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SEQUENTIAL CALCULI FOR MANY-VALUED LOGICS WITH EQUALITY DETERMINANT

Abstract

We propose a general method of constructing sequential calculi with cut elimination property for propositional finitely-valued logics with equality determinant. We then prove the non-algebraizability of the consequence operations of cut-free versions of such sequential calculi.

Key words and phrases: many-valued logic, equality determinant, sequential calculus, cut elimination, algebraizable sequential consequence operation.

One of the main issues concerning many-valued logics is to find their appropriate useful axiomatizations. Since the development of the formalism of *many-place sequents* in [13] which enabled one to axiomatize arbitrary finitely-valued logics, the main emphasis within the topic has been laid on developing generic approaches dealing with variations of the approach [13]. However, the possibility of using standard Gentzen (two-place) calculi for finitely-valued logics has deserved much less attention, especially, in connection with developing general approaches. Nevertheless, such possibility does exist, at least, provided some restrictions are laid on finitely-valued logics under consideration. In our paper, we restrict our consideration by finitely many-valued logics having what we call here *equality determinant.*

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Throughout the paper, we follow standard set-theoretical, lattice-theoretical, algebraic and logical conventions (which we do not specify here explicitly) as well as the terminology and notations adopted in the end of Section 1 and in Section 4 of [12] (which we do not repeat here), except that propositional (i.e., algebraic) signatures are called *(propositional) languages* while L-matrices (where L is a language), are referred to as *(propositional many-valued) L-logics*.

Throughout the paper, unless otherwise specified, fix any language L, the set of all nullary connectives of which is denoted by L_0, any $k, l \in \{0, 1\}$, any denumerable set $\mathrm{Var} := \{p, q\} \cup \{p_i, q_i \mid i \geq 1\}$ of (propositional) variables and any many-valued L-logic $\mathcal{M} = \langle \mathcal{A}, D \rangle$ where (just recall it) $\mathcal{A}$ is an L-algebra, which is called *the underlying algebra of* $\mathcal{M}$ and the elements of which are referred to as *(truth) values of* $\mathcal{M}$, and D is a subset of A, the elements of which are referred to as *distinguished values of* $\mathcal{M}$.

A 0-*(*1-*)subalgebra of* $\mathcal{M}$ is a subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $B \cap D = \emptyset$ ($B \subseteq D$). (Clearly, the problem of determining whether $\mathcal{M}$ has either of such subalgebras is decidable whenever L is finite.)

Proposition 1. *Let $X \subseteq \mathrm{Seq}^{(1,0)}$ ($\subseteq \mathrm{Seq}^{(0,1)}$) and $\Gamma \vdash \Delta \in \mathrm{Cn}^{(0,0)}_{\mathcal{M}}(X)$. Assume $\mathcal{M}$ has a* 0-*(*1-*)subalgebra. Then, $\Gamma \vdash \Delta \in \mathrm{Seq}^{(1,0)}$ ($\in \mathrm{Seq}^{(0,1)}$).*

Proof: By contradiction. Suppose that $\Gamma \vdash \Delta \notin \mathrm{Seq}^{(1,0)}$ ($\notin \mathrm{Seq}^{(0,1)}$). Then, the sequence Γ (Δ) is empty. Consider any 0-(1-)subalgebra $\mathcal{B}$ of $\mathcal{M}$ and any $h \in \hom(\mathcal{F}m_L, \mathcal{B})$. Then, $\Theta \models^h_{\mathcal{M}} \Xi$ for each $\Theta \vdash \Xi \in X$, since the sequence Θ (Ξ) is not empty. However, $\Gamma \not\models^h_{\mathcal{M}} \Delta$. This contradiction shows that $\Gamma \vdash \Delta \in \mathrm{Seq}^{(1,0)}$ ($\in \mathrm{Seq}^{(0,1)}$). ■

In particular, this means that $\mathrm{Cn}^{(0,0)}_{\mathcal{M}}(\emptyset) \subseteq \mathrm{Seq}^{(1,0)}_L$ ($\mathrm{Cn}^{(0,0)}_{\mathcal{M}}(\emptyset) \subseteq \mathrm{Seq}^{(0,1)}_L$) whenever $\mathcal{M}$ has a 0-(1-)subalgebra. This justifies the following restriction of the standard sequential language $\mathrm{Seq}^{(0,0)}_L$ by sequents of rank (k, l) adopted in the present paper. From now on, we assume that $k = 0$ ($l = 0$) whenever $\mathcal{M}$ has no 0-(1-)subalgebra.

