On the Equal-height Elements of Fuzzy AG-subgroups

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Abstract: In this paper we introduce the left (right) equal-height elements of a fuzzy power set. We show that both left and right equal-height elements coincide in fuzzy AG-subgroups. We investigate that the collection of left (right) equal-height elements of AG-group G form an AG-subgroup of G. We also establish a relation between the left equal-height elements and left cosets as well as the right equal-height elements and right cosets of an AG-group G.

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1. Introduction

A fuzzy subset μ of a set X is a function from X to the unit closed interval [0, 1]. For the first time this concept was introduced by Zadeh in 1965 [1]. In 1971, Rosenfeld introduced the concept of fuzzy subgroups which explore the study of the algebraic structures [2]. Wu [3, 4] and Kumar [5] studied normal fuzzy subgroups. Chenh-yi [6] introduced the concept on the equal height-elements in groups. On the other hand in 1993, Kamran extended the idea of AG-groupoid to AG-groups [7]. In 1996, he provided various important results on AG-groups [8]. In 2011, the 3rd author reconsidered the study of AG-groups and defined normality in AG-groups [9, 10]. In 2012, the authors of this paper introduced new concepts on fuzzy AG-subgroups [11]. The present work is a continuation of the concepts given in [11].

In this paper we introduce the concepts of the equal-height elements of a fuzzy AG-subgroup. An AG-groupoid is a groupoid satisfying the left invertive law: $(ab)c=(cb)a$. AG-groupoid satisfies the medial law: $(ab)(cd)=(ac)(bd)$, and if $e\in G$ then it also satisfies paramedical law: $(ab)(cd)=(db)(ca)$. AG-groupoid $(G,\cdot)$ is called an AG-group or a left almost group (LA-group), if
(i) There exists left identity $e\in G$ (that is, $ea=a$ for all $a\in G$). (ii) For all $a\in G$ there exists a^{-1} in G, such that $a^{-1}a=e=aa^{-1}$.

2. Preliminaries

A fuzzy subset μ is a mapping $\mu : X\rightarrow[0,1]$. The set of all fuzzy subsets of X is called the fuzzy power set of X and is denoted by $FP(X)$. One of the most important concept of fuzzy set μ is the concept of $\alpha-$cut and its variant, a strong $\alpha-$cut. Given a fuzzy set μ defined on X and any number $\alpha\in[0,1]$, the $\alpha-$cut, and the strong $\alpha-$cut, are the crisp sets

$$\mu_\alpha=\{x:x\in X,\ \mu(x)\geq\alpha\}$$

and

$$\mu_{\alpha^+}=\{x:x\in X,\ \mu(x)>\alpha\}$$

that is, the $\alpha-$cut (or strong $\alpha-$cut) of a fuzzy set μ_α (or the crisp set μ_{α^+}) that contains all the elements of the universal set X whose membership grades in μ are greater than or equal to (or only greater than) the specified value of α.

The support of a fuzzy set μ within a universal set X is the crisp set that contains all the elements of X that have non-zero membership grades in μ. Clearly, the support of μ is exactly the same as the strong $\alpha-$cut for $\alpha=0$. Although special symbols, such as μ^* or Supp(μ), are often used in the literature to denote the support of μ. We prefer to use the natural symbol μ_{0^+}.

The height $H_e(\mu)$, of a fuzzy set μ is the largest membership grade obtained by any element in that set. Symbolically it is represented by;

$$H_e(\mu)=\underset{x\in X}{Sup}\ \mu(x) \quad \text{or}$$

$$H_e(\mu)=\vee\{\lambda:\lambda\in \operatorname{Im}(\mu)\}.$$

The height of μ may also be viewed as the supremum of α for $\mu_\alpha\neq\phi$. If $x\in X$, such that

$H_e(\mu) = \mu(x)$, then x is called a top-element. If $x, y \in X$, $\mu(x) = \mu(y)$, then we call x and y are of equal-heights [6].

