_x\geq t_0$ and a non-trivial real-valued function $x(t)\in C^1_{rd}[t_x,\infty)_{\mathbb{T}}$ such that $r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\in C^1_{rd}[t_x,\infty)_{\mathbb{T}}$ and satisfies equation (1) on $[t_x,\infty)_{\mathbb{T}}$. Our attention is restricted to the those solutions of (1) which exist on the half-line $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)|:t>t_1\}>0$ for any $t_1\geq t_x$. A solution $x(t)$ of (1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative; otherwise it is called *nonoscillatory*. Equation (1) is said to be oscillatory if all its solutions are oscillatory. The monographs of Bohner and Peterson ([4], [5]) summarize and organize much of the time scale calculus.

Recently, there has been increasing interest in obtaining sufficient conditions for oscillation of the solutions of different classes of dynamic equations with or without deviating arguments on time scales. For recent contributions we refer the reader to the papers ([1]–[14]) and the references cited therein. The majority of these results are obtained for particular cases of equation (1). For example, Agarwal et al. [1] considered the second order linear delay dynamic equation

$$x^{\Delta\Delta}(t)+q(t)x(\tau(t))=0,\quad \text{for } t\in\mathbb{T}, \tag{4}$$

and established some sufficient conditions for oscillation of (4). Sahiner [23] considered the second-order nonlinear delay dynamic equation

$$x^{\Delta\Delta}(t)+q(t)f(x(\tau(t)))=0,\quad \text{for } t\in\mathbb{T}, \tag{5}$$

and obtained some sufficient conditions for oscillation of (5) by using a Riccati type transformation. Han et al. [17] extended the results in Agarwal et al. [1] to the second-order Emden-Fowler delay dynamic equation

$$x^{\Delta\Delta}(t)+q(t)x^{\gamma}(\tau(t))=0,\quad \text{for } t\in\mathbb{T}, \tag{6}$$

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where γ is a quotient of odd positive integers.

Erbe et al. [12] considered the second-order nonlinear delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)f(x(\tau(t))) = 0, \quad \text{for } t \in \mathbb{T}, \tag{7}$$

and gave some oscillation results that improve the results established by Zhang and Shanliang [27] and Sahiner [23].

Han et al. [18] considered the second-order nonlinear delay dynamic equation

$$(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + q(t)f(x(\tau(t))) = 0, \quad \text{for } t \in \mathbb{T}, \tag{8}$$

and established some oscillation results for $\gamma \geq 1$ being an odd positive integer that improve and extend the results of Saker ([24], [25]) and Sahiner [23].

Very recently, Chen [7] considered the second-order half-linear dynamic equation

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1} x^{\Delta}(t)\right)^{\Delta} + q(t)\left|x(t)\right|^{\gamma-1} x(t) = 0, \quad \text{for } t \in \mathbb{T}, \tag{9}$$

and obtained some sufficient conditions for the oscillation of the equation (9) that improve and extend the results of Saker [25], Agarwal et al. [3] and Hassan [20].

Motivated by the work in [1, 23, 17, 12, 18, 7] mentioned above, using Riccati type transformations we establish some sufficient conditions guaranteeing the oscillation of solutions of Eq. (1). It should be noted that Agarwal et al. [1], Sahiner [23], Han et al. [17], Erbe et al. [12], and Han et al. [18] only discussed the oscillation of solutions in the delay case $\tau(t) \leq t$. Here, our results also can be applied to the advanced case $\tau(t) \geq t$ as well. Finally, some example are given to illustrate our results.

The usual notation and concepts from the time scale calculus as can be found in Bohner and Peterson ([4],[5]) will be used throughout the paper without further mention.

2. Some Lemmas

In this section, we give some lemmas that will be used in the proofs of our main results. We also need the expression

$$(x^{\gamma}(t))^{\Delta} = \gamma \left\{ \int_0^1 \left[(1-h)x(t) + hx^{\sigma}(t)\right]^{\gamma-1} dh \right\} x^{\Delta}(t), \tag{10}$$

which is a simple consequence of Keller's chain rule [4, Theorem 1.90].