In addition, from now on, we suppose that $|A| > 1$ and $\mathcal{M}$ has an *equality determinant*, that is, an arbitrary finite $\Im(p) \subseteq Fm_L$ such that $p \in \Im$, $\Im(p) \cap \Im(q) = \emptyset$ and, for all $a, b \in A$, $a = b$ whenever, for each $\iota \in \Im$, $\iota^{\mathcal{A}}(a) \in D \Leftrightarrow \iota^{\mathcal{A}}(b) \in D$. The finiteness of $\Im$ implies the finiteness of A, whereas $|A| > 1$ implies $\emptyset \neq D \neq A$.

Let $X \subseteq Fm_L$. Put $\Im(X) := \{\iota(\phi) \mid \iota \in \Im, \phi \in X\}$. An L-sequent $\Gamma \vdash \Delta$ is called X*-simple* provided $\Gamma, \Delta \in X^*$ and the sequence (Γ, Δ) has no repetitions.

By an $(\Im, L)$*-type* we mean any expression of the form $\iota(F)$, where $\iota \in \Im$ and $F \in L \backslash L_0$. It is said to be $\Im$*-complex* provided $\iota(F(p_1, \ldots, p_n)) \notin \Im(\mathrm{Var})$, where n is the arity of F.

By an *L-sequential $\Im$-table of rank (k, l) for $\mathcal{M}$* we mean any pair of the form $\mathcal{T} = \langle \lambda_{\mathcal{T}}, \rho_{\mathcal{T}} \rangle$, where $\lambda_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$ are mappings from the set of all $\Im$-complex $(\Im, L)$-types to the set of all finite sets of L-sequents of rank $(0, 0)$ such that, for each $\Im$-complex $(\Im, L)$-type $\iota(F)$, where F is n-ary, $\lambda_{\mathcal{T}}(\iota(F))$ and $\rho_{\mathcal{T}}(\iota(F))$ consist of $\Im(\{p_1, \ldots, p_n\})$-simple L-sequents of ranks $(k, 0)$ and $(0, l)$, respectively, such that

$$\begin{aligned}
\mathrm{Cn}^{(k,0)}_{\mathcal{M}}(\iota(F(p_1, \ldots, p_n)) \vdash\ ) &= \mathrm{Cn}^{(k,0)}_{\mathcal{M}}(\lambda_{\mathcal{T}}(\iota(F))) \qquad (1)\\
\mathrm{Cn}^{(0,l)}_{\mathcal{M}}(\ \vdash \iota(F(p_1, \ldots, p_n))) &= \mathrm{Cn}^{(0,l)}_{\mathcal{M}}(\rho_{\mathcal{T}}(\iota(F))) \qquad (2)
\end{aligned}$$

THEOREM 1. *There is an L-sequential $\Im$-table of rank (k, l) for $\mathcal{M}$.*

PROOF: Consider any $F \in L$ of arity $n \geq 1$ and any $\iota \in \Im$ such that the $(\Im, L)$-type $\iota(F)$ is $\Im$-complex. For every $a \in A$, set $\Im^+_a := \{\mu \in \Im \mid \mu^{\mathcal{A}}(a) \in D\}$ and $\Im^-_a := \{\mu \in \Im \mid \mu^{\mathcal{A}}(a) \notin D\}$. Choose arbitrary enumerations Θ^+_a and Θ^-_a of the sets $\Im^+_a$ and $\Im^-_a$, respectively. Since $\Im$ is an equality determinant for $\mathcal{M}$ and $\Im^+_a \cup \Im^-_a = \Im$ for all $a \in A$, we have

$$\forall a, b \in A : a = b \Leftrightarrow \forall \mu \in \Im^+_a : \mu^{\mathcal{A}}(b) \in D \text{ and } \forall \mu \in \Im^-_a : \mu^{\mathcal{A}}(b) \notin D. \quad (3)$$

