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Interval Domains and Computable Sequences: A Case Study of Domain Reductions

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The interval domain as a model of approximations of real numbers is not unique, in fact, there are many variations of the interval domain. We study these variations with respect to domain reductions. The effectivity theory induced by these variations is not stable, and this paper investigates some of the rich structure found. We follow Mostowski (On computable sequences. *Fund. Math.*, 44, 37–51) and use computable sequences to exhibit this structure.

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1. INTRODUCTION

The interval domain, proposed by Scott [1], is a convenient model of approximations of real numbers. It has been used for semantics of computable reals, interval arithmetic and constraint satisfaction over the reals.

Computations over uncountable spaces, such as the reals, but also many function spaces, will for cardinality reasons have to be performed on approximations. The interval domain serves as a template for computing with such approximations.

While the interval domain is often seen as a single entity, it is, in fact, possible to consider many variations. There are many equivalent versions of the interval domain, even in the context of effectivity, but there are also infinitely many inequivalent interval domains. Exhibiting this rich set of interval domains is one of the aims of the paper.

In order to give sharp results in this direction, it is not enough to consider only the data. An algebraic structure is needed to exhibit the differences. Following a pattern established more than half a century ago, we will use sequences of real numbers as our algebraic structure. In particular, we will follow Mostowski [2].

By giving all the classes of computable sequences considered in [2] domain representations, we can use his results to understand the plethora of interval domains better. We are also further convinced that any approximation structure is naturally modelled as domains.

Finally, we give a short historical reflection on the importance of studying computability not merely as effectively generated data objects, but rather as effective data with effective operations, i.e. as effective algebras [3]. The inspiration for this view is largely due to frequent conversations with John V. Tucker.

2. BACKGROUND

2.1. Domains

We will briefly give some background to domain theory. For a complete background on domains, we refer the readers to [4, 5].

Let $D = (D, \sqsubseteq)$ be a partially ordered set. A subset $A \subseteq D$ is an *upper set* (dual *lower set*) if $x \in A$ and $x \sqsubseteq y$ implies $y \in A$. Let $\uparrow A = \{y \in D : \exists x \in A(x \sqsubseteq y)\}$. We will abbreviate $\uparrow\{x\}$ by $\uparrow x$. A subset $A \subseteq D$ is *directed* if $A \neq \emptyset$ and whenever $x, y \in A$ then there is $z \in A$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. The supremum, or least upper bound, of A (if it exists) is denoted by $\bigsqcup A$.

A (*directed*) *complete partial order*, abbreviated *CPO*, is a partial order, $D = (D; \sqsubseteq, \bot)$, such that $\bot$ is the least element in D and any directed set $A \subseteq D$ has a supremum, $\bigsqcup A$.

Let D be a CPO. An element $c \in D$ is *compact* if, for each directed $A \subseteq D$,

$$c \sqsubseteq \bigsqcup A \implies (\exists a \in A)(c \sqsubseteq a).$$

The set of compact elements of D is denoted by D_c. A domain D is *algebraic* if, for all $x \in D$, $\mathrm{approx}(x) = \{a \in D_c : a \sqsubseteq x\}$ is directed and $\bigsqcup \mathrm{approx}(x) = x$.

A CPO D is *consistently complete* if $\bigsqcup A$ exists in D whenever $A \subseteq D$ is a consistent set, i.e. has an upper bound.

Definition 2.1. *A Scott–Ershov domain, or simply a domain, is a consistently complete algebraic CPO.*

The topology normally used on domains is called the Scott topology. Let D be an algebraic CPO. A subset U of D is open if

(i) U is an upper set and
(ii) $x \in U$ implies that there exists $a \in \mathrm{approx}(x)$ such that $a \in U$.

An easy observation is that the Scott topology on a domain is T_0. However, the Scott topology fails to be T_1 on all domains except the trivial domain consisting of a single element.

The sets $\uparrow a$, for $a \in D_c$, constitute a base for the Scott topology on a domain D.

Let D and E be domains. A function $f : D \to E$ is Scott continuous if f is monotone and

$$f\left(\bigsqcup A\right) = \bigsqcup f[A]$$

for any directed $A \subseteq D$. The notion of Scott continuity coincides with the notion of continuity induced from the Scott topology on the domains.

Any continuous function between domains is determined by its values on the compact elements. Let D be an algebraic CPO, E be a CPO and let $f : D_c \to E$ be a monotone function. Then there exists a unique continuous extension $g : D \to E$ of f such that $f = g|_{D_c}$.

Domains are often constructed as the completion of some underlying structure. We present here the type of structure from which Scott–Ershov domains are constructed.

