On the Ruelle eigenvalue sequence

Oscar Bandtlow and Oliver Jenkinson

Abstract. For certain real analytic data, we show that the eigenvalue sequence of the associated transfer operator $\mathcal{L}$ is insensitive to the holomorphic function space on which $\mathcal{L}$ acts. Explicit bounds on this eigenvalue sequence are established.

1. Introduction

For compact $X \subset \mathbb{C}^d$, and appropriate real analytic $T_i : X \to X$ and $w_i : X \to \mathbb{C}$, Ruelle [**Rue1**] considered the action of the transfer operator $\mathcal{L}f := \sum_i w_i \cdot f \circ T_i$ on $U(D)$, where D is a common domain of holomorphy for the T_i and w_i, and $U(D)$ consists of those holomorphic functions on D which extend continuously to $\bar{D}$. Ruelle proved that $\mathcal{L} : U(D) \to U(D)$ is nuclear, hence in particular compact, and that its eigenvalue sequence $\{\lambda_n(\mathcal{L})\}_{n=1}^\infty$, henceforth referred to as the *Ruelle eigenvalue sequence*, is given by the reciprocals of the zeros of a dynamical determinant Δ (see (9) for the definition).

In view of its various interpretations and applications (e.g. correlation decay rates [**Bal, CPR**], Fourier resonances [**Rue2**], Laplacians for hyperbolic surfaces [**Pol1, PR**], Feigenbaum period-doubling [**AAC, CCR, JMS, Pol2**]), it is desirable to establish explicit bounds on the Ruelle eigenvalue sequence. In the case where D may be chosen as a ball, and the T_i all map D within the concentric ball whose radius is $r < 1$ times that of D, we establish (Theorem 3.2) the stretched-exponential bound

$$|\lambda_n(\mathcal{L})| < \frac{W}{r^d} n^{1/2} \, r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}} \quad \text{for all } n \geq 1\,, \tag{1}$$

where $W := \sup_{z \in D} \sum_i |w_i(z)|$.

We go on to investigate properties of transfer operators acting on other spaces of holomorphic functions, and prove (Theorem 4.2) that the Ruelle eigenvalue sequence is in a sense *universal*: for a wide range of domains of holomorphy D, and a broad class of spaces $A(D)$ of holomorphic functions on D, the eigenvalue sequence of $\mathcal{L} : A(D) \to A(D)$ is precisely the Ruelle eigenvalue sequence. This universality suggests the possibility of sharpening the estimate (1), by adapting the proof of Theorem 3.2 to some other space $A(D)$. In particular, the choice of $A(D)$ as the Hardy space $H^2(D)$ is known to yield a concrete eigenvalue bound for $\mathcal{L} : H^2(D) \to H^2(D)$ (see [**BJ**]). Intriguingly, this bound turns out to be *complementary* to (1): in every dimension d, and for every $r < 1$, (1) is superior for sufficiently small n, while the Hardy space bound is superior for sufficiently large n. If $N(r,d)$ denotes the integer such that (1) gives the sharper bound on $|\lambda_n(\mathcal{L})|$ precisely for $1 \leq n \leq N(r,d)$, then both $r \mapsto N(r,d)$ and $d \mapsto N(r,d)$ are increasing (cf. Corollary 4.4, Remark 4.5); in other words, (1) is more useful if the T_i are weakly contracting, or if the ambient dimension is high.

2. Transfer operators on favourable spaces of holomorphic functions

2.1. Notation. Let $\mathbb{N}$ denote the set of strictly positive integers, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$, equip $\mathbb{C}^d$ with the Euclidean inner product $(\cdot,\cdot)_{\mathbb{C}^d}$, the corresponding norm $\|\cdot\|_{\mathbb{C}^d}$, and the induced Euclidean metric, denoted δ. For $X \subset \mathbb{C}^d$ we use $\Delta_\varepsilon(X) = \{z \in \mathbb{C}^d \mid \delta(z,X) < \varepsilon\}$ for the Euclidean ε-neighbourhood of X. The set of all bounded domains (non-empty connected open subsets) in $\mathbb{C}^d$ will be denoted by $\mathcal{D}_d$. For two bounded open sets $\Delta_1, \Delta_2 \subset \mathbb{C}^d$ we write $\Delta_1 \subset\subset \Delta_2$ to mean that $\overline{\Delta_1} \subset \Delta_2$.

Let $B = (B, \|\cdot\|_B)$ be a Banach space. We often write $\|\cdot\|$ instead of $\|\cdot\|_B$ whenever this does not lead to confusion. For $X \subset \mathbb{C}^d$ compact and $D \in \mathcal{D}_d$ define

$$\begin{aligned} Hol(D,B) &:= \{f : D \to B \mid f \text{ holomorphic}\} \\ C(X,B) &:= \{f : X \to B \mid f \text{ continuous}\}, \ \|f\|_{C(X,B)} := \sup_{x\in X} \|f(x)\|_B \\ U(D,B) &:= \left\{f : \overline{D} \to B \mid f \in C(\overline{D},B) \cap Hol(D,B)\right\}, \ \|f\|_{U(D,B)} := \sup_{z\in\overline{D}} \|f(z)\|_B \,. \end{aligned}$$

Note that $C(X,B)$ and $U(D,B)$ are Banach spaces when equipped with the indicated norms, while $Hol(D,B)$ is a Fréchet space when equipped with the topology of uniform convergence on compact subsets of D. If $(B, \|\cdot\|) = (\mathbb{C}, |\cdot|)$ then we use $C(X)$, $Hol(D)$, and $U(D)$ to denote $C(X,\mathbb{C})$, $Hol(D,\mathbb{C})$, and $U(D,\mathbb{C})$ respectively.