Using (3), Proposition 1, the finiteness of A, the fact that $\Im^+_a \cap \Im^-_a = \emptyset$ for all $a \in A$ and the fact that $\Im(p) \cap \Im(q) = \emptyset$, it is easy to check that

$$\begin{aligned}
\lambda_{\mathcal{T}}(\iota(F)) &:= \{\Theta^+_{a_1}(p_1), \ldots, \Theta^+_{a_n}(p_n) \vdash \Theta^-_{a_1}(p_1), \ldots, \Theta^-_{a_n}(p_n) \mid \\
&\qquad a_1, \ldots, a_n \in A, \iota^{\mathcal{A}}(F^{\mathcal{A}}(a_1, \ldots, a_n)) \in D\}\\
\rho_{\mathcal{T}}(\iota(F)) &:= \{\Theta^+_{a_1}(p_1), \ldots, \Theta^+_{a_n}(p_n) \vdash \Theta^-_{a_1}(p_1), \ldots, \Theta^-_{a_n}(p_n) \mid \\
&\qquad a_1, \ldots, a_n \in A, \iota^{\mathcal{A}}(F^{\mathcal{A}}(a_1, \ldots, a_n)) \notin D\}
\end{aligned}$$

are finite sets of $\Im(\{p_1, \ldots, p_n\})$-simple L-sequents of ranks $(k, 0)$ and $(0, l)$, respectively, such that (1) and (2) hold. Thus, we get an L-sequential $\Im$-table $\mathcal{T}$ of rank (k, l) for $\mathcal{M}$. ■

Remark that the proof of Theorem 1 is constructive and provides an effective method of constructing an L-sequential $\Im$-table of rank (k,l) for $\mathcal{M}$ whenever L is finite.

From now on, we suppose that $\mathcal{T}$ is an arbitrary L-sequential $\Im$-table of rank (k,l) for $\mathcal{M}$.

Definition 1. $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ is the sequential L-calculus of rank (k,l) consisting of the following axioms and rules:

(i) all axioms of the form $\Gamma, \phi, \Theta \vdash \Delta, \phi, \Xi$, where $\Gamma, \Delta, \Theta, \Xi \in \Im(\mathrm{Var} \cup L_0)^*$ and $\phi \in \Im(\mathrm{Var})$;

(ii) all axioms of the form $\Gamma \vdash \Delta$, where $\Gamma, \Delta \in \Im(\mathrm{Var} \cup L_0)^*$, such that there are an $\Im$-simple $\Theta \vdash \Xi \in \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset)$ and a $v \in \mathrm{Var}$ such that $\Theta[p/v]$ ($\Xi[p/v]$) is a subsequence of Γ (Δ);

(iii) for each $c \in L_0$ and every $\iota \in \Im$ such that $\iota^{\mathcal{A}}(c^{\mathcal{A}}) \notin D$, all axioms of the form $\Gamma, \iota(c), \Theta \vdash \Delta$, where $\Gamma, \Theta, \Delta \in \Im(\mathrm{Var} \cup L_0)^*$ and $|\Delta| \geq l$;

(iv) for each $c \in L_0$ and every $\iota \in \Im$ such that $\iota^{\mathcal{A}}(c^{\mathcal{A}}) \in D$, all axioms of the form $\Gamma \vdash \Delta, \iota(c), \Xi$, where $\Gamma, \Delta, \Xi \in \Im(\mathrm{Var} \cup L_0)^*$ and $|\Gamma| \geq k$;

(v) for each $\Im$-complex $(\Im, L)$-type $\iota(F)$, where F is n-ary, all rules of the form

$$\frac{\{\Gamma, (\Gamma'[p_i/\varphi_i]_{1\leq i\leq n}), \Theta \vdash \Delta, (\Delta'[p_i/\varphi_i]_{1\leq i\leq n}) \mid \Gamma' \vdash \Delta' \in \lambda_{\mathcal{T}}(\iota(F))\}}{\Gamma, \iota(F(\varphi_1, \ldots, \varphi_n)), \Theta \vdash \Delta}$$

where $\varphi_1, \ldots, \varphi_n \in Fm_L$, $\Gamma, \Delta \in Fm^*_L$, $\Theta \in \Im(\mathrm{Var} \cup L_0)^*$ and $|\Delta| \geq l$, and of the form

$$\frac{\{\Gamma, (\Gamma'[p_i/\varphi_i]_{1\leq i\leq n}) \vdash \Delta, (\Delta'[p_i/\varphi_i]_{1\leq i\leq n}), \Xi \mid \Gamma' \vdash \Delta' \in \rho_{\mathcal{T}}(\iota(F))\}}{\Gamma \vdash \Delta, \iota(F(\varphi_1, \ldots, \varphi_n)), \Xi}$$

where $\varphi_1, \ldots, \varphi_n \in Fm_L$, $\Gamma, \Delta \in Fm^*_L$, $\Xi \in \Im(\mathrm{Var} \cup L_0)^*$ and $|\Gamma| \geq k$.