In the rest of this paper G will denote an AG-group otherwise stated and e will denote the left identity of G.

3. Fuzzy AG-subgroup

Definition 1. Let S be a groupoid, i.e. a set which is closed under a binary operation of multiplication (let say). A mapping $\mu : S \to [0,1]$ is called fuzzy sub-groupoid if

$$\mu(xy) \geq \mu(x) \wedge \mu(y) \quad \forall\, x, y \in S.$$

Remark 1. If $x_1, x_2, \cdots, x_n \in S$, then for a fuzzy sub-groupoid, it follows from the definition that

$$\mu(x_1 \cdot x_2 \cdot \cdots \cdot x_n) \geq \min\{\mu(x_i) : 1 \leq i \leq n\}.$$

Definition 2. Let $\mu \in FP(G)$, then μ is called a fuzzy AG-subgroup of G if for all $x, y \in G$;

(i). $\mu(xy) \geq \mu(x) \wedge \mu(y)$,

(ii). $\mu(x^{-1}) \geq \mu(x)$.

We will denote the set of all fuzzy AG-subgroups of G briefly by $F(G)$.

Lemma 1. [11, Lemma 1.2.5] For an AG-group G. Let $\mu \in F(G)$. Then for all $x \in G$,

(i). $\mu(e) \geq \mu(x)$,

(ii). $\mu(x) = \mu(x^{-1})$.

Theorem 1. [11] Let $\mu \in F(G)$. Then the following assertions holds; for all $x, y \in G$,

(i). $\mu(xy) = \mu(yx)$,

(ii). $\mu(ye) = \mu(y)$,

(iii). $\mu(ye) \geq \mu(y)$,

(iv). $\mu(ye) \leq \mu(y)$.

Definition 3. Let G be an AG-group and μ be a fuzzy subset of G. Then for $a, b \in G$, we define a to be left equivalent to b, written as $a \sim_L b$, if and only if

$$\mu(ax) = \mu(bx) \quad \text{for all} \quad x \in G.$$

Similarly a to be right equivalent to b, written as $a \sim_R b$, if and only if

$$\mu(xa) = \mu(xb) \quad \text{for all} \quad x \in G.$$

Remark 2. The left relation "$\sim_L$" as well as the right relation "$\sim_R$" are equivalence relations on G.

Corollary 1. Let G be an AG-group and μ be a fuzzy AG-subgroup of G. For $a, b \in G$; $a \sim_L b$, then $\mu(a) = \mu(b)$, that is a and b are equal-height elements.

Proof. Let for $a, b \in G$; $a \sim_L b$, then by definition we get;

$$\mu(ax) = \mu(bx) \quad \text{for all} \quad x \in G$$

Substitute instead of x the left identity e of G, we get

$$(ae) = \mu(be)$$

$$\Rightarrow \mu(ea) = \mu(eb); \quad \text{(by Theorem 1(i))}$$

$$\Rightarrow \mu(a) = \mu(b).$$

With the help of the following example we show that the converse is not true in general in AG-groups. Consider the AG-group of order 5:

.	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Define the fuzzy subset $\mu : G \to [0,1]$ as follows:

$$\mu(0) = 1 \text{ and } \mu(1) = \mu(2) = \mu(3) = \mu(4) = \frac{1}{2}.$$

Now $\mu(1) = \mu(2)$, that is, the height of 1 and 2 are equal under μ, but $\mu(12) \neq \mu(22)$ so $\mu(1x) \neq \mu(2x)$, $\forall\, x \in G$. Therefore, $1 \not\sim_L 2$.

Theorem 2. Let μ be any fuzzy AG-subgroup of G. Then for $a, b \in G$; $a \sim_L b \Leftrightarrow a \sim_R b$.