Lemma 2.1. (Mean Value Theorem on time scales [5, 15]) *If f is a continuous function on $[a,b]$ and is Δ-differentiable on $[a,b)$, then there exist $\xi, \eta \in [a,b)$ such that*

$$f^{\Delta}(\eta)(b-a) \leq f(b) - f(a) \leq f^{\Delta}(\xi)(b-a). \tag{11}$$

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Lemma 2.2. ([19]) *If X and Y are nonnegative and $\lambda > 1$, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda},$$

where equality holds if and only if $X = Y$.

For convenience in what follows we let $\tau_*(t) = \min\{t, \tau(t)\}$.

Lemma 2.3. *Suppose that the following conditions are satisfied:*

(C1) $u \in C^2_{rd}(I, \mathbb{R})$ *where* $I = [T, \infty)_{\mathbb{T}} \subset \mathbb{T}$ *for some* $T > 0$*;*

(C2) $u(t) > 0$, $u^{\Delta}(t) > 0$, *and* $u^{\Delta\Delta}(t) \leq 0$ *for* $t \geq T$.

Then, for each $0 < k < 1$, there is a $T_k \geq T$ such that

$$u(\tau(t)) \geq k u(t) \frac{\tau_*(t)}{t} \quad \text{for} \quad t \geq T_k. \tag{12}$$

Proof. We consider the two cases: (i) $\tau(t) \leq t$ and (ii) $\tau(t) \geq t$.

Case (i): $\tau(t) \leq t$. If $\tau(t) = t$, (12) clearly holds, so it suffices to consider only those t for which $\tau(t) < t$. Let $T_1 \geq T$ be such that $\tau(t) \geq T$ for $t \geq T_1$. By the Mean Value Theorem on time scales and the monotone property of u^{Δ}, for each $t > T_1$, there exists $\xi_1 \in [\tau(t), t)$ such that

$$u(t) - u(\tau(t)) \leq u^{\Delta}(\xi_1)(t - \tau(t)) \leq u^{\Delta}(\tau(t))(t - \tau(t)).$$

Since $u(t) > 0$, we have

$$\frac{u(t)}{u(\tau(t))} \leq 1 + \frac{u^{\Delta}(\tau(t))}{u(\tau(t))}(t - \tau(t)) \quad \text{for } t > \tau(t) \geq T_1. \tag{13}$$

Similarly, there exists $\xi_2 \in [T_1, \tau(t))$ such that

$$u(\tau(t)) - u(T_1) \geq u^{\Delta}(\xi_2)(\tau(t) - T_1) \geq u^{\Delta}(\tau(t))(\tau(t) - T_1),$$

so

$$\frac{u(\tau(t))}{u^{\Delta}(\tau(t))} \geq \tau(t) - T_1. \tag{14}$$

Let $k \in (0,1)$. Then for $t \geq T_1/(1-k) = T_k \geq T$ we have $t - T_1 \geq kt$ and $\tau(t) - T_1 \geq k\tau(t)$. Now, (14) implies

$$\frac{u(\tau(t))}{u^{\Delta}(\tau(t))} \geq k\tau(t) \quad \text{for} \quad t \geq T_k. \tag{15}$$

From (13) and (15), we obtain

$$\frac{u(t)}{u(\tau(t))} \leq 1 + \frac{u^{\Delta}(\tau(t))}{u(\tau(t))}(t - \tau(t))$$

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$$\begin{aligned} &\leq 1+\frac{t-\tau(t)}{k\tau(t)} \\ &= \frac{t+(k-1)\tau(t)}{k\tau(t)} \\ &\leq \frac{t}{k\tau(t)} = \frac{t}{k\tau_*(t)} \end{aligned}$$

for $t \geq T_k$, which is inequality (12).