By $\widehat{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ ($\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$) denote the sequential L-calculus of rank (k,l) obtained from $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ by adding all sequential L-rules of rank (k,l) which are instances of Contraction (Cut, respectively), Enlargement and Exchange.

Theorem 2. $\mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset) = \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset)$.

PROOF: Clearly, every rule in $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ is satisfied in $\mathrm{Cn}^{(k,l)}_{\mathcal{M}}$. Therefore, by induction on the length of $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$-derivations from $\emptyset$, one can easily check that $\mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset) \subseteq \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset)$. The converse inclusion is proved by induction on the *degree* $\partial(\Gamma \vdash \Delta)$ of an arbitrary $\Gamma \vdash \Delta \in \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset)$ which is defined as follows. First, by induction on the length of L-formulas, define a mapping $\partial_{|\Im|} : Fm_L \to \omega$ setting, for each $\varphi \in \mathrm{Var} \cup L_0$, $\partial_{|\Im|}(\varphi) := 0$ and, for each $F \in L$ of arity $n \geq 1$ and all $\psi_1, \dots, \psi_n \in Fm_L$, $\partial_{|\Im|}(F(\psi_1, \dots, \psi_n)) := 1 + |\Im| \cdot \sum_{i=1}^{n} \partial_{|\Im|}(\psi_i)$. (Remark that $|\Im| \geq 1$, since $p \in \Im$. Therefore, for every $\varphi \in Fm_L$, $\partial_{|\Im|}(\varphi) = 0 \Leftrightarrow \varphi \in \mathrm{Var} \cup L_0$.) Next, we define a mapping $\partial^{\Im} : Fm_L \to \omega$ putting $\partial^{\Im}(\varphi) := \min\{\partial_{|\Im|}(\psi) \mid \psi \in Fm_L, \varphi = \iota(\psi), \iota \in \Im\}$ for all $\varphi \in Fm_L$. (Since $p \in \Im$, the minimum is taken over a non-empty set of natural numbers, and so $\partial^{\Im}(\varphi)$ is defined correctly. It is easy to see that $\partial^{\Im}(\varphi) = 0 \Leftrightarrow \varphi \in \Im(\mathrm{Var} \cup L_0)$.) Finally, define a mapping $\partial : \mathrm{Seq}^{(k,l)}_L \to \omega$ setting $\partial(\phi_1, \dots, \phi_m \vdash \psi_1, \dots, \psi_n) := \sum_{1 \leq i \leq m} \partial^{\Im}(\phi_i) + \sum_{1 \leq j \leq n} \partial^{\Im}(\psi_j)$ for all $\phi_1, \dots, \phi_m \vdash \psi_1, \dots, \psi_n \in \mathrm{Seq}^{(k,l)}_L$.

Assume $\partial(\Gamma \vdash \Delta) = 0$. Consider the following 4 cases:

1. Γ and Δ have a common L-formula in $\Im(\mathrm{Var})$.
   Then, by Definition 1(i), $\Gamma \vdash \Delta \in \mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset)$.
2. Γ contains an L-formula of the form $\iota(c)$, where $c \in L_0$, $\iota \in \Im$ and $\iota^{\mathcal{A}}(c^{\mathcal{A}}) \notin D$.
   Then, by Definition 1(iii), $\Gamma \vdash \Delta \in \mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset)$.
3. Δ contains an L-formula of the form $\iota(c)$, where $c \in L_0$, $\iota \in \Im$ and $\iota^{\mathcal{A}}(c^{\mathcal{A}}) \in D$.
   Then, by Definition 1(iv), $\Gamma \vdash \Delta \in \mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset)$.
4. Neither 1 nor 2 nor 3 holds.
   For each $v \in \mathrm{Var}$, by induction on the length of an arbitrary $\Theta \in \Im(\mathrm{Var} \cup L_0)^*$ define $\Theta{\downarrow}v \in \Im(v)^*$ putting $\emptyset{\downarrow}v := \emptyset$ and

$$\langle \Xi, \xi \rangle{\downarrow}v := \begin{cases} \langle \Xi{\downarrow}v, \xi \rangle & \text{if } \xi \in \Im(v) \text{ and } \xi \notin \Xi{\downarrow}v, \\ \Xi{\downarrow}v & \text{otherwise,} \end{cases}$$