Proof. Let $\mu \in F(G)$ and $a, b \in G$; then $a \sim_L b$ if and only if

$$\mu(ax) = \mu(bx) \quad \forall\, x \in G$$

$$\Leftrightarrow \mu(xa) = \mu(xb) \quad \text{(by Theorem 1(i))}$$

$$\Leftrightarrow a \sim_R b.$$

Hence both left and right equivalence relations coincide.

Definition 4. Let μ be any fuzzy subset of G. Then the collection

$$H_L = \{a \in G : \mu(ax) = \mu(x), \ \forall\ x \in G\}$$

is called the set of left equal height elements of G. Similarly the collection

$$H_R = \{a \in G : \mu(xa) = \mu(x), \ \forall\ x \in G\}$$

is called the set of right equal height elements of G.

Theorem 3. Let μ be a fuzzy AG-subgroup of G. Then H_L is an AG-subgroup of G.

Proof. Clearly H_L is non-empty; as for all $x \in G$; $\mu(ex) = \mu(x)$, this implies that $e \in H_L$.

Now for any $a, b \in H_L$, let

$$\mu((ab)x) = \mu((xb)a) \quad \text{(using left invertive law)}$$

$$= \mu(a \cdot xb) \quad \text{(using Theorem 1(i))}$$

$$=\mu(xb) \qquad (a \in H_L)$$
$$=\mu(bx) \qquad \text{(using Theorem 1(i))}$$
$$=\mu(x) \qquad (b \in H_L)$$

Thus $ab \in H_L$.

Next for any $a \in H_L$, let

$$\mu(a^{-1}x) = \mu(ax^{-1})^{-1} \quad (\text{in } G;\ (ab)^{-1} = a^{-1}b^{-1})$$
$$= \mu(ax^{-1}) \qquad \text{(by Lemma 1(ii))}$$
$$= \mu(x^{-1}) \qquad (a \in H_L)$$
$$= \mu(x) \qquad \text{(by Lemma 1(ii))}$$

Thus $a^{-1} \in H_L$. Hence H_L is an AG-subgroup G.

Remarks 3. Let μ be a fuzzy AG-subgroup of G. Then H_R is also an AG-subgroup of G; as

$$\mu(ax) = \mu(xa) = \mu(x) \quad \forall\ x \in G.$$

Theorem 4. Let μ be a fuzzy AG-subgroup of G, and $H = \{a \in G : \mu(a) = \mu(e) = H_e(\mu)\}$. Then $H = H_L$.

Proof. Let μ be a fuzzy AG-subgroup of G, we show that $H = H_L$.

Let $a \in H_L$ then $\mu(ax) = \mu(x)$,

by putting $x = e$, we get

$$\mu(ae) = \mu(ea) = \mu(e) \Rightarrow \mu(a) = \mu(e)$$
$$\Rightarrow a \in H \Rightarrow H_L \subseteq H.$$

Conversely, let $a \in H$. Then according to the condition of H; for all x in G, we have $\mu(a) = \mu(e) \geq \mu(x)$. Consider

$$\mu(ax) \geq \mu(a) \wedge \mu(x)$$
$$= \mu(x)$$
$$= \mu(ex) = \mu(a^{-1}a \cdot x)$$
$$= \mu(xa \cdot a^{-1}) \quad \text{(by left invertive law)}$$
$$\geq \mu(xa) \wedge \mu(a^{-1})$$
$$= \mu(ax) \wedge \mu(a) \quad (\mu \in F(G))$$
$$= \mu(ax) \wedge \mu(e)$$
$$= \mu(ax).$$

Consequently for all $x \in G$;

$$\mu(ax) \geq \mu(x) \geq \mu(ax) \Rightarrow \mu(ax) = \mu(x) \Rightarrow a \in H_L.$$

Hence $H = H_L$.