The compact elements D_c of a Scott–Ershov domain D form a conditional upper semilattice with least element, abbreviated *CUSL*. That is, a CUSL is a partially ordered set where a least upper bound exists for every pair of elements that have an upper bound.

An *ideal* is a directed lower set. The ideal completion over a CUSL P is the set of all ideals over P, denoted by $\mathrm{Idl}(P)$. When ordered by set inclusion, the ideal completion of a CUSL forms a Scott–Ershov domain. For a in a CUSL P, $\downarrow a$ is an ideal, the *principal ideal* generated by a. The compact elements of $\mathrm{Idl}(P)$ are the principal ideals $\downarrow a$, for $a \in P$.

The representation theorem for Scott–Ershov domains tells us that any Scott–Ershov domain is the ideal completion of a CUSL.

Theorem 2.1. *Let D be a Scott–Ershov domain. Then* $\mathrm{Idl}(D_c) \cong D$.

We clearly have the following equivalence, for $I \in \mathrm{Idl}(P)$,

$$\downarrow a \subseteq I \iff a \in I.$$

Thus, the sets $B_a = \{I \in \mathrm{Idl}(P) : a \in I\}$ for $a \in P$ form a base for the Scott topology on $\mathrm{Idl}(P)$.

Definition 2.2. *A domain D is effective if there exists a numbering $\alpha : \Omega_\alpha \to D_c$, where $\Omega_\alpha \subseteq \mathbb{N}$, making the structure* $(D_c, \sqsubseteq, \mathrm{Cons}, \sqcup, \bot)$ *computable.*

Let D and E be effective domains with numberings α and β, respectively. A domain function $f : D \to E$ is *effective* if there exists a computable function $\bar{f}$ that tracks f with respect to α and β.

2.2. Domain representations

We give some background on domain representations of topological spaces.

Definition 2.3. *A (domain) representation of a topological space X is a triple (D, D^R, ρ), where D is a domain, $D^R \subseteq D$ with the subspace topology and $\rho : D^R \to X$ is continuous and onto.*

The set D^R above will be called the set of *representing elements*. For a domain-like structure D, the set D^R is also known as a *totality* on D. The ordering of the domain D can be interpreted as an information ordering. With this interpretation, the domain contains both proper approximations and total or complete representations of elements of X, the latter constituting the set D^R. Intuitively, D^R consists of those domain elements that contain sufficient information to completely determine an element in X via ρ.

Definition 2.4. *An effective domain representation is a domain representation (D, D^R, ρ) where the domain D is effective. Let $D_k \subseteq D$ denote the computable elements of D, i.e. $D_k = \{d \in D : \mathrm{approx}(d)$ is c.e.$\}$, and let $D^R_k = D^R \cap D_k$.*

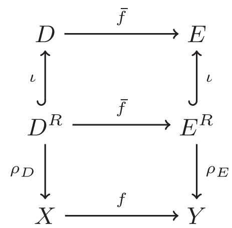
Let the represented space be X, then $X_k = \rho[D^R_k]$ denotes the computable elements of X induced by the domain representation.

The following is a stronger version of domain representability.

Definition 2.5. *A retract representation of X is a quadruple (D, D^R, ρ, η) where (D, D^R, ρ) is a representation, and $\eta : X \to D^R$ is a continuous function such that* $\rho\eta = \mathrm{id}_X$.

For a retract representation $(D, D^{\mathrm{R}}, \rho, \eta)$, we have that ρ is a quotient, and that $\eta\rho$ is a retraction on D^{R}. In fact, X will be homeomorphic to the retract of D^{R}. In a retract representation, a canonical representative can be found continuously from any representation of an element of X.

DEFINITION 2.6. *Let $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be representations of X and Y, respectively. A function $f : X \to Y$ is represented by a continuous function $\bar{f} : D \to E$ if $\rho_E \bar{f}(x) = f\rho_D(x)$, for all $x \in D^{\mathrm{R}}$ (in particular, $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$).*



The functions between the subsets of representing elements are restrictions of functions. To avoid clumsy explicit restriction notation, as in $\bar{f}|_{D^{\mathrm{R}}} : D^{\mathrm{R}} \to E^{\mathrm{R}}$, we write $\bar{f} : D^{\mathrm{R}} \to E^{\mathrm{R}}$ and trust the reader to understand this as the restriction to the indicated domain of the function.

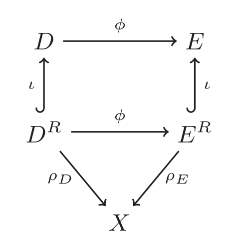
Let $(D, D^{\mathrm{R}}, \rho_D)$ and $(E, E^{\mathrm{R}}, \rho_E)$ be representations of X and Y, respectively, and let $\bar{f} : D \to E$ be continuous such that $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$. If $\bar{f}$ respects the equivalence relations induced by ρ_D and ρ_E, then $\bar{f}$ represents a well-defined function $f : X \to Y$. Furthermore, if ρ_D is a quotient map, then f is continuous, since then the topology of X is fine enough.