We use $L(B)$ to denote the space of bounded linear operators from a Banach space $(B, \|\cdot\|)$ to itself, always equipped with the induced operator norm.

If T is holomorphic on some $D \in \mathcal{D}_d$, its derivative at $z \in D$ is denoted by $T'(z)$.

2.2. Definition. Let $\mathcal{I}$ be a non-empty countable set. For $D \in \mathcal{D}_d$, a collection $(T_i)_{i\in\mathcal{I}} = (T_i, D)_{i\in\mathcal{I}}$ of holomorphic maps $T_i \in U(D, \mathbb{C}^d)$ is called a *holomorphic map system* (on D) if $\cup_{i\in\mathcal{I}} T_i(D) \subset\subset D$.

Write $T_{\underline{i}} := T_{i_n} \circ \cdots \circ T_{i_1}$ for $\underline{i} = (i_1, \ldots, i_n) \in \mathcal{I}^n$, $n \in \mathbb{N}$.

For $X \subset \mathbb{C}^d$ compact, a collection $(T_i)_{i\in\mathcal{I}} = (T_i, X)_{i\in\mathcal{I}}$ of maps $T_i : X \to X$ is a C^ω *map system* (on X) if there exists $D \in \mathcal{D}_d$ with $X \subset\subset D$ such that each T_i extends holomorphically to D and $(T_i, D)_{i\in\mathcal{I}}$ is a holomorphic map system. Any such D is called *admissible* for the C^ω map system $(T_i, X)_{i\in\mathcal{I}}$.

For $n \in \mathbb{N}$, a C^ω map system $(T_i, X)_{i\in\mathcal{I}}$ is called *complex n-contracting* (or simply *complex contracting*) if there exists $D \in \mathcal{D}_d$ with $X \subset\subset D$, such that $T'_{\underline{i}} \in U(D, L(\mathbb{C}^d))$ for every $\underline{i} \in \mathcal{I}^n$ and

$$\sup_{\underline{i}\in\mathcal{I}^n} \left\|T'_{\underline{i}}\right\|_{U(D,L(\mathbb{C}^d))} < 1 \,. \tag{2}$$

Note that if $\mathcal{I}$ is finite then (2) is implied by the condition $\sup_{\underline{i}\in\mathcal{I}^n} \|T'_{\underline{i}}\|_{C(X,L(\mathbb{C}^d))} < 1$.

2.3. Example. If $X = [0,1] \subset \mathbb{C}$, define the *Gauss map system* $(T_i)_{i\in\mathbb{N}}$ by $T_i(x) = 1/(i+x)$ (the T_i are the inverse branches to the Gauss map $x \mapsto 1/x \pmod 1$ on X). This is a C^ω map system on X: for example if $D \subset \mathbb{C}$ is the open disc of radius $3/2$ centred at the point 1 then $(T_i, D)_{i\in\mathcal{I}}$ is a holomorphic map system. The system is also complex contracting, because $\sup_{\underline{i}\in\mathcal{I}^2} \|T'_{\underline{i}}\|_{U(D)} = |T'_{(1,1)}(-1/2)| = 4/9 < 1$ (note we cannot choose $n = 1$ in (2), because $T'_1(0) = -1$).

Complex contraction guarantees the existence of an admissible domain, and this domain may be chosen arbitrarily close to X:

2.4. Lemma. *If a C^ω map system on X is complex contracting then there exists a family $\{D_\theta\}_{\theta\in(0,\Theta)}$ of admissible domains, such that $\cap_{\theta\in(0,\Theta)} D_\theta = X$.*

Proof. Let $(T_i)_{i\in\mathcal{I}}$ denote the C^ω map system on X. Choose $n \in \mathbb{N}$ and $D \in \mathcal{D}_d$ such that $\gamma := \sup_{\underline{i}\in\mathcal{I}^n} \|T'_{\underline{i}}\|_{U(D,L(\mathbb{C}^d))} < 1$. From the several variables mean value theorem [**Ave**, Thm. 2.3], for each $\underline{i} \in \mathcal{I}^n$, the map $T_{\underline{i}}$ is γ-Lipschitz, with respect to Euclidean distance δ, on any convex subset of D. Now set $\beta := \gamma^{1/n} < 1$, and define the distance

$$\mathrm{dist}(x,y) = \sup_{\underline{i}\in\mathcal{I}^{n-1}} \sum_{k=0}^{n-1} \beta^{n-1-k}\delta(T_{P_k(\underline{i})}(x), T_{P_k(\underline{i})}(y))\,,$$

where for $1 \le k \le n-1$, $P_k : \mathcal{I}^{n-1} \to \mathcal{I}^k$ denotes the projection $P_k\underline{i} = (i_1,\dots,i_k)$ onto the first k coordinates, with the convention that $T_{P_0\underline{i}} = \mathrm{id}$. Note that for each $i \in \mathcal{I}$, the map T_i is β-Lipschitz, with respect to dist, on any convex subset of D. Moreover, dist and δ generate the same topology on δ-compact subsets of D. To see this observe that on the one hand we clearly have $\delta(x,y) \le \beta^{1-n}\mathrm{dist}(x,y)$ for every $x,y \in D$. On the other hand, if K is a δ-compact convex subset of D, then by Cauchy's theorem there is $C > 0$ such that $\|T'_{\underline{i}}\|_{C(K,L(\mathbb{C}^d))} \le C$ for every $\underline{i} \in \mathcal{I}^k$, $1 \le k \le n-1$. Thus by the mean value theorem $\mathrm{dist}(x,y) \le C\sum_{k=0}^{n-1}\beta^{n-1-k}\delta(x,y)$ for every $x,y \in K$.