   where $\xi \in \Im(\mathrm{Var} \cup L_0)$ and $\Xi \in \Im(\mathrm{Var} \cup L_0)^*$. By contradiction prove that there is some $v \in \mathrm{Var}$ such that $(\Gamma{\downarrow}v) \vdash (\Delta{\downarrow}v) \in \mathrm{Cn}^{(0,0)}_{\mathcal{M}}(\emptyset)$. Suppose that, for all $v \in \mathrm{Var}$, there is some $h_v \in \hom(\mathcal{F}m_L, \mathcal{A})$ such that $\Gamma{\downarrow}v \not\models^{h_v}_{\mathcal{M}} \Delta{\downarrow}v$. Define an $h \in \hom(\mathcal{F}m_L, \mathcal{A})$ setting $hv := h_v v$

for all $v \in \mathrm{Var}$. Then, $\Gamma \not\models^h_{\mathcal{M}} \Delta$. This contradicts the assumption $\Gamma \vdash \Delta \in \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset)$. Thus, there is some $v \in \mathrm{Var}$ such that $(\Gamma{\downarrow}v) \vdash (\Delta{\downarrow}v) \in \mathrm{Cn}^{(0,0)}_{\mathcal{M}}(\emptyset)$. Hence, $\Theta \vdash \Xi := (\Gamma{\downarrow}v)[v/p] \vdash (\Delta{\downarrow}v)[v/p] \in \mathrm{Cn}^{(0,0)}_{\mathcal{M}}(\emptyset)$. Applying Proposition 1 with $X = \emptyset$, we conclude that $\Theta \vdash \Xi \in \mathrm{Seq}^{(k,l)}$, and so $\Theta \vdash \Xi \in \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset)$. Moreover, $\Theta \vdash \Xi$ is $\Im$-simple. Finally, $\Theta[p/v] = \Gamma{\downarrow}v$ ($\Xi[p/v] = \Delta{\downarrow}v$) is a subsequence of Γ (Δ). Then, by Definition 1(ii), $\Gamma \vdash \Delta \in \mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset)$.

Now assume that $\partial(\Gamma \vdash \Delta) > 0$. Then, there is some $\xi \in Fm_L$ belonging to either Γ or Δ such that $\partial^{\Im}(\xi) > 0$. First, suppose that $\xi \in \Gamma$. Then, $\Gamma = (\Theta, \varphi, \Xi)$, where $\Theta \in Fm^*_L$, $\Xi \in \Im(\mathrm{Var} \cup L_0)^*$, $\varphi \in Fm_L$ and $\partial^{\Im}(\varphi) > 0$. By definition of $\partial^{\Im}$, there are some $\phi \in Fm_L$ and $\iota \in \Im$ such that $\varphi = \iota(\phi)$ and $\partial^{\Im}(\varphi) = \partial_{|\Im|}(\phi)$. Then, $\partial_{|\Im|}(\phi) > 0$, and so $\phi = F(\psi_1, \ldots, \psi_n)$, where $F \in L$, $n \geq 1$ is the arity of F and $\psi_1, \ldots, \psi_n \in Fm_L$. By contradiction prove that $\iota(F(p_1, \ldots, p_n)) \notin \Im(\mathrm{Var})$. Suppose that $\iota(F(p_1, \ldots, p_n)) \in \Im(\mathrm{Var})$. Then, $n = 1$. In that case $\iota(F(p)) \in \Im$, $\phi = F(\psi_1)$ and $\varphi = \iota(F(\psi_1))$. Hence, $\partial^{\Im}(\varphi) \leq \partial_{|\Im|}(\psi_1) < \partial_{|\Im|}(\phi) = \partial^{\Im}(\varphi)$. This contradiction shows that $\iota(F(p_1, \ldots, p_n)) \notin \Im(\mathrm{Var})$. Hence, the $(\Im, L)$-type $\iota(F)$ is $\Im$-complex. Take any $\Gamma' \vdash \Delta' \in \lambda_{\mathcal{T}}(\iota(F))$. By (1), we have $\Theta, (\Gamma'[p_i/\psi_i]_{1 \leq i \leq n}), \Xi \vdash \Delta, (\Delta'[p_i/\psi_i]_{1 \leq i \leq n}) \in \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset)$. Moreover,