Case (ii): $\tau(t) \geq t$. Since $u^{\Delta}(t) > 0$, we have

$$u(\tau(t)) \geq u(t) \geq ku(t) = ku(t)\frac{\tau_*(t)}{t} \quad \text{for} \quad t \geq T_k,$$

which again is (12). This completes the proof of the lemma. □

Lemma 2.4. *Assume that (2) holds,* $r^{\Delta}(t) \geq 0$,

$$\operatorname{sgn} F(t,x,u,v,w) = \operatorname{sgn} x \quad \textit{for} \quad t \in [t_0,\infty)_{\mathbb{T}} \textit{ and } x,u,v,w \in \mathbb{R}, \tag{16}$$

and x is an eventually positive solution of (1). Then, there exist $T \geq t_0$ such that

$$x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad \textit{and} \quad \left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta} < 0 \tag{17}$$

for $t \geq T$.

Proof. Since $x(t)$ is an eventually positive solution of (1), there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \geq t_1$. From (1) and (16), we have

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta} = -F(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t))) < 0 \tag{18}$$

for $t \geq t_1$, so $r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)$ is eventually decreasing, say for $t \in [t_2,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$. We claim that

$$x^{\Delta}(t) > 0 \text{ for } t \geq t_2. \tag{19}$$

If this is not so, then there exists $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $x^{\Delta}(t_3) \leq 0$. In view of (18), there is a $t_4 \geq t_3$ such that

$$r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t) \leq r(t_4)\left|x^{\Delta}(t_4)\right|^{\gamma-1}x^{\Delta}(t_4) := c < 0$$

for $t \in [t_4,\infty)_{\mathbb{T}}$. Hence,

$$x^{\Delta}(t) \leq -(-c)^{1/\gamma}\frac{1}{r^{1/\gamma}(t)}, \tag{20}$$

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for $t \geq t_4$, and so from (2),

$$x(t) \leq x(t_4) - (-c)^{1/\gamma} \int_{t_4}^{t} \frac{\Delta s}{r^{1/\gamma}(s)} \to -\infty \text{ as } t \to \infty,$$

which contradicts the fact that $x(t) > 0$ for $t \geq t_1$. Hence, (19) holds.

From (18) and (19), we see that

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} = -F(t, x(t), x(\tau(t)), x^{\Delta}(t), x^{\Delta}(\tau(t))) < 0 \tag{21}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. Thus, $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing on $[t_2, \infty)_{\mathbb{T}}$.

Finally, we want to show that

$$x^{\Delta\Delta}(t) < 0 \quad \text{for } t \geq t_2. \tag{22}$$

Assume that $x^{\Delta\Delta}(t) \geq 0$ for $t \geq t_2$. Then, $x^{\Delta}(t)$ is nondecreasing and so $x^{\Delta}(t) \leq x^{\Delta}(\sigma(t))$ for $t \geq t_2$. This, together with the fact that $r^{\Delta}(t) \geq 0$, implies

$$r(t)(x^{\Delta}(t))^{\gamma} \leq r(\sigma(t))(x^{\Delta}(t))^{\gamma} \leq r(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma},$$

which contradicts $r(t)(x^{\Delta}(t))^{\gamma}$ being decreasing on $[t_2, \infty)_{\mathbb{T}}$. Thus, (22) holds and this completes the proof of the lemma. □

3. Main Results

In this section, we present our main oscillation results. For any continuous function $u(t)$ we set $u(t)_+ = \max\{u(t), 0\}$ and $u(t)_- = \max\{-u(t), 0\}$ so that $u(t) = u(t)_+ - u(t)_-$.

Theorem 3.5. *In addition to condition (2), assume there are positive functions $\delta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$F(t, x, u, v, w)/\left|x\right|^{\gamma-1} x \geq p(t) \tag{23}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$, $x \in \mathbb{R} \setminus \{0\}$, and u, v, $w \in \mathbb{R}$, and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left\{ \delta(s)p(s) - \frac{r(s)\left[\left(\delta^{\Delta}(s)\right)_+\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \right\} \Delta s = \infty. \tag{24}$$

Then every solution of equation (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose, to the contrary, that Eq. (1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$, say $x(t)>0$ and $x(\tau(t))>0$ on $[t_1,\infty)_{\mathbb{T}}$ for some $t_1\in[t_0,\infty)_{\mathbb{T}}$. From (1) and (23), we have