Since X is compactly contained in the domain D, there exists $\varepsilon > 0$ such that the Euclidean neighbourhood $\Delta_\varepsilon(X)$ is contained in D. Setting $\Theta := \varepsilon\beta^{n-1}$, we see that $D_\theta := \left\{z \in \mathbb{C}^d \,|\, \mathrm{dist}(z,X) < \theta\right\} \subset \Delta_\varepsilon(X)$ for all $\theta \in (0,\Theta]$, and that $\cap_{\theta\in(0,\Theta]}D_\theta = X$. If $z \in D_\theta$, and $x \in X$ satisfies $\mathrm{dist}(z,X) = \mathrm{dist}(z,x)$, then $x,z \in \Delta_\varepsilon(x)$, a convex subset of D, so $\mathrm{dist}(T_i(z),T_i(x)) \le \beta\,\mathrm{dist}(z,x)$. Therefore $\mathrm{dist}(T_i(z),X) \le \mathrm{dist}(T_i(z),T_i(x)) \le \beta\,\mathrm{dist}(z,x) = \beta\,\mathrm{dist}(z,X)$, and hence $\cup_{i\in\mathcal{I}}T_i(D_\theta) \subset D_{\beta\theta} \Subset D_\theta$, so D_θ is admissible for $(T_i,X)_{i\in\mathcal{I}}$. □

2.5. Remark. The above proof shows that if a C^ω map system on X is complex 1-contracting then all sufficiently small Euclidean ε-neighbourhoods $\Delta_\varepsilon(X)$ are admissible. This is not the case for the Gauss map system $T_i(z) = 1/(z+i)$ on $X = [0,1] \subset \mathbb{C}$: no Euclidean ε-neighbourhood is admissible, since $\delta(T_1(-\varepsilon),X) > \varepsilon$.

2.6. Definition. Let $\mathcal{I}$ be a non-empty countable set. A *holomorphic weight system* on $D \in \mathcal{D}_d$ is a collection $(w_i)_{i\in\mathcal{I}} = (w_i,D)_{i\in\mathcal{I}}$ of holomorphic functions (called *weight functions*) $w_i \in U(D)$ such that $\sum_{i\in\mathcal{I}}\|w_i\|_{U(D)} < \infty$.

For $X \subset \mathbb{C}^d$ compact, a collection $(w_i)_{i\in\mathcal{I}} = (w_i,X)_{i\in\mathcal{I}}$ of maps $w_i : X \to \mathbb{C}$ is a C^ω *weight system* (on X) if there exists $D \in \mathcal{D}_d$ with $X \Subset D$ such that $(w_i,D)_{i\in\mathcal{I}}$ is a holomorphic weight system. Any such D is called *admissible* for $(w_i,X)_{i\in\mathcal{I}}$.

If $(T_i)_{i\in\mathcal{I}}$ is a holomorphic (respectively, C^ω) map system and $(w_i)_{i\in\mathcal{I}}$ is a holomorphic (respectively, C^ω) weight system then $(T_i,w_i)_{i\in\mathcal{I}}$ is called a *holomorphic (respectively, C^ω) map-weight system*. A domain $D \in \mathcal{D}_d$ is called *admissible* for a C^ω map-weight system $(T_i,w_i)_{i\in\mathcal{I}}$ if it is admissible for both $(T_i,X)_{i\in\mathcal{I}}$ and $(w_i,X)_{i\in\mathcal{I}}$.

With each holomorphic map-weight system $(T_i,w_i)_{i\in\mathcal{I}}$ we associate a linear operator,

$$\mathcal{L}f = \sum_{i\in\mathcal{I}} w_i \cdot f\circ T_i\,, \tag{3}$$

called the *transfer operator*. It will be seen that the transfer operator $\mathcal{L}$ preserves, and acts compactly upon, the following class of spaces of holomorphic functions.

2.7. Definition. For $D \in \mathcal{D}_d$, a Banach space $A = A(D)$ of functions $f : D \to \mathbb{C}$ holomorphic on D is called a *favourable space of holomorphic functions (on D)* if
(i) for each $z \in D$, the point evaluation functional $f \mapsto f(z)$ is continuous on A, and
(ii) A contains $U(D)$, and the natural embedding[1] $U(D) \hookrightarrow A$ has norm 1.

2.8. Remark. Let $D \in \mathcal{D}_d$. Then $U(D)$ is trivially a favourable space of holomorphic functions. Other examples include, for $p \in [1,\infty]$, *Bergman spaces* $L^p_{Hol}(D)$ (see [**Ran**, Ch. I, Cor. 1.7, 1.10]) and *Hardy spaces* $H^p(D)$ (see [**Kra**, Ch. 8.3]). If $p = 2$ and D has C^2 boundary, then $H^2(D)$ can be identified with the $L^2(\partial D,\sigma)$-closure of $U(D)$, where σ

[1] The embedding $U(D) \hookrightarrow A$ is automatically continuous: continuity of point evaluation on both A and $U(D)$ implies that it has closed graph; cf. the proof of Lemma 2.9.

denotes $(2d-1)$-dimensional Lebesgue on the boundary ∂D, normalised so that $\sigma(\partial D) = 1$. In particular, $H^2(D)$ is a Hilbert space with inner product given by $(f, g) = \int_{\partial D} f^* \overline{g^*}\, d\sigma$, where, for $h \in H^2(D)$, the symbol h^* denotes the corresponding nontangential limit function in $L^2(\partial D, \sigma)$ — see [**Kra**, Ch. 1.5 and 8].