$$\begin{aligned}
\partial(\Theta, (\Gamma'[p_i/\psi_i]_{1 \leq i \leq n}), \Xi \vdash \Delta, (\Delta'[p_i/\psi_i]_{1 \leq i \leq n})) &= \\
\partial(\Theta, \Xi \vdash \Delta) + \partial((\Gamma' \vdash \Delta')[p_i/\psi_i]_{1 \leq i \leq n}) &\leq \\
\partial(\Theta, \Xi \vdash \Delta) + \sum_{\iota \in \Im} \sum_{i=1}^{n} \partial^{\Im}(\iota(\psi_i)) &\leq \\
\partial(\Theta, \Xi \vdash \Delta) + |\Im| \cdot \sum_{i=1}^{n} \partial_{|\Im|}(\psi_i) &< \\
\partial(\Theta, \Xi \vdash \Delta) + \partial_{|\Im|}(\phi) &= \\
\partial(\Theta, \Xi \vdash \Delta) + \partial^{\Im}(\varphi) &= \partial(\Gamma \vdash \Delta).
\end{aligned}$$

By the induction hypothesis, $\Theta, (\Gamma'[p_i/\psi_i]_{1 \leq i \leq n}), \Xi \vdash \Delta, (\Delta'[p_i/\psi_i]_{1 \leq i \leq n}) \in \mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset)$ for all $\Gamma' \vdash \Delta' \in \lambda_{\mathcal{T}}(\iota(F))$. By Definition 1(v), this yields $\Gamma \vdash \Delta \in \mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset)$. The case $\xi \in \Delta$ is analyzed in a similar way but with using (2). ∎

Remark that the proof of Theorem 2 generalizing the proof of Theorem 3.2 of [10], which in its turn goes back to [9] and which has been used in [12], is constructive and, provided L is finite, gives an effective procedure which uses a given $\mathcal{T}$ and enables us to determine the derivability of sequents in $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ and (in case of the derivability) to search a derivation.

By Proposition 4.3 of [12] and Theorem 2, we get

COROLLARY 1. *Structural rules (including Contraction) are admissible in* $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$*, and so* $\mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset) = \mathrm{Cn}_{\widehat{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset) = \mathrm{Cn}_{\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}}(\emptyset)$*. In particular, Cut is admissible in* $\widehat{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ *and is eliminable from* $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$.

It is easy to check that the sequential consequence $\mathrm{Cn}_{\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}}$ is enlargable in the sense of p. 78 of [12]. Therefore, by Theorem 4.2 and Proposition 4.3 of [12], taking the finitarity of $\mathrm{Cn}_{\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}}$ (see p. 77 of [12]) and $\mathrm{Cn}^{(k,l)}_{\mathcal{M}}$ (see p. 79 of [12]) into account, Theorem 2 and Corollary 1 yield

COROLLARY 2. $\mathrm{Cn}_{\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}} = \mathrm{Cn}^{(k,l)}_{\mathcal{M}}$.

Since $\emptyset \neq D \neq A$, $\{\,\vdash\,; p \vdash\,;\, \vdash p\} \cap \mathrm{Cn}^{(k,l)}_{\mathcal{M}}(\emptyset) = \emptyset$. Therefore, $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ and $\widehat{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ consist of connective-introducing rules (see Subsection 3.6 of [11]) and some structural rules except for Cut. Then, by Theorem 3.30 of [11], we have

PROPOSITION 2. *Neither* $\mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}$ *nor* $\mathrm{Cn}_{\widehat{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}}$ *is algebraizable.*

Moreover, by Lemma 3.31 of [11], the instance $\dfrac{p \vdash q, p_1 \quad p_1, p \vdash q}{p \vdash q}$ of Cut is derivable neither in $\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ nor in $\widehat{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$. Hence, we get

PROPOSITION 3. $\mathrm{Cn}_{\widehat{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}} \neq \mathrm{Cn}_{\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}} \neq \mathrm{Cn}_{\mathcal{S}^{(k,l)}_{\mathcal{M},\mathcal{T}}}$.

Clearly, every functionally-complete finitely-valued logic having both distinguished and non-distinguished values (in particular, every finitely-valued logic of Post) has an equality determinant. (Notice that such a logic has neither 0- nor 1-subalgebras, and so $k = l = 0$.) Let us mention more denumerable classes of finitely-valued logics having an equality determinant.

