A COGNITIVE STYLE AND AGGREGATION OPERATOR MODEL: A LINGUISTIC APPROACH FOR CLASSIFICATION AND SELECTION OF THE AGGREGATION OPERATORS

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Abstract. Aggregation operators (AOs) have been studied by many scholars. As many AOs are proposed, there is still lacking approach to classify the categories of AO, and to select the appropriate AO within the AO candidates. In this research, each AO can be regarded as a cognitive style or individual difference. A Cognitive Style and Aggregation Operator (CSAO) model is proposed to analyze the mapping relationship between the aggregation operators and the cognitive styles represented by the decision attitudes. Four algorithms are proposed for CSAO: CSAO-1, CSAO-2 and two selection strategies on the basis of CSAO-1 and CSAO-2. The numerical examples illustrate how the choice of the aggregation operators on the basis of the decision attitudes can be determined by the selection strategies of CSAO-1 and CSAO-2. The CSAO model can be applied to decision making systems with the selection problems of the appropriate aggregation operators with consideration of the cognitive styles of the decision makers.

1. Introduction

Aggregation Operators (AOs) are applied in many domains on problems concerning the fusion of a collection of information granules. These domains include mathematics, physics, engineering, economics, business, management, and social sciences. Although the discussions of AOs are very broad [1, 3, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 40, 41], there is a lack of research about the best practice in choosing aggregation operators. The selection of the AOs can make use of the theory of cognitive style, but it seems no research has investigated the relationship between aggregation operators and the cognitive styles. Cognitive styles can be used to select the best individual for decision making.

The term 'cognitive style', was used by Allport [2], and has been described as a person's typical or habitual mode of problem solving, thinking, perceiving and remembering [22]. A style is considered to be a fairly fixed characteristic of an individual [22]. Studies in cognitive styles initially developed as a result of interest in individual differences, particularly during the 1960's [22]. Since the early 1970s, they have been more seriously considered by the teaching and training world [22]. In

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this research, the cognitive styles are associated with the development of artificial intelligence. The new motivation could be called computational cognitive style, which is to classify the individual styles of the algorithms or functions under the same objective. This research shows the classification of the aggregation operators using the cognitive styles.

Different researchers have used a variety of labels for the styles they have investigated. For example, Rading and Cheema [22] suggested that the labels be grouped into two principal cognitive styles. These were labeled the Wholist-Analytic and Verbialiser-Imager dimensions. Most researchers apply a set of uni-dimensional labels, which are postulated in the individual preferences, for the quantitative research of the cognitive style. This leads not to having a formal definition of the labels of the cognitive style. In this research, the cognitive style is described by a decision attitude variable which includes three basic linguistic labels: pessimistic, neural, and optimistic.

Many studies use linguistic methods in the decision making, for example [6, 12, 17, 18, 21, 35, 38, 39]. In fuzzy decision making process, a linguistic label is usually represented by a fuzzy number. Expert uses the linguistic label sets to access the candidates with respect to a series of criteria. An assessment result that is the selected fuzzy number assigned to the candidate under a criterion is called an information granule. The aggregation operator takes a set of information granules as input to yield a final meaningful result. There are many aggregation operators which lead to different final results. It is difficult to evaluate which aggregation operator performs better than others. This research suggested that an aggregation operator is analogue to the cognitive style of a human expert, and such cognitive style can be represented by a linguistic variable which is represented by a native fuzzy number. However, it seems that none of research discusses the use of linguistic method for such classification and selection of the aggregation operators.

The aggregation operator is a function or an algorithm to process information, analogous to the humans information process, which should be related to the scope of Cognitive psychology. Cognitive psychology is committed to using computers as a tool for aiding understanding of the mind [4]. Computational intelligence is one of studies of cognitive psychology. Cognitive style is the individual differences of the information processes of the mind. As there are similar relationships between the attributes of the aggregation operators and the cognitive styles, this paper proposes the Cognitive Style and Aggregation Operator (CSAO) model, which includes several algorithms to classify the individual styles of the AOs using the linguistic approach. In this paper, CSAO is the extension of Yuens work [36], which established the foundation for CSAO-1.