DEFINITION 2.7. *For a topological space X, let* $\mathrm{DRep}(X)$ *denote the class of all domain representations $(D, D^{\mathrm{R}}, \rho)$ of X.*

2.3. Reducibility

We give a short summary of the notion of reducibility between domain representations used in [6]. The concept is closely related to reductions in type-2 theory of effectivity (TTE) [7].

For representations D and E of a space X, we have that D reduces to E if the representation function of D factors through the representation function of E, i.e. if there is a function $\phi : D \to E$ such that the following diagram commutes:



An equivalent formulation is the existence of a domain function from D to E that induces the identity on X.

DEFINITION 2.8. *Let $D = (D, D^{\mathrm{R}}, \rho_D)$ and $E = (E, E^{\mathrm{R}}, \rho_E)$ be representations of a space X. A reduction of D to E is a function $\phi : D \to E$ such that $\phi[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ and $\rho_D = \rho_E\phi$.*

The existence of reductions depend on the class of functions from which ϕ has to be taken. Our interest is in *effective reductions* by effective domain functions. Let $D \leq_e E$ denote that D reduces to E effectively. Effective reductions form a preorder on $\mathrm{DRep}(X)$. We denote the induced equivalence relation by $\equiv_e$.

DEFINITION 2.9. *An e-spectrum over a topological space X, written* $\mathrm{Spec}(X, \mathcal{D}, \leq_e)$, *is the quotient $\mathcal{D}/\equiv_e$ ordered by $\leq_e$, where $\mathcal{D}$ is a class of representations of the space X.*

DEFINITION 2.10. *A representation D is e-universal in a class $\mathcal{D}$ if $E \leq_e D$ for all $E \in \mathcal{D}$.*

We will occasionally consider continuous reductions as well, where c will replace e in all definitions above.

2.4. Mostowski's classes of computable sequences

Mostowski [2] defined a number of classes of computable sequences of real numbers. It was at the time known that a number of definitions of the set of computable real numbers coincided. This was first observed by Robinson [8].

It seems that Mostowski's motivation was to differentiate among the possible definitions of the computable reals.

> There is no doubt that of these various definitions the one which best expresses the existence of an algorithm permitting one to calculate uniformly the terms of a sequence with any desired degree of accuracy is that which corresponds to $[\mathcal{C}_1]$.

The reason for looking at sequences of real numbers was, in retrospect, that some structure is needed to differentiate between the various definitions of computable real numbers. It is enough to look at the field operation of real numbers to distinguish some of them, but by looking at sequences he could exhibit a number of strict inclusions.

Mostowski defined classes of computable sequences of reals where the following conditions hold:

- $\mathcal{C}_1$ are computable Cauchy sequences with a known modulus,
- $\mathcal{C}_{2\beta}$ are computable expansions in base β,
- $\mathcal{C}_3$ are computable expansions in every base $\beta \geq 2$,
- $\mathcal{C}_4$ are decidable left Dedekind cuts and
- $\mathcal{C}_5$ are decidable right Dedekind cuts,

all computable uniformly in the index. The formal definitions follows. Define subsets $\mathcal{C}_i$, $i = 1, 2\beta, 3, 4, 5$, of real-valued sequences, or equivalently, functions from $\mathbb{N}$ to $\mathbb{R}$, i.e. $\mathcal{C}_i \subseteq \mathbb{R}^{\mathbb{N}}$.

Let φ range over total recursive functions where we assume that it has the correct arity, and let ν be a standard enumeration of the rationals $\mathbb{Q}$. Then

$$\mathcal{C}_i = \{(x_k)_k \in \mathbb{R}^{\mathbb{N}} : \exists\varphi\ \forall k \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Psi_i(\varphi, k, n)\},$$

where

$$\Psi_1(\varphi, k, n) \iff \left| x_k - \frac{\varphi(k,n)}{n+1} \right| < \frac{1}{n+1},$$

$$\Psi_{2\beta}(\varphi, k, n) \iff x_k = \sum_{n=0}^{\infty} \varphi(k,n)\beta^{-n} \wedge n \geq 1 \implies 0 \leq \varphi(k,n) < \beta,$$

$$\Psi_3(\varphi, k, n) \iff \forall \beta \geq 2 \left(x_k = \sum_{n=0}^{\infty} \varphi(k,n,\beta)\beta^{-n} \wedge n \geq 1 \implies 0 \leq \varphi(k,n,\beta) < \beta \right),$$

$$\Psi_4(\varphi, k, n) \iff (\varphi(k,n) = 1 \iff \nu(n) < x_k),$$

$$\Psi_5(\varphi, k, n) \iff (\varphi(k,n) = 1 \iff \nu(n) > x_k).$$

We have chosen not to use the original notation above. We have also made the trivial extension from sequences over the unit interval to sequences over the real line rather than that over the unit interval. For $\mathcal{C}_{2\beta}$ and $\mathcal{C}_3$, the integral part of the real number x_k is computed by $\varphi(k, 0)$ and $\varphi(k, 0, \beta)$, respectively.