Recall (see e.g. [**Pie**, 1.7.1]) that a linear operator $L : B \to B$ on a Banach space[2] B is *p-nuclear* if there exist sequences $b_i \in B$ and $l_i \in B^*$ (the strong dual of B) with $\sum_i (\|b_i\|\,\|l_i\|)^p < \infty$, such that $L(b) = \sum_{i=1}^\infty l_i(b) b_i$ for all $b \in B$. The operator is *strongly nuclear* (or *nuclear of order zero*) if it is p-nuclear for every $p > 0$. It turns out that certain natural embeddings between favourable spaces are strongly nuclear:

2.9. Lemma. *Let D and D' be domains in $\mathbb{C}^d$ such that $D' \subset\subset D$. Let A and A' be favourable Banach spaces of holomorphic functions on D and D' respectively. Then $A \subset A'$, and the natural embedding $J : A \hookrightarrow A'$, defined by $Jf = f|_{D'}$, is strongly nuclear.*

Proof. Choose $D'' \in \mathcal{D}_d$ with $D' \subset\subset D'' \subset\subset D$, and consider the natural embeddings

$$A \overset{J_1}{\hookrightarrow} Hol(D'') \overset{J_2}{\hookrightarrow} U(D') \overset{J_3}{\hookrightarrow} A'\,.$$

Clearly $J = J_3 J_2 J_1$. The unit ball of $U(D')$ is a neighbourhood in $Hol(D'')$, so the map J_2 is bounded. But the Fréchet space $Hol(D'')$ is nuclear [**Gro**, II, Cor., p. 56], so J_2 is p-nuclear for every $p > 0$ by [**Gro**, II, Cor. 4, p. 39, Cor. 2, p. 61]. It thus suffices to show that J_1 and J_3 are continuous by [**Gro**, I, p. 84, II, p. 9].

Now, J_3 is continuous since A' is favourable. Finally, to see that J_1 is continuous we note that, by the closed graph theorem (see e.g. [**Scha**, Ch. III, 2.3]), it is enough to show that if $f_n \to f$ in A, and $J_1 f_n \to g$ in $Hol(D'')$, then $g = J_1 f = f|_{D''}$. Since point evaluation is continuous on A, $f_n(z) \to f(z)$ for all $z \in D$ and in particular for all $z \in D''$. But point evaluation is also continuous on $Hol(D'')$, so $f_n(z) = J_1 f_n(z) \to g(z)$ as $n \to \infty$ for all $z \in D''$. Therefore $g = f|_{D''}$. □

Favourable spaces A are always invariant under the transfer operator $\mathcal{L}$, and the restricted operator (henceforth denoted by $\mathcal{L}_A$) is always compact, indeed strongly nuclear:

2.10. Proposition. *Let $(T_i, w_i, D)_{i\in\mathcal{I}}$ be a holomorphic map-weight system. The corresponding transfer operator leaves invariant every favourable space A of holomorphic functions on D, and $\mathcal{L}_A : A \to A$ is strongly nuclear.*

Proof. Choose $D' \in \mathcal{D}_d$ with $\cup_{i\in\mathcal{I}} T_i(D) \subset\subset D' \subset\subset D$. We claim that $\hat{\mathcal{L}} f := \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i$ defines a continuous operator $\hat{\mathcal{L}} : U(D') \to U(D)$. To see this, fix $f \in U(D')$ and note that $w_i \cdot f \circ T_i \in U(D)$ with $\|w_i \cdot f \circ T_i\|_{U(D)} \le \|w_i\|_{U(D)} \|f\|_{U(D')}$ for every $i \in \mathcal{I}$. But since $\|\hat{\mathcal{L}} f\|_{U(D)} \le \sum_{i\in\mathcal{I}} \|w_i\|_{U(D)} \|f\|_{U(D)}$ and $\sum_{i\in\mathcal{I}} \|w_i\|_{U(D)} < \infty$ by hypothesis, we conclude that $\hat{\mathcal{L}} f \in U(D)$ and that $\hat{\mathcal{L}}$ is continuous.

Since A is favourable, the embedding $\hat{J} : U(D) \hookrightarrow A$ is continuous, and $J : A \hookrightarrow U(D')$ is p-nuclear for every $p > 0$ by Lemma 2.9. Moreover, if $f \in A$ then $\mathcal{L} f = \hat{J}\hat{\mathcal{L}} J f \in A$. Thus A is $\mathcal{L}$-invariant, and the operator $\mathcal{L}_A = \hat{J}\hat{\mathcal{L}} J$ is p-nuclear for any $p > 0$. □

2.11. Remark. Strong nuclearity of the transfer operator on spaces of holomorphic functions is not new (the original result of this kind is [**Rue1**], but see also e.g. [**GLZ, JP, May3**]); the novelty of Proposition 2.10 is in the breadth of spaces covered.[3]

[2] See [**Gro**, II, Déf. 1, p. 3] for the generalisation to locally convex spaces.

[3] Actually the result can be further extended to certain locally convex spaces of holomorphic functions, including $Hol(D)$.