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta}\leq -p(t)x^{\gamma}(t)<0, \tag{25}$$

for all $t\geq t_1$, and so $r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. As in the proof of Lemma 2.4 we can show that

$$x^{\Delta}(t)>0 \quad \text{for} \quad t\in[t_2,\infty)_{\mathbb{T}} \tag{26}$$

for some $t_2\geq t_1$. In view of (25) and (26), we see that

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\leq -p(t)x^{\gamma}(t)<0 \quad \text{for} \quad t\in[t_2,\infty)_{\mathbb{T}}. \tag{27}$$

Now consider the generalized Riccati substitution

$$w(t)=\delta(t)\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)} \quad \text{for} \quad t\geq t_2. \tag{28}$$

Clearly, $w(t)>0$, and

$$\begin{aligned}
w^{\Delta}(t) &= \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\frac{\delta(t)}{x^{\gamma}(t)}+\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(\frac{\delta(t)}{x^{\gamma}(t)}\right)^{\Delta}\\
&\leq -\delta(t)p(t)+\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(\frac{\delta^{\Delta}(t)}{x^{\gamma}(\sigma(t))}-\frac{\delta(t)\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right)\\
&= -\delta(t)p(t)+\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\\
&\leq -\delta(t)p(t)+\frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t)-\delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}.
\end{aligned} \tag{29}$$

From (10) and (26), we obtain

$$\begin{aligned}
\left(x^{\gamma}(t)\right)^{\Delta} &= \gamma\left\{\int_0^1\left[(1-h)x(t)+hx^{\sigma}(t)\right]^{\gamma-1}dh\right\}x^{\Delta}(t)\\
&\geq \begin{cases}\gamma\left(x^{\sigma}(t)\right)^{\gamma-1}x^{\Delta}(t), & 0<\gamma\leq 1,\\ \gamma\left(x(t)\right)^{\gamma-1}x^{\Delta}(t), & \gamma>1\end{cases}
\end{aligned} \tag{30}$$

If $0<\gamma\leq 1$, then (29) and (30) imply

$$w^{\Delta}(t)\leq \quad - \quad \delta(t)p(t)+\frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t)$$

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$$
\begin{aligned}
& - \quad \delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\gamma\left(x^{\sigma}(t)\right)^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))} \\
= & - \quad \delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t) \\
& - \quad \gamma\delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{x^{\gamma+1}(\sigma(t))}\frac{x^{\gamma}(\sigma(t))}{x^{\gamma}(t)}x^{\Delta}(t). \qquad (31)
\end{aligned}
$$

If $\gamma > 1$, (29) and (30) imply

$$
\begin{aligned}
w^{\Delta}(t) \leq & - \quad \delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t) \\
& - \quad \delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\gamma\left(x(t)\right)^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))} \\
= & - \quad \delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t) \\
& - \quad \gamma\delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{x^{\gamma+1}(\sigma(t))}\frac{x(\sigma(t))}{x(t)}x^{\Delta}(t). \qquad (32)
\end{aligned}
$$

Since $t \leq \sigma(t)$ and $x(t)$ is increasing on $[t_2, \infty)_{\mathbb{T}}$, we have $x(t) \leq x(\sigma(t))$. Therefore, (31) and (32) yield

$$
w^{\Delta}(t) \leq -\delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t) - \gamma\delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}}{x^{\gamma+1}(\sigma(t))}x^{\Delta}(t) \qquad (33)
$$

on $[t_2, \infty)_{\mathbb{T}}$ for $\gamma > 0$.