Semidecidable Dedekind cuts correspond to left- and right-computable reals, respectively. These classes are larger than the class of computable reals. Moreover, these classes are not naturally related to interval domains, but rather to the continuous domains obtained by ordering the reals by $<$ and $>$, respectively.

Mostowski showed the following inclusions for all $\beta \geq 2$.

$$\begin{matrix} \mathcal{C}_4 & & & & & & \\ & \subsetneq & & & & & \\ & & \mathcal{C}_3 & \subseteq & \mathcal{C}_{2\beta} & \subseteq & \mathcal{C}_1 \\ & \subsetneq & & & & & \\ \mathcal{C}_5 & & & & & & \end{matrix}$$

These inclusions are all straightforward since it is possible to effectively translate from one representation to its superclass. In fact, it is enough to have primitive recursion. Furthermore, he showed $\mathcal{C}_{2\beta} \subseteq \mathcal{C}_{2\beta'}$ if, and only if, $\beta' | \beta^k$ for some k.

Mostowski then gives counterexamples to the reverse inclusions, thereby showing the inclusions to be strict. He also shows that $\mathcal{C}_4$ and $\mathcal{C}_5$ are distinct classes. To do this, he uses the added structure of sequences to show his results.

We recount the informal sketch of the proof that $\mathcal{C}_1 \neq \mathcal{C}_{2\beta}$. Let X_1 and X_2 be disjoint c.e. sets that are recursively inseparable. This is equivalent to the existence of computable total functions $\varphi_i(k, n)$, $i = 1, 2$ such that $k \in X_i \iff \exists n\, (\varphi_i(k,n) = 0)$.

Let $a_k = \lim_{n\to\infty} a_{k,n}$, where

$$a_{k,n} = \begin{cases} \beta^{-1} & \text{if } \forall m \leq n\ (\varphi_i(k,m) \neq 0), \\ \beta^{-1} + \beta^{-m_0-2} & \text{if } m_0 \leq n \text{ is the least number such that } \varphi_1(k, m_0) = 0, \\ \beta^{-1} - \beta^{-m_0-2} & \text{if } m_0 \leq n \text{ is the least number such that } \varphi_2(k, m_0) = 0. \end{cases}$$

The first digit in the base β expansion of a_k is 1 if $k \in X_1$, and 0 if $k \in X_2$. By assumption, there cannot exist a computable φ giving the first digit of each number in the sequence. Clearly, the sequence $(a_k)_k \in \mathcal{C}_1$, and hence $\mathcal{C}_1 \neq \mathcal{C}_{2\beta}$.

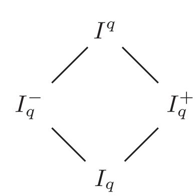
3. INTERVAL DOMAINS

We will now introduce the interval domain and consider the effectivity theory this introduces on the reals. Let us start with an all encompassing (algebraic) interval domain. First, we need to choose a set of approximations rich enough to distinguish our data points of interest. We will use intervals for the reals, hence the name 'interval domain'. Let P be the set of all closed rational intervals, i.e. $P = \{[a,b] : a, b \in \mathbb{Q}\} \cup \{\mathbb{R}\}$, and let $D = \mathrm{Idl}(P)$, the ideal completion of P under reverse inclusion.

An ideal I represents a real number r if $\bigcap I = \{r\}$. It is easy to verify that I is a representing ideal if, and only if, for all $\varepsilon > 0$ there exists $[a,b] \in I$ such that $b - a < \varepsilon$. Let D^{R} be the subset of representing elements of D, and let $\rho : D^{\mathrm{R}} \to \mathbb{R}$ be the obvious representation map. Clearly, the ideal $I_{\mathrm{r}} = \{[a,b] : a < r < b\}$ represents the real number r. Thus, the interval domain contains representations of all real numbers.