3. Eigenvalue bounds

For favourable A, the compactness of $\mathcal{L}_A$ means its spectrum consists of a countable set of eigenvalues, each with finite algebraic multiplicity, together with a possible accumulation point at 0. We wish to obtain bounds on the *eigenvalue sequence* $\lambda(\mathcal{L}_A) := \{\lambda_n(\mathcal{L}_A)\}_{n=1}^\infty$, i.e. the sequence of all eigenvalues of $\mathcal{L}_A$ counting algebraic multiplicities and ordered by decreasing modulus.[4]

If $L : B_1 \to B_2$ is a continuous operator between Banach spaces then for $k \geq 1$, its *k-th approximation number* $a_k(L)$ is defined as

$$a_k(L) = \inf\{\|L-K\| \mid K : B_1 \to B_2 \text{ linear with } \operatorname{rank}(K) < k\}\ .$$

3.1. Proposition. *For a C^ω map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$ such that $(T_i)_{i\in\mathcal{I}}$ is complex contracting, and a favourable space $A = A(D)$ such that $D \in \mathcal{D}_d$ is admissible,*

$$|\lambda_n(\mathcal{L}_A)| \leq W n^{1/2} \prod_{k=1}^n a_k(J)^{1/n} \quad \textit{for all } n \geq 1\,, \tag{4}$$

where $W := \sup_{z\in D}\sum_{i\in\mathcal{I}} |w_i(z)|$, $D' \in \mathcal{D}_d$ is such that $\cup_{i\in\mathcal{I}} T_i(D) \subset D' \subset\subset D$, and $J : A(D) \hookrightarrow U(D')$ is the canonical embedding.

Proof. Since $A(D)$ is favourable, the embedding $\hat{J} : U(D) \hookrightarrow A(D)$ is continuous of norm 1. Observe that $\hat{\mathcal{L}} f = \sum_{i\in\mathcal{I}} w_i \cdot f \circ T_i$ defines a continuous operator $\hat{\mathcal{L}} : U(D') \to U(D)$ (see the proof of Proposition 2.10) with $\|\hat{\mathcal{L}}\| \leq W$. To see the latter note that for $f \in U(D')$ we have $|f(T_i(z))| \leq \|f\|_{U(D)}$ for every $z \in D$, $i \in \mathcal{I}$; thus by the maximum principle $\|\hat{\mathcal{L}} f\|_{U(D)} = \sup_{z\in D} |(\hat{\mathcal{L}} f)(z)| \leq \sup_{z\in D} \sum_{i\in\mathcal{I}} |w_i(z)|\ |f(T_i(z))| \leq W\,\|f\|_{U(D)}$.

Now clearly $\mathcal{L}_A = \hat{J}\hat{\mathcal{L}} J$, so

$$a_k(\mathcal{L}_A) \leq \|\hat{J}\hat{\mathcal{L}}\| a_k(J) \leq W a_k(J) \quad \text{for all } k \geq 1\,, \tag{5}$$

since in general $a_k(L_1L_2) \leq \|L_1\|\, a_k(L_2)$ whenever L_1 and L_2 are bounded operators between Banach spaces (see [**Pie**, 2.2]). Moreover, since $\mathcal{L}_A$ is compact, Weyl's inequality[5] (see e.g. [**Hin**]) asserts that $\prod_{k=1}^n |\lambda_k(\mathcal{L}_A)| \leq n^{n/2} \prod_{k=1}^n a_k(\mathcal{L}_A)$ for every $n \in \mathbb{N}$. Together with (5) this yields (4), because $|\lambda_n(\mathcal{L}_A)| \leq \prod_{k=1}^n |\lambda_k(\mathcal{L}_A)|^{1/n}$. □

Taking $A(D) = U(D)$, the *Ruelle eigenvalue sequence* $\lambda(\mathcal{L}_{U(D)})$ can be bounded as follows:

3.2. Theorem. *Suppose the Euclidean ball $D \subset \mathbb{C}^d$ is an admissible domain for a C^ω map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$, and that $\cup_{i\in\mathcal{I}} T_i(D)$ is contained in the concentric ball whose radius is $r < 1$ times that of D. Setting $W := \sup_{z\in B}\sum_{i\in\mathcal{I}} |w_i(z)|$, the Ruelle eigenvalue sequence $\lambda(\mathcal{L}_{U(D)})$ can be bounded by*

$$|\lambda_n(\mathcal{L}_{U(D)})| < \frac{W}{r^d} n^{1/2}\ r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}} \quad \textit{for all } n \geq 1\,. \tag{6}$$

If $d = 1$ then

$$|\lambda_n(\mathcal{L}_{U(D)})| \leq W n^{1/2}\ r^{(n-1)/2} \quad \textit{for all } n \geq 1\,. \tag{7}$$

Proof. Without loss of generality let $D = D_1$ be the open unit ball, and let the smaller concentric ball be D_r, the ball of radius r centred at 0. Let $J : U(D_1) \hookrightarrow U(D_r)$ be the canonical embedding. From [**Far**, Prop. 2.1 (a)] it follows that $a_l(J) \leq r^{t_l}$, where $t_l := k$ for $\binom{k-1+d}{d} < l \leq \binom{k+d}{d}$, hence $\prod_{l=1}^n a_l(J)^{1/n} \leq r^{\frac{1}{n}\sum_{l=1}^n t_l}$. If $d = 1$ then $\frac{1}{n}\sum_{l=1}^n t_l =$

[4]By convention distinct eigenvalues with the same modulus can be written in any order (see e.g. [**Pie**, 3.2.20]).

[5]This is a Banach space version of Weyl's original inequality in Hilbert space (see [**Wey**]).