Since $r(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is decreasing, we have

$$
r(t)\left(x^{\Delta}(t)\right)^{\gamma} \geq \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma},
$$

so

$$
x^{\Delta}(t) \geq \frac{\left[\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\right]^{1/\gamma}}{r^{1/\gamma}(t)}. \qquad (34)
$$

Using (34) in (33), we obtain

$$
\begin{aligned}
w^{\Delta}(t) \leq & -\delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t) \\
& - \gamma\delta(t)r^{-1/\gamma}(t)\frac{\left[\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\right]^{(\gamma+1)/\gamma}}{x^{\gamma+1}(\sigma(t))}. \qquad (35)
\end{aligned}
$$

From (28) and (35), we conclude that

$$w^{\Delta}(t) \le -\delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_+}{\delta^{\sigma}(t)} w^{\sigma}(t) - \gamma\delta(t) r^{-1/\gamma}(t) \left(\delta^{\sigma}(t)\right)^{-(\gamma+1)/\gamma} \left(w^{\sigma}(t)\right)^{(\gamma+1)/\gamma}. \quad (36)$$

Letting

$$X = \frac{(\gamma\delta(t))^{\gamma/(\gamma+1)} w^{\sigma}(t)}{r^{1/(\gamma+1)}(t)\delta^{\sigma}(t)}, \quad \lambda = \frac{\gamma+1}{\gamma},$$

and

$$Y = \frac{r^{\gamma/(\gamma+1)}(t)\left(\left(\delta^{\Delta}(t)\right)_+\right)^{\gamma}}{\lambda^{\gamma}\left(\gamma\delta(t)\right)^{\gamma/\lambda}},$$

in Lemma 2.2, (36) implies

$$w^{\Delta}(t) \le -\delta(t)p(t) + \frac{r(t)\left[\left(\delta^{\Delta}(t)\right)_+\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)} \quad \text{for} \quad t \in [t_2, \infty)_{\mathbb{T}}. \quad (37)$$

Integrating (37) from t_2 to t, we obtain

$$\int_{t_2}^{t} \left\{ \delta(s)p(s) - \frac{r(s)\left[\left(\delta^{\Delta}(s)\right)_+\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \right\} \Delta s \le -w(t) + w(t_2) \le w(t_2),$$

which contradicts condition (24). Therefore, equation (1) is oscillatory. □

Theorem 3.6. *Assume that conditions (2) and* (16) *hold,* $r^{\Delta}(t) \ge 0$, *and there are a positive functions* $\delta \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *and* $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *and a constant* $k \in (0, 1)$ *such that*

$$F(t, x, u, v, w)/|u|^{\gamma-1} u \ge p(t) \quad (38)$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, x, $u \in \mathbb{R} \setminus \{0\}$, *and* v, $w \in \mathbb{R}$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left\{ \delta(s)p(s)\left[\frac{k\tau_*(s)}{s}\right]^{\gamma} - \frac{r(s)\left[\left(\delta^{\Delta}(s)\right)_+\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \right\} \Delta s = \infty, \quad (39)$$

then every solution of equation (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose that Eq. (1) has a nonoscillatory solution $x(t)$, say $x(t) > 0$ and $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$. From Lemma 2.4, we eventually have $x^{\Delta}(t) > 0$ and $x^{\Delta\Delta}(t) < 0$. Hence, in view of Lemma 2.3, there exits $t_2 \ge t_1$, such that

$$x(\tau(t)) \ge \frac{k\tau_*(t)}{t} x(t) \quad \text{for all} \quad t \ge t_2. \quad (40)$$

Defining $w(t)$ as in the proof of Theorem 3.5, we have

$$\begin{aligned} w^{\Delta}(t) &= \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\frac{\delta(t)}{x^{\gamma}(t)}+\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(\frac{\delta(t)}{x^{\gamma}(t)}\right)^{\Delta} \\ &= -\delta(t)\frac{F(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))}{x^{\gamma}(t)} \\ &\quad +\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(\frac{\delta^{\Delta}(t)}{x^{\gamma}(\sigma(t))}-\frac{\delta(t)\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right) \\ &= -\delta(t)\frac{F(t,x(t),x(\tau(t)),x^{\Delta}(t),x^{\Delta}(\tau(t)))}{(x(\tau(t)))^{\gamma}}\frac{(x(\tau(t)))^{\gamma}}{x^{\gamma}(t)} \\ &\quad +\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))} \\ &\le -\delta(t)p(t)\frac{(x(\tau(t)))^{\gamma}}{x^{\gamma}(t)}+\frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t) \\ &\quad -\delta(t)\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \end{aligned} \tag{41}$$