The extended real line $\mathbb{R} \cup \{-\infty, \infty\}$ can be represented by allowing the endpoints of the intervals to be in the set $\{-\infty, \infty\}$. That is, starting with $P = \{[a,b] : a, b \in \mathbb{Q} \cup \{-\infty, \infty\}\}$. For an ideal I to represent ∞, we require for all $n \in \mathbb{N}$ that there exists $[a, \infty) \in I$ where $a > n$, and similarly for $-\infty$. This change can be done regardless of the particular choice of approximations chosen in the following.

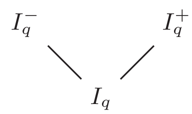
One of the characteristic properties of the algebraic interval domain D is that for each rational point $q \in \mathbb{Q}$ there will be four ideal representing it, I^q, I_q, I_q^+ and I_q^-. These ideals are ordered as follows:



The ideals higher up contain more information which here is finite (compact) information saying that the number is $\geq q$ or $\leq q$.

The continuous interval domain is a retract of the algebraic interval domain that eliminates these extra ideals. There exists a topological embedding of the reals into the maximal elements of the continuous interval domain. This is desirable but unfortunately the computability theory of continuous domains is not as simple as the corresponding theory for algebraic domains. We will not further consider continuous interval domains.

We also note the interval domain D' obtained by starting with open rational intervals. This gives a similar structure, but the ideals representing a rational point $q \in \mathbb{Q}$ are now just three: I_q, I_q^+ and I_q^-, ordered as before.



The representing ideals of D' is therefore not *co-dense*, since there are inconsistent ideals representing the same element.

It is common to look at some substructure of the interval domain. For example, one can consider the dyadic interval domain, where the endpoints of compact intervals are dyadic. This domain is relevant for computations since computations over dyadic numbers are more efficient compared with computations over the rationals.

3.1. Variations of the interval domain

We aim to capture the classes considered by Mostowski. We start by creating a number of variations of the standard interval domain and will later build domain representations of Mostowski's classes from function spaces of domain functions. Let

$$\begin{aligned}
P_1 &= \left\{\left[\frac{a-1}{n}, \frac{a+1}{n}\right] : a \in \mathbb{Z}, n \in \mathbb{N}\right\} \cup \{\mathbb{R}\},\\
P_{2\beta} &= \{[a\beta^{-n}, (a+1)\beta^{-n}) : a \in \mathbb{Z}, n \in \mathbb{N}\} \cup \{\mathbb{R}\},\\
P_3 &= \bigcup_{\beta \geq 2} \{[a\beta^{-n}, (a+1)\beta^{-n}) : a \in \mathbb{Z}, n \in \mathbb{N}\} \cup \{\mathbb{R}\},\\
P_4 &= \{(p, q) : p, q \in \mathbb{Q} \cup \{\infty, -\infty\}\},\\
P_5 &= \{(p, q) : p, q \in \mathbb{Q} \cup \{\infty, -\infty\}\}.
\end{aligned}$$

Note that the sets P_i consist only of (possibly infinite) intervals with rational endpoints. Let $D_i = \mathrm{Idl}(P_i)$, the ideal completion of P_i under reverse inclusion. It may be surprising that the intervals used for P_4 and P_5 are symmetric, since the intuitive feeling is that we should use half-open intervals as our compact elements. This will be explained below when the domain representations are constructed.

To construct domain representations of the reals from these domains, it remains to give the subset of representing elements and the representing function.

THEOREM 3.1. *$D_1 = (D_1, D_1^{\mathrm{R}}, \rho_1) \in \mathrm{DRep}(\mathbb{R})$ and $D_1 \equiv_e D$.*

Proof. The representing ideals D_1^{R} is defined as for D, i.e.

$$D_1^{\mathrm{R}} = \left\{ I \in D_1 : \bigcap I = \{x\}, \text{ some } x \in \mathbb{R} \right\}.$$

The representation map $\rho_1 : D_1^{\mathrm{R}} \to \mathbb{R}$ is the obvious one mapping a representing ideal to the singleton element of its intersection. Clearly, ρ_1 is surjective, so $D_1 \in \mathrm{DRep}(\mathbb{R})$.

Define $f : D_1 \to D$ and $g : D \to D_1$ on compact elements by

$$\begin{aligned}
f([a, b]) &= \bigsqcup \{[c, d] \in P : c < a \wedge b < d\},\\
g([a, b]) &= \bigsqcup \{[c, d] \in P_1 : c < a \wedge b < d\},
\end{aligned}$$

and extend them to continuous domain functions. Clearly, the maps are effective, since ordering is decidable for rationals. It is straightforward to show that representing ideals are mapped to representing ideals and that the maps induce the identity. Thus, $D_1 \equiv_e D$. □

The structure of $P_{2\beta}$ under reverse inclusion are trees with branching factor β except for the root node which have countably many children. The expansion in base β can be read from any infinite path through the tree, and the representing elements are ideals that contain such an infinite path.