$\frac{1}{n}\sum_{l=1}^n(l-1)=(n-1)/2$, and (7) follows from (4). More generally $t_l \geq (d!)^{1/d}l^{1/d}-d$, so that

$$\frac{1}{n}\sum_{l=1}^n t_l \geq -d+(d!)^{1/d}\frac{1}{n}\sum_{l=1}^n l^{1/d} > -d+(d!)^{1/d}\frac{d}{d+1}n^{1/d}$$

using the estimate $\sum_{l=1}^n l^{1/d} > \int_{x=0}^n x^{1/d} = \frac{d}{d+1}n^{1+1/d}$, and (6) follows from (4). □

4. Universality of the Ruelle eigenvalue sequence

If $(T_i, w_i)_{i\in\mathcal{I}}$ is a C^ω map-weight system with complex contracting $(T_i)_{i\in\mathcal{I}}$ then, in view of Lemma 2.4 and Proposition 2.10, there is some freedom in the choice of an admissible D, and a favourable space $A = A(D)$ on which to consider the transfer operator $\mathcal{L}_A$. The purpose of this section is to show that the eigenvalue sequence of $\mathcal{L}_A$ is in fact independent of A: it is always equal to the Ruelle eigenvalue sequence $\lambda(\mathcal{L}_{U(D)})$ (see Corollary 4.3). For this, we first require some facts from the Fredholm theory originally developed by Grothendieck [**Gro**]. If B is a Banach space, we denote by $N_p(B)$ $(p>0)$ the quasi-Banach operator ideal of p-nuclear operators on B (cf. [**Pie**, D.1.4, 1.7.1]). If $p \leq 2/3$ then $N_p(B)$ admits a unique continuous trace τ and a unique continuous determinant det (see [**Pie**, 1.7.13, 4.7.8, 4.7.11]), related for a fixed $L \in N_p(B)$ by

$$\det(I-zL) = \exp\left(-\sum_{n=1}^\infty \frac{z^n}{n}\tau(L^n)\right), \tag{8}$$

for all $z \in \mathbb{C}$ in a suitable neighbourhood of 0 (see [**Pie**, 4.6.2]). Moreover, both τ and det are spectral, which means that $\tau(L) = \sum_{n=1}^\infty \lambda_n(L)$ and that, counting multiplicities, the zeros of the entire function $z \mapsto \det(I-zL)$ are precisely the reciprocals of the eigenvalues of L (see [**Pie**, 4.7.14, 4.7.15]).

4.1. Definition. To any holomorphic map-weight system $(T_i, w_i)_{i\in\mathcal{I}}$, the associated *dynamical determinant* is the entire function $\Delta : \mathbb{C} \to \mathbb{C}$, defined for all z of sufficiently small modulus by

$$\Delta(z) = \exp\left(-\sum_{n\in\mathbb{N}}\frac{z^n}{n}\sum_{\underline{i}\in\mathcal{I}^n}\frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I-T'_{\underline{i}}(z_{\underline{i}}))}\right), \tag{9}$$

where $w_{\underline{i}} := \prod_{k=1}^n w_{i_k}\circ T_{P_{k-1}\underline{i}}$, $P_k : \mathcal{I}^n \to \mathcal{I}^k$ denotes the projection $P_k\underline{i} = (i_1, \ldots, i_k)$ with the convention that $T_{P_0\underline{i}} = \mathrm{id}$, and $z_{\underline{i}}$ denotes the (unique, by [**EH**]) fixed-point of $T_{\underline{i}}$ in D.

Ruelle [**Rue1**] showed that Δ is the determinant of the strongly nuclear operator $\mathcal{L} : U(D) \to U(D)$. Therefore, if the zeros $z_1, z_2, \ldots$ of Δ are listed according to increasing modulus and counting multiplicity, then the reciprocal sequence $\{z_n^{-1}\}_{n=1}^\infty$ is precisely the Ruelle eigenvalue sequence.

4.2. Theorem. *Let $(T_i, w_i, D)_{i\in\mathcal{I}}$ be a holomorphic map-weight system. Then the associated transfer operator preserves every favourable space of holomorphic functions on D, and its determinant on each of these spaces is precisely the dynamical determinant Δ.*

Proof. Comparison of (8) and (9) means we require the trace formula[6]

$$\tau(\mathcal{L}_A^n) = \sum_{\underline{i}\in\mathcal{I}^n}\frac{w_{\underline{i}}(z_{\underline{i}})}{\det(I-T'_{\underline{i}}(z_{\underline{i}}))} \quad \text{for all } n \geq 1, \tag{10}$$

for every favourable space A on the admissible domain D.

[6]This trace formula (10) generalises the original one of Ruelle [**Rue1**] for $A = U(D)$, as well as that of Mayer [**May1, May2, May3**]. Our method of proof is rather direct, reducing to a simple Hilbert space computation; in particular, we do not need to explicitly evaluate the eigenvalues of each weighted composition operator $f \mapsto w_{\underline{i}} \cdot f \circ T_{\underline{i}}$ (a more complicated procedure, particularly in higher dimensions, cf. [**May2**, §III]).

First consider the holomorphic map-weight system (T, w, D) consisting of a single map and weight. Since $T(D) \Subset D$, the Earle-Hamilton theorem [**EH**] implies that T has a unique fixed-point $z_0 \in D$, and the eigenvalues of $T'(z_0)$ lie in the open unit disc [**May2**, Thm. 1]. If $\mathcal{L}f = w \cdot f \circ T$ is the corresponding transfer operator, we claim that

$$\tau(\mathcal{L}_A) = \frac{w(z_0)}{\det(I - T'(z_0))} . \tag{11}$$

The admissibility of D and favourability of $A = A(D)$ are invariant under affine coordinate changes, and τ is invariant under continuous similarities, so we may assume that $z_0 = 0$ and $\|T'(0)\|_{L(\mathbb{C}^d)} < 1$. Therefore, by Lemma 2.4 and Remark 2.5, there exists $R > 0$ such that, for $r \in (0, R)$, the radius-r Euclidean ball B_r centred at 0 is admissible.