As in the proof of Theorem 3.5, we conclude from (41) that

$$\begin{aligned} w^{\Delta}(t) \le -\delta(t)p(t)\left[\frac{k\tau_{*}(t)}{t}\right]^{\gamma}+\frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}(t)}w^{\sigma}(t) \\ -\gamma\delta(t)r^{-1/\gamma}(t)\left(\delta^{\sigma}(t)\right)^{-(\gamma+1)/\gamma}\left(w^{\sigma}(t)\right)^{(\gamma+1)/\gamma}. \end{aligned}$$

The remainder of the proof is similar to that of Theorem 3.5 and is omitted. □

Next, if (3) holds, we establish some sufficient conditions that guarantee a solution $x(t)$ of Eq. (1) either oscillates or converges to zero.

Theorem 3.7. *Assume that (3) holds and let δ and p be defined as in Theorem 3.5 so that* (23) *and* (24) *hold. If*

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{1/\gamma}\Delta t=\infty, \tag{42}$$

then a solution of Eq. (1) is either oscillatory or converges to zero.

Proof. Proceeding as in the proof of Theorem 3.5, we let $x(t)$ be a nonoscillatory solution of (1) with $x(t)>0$ and $x(\tau(t))>0$ on $[t_1,\infty)_{\mathbb{T}}$ for some $t_1\in[t_0,\infty)_{\mathbb{T}}$. There are two possible cases for the sign of $x^{\Delta}(t)$. The proof if $x^{\Delta}(t)$ is eventually positive is similar to that in the proof of Theorem 3.5, and hence is omitted.

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Now assume that $x^{\Delta}(t) \leq 0$ for $t \geq t_1 \geq t_0$. Then $x(t)$ is decreasing and $\lim_{t\to\infty} x(t) = b \geq 0$ exists. If $b > 0$, then $x(t) > b > 0$ for $t \geq t_2 \geq t_1$. So, in view of (25) and the fact that $x^{\Delta}(t) \leq 0$, we see that

$$-\left(r(t)(-x^{\Delta}(t))^{\gamma}\right)^{\Delta} \leq -b^{\gamma} p(t)$$

for all $t \in [t_2, \infty)_{\mathbb{T}}$. Integrating from t_2 to t, we obtain

$$x^{\Delta}(t) \leq -b\left[\frac{1}{r(t)}\int_{t_2}^{t} p(s)\Delta s\right]^{1/\gamma}.$$

A second integration yields

$$x(t) \leq x(t_2) - b\int_{t_2}^{t}\left[\frac{1}{r(u)}\int_{t_2}^{u} p(s)\Delta s\right]^{1/\gamma}\Delta u.$$

Letting $t \to \infty$, we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$ for all $t \geq t_1$. Thus, $b = 0$ and $x(t) \to 0$ as $t \to \infty$. This completes the proof of the theorem. □

The proof of our next and final result is similar to the proof of Theorem 3.7 and so we omit the details.

Theorem 3.8. *Assume that (3) and* (16) *hold,* $r^{\Delta}(t) \geq 0$, *and* $\tau(t) \leq t$. *Let* δ *and* p *be defined as in Theorem 3.6 such that* (38) *and* (39) *hold for some constant* $k \in (0, 1)$. *If* (42) *holds, then a solution of Eq. (1) is either oscillatory or converges to zero.*

4. Examples

In this section, we give some examples to illustrate our main results.

Example 4.9. Consider the second-order dynamic equation

$$\left(t^{\gamma}\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta} + \left(t + \frac{\beta}{t}\right)|x(t)|^{\gamma-1}x(t) = 0, \tag{43}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 > 0$, where $\gamma > 0$ and $\beta > 0$ are constants, $p(t) = t + \frac{\beta}{t}$, and $r(t) = t^{\gamma}$. Then,

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \int_{t_0}^{\infty} \frac{\Delta t}{t} = \infty,$$

so (2) holds. With $\delta(t)=t$, condition (24) becomes

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left\{\delta(s)p(s)-\frac{r(s)\left[\left(\delta^{\Delta}(s)\right)_+\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right\}\Delta s\\ &=\limsup_{t\to\infty}\int_{t_0}^{t}\left\{s\left(s+\frac{\beta}{s}\right)-\frac{s^{\gamma}}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right\}\Delta s\\ &=\limsup_{t\to\infty}\int_{t_0}^{t}\left\{s^2+\beta-\frac{1}{(\gamma+1)^{\gamma+1}}\right\}\Delta s=\infty.\end{aligned}$$

So every solution of (43) is oscillatory by Theorem 3.5.