THEOREM 3.2. *For all $\beta \geq 2$, $D_{2\beta} = (D_{2\beta}, D_{2\beta}^{\mathrm{R}}, \rho_{2\beta}) \in \mathrm{DRep}(\mathbb{R})$ and $D_{2\beta} <_e D_1$.*

Proof. Again, the representing ideals of $D_{2\beta}^{\mathrm{R}}$ are ideals with a singleton intersection. There exists an effective domain function $f : D_{2\beta} \to D_1$, tracking the identity on the reals, defined on compact elements by

$$f([a, b)) = \bigsqcup \{[c, d] \in P_1 : c < a \wedge b < d\}.$$

Thus, $D_{2\beta} \in \mathrm{DRep}(\mathbb{R})$ and $D_{2\beta} \leq_e D_1$.

In the other direction, we have the stronger $D_1 \not\leq_c D_{2\beta}$. To see this, consider the ideal I_0. There exist representing ideals above every compact element of I_0 representing both positive and negative numbers. Thus, there is no continuous function giving the integer part. □

THEOREM 3.3. *$D_{2\beta} \leq_e D_{2\beta'}$ if, and only if, $\beta' | \beta^k$ for some k.*

Proof. The argument provided by Mostowski for [2, Theorem 3] can be adapted to show that $\beta' | \beta^k$ implies $D_{2\beta} \leq_e D_{2\beta'}$.

An equivalent formulation of the condition $\beta' | \beta^k$ for some k is that all prime divisors of β' are divisors of β. Now [6, Theorem 6.7] implies the stronger $D_{2\beta'} \not\leq_c D_{2\beta}$. □

THEOREM 3.4. *$D_3 = (D_3, D_3^{\mathrm{R}}, \rho_3) \in \mathrm{DRep}(\mathbb{R})$ and for all $\beta \geq 2$, $D_3 <_e D_{2\beta}$.*

Proof. The same definition of representing ideals and representation map as in the previous theorem gives the first part.

The structure of P_3 is an amalgamation of all the $P_{2\beta}$. If $I \in D_3^{\mathrm{R}}$, then one can compute the expansion of the represented real number in every base β. Thus, $D_3 \leq_e D_{2\beta}$.

Clearly, the converse is not true, it is impossible to effectively find, for example, the ternary expansion from the binary expansion. Any finite prefix of the binary expansion $0.010101\ldots$ does not allow us to determine whether the first fractional digit in base 3 is 0 or 1. □

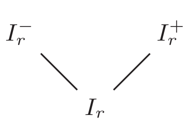
The domain representations D_4 and D_5 are conceptually a bit harder. The reason for this is that the natural effectivity theory for domains is built around c.e. ideals. To capture the classes $\mathcal{C}_4$ and $\mathcal{C}_5$, we are looking for decidable Dedekind cuts. We achieve this by representing both the cut and its complement. If both these sets are c.e., then they are in fact computable, i.e. the cut is decidable.

The finite information we can have about a left Dedekind cut C is either $a \in C$, i.e. a is a (strict) lower bound of the real, or $b \notin C$, i.e. b is an upper bound of the real. Combining two lower bounds $a \in C$ and $a' \in C$ is simply $\max\{a, a'\} \in C$. Similarly, for upper bounds. To build a domain, we also need to be able to combine the information of a lower bound $a \in C$ and an upper bound $b \notin C$. This is encoded in the pair (a, b). This can be seen as an interval as well. It might seem natural to capture the asymmetric property of Dedekind cuts by interpreting the pair as an half-open interval, but the information of a pair (a, b) about a (left) Dedekind cut is that $(-\infty, a] \subseteq C$ and $[b, \infty) \cap C = \emptyset$. Hence, we think of the pair (a, b) as the open interval (a, b) where it is still unknown whether the points belong to C or not.

It is not enough for an ideal $I \in D_4$ to contain arbitrarily short intervals to represent a decidable Dedekind cut. For example, given the ideal I_q, $q \in \mathbb{Q}$, we cannot decide if q is in the cut or not, but given I_q^- we can decide that q is not in the cut, since $(p, q) \in I_q^-$, for all $p < q$.

Theorem 3.5. *Let* $\mathcal{D}_4 = (D_4, D_4^{\mathrm{R}}, \rho_4)$ *and* $\mathcal{D}_5 = (D_5, D_5^{\mathrm{R}}, \rho_5)$, *then* $\mathcal{D}_4, \mathcal{D}_5 \in \mathrm{DRep}(\mathbb{R})$, *and* $\mathcal{D}_4 <_e \mathcal{D}_3$ *and* $\mathcal{D}_5 <_e \mathcal{D}_3$.