Let $H^2_r = H^2(B_r)$ denote the Hardy space on B_r, a favourable Hilbert space (see Remark 2.8) with inner product $(f, g)_{H^2_r} = \int_{S_r} f\,\overline{g}\,d\sigma_r$, where $S_r = \partial B_r$, $\sigma_r(S_r) = 1$, and with orthonormal basis (cf. [**Rud**, Prop. 1.4.8, 1.4.9]) $\{p_{\underline{n},r} \mid \underline{n} \in \mathbb{N}_0^d\}$, where $p_{\underline{n},r}(z) = K_{\underline{n}} r^{-|\underline{n}|} z^{\underline{n}}$ and $K_{\underline{n}} = \sqrt{\frac{(|\underline{n}|+d-1)!}{(d-1)!\,\underline{n}!}}$, $\underline{n} = (n_1, \ldots, n_d)$, $z^{\underline{n}} = z_1^{n_1} \cdots z_d^{n_d}$, $\underline{n}! = n_1! \cdots n_d!$, $|\underline{n}| = n_1 + \cdots + n_d$.

The canonical embedding $J_r : A \hookrightarrow H^2_r$ has dense range, because complex polynomials are dense in H^2_r, and $J_r \mathcal{L}_A = \mathcal{L}_{H^2_r} J_r$. An intertwining argument of Grabiner [**Gra**, Lem. 2.3] then implies that $\lambda(\mathcal{L}_A) = \lambda(\mathcal{L}_{H^2_r})$, and hence that $\tau(\mathcal{L}_A) = \tau(\mathcal{L}_{H^2_r})$ because τ is spectral. The strong nuclearity of $\mathcal{L}_{H^2_r}$ means it is trace class, so $\tau(\mathcal{L}_{H^2_r})$ equals the sum of the diagonal entries of the matrix representation of $\mathcal{L}_{H^2_r}$ with respect to an orthonormal basis. Thus, for any $r \in (0, R)$,

$$\begin{aligned}
\tau(\mathcal{L}_A) = \tau(\mathcal{L}_{H^2_r}) &= \sum_{\underline{n} \in \mathbb{N}_0^d} (\mathcal{L} p_{\underline{n},r}, p_{\underline{n},r})_{H^2_r} = \int_{S_r} w(z) \sum_{\underline{n} \in \mathbb{N}_0^d} K_{\underline{n}}^2 r^{-2|\underline{n}|} T(z)^{\underline{n}}\, \overline{z}^{\underline{n}}\, d\sigma_r(z) \\
&= \int_{S_r} \frac{w(z)}{(1 - (r^{-1}T(z), r^{-1}z)_{\mathbb{C}^d})^d}\, d\sigma_r(z) = \int_{S_1} \frac{w(rz)}{(1 - (r^{-1}T(rz), z)_{\mathbb{C}^d})^d}\, d\sigma_1(z) .
\end{aligned}$$

Letting $r \to 0$ gives

$$\tau(\mathcal{L}_A) = \int_{S_1} \frac{w(0)}{(1 - (T'(0)z, z)_{\mathbb{C}^d})^d}\, d\sigma_1(z) = \frac{w(0)}{\det(I - T'(0))}$$

by an elementary integration, and (11) is proved.

Returning to the case of the holomorphic map-weight system $(T_i, w_i)_{i \in \mathcal{I}}$, the factorisation argument used in the proof of Proposition 2.10 shows that for $n \in \mathbb{N}$, the series $\sum_{\underline{i} \in \mathcal{I}^n} \mathcal{L}_{\underline{i}}$ converges in $N_{2/3}(A)$ to $\mathcal{L}_A^n$, where $\mathcal{L}_{\underline{i}} : A \to A$ is given by $\mathcal{L}_{\underline{i}} f = w_{\underline{i}} \cdot f \circ T_{\underline{i}}$. Since τ is continuous, $\tau(\mathcal{L}_A^n) = \sum_{\underline{i} \in \mathcal{I}^n} \tau(\mathcal{L}_{\underline{i}})$, and the required trace formula (10) follows from (11). □

4.3. Corollary. *Let $(T_i, w_i)_{i \in \mathcal{I}}$ be a C^ω map-weight system such that $(T_i)_{i \in \mathcal{I}}$ is complex contracting. Then the associated transfer operator preserves every favourable space on every admissible domain, and its eigenvalue sequence on each of these spaces is precisely the Ruelle eigenvalue sequence.*

In view of Corollary 4.3, the Ruelle eigenvalue sequence associated with a complex contracting C^ω map-weight system will henceforth be denoted simply by $\lambda(\mathcal{L}) = \{\lambda_n(\mathcal{L})\}_{n=1}^\infty$.

4.4. Corollary. *Under the hypotheses of Theorem 3.2, the Ruelle eigenvalue sequence $\lambda(\mathcal{L})$ can be bounded by*

$$|\lambda_n(\mathcal{L})| < \min\left(n^{1/2}\ ,\ \frac{\sqrt{d}}{(1-r^2)^{d/2}}\ n^{(d-1)/(2d)}\right) \frac{W}{r^d}\, r^{\frac{d}{d+1}(d!)^{1/d} n^{1/d}} . \tag{12}$$

Proof. Hardy space $H^2(D)$ is favourable, so Corollary 4.3 implies that $\lambda(\mathcal{L}_{H^2(D)})$ is the Ruelle eigenvalue sequence. The bound

$$|\lambda_n(\mathcal{L})| < \frac{W\sqrt{d}}{r^d(1-r^2)^{d/2}}\ n^{(d-1)/(2d)}\ r^{\frac{d}{d+1}(d!)^{1/d}n^{1/d}}$$

then follows from [**BJ**, Thm. 1]. The other part of (12) is immediate from Theorem 3.2. □

4.5. Remark. For a given $(T_i, w_i)_{i\in\mathcal{I}}$, if $r < 1$ is chosen as small as possible then the part of (12) arising from [**BJ**] is asymptotically superior as $n \to \infty$. For sufficiently small n, the part of (12) arising from Theorem 3.2 is sharper. For example, in dimension $d = 1$ this latter bound on $|\lambda_n(\mathcal{L})|$ is superior whenever $n^2 < 1/(1-r^2)$; this is always the case for $n = 1$, and may be true for many n if r is large (i.e. the map system is weakly contracting).