Example 4.10. Consider the dynamic equation

$$\left(t^{\gamma-1}\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta}+(t+\sigma(t))\left|x(\tau(t))\right|^{\gamma-1}x(\tau(t))=0, \tag{44}$$

for $t\in[1,\infty)_{\mathbb{T}}$, where $r(t)=t^{\gamma-1}$, $p(t)=t+\sigma(t)$, $\gamma>1$, and $\tau(t)\geq t$. It is clear that

$$\int_1^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}=\int_1^{\infty}\frac{\Delta t}{t^{1-1/\gamma}}=\infty.$$

For $\delta(t)=1$, we have

$$\begin{aligned}&\limsup_{t\to\infty}\int_1^{t}\left\{\delta(s)p(s)\left[\frac{k\tau_*(s)}{s}\right]^{\gamma}-\frac{r(s)\left[\left(\delta^{\Delta}(s)\right)_+\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right\}\Delta s\\ =\quad&\limsup_{t\to\infty}\int_1^{t}(s+\sigma(s))\left[\frac{k\tau_*(s)}{s}\right]^{\gamma}\Delta s\\ =\quad&k^{\gamma}\limsup_{t\to\infty}\int_1^{t}(s+\sigma(s))\Delta s\\ =\quad&k^{\gamma}\limsup_{t\to\infty}\int_1^{t}\left(s^2\right)^{\Delta}\Delta s=k^{\gamma}\limsup_{t\to\infty}\left(t^2-1\right)=\infty.\end{aligned}$$

Therefore, every solution of (44) is oscillatory by Theorem 3.6.

Example 4.11. Consider the second-order dynamic equation

$$\left(t^{\gamma+1}\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta}+\alpha\left|x\left(\frac{t}{2}\right)\right|^{\gamma-1}x\left(\frac{t}{2}\right)=0, \tag{45}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 > 0$, where $\gamma > 0$ is constant, $p(t) = \alpha > 0$, $r(t) = t^{\gamma+1}$, and $\tau(t) = t/2$. Then,

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}=\int_{t_0}^{\infty}\frac{\Delta t}{t^{\frac{\gamma+1}{\gamma}}}<\infty,$$

so (3) holds. To apply Theorem 3.8, it remains to show that conditions (39) and (42) hold. To see this, note that if $\delta(t) = 1$, then

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left\{\delta(s)p(s)\left[\frac{k\tau_*(s)}{s}\right]^{\gamma}-\frac{r(s)\left[\left(\delta^{\Delta}(s)\right)_+\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right\}\Delta s$$
$$=\limsup_{t\to\infty}\int_{t_0}^{t}\alpha\left(\frac{k}{2}\right)^{\gamma}\Delta s=\infty,$$

which implies that (39) holds. Note that $t - t_0 \geq t/2$ if $t \in [2t_0, \infty)_{\mathbb{T}}$. Thus,

$$\begin{aligned}\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{1/\gamma}\Delta t &= \int_{t_0}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_0}^{t}\alpha\Delta s\right]^{1/\gamma}\Delta t\\ &= \int_{t_0}^{\infty}\left(\frac{\alpha(t-t_0)}{t^{\gamma+1}}\right)^{1/\gamma}\Delta t\\ &\geq (\alpha/2)^{1/\gamma}\int_{2t_0}^{\infty}\frac{\Delta t}{t}=\infty,\end{aligned}$$

so (42) holds. Therefore, by Theorem 3.8, a solution of (45) is either oscillatory or converges to zero.

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