Example 1. **(Two-valued logics)** Let $A := \{\mathsf{f}, \mathsf{t}\}$, where f and t are different truth values, and $D := \{\mathsf{t}\}$. Then, $\Im := \{p\}$ is an equality determinant for $\mathcal{M}$.

Example 1 covers the classical logic and its fragments.

Example 2. **(Three- and four-valued logics with constructive negation)** Suppose that L contains a unary connective $\neg$ treated as *constructive negation.* Let $\{\mathsf{f}, \mathsf{t}\} \subset A \subseteq \{\mathsf{f}, \mathsf{t}, \mathsf{n}, \mathsf{b}\}$, where $\mathsf{f}, \mathsf{t}, \mathsf{n}, \mathsf{b}$ are different truth values, $D := A \cap \{\mathsf{t}, \mathsf{b}\}$, $\neg\mathsf{f} \in D$, $\neg\mathsf{t} \notin D$, $\mathsf{n} \in A \Rightarrow \neg\mathsf{n} \notin D$ and $\mathsf{b} \in A \Rightarrow \neg\mathsf{b} \in D$. Then, $\Im := \{p, \neg p\}$ is an equality determinant for $\mathcal{M}$.

Example 2 covers, in particular, 12 four-valued logics studied in [12], [11], [10][1], including the logic of first-degree entailments in the relevance system [4], [5] known also as Dunn-Belnap's four-valued logic [4], [1], as well as three-valued logics studied in [11], [9], including the logic of first-degree entailments (in Dunn's sense [4], [5]) in Dummett's *LC* [2] (cf. Definition 4.80 of [11] and the paragraph after it). Notice that the calculi introduced in [12], [11], [10], [9] can be constructed upon the basis of some sequential tables with using Definition 1. And what is more, Theorem 2, Corollaries 1 and 2 as well as Propositions 2 and 3 generalize corresponding results of [12], [11], [10], [9]. As a one more logic covered by Example 2, we should like to highlight Dunn's *RM*3 [3], for which no appropriate cut-free Gentzen-style calculus has been known until the present paper.

Example 3. **(Finitely-valued Łukasiewicz logics)** [7] Let $L := \{\wedge, \vee, \supset, \neg\}$, where $\wedge$ (conjunction), $\vee$ (disjunction) and $\supset$ (implication) are binary infix connectives and $\neg$ (negation) is a unary connective, $n \geq 2$, $A := \{\frac{i}{n-1} \mid i < n\}$, $D := \{1\}$, $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$, $a \supset b := \min(1, 1 - a + b)$ and $\neg a := 1 - a$ for all $a, b \in A$. In case $n \geq 4$, $\mathcal{M}$ falls into neither Example 1 nor Example 2. Nevertheless, $\mathcal{M}$ has an equality determinant consisting of $n - 1$ elements. Take any $0 < i < n - 1$. Consider the function $f_i : [0, 1] \to \mathbf{R}$ given by $f_i(x) := (n - 1)x - (i - 1)$ for all $x \in [0, 1]$. Then, by McNaughton's Lemma [8], there is some $\gamma_i(p) \in Fm_L$ such that $\gamma_i^{\mathcal{A}}(a) = \min(\max(f_i(a), 0), 1)$ for all $a \in A$. Remark that, for all $j < n$, $\gamma_i^{\mathcal{A}}(\frac{j}{n-1}) \in D \Leftrightarrow i \leq j$. Finally, it is easy to check that $\Im := \{p\} \cup \{\gamma_i : 0 < i < n - 1\}$ is an equality determinant for $\mathcal{M}$ con-

[1] In [12] the notations $\sim, 0, 1, \bot, \top$ are used instead of $\neg, \mathsf{f}, \mathsf{t}, \mathsf{n}, \mathsf{b}$, respectively.

sisting of $n-1$ elements. Notice that $\mathcal{M}$ has neither 0- nor 1-subalgebras. Therefore, $k = l = 0$.

EXAMPLE 4. As logics with equality determinant covered by neither Example 1 nor Example 2 nor Example 3, we should like to highlight arbitrary fragments and expansions of the 16-valued trilattice logic suggested by Dunn, et al., in [6] with $D = \{a \in A \mid \mathbf{T} \in a\}$, $L \supseteq \{\sim_t, \sim_c\}$ and $\Im = \{p, \sim_t p, \sim_c p, \sim_t \sim_c p\}$.

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