Proof. Assume that $I = \{(a_i, b_i) : i \in J\}$, for some index set J. Clearly, we want the real represented by I to be $r = \sup_{i \in J} a_i$. Now, the formal requirement for I to be representing is that $\inf_{i \in J} b_i = r$ and furthermore if r happens to be rational, then there must exist $i \in J$ such that $b_i = r$. Thus, an irrational r is represented by the only possible choice I_{r}, but for a rational r the only representing ideal is I_{r}^-. That is, only the left ideal I_{r}^- out of the open interval ideals representing r.



We have that $(D_4, D_4^{\mathrm{R}}, \rho_4)$ is a domain representation of the left Dedekind cuts, i.e. the reals.

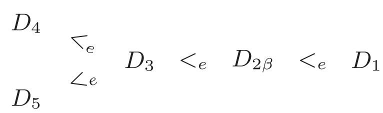
Using the same argument, the same domain, but only including the ideals I_q^+ as representing ideals for rational q, will be a domain representation $(D_5, D_5^{\mathrm{R}}, \rho_5)$ of the right Dedekind cuts.

Since the Dedekind cuts of $\mathcal{D}_4$ and $\mathcal{D}_5$ are decidable it is possible to effectively generate the expansion in any base $\beta \geq 2$. For a rational $q = m\beta^{-n}$, we would always get a β-expansion ending with infinitely many zeros from $\mathcal{D}_4$, and ending with infinitely many $\beta - 1$ from $\mathcal{D}_5$. We leave the details of showing $\mathcal{D}_4, \mathcal{D}_5 \leq_e \mathcal{D}_3$ to the reader. □

Corollary 3.1. *The domain representations* $(D_i, D_i^{\mathrm{R}}, \rho_i)$, *where* $i = 1, 2\beta, 3, 4, 5$, *represent the real numbers. Moreover, the image of the computable elements of* D_i^{R} *under* ρ_i *is exactly the computable reals.*

Recall that the set of computable reals have been shown to be the same regardless of which of the above constructions have been used to construct the reals.

Corollary 3.2. *There exist effective domain reductions as indicated.*



Note that by Theorem 3.3 there is a preorder on the $\mathcal{D}_{2\beta}$ for varying β, so the structure of $\mathrm{Spec}(\mathbb{R}, \mathrm{DRep}(\mathbb{R}), \leq_e)$ is much richer than depicted.

Theorem 3.6. *The spectrum* $\mathrm{Spec}(\mathbb{R}, \mathrm{DRep}(\mathbb{R}), \leq_e)$ *contains infinitely many unrelated elements and an infinite strict chain.*

Proof. Let $p_1, p_2, \ldots$ be an infinite enumeration of the prime numbers. Then $\mathcal{D}_{2p_m}$ and $\mathcal{D}_{2p_n}$ are unrelated if $p_m \neq p_n$.

Let $r_n = \prod_{i=1}^{n} p_i$. Then $\mathcal{D}_{2r_m} <_e \mathcal{D}_{2r_n}$ if $n < m$. □

Also note that two effectively equivalent interval domain representations may still differ in how efficient operations can be computed. For example, dyadic numbers are much more efficient than rational numbers.

3.2. Domain representations of sequences

Let $\mathcal{D} = (D, D^{\mathrm{R}}, \rho)$ be a domain representation of X, and let $\mathbb{N}_\perp = (\mathbb{N}_\perp, \mathbb{N}_\perp^{\mathrm{R}}, \nu)$ be the obvious domain representation of $\mathbb{N}$. Then the function space $[\mathbb{N}_\perp \to D]$ of continuous functions from $\mathbb{N}_\perp$ to D again forms a domain. The domain $[\mathbb{N}_\perp \to D]$ is effective if D is effective. Let

$$[\mathbb{N}_\perp \to D]^{\mathrm{R}} = \{f \in [\mathbb{N}_\perp \to D] : f[\mathbb{N}_\perp^{\mathrm{R}}] \subseteq D^{\mathrm{R}} \text{ and } \forall m, n \in \mathbb{N}_\perp^{\mathrm{R}}(\nu m = \nu n \implies \rho(fm) = \rho(fn))\},$$

and let $\psi : [\mathbb{N}_\perp \to D]^{\mathrm{R}} \to X^{\mathbb{N}}$ be defined by $\psi(f) = s$, where s is a sequence of elements from X defined for all $n \in \mathbb{N}$ by

$$s(n) = \rho(fn).$$

We have shown the following.