In the following theorems we establish a relation between the left (right) equal-height elements and the left (right) cosets respectively. Like a group cosets in AG-group are defined as follows; let H be an AG-subgroup of an AG-group G, and let $a \in G$. Then $aH = \{ah : h \in H\}$ and $Ha = \{ha : h \in H\}$ are the corresponding left and right cosets of H in G, in which the following property contrary to the group concept holds;

$$Ha = Hb \iff b^{-1}a \in H \quad [10].$$

Theorem 5. Let μ be a fuzzy AG-subgroup of G and

$$H = \{a \in G : \mu(a) = \mu(e) = H_e(\mu)\}.$$

Then $a \sim_L b$ if and only if $Ha = Hb$.

Proof. Let $a \sim_L b$. Then for any $x \in G$,

$$\mu(ax) = \mu(bx)$$
$$\Rightarrow \mu(ab^{-1}) = \mu(bb^{-1}) \ (\text{replacing } x \text{ by } b^{-1})$$
$$\Rightarrow \mu(b^{-1}a) = \mu(e) \qquad \text{(by Theorem 1(i))}$$
$$\Rightarrow b^{-1}a \in H$$
$$\Rightarrow Ha = Hb.$$

Conversely, let $Ha = Hb$ then $b^{-1}a \in H$. Now

$$\mu(ax) = \mu(xa) \qquad \text{(by Theorem 1(i))}$$
$$=\mu(e \cdot xa)$$
$$= \mu(bb^{-1} \cdot xa)$$
$$= \mu(ab^{-1} \cdot xb) \quad \text{(by paramedial law)}$$
$$\geq \mu(ab^{-1}) \wedge \mu(xb)$$
$$= \mu(b^{-1}a) \wedge \mu(bx) \qquad (\mu \in F(G))$$
$$= \mu(e) \wedge \mu(bx) \qquad (b^{-1}a \in H)$$
$$= \mu(bx)$$
$$= \mu(e \cdot bx)$$
$$= \mu(a^{-1}a \cdot bx)$$
$$= \mu(a^{-1}b \cdot ax) \qquad \text{(by medial law)}$$
$$\geq \mu(a^{-1}b) \wedge \mu(ax)$$
$$= \mu((ab^{-1})^{-1}) \wedge \mu(ax)$$
$$= \mu(ab^{-1}) \wedge \mu(ax)$$
$$= \mu(b^{-1}a) \wedge \mu(ax)$$
$$= \mu(e) \wedge \mu(ax) \qquad (b^{-1}a \in H)$$
$$= \mu(ax).$$

Consequently $\forall\ x \in G$;

$$\mu(ax) \geq \mu(bx) \geq \mu(ax) \Rightarrow \mu(ax) = \mu(bx).$$

Hence $a \sim_L b$.

Similarly, we can establish a relation between right equivalent elements and right cosets as follows:

Theorem 6. Let μ be a fuzzy AG-subgroup of G and

$$H = \{a \in G : \mu(a) = \mu(e) = H_e(\mu)\}.$$

Then $a \sim_R b$ if and only if $aH = bH$.

Definition 5. Let G be an AG-group and μ be a fuzzy subset of G. Then for any $a, b \in G$, we define a is equivalent to b, written as $a \sim b$ if and only if

$$\mu(x(ay)) = \mu(x(by)) \qquad \forall\ x, y \in G.$$

Proposition 1. Let $\mu \in FP(G)$. If $a \sim b$ then

$$a \sim_L b.$$

Proof. Let $a \sim b$ then for any $\mu \in FP(G)$ we have to show that a is left equivalent to b, that is, $a \sim_L b$. Let for any $x, y \in G$;

$$\mu(x(ay)) = \mu(x(by))$$
$$\Rightarrow \mu(a(xy)) = \mu(b(xy)) \text{ (in } G; a(bc) = b(ac) \text{ [12])}$$
$$\Rightarrow a \sim_L b.$$

Proposition 2. Let $\mu \in F(G)$. If $a \sim b$ then $a \sim_L b$ and $a \sim_R b$.

Proof. The proof follows by Theorem 2.

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