Acknowledgements. We are grateful to K. Osipenko, A. Pinkus and M. Stessin for bibliographical assistance concerning the approximation numbers for the embedding of $U(B_1)$ in $U(B_r)$. Both authors were partially supported by EPSRC grants GR/R64650/01 (2001–2003) and GR/S50991/01 (2004–2007), and the second author was partially supported by an EPSRC Advanced Research Fellowship.

References

[AAC] R. Artuso, E. Aurell, P. Cvitanović, Recycling of strange sets. II. Applications, Nonlinearity, **3** (1990), 361–386.

[Ave] A. Avez, *Differential Calculus*, John Wiley and Sons, Chichester, 1986.

[Bal] V. Baladi, *Positive transfer operators and decay of correlations*, Advanced series in nonlinear dynamics vol. 16, World Scientific, Singapore-New Jersey-London-Hong Kong, 2000.

[BJ] O. Bandtlow & O. Jenkinson, Explicit a priori bounds on transfer operator eigenvalues, *Comm. Math. Phys.*, to appear.

[CCR] F. Christiansen, P. Cvitanović, & H.-H. Rugh, The spectrum of the period-doubling operator in terms of cycles, *J. Phys. A*, **23** (1990), L713–L717.

[CPR] F. Christiansen, G. Paladin & H. H. Rugh, Determination of correlation spectra in chaotic systems, *Phys. Rev. Lett.*, **65** (1990), 2087–2090.

[CowM] C. Cowen & B. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.

[EH] C. J. Earle & R. S. Hamilton, A fixed point theorem for holomorphic mappings, in *Global Analysis* (S. Chern & S. Smale, Eds.), Proc. Symp. Pure Math., Vol. XVI, pp. 61–65, American Mathematical Society, Providence R.I., 1970.

[Far] Yu. A. Farkov, The N-widths of Hardy-Sobolev spaces of several complex variables, *J. Approx. Theory*, **75** (1993), 183–197.

[Fri] D. Fried, Zeta functions of Ruelle and Selberg I, *Ann. Sci. Ec. Norm. Sup.*, **9** (1986) 491–517.

[Gra] S. Grabiner, Spectral consequences of the existence of intertwining operators, *Ann. Soc. Math. Polonae Ser. 1*, **22** (1970), 227–238.

[Gro] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16**, 1955.

[GLZ] L. Guillopé, K. Lin & M. Zworski, The Selberg zeta function for convex co-compact Schottky groups, *Comm. Math. Phys.*, **245** (2004), 149–176.

[Hin] A. Hinrichs, Optimal Weyl inequality in Banach spaces, *Proc. Amer. Math. Soc.*, **134** (2006), 731–735.

[JMS] Y. Jiang, T. Morita & D. Sullivan, Expanding direction of the period-doubling operator, *Comm. Math. Phys.*, **144** (1992), 509–520.

[JP] O. Jenkinson & M. Pollicott, Orthonormal expansions of invariant densities for expanding maps, *Adv. Math.*, **192** (2005), 1-34.

[Kra] S. G. Krantz, *Function theory of several complex variables*, 2nd edition, AMS Chelsea, 1992.

[May1] D. H. Mayer, On a ζ function related to the continued fraction transformation, *Bull. Soc. Math. France*, **104** (1976), 195–203.

[May2] D. H. Mayer, On composition operators on Banach spaces of holomorphic functions, *J. Funct. Anal.*, **35** (1980), 191–206.

[May3] D. H. Mayer, Continued fractions and related transformations, in *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces* (T. Bedford, M. Keane & C. Series Eds.), Oxford, Oxford University Press, 1991.

[Pie] A. Pietsch, *Eigenvalues and s-numbers*, CUP, Cambridge, 1987.

[Pol1] M. Pollicott, Some applications of thermodynamic formalism to manifolds of constant negative curvature, *Adv. Math.*, **85** (1991), 161–192.

[Pol2] M. Pollicott, A note on the Artuso-Aurell-Cvitanovic approach to the Feigenbaum tangent operator, *J. Stat. Phys.*, **62** (1991), 257–267.

[PR] M. Pollicott & A. Rocha, A remarkable formula for the determinant of the Laplacian, *Invent. Math.*, **130** (1997), 399–414.

[Ran] R. M. Range, *Holomorphic functions and integral representations in several complex variables*, Springer, New York, 1986.

[Rud] W. Rudin, *Function theory in the unit ball of* C^n, Grundlehren der Mathematischen Wissenschaften 241, Springer-Verlag, New York-Berlin, 1980.

[Rue1] D. Ruelle, Zeta-functions for expanding maps and Anosov flows, *Invent. Math.*, **34** (1976), 231–242

[Rue2] D. Ruelle, Resonances for Axiom A flows, *J. Diff. Geom.*, **25** (1987), 117–137.

[Scha] H. H. Schaefer, *Topological vector spaces*, Springer-Verlag, New York, 1971.

[Wey] H. Weyl, Inequalities between two kinds of eigenvalues of a linear transformation, *Proc. Nat. Acad. Sci. USA*, **35** (1949), 408–411.

Oscar Bandtlow; School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London, E1 4NS, UK.
ob@maths.qmul.ac.uk
www.maths.qmul.ac.uk/∼ob

Oliver Jenkinson; School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London, E1 4NS, UK.
omj@maths.qmul.ac.uk
www.maths.qmul.ac.uk/∼omj