THEOREM 3.7. *Let $D = (D, D^{\mathrm{R}}, \rho)$ be an effective domain representation of X. Then $[\mathbb{N}_\perp \to D] = ([\mathbb{N}_\perp \to D], [\mathbb{N}_\perp \to D]^{\mathrm{R}}, \psi)$ is an effective domain representation of $X^{\mathbb{N}}$.*

COROLLARY 3.3. *For $i = 1, 2\beta, 3, 4, 5$,*

$$([\mathbb{N}_\perp \to D_i], [\mathbb{N}_\perp \to D_i]^{\mathrm{R}}, \psi_i)$$

are effective domain representations of $\mathbb{R}^{\mathbb{N}}$.

THEOREM 3.8. *For $i = 1, 2\beta, 3, 4, 5$,*

$$\mathcal{C}_i = \psi_i[[\mathbb{N}_\perp \to D_i]_k].$$

Proof. The approximations used in the constructions of the domains D_i has been taken to reflect the information that the stipulated computable functions φ contain in the definitions of $\mathcal{C}_i$. A computable element in $[\mathbb{N}_\perp \to D_i]^{\mathrm{R}}$ is a computable function that can generate that information, hence the result. □

The inclusions between the $\mathcal{C}_i$ that Mostowski proved can now be shown by composing the computable function representing the sequence with the effective reductions of Corollary 3.2.

On the other hand, Mostowski's non-inclusion results can be used to give counter-examples about domain representations.

EXAMPLE 3.1. We have that $\rho_1[D_{1k}^{\mathrm{R}}] = \rho_3[D_{3k}^{\mathrm{R}}]$. However, since $\mathcal{C}_1 \not\subseteq \mathcal{C}_3$, we have

$$\psi_1[[\mathbb{N}_\perp \to D_1]_k] \not\subseteq \psi_3[[\mathbb{N}_\perp \to D_3]_k].$$

So, although two effective representations have the same set of effectively represented points, it does not follow that the sets of effectively represented sequences are the same.

4. SUMMARY

Using reductions to study domain representations of real sequences, we have established a fine grained structure of different interval domains. Sequences have proved to be very powerful in this process, but let us briefly consider the more common field operations over real numbers.

While all field operations can be lifted to computable domain operations on D_1, it is well known that the $D_{2\beta}$ domain representations allow neither addition nor multiplication to be continuously represented. On the other hand, addition has computable domain representations for D_4 and D_5, but again multiplication cannot be continuously represented (neither can negation).

The computability of an operation is *not* preserved by effective reductions. In fact, computability of an operation is not even monotone with respect to the reduction ordering, as exemplified by addition being computable over D_1, D_4 and D_5, but not over $D_{2\beta}$ and D_3. Thus, computability of continuous data cannot be studied on the data alone, but must be considered in the context of the algebraic structure needed.

5. HISTORICAL REFLECTION

To a modern reader of Mostowski [2], it seems curious that existence/non-existence of effective reductions between the various definitions of computable numbers seemingly have been overlooked as a tool for distinguishing the definitions. Taken further this idea ultimately leads to the notion of universal (domain) representations [6] and the closely related notion of admissibility, introduced by Weihrauch [7] and studied by Schröder [9] (for TTE) and Hamrin [10] (for domain representations).

Another general tool for distinguishing between different definitions of computability for continuous data is to consider the topological properties of the representation map, as the author has often done [11]. Topology is an additional algebraic structure over the data. In fact, any attempt at introducing computability via approximations will also induce a topology. In particular, if the represented space is a retract, then this will help in lifting operations to the representation. The relationship of the above two methods was the main motivation behind [6].

Nevertheless, Mostowski's paper is an important step towards understanding computability over continuous data. I believe that it has not been given the due recognition that it deserves. At the time, it is common to see that the various ways of formulating computability on reals are *equivalent*, see [8, 12, 13]. Today, we view these different approaches to real computability as very different even though they do induce the same set of computable numbers.

The pivotal step in Mostowski's paper is the use of algebraic structure over the data in order to investigate representations of continuous data. Mostowski acknowledges the influence of Specker [14] to use sequences. Specker used sequences to exhibit counter examples to, for example, the supremum principle of real analysis. This result holds for the $\mathcal{C}_i$ classes of sequences for $i = 1, 2\beta, 3, 4, 5$. Mostowski's contribution is that he has exhibited stark examples showing that computability is dependent on the representation in the context of algebraic structure. It can be noted that the supremum of a computable bounded monotone real sequence is computable if, for example, the left-recursive reals are used.

In the Turing centenary 2012, one must of course observe that one of the main corrections in Turing [15] compared with the original paper [16] is that he replaced binary expansion by nested intervals in his definition of 'computable number'. Undoubtedly, he observed the problems of computing field operations with the binary expansion version of computable real numbers. Thus, the use of algebra in order to understand computability could be said to reach all the way back to the inception of computability.

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