The paper is organized as follows. Section 2 defines the properties of aggregation operators whilst section 3 reviews different categories of the aggregation operators. On the basis of reviewed aggregation operators and first order linguistic ordinal scale using for cognitive style linguistic terms, section 4 proposes a CSAO-1 model. Section 5 introduces the Compound Linguistic Ordinal Scale (CLOS), which is the second order linguistic ordinal scale. Section 6 proposes a CSAO-2 model based on CLOS and CSAO-1. The numerical analyses are illustrated in section 7.

2. **Fundamental Definitions of Aggregation Operators**

The formal definitions of aggregation operators are as follows.

Definition 2.1. A generic aggregation operator *Agg* is a function which aggregates a set of granules $X = (x_1, \ldots, x_i, \ldots, x_n)$ into an aggregated value y. It has the form:

$$y = Agg_{(n)}^{(t)}\left(\alpha; (x_1, \ldots, x_i, \ldots, x_n)\right) = Agg_{(n)}^{(t)}\left(\alpha; X\right) \tag{1}$$

t is the length of tuple(s) of x_i and n is the number of the granules. α is a construct parameter or a bag of construct parameters to scale *Agg*.

Sometimes, α is not shown if the information of α is not important for discussion in some scenarios. Likewise, AO can be simplified as the notations such as *Agg*, $Agg\left(\alpha; X\right)$, $Agg^{(t)}\left(\alpha; X\right)$ or $Agg_{(n)}^{(t)}\left(\alpha; X\right)$. This research is only interested in $t \in \{1, 2\}$. To extend Definition 2.1, the following definition is proposed.

Definition 2.2. *Agg* is a non-weighted AO such that $x_i = c_i$ where $c_i \in C$ is a single element, or 1-tuple element. It has the form:

$$Agg^{(1)}\left(\alpha; X\right) = Agg^{(1)}\left(\alpha; (c_1, \ldots, c_i, \ldots, c_n)\right) = Agg^{(1)}\left(\alpha; C\right) \tag{2}$$

Definition 2.3. *Agg* is a weighted AO such that $x_i = \{c_i, v_i\}$ where $v_i \in V = (v_1, \ldots, v_n)$ is a utility weight. Thus x_i is a pair (or 2-tuple). The weighted AO is of the form:

$$Agg^{(2)}\left(\alpha; X\right) = Agg^{(2)}\left(\alpha; (\{c_1, v_1\}, \ldots, \{c_i, v_i\}, \ldots, \{c_n, v_n\})\right) \tag{3}$$

Definition 2.4. If $w_i = \frac{v_i}{\sum_{i=1}^{n} v_i}$, then $w_i \in W = (w_1, \ldots, w_n)$ is the probability weight such that $\sum\limits_{i \in \{1, \ldots, n\}} w_i = 1$. Thus A is a normalized weighted AO of the form:

$$Agg^{(2)}\left(\alpha; X\right) = Agg^{(2)}\left(\alpha; (\{c_1, w_1\}, \ldots, \{c_i, w_i\}, \ldots, \{c_n, w_n\})\right) \tag{4}$$

This paper focuses on discussion of normalized weighted AO.

Let y be the output of AO of X. Usually $y$and c_i have a fix interval $I' = [a, b] \subseteq [-\infty, \infty]$. Many studies used the fix interval $I = [0, 1]$ for discussion. This is the only mathematical matter of scaling or normalizing I' into I. To merge the discussion with other studies, and to associate membership theory with the aggregation problems (as the membership value also belongs to $[0, 1]$), this research uses a fix interval $I = [0, 1]$. The scaling functions of I' into I are beyond the research topic here. Now let X and y be scaled, and the extension of Definition 2.5 be as follows.

Definition 2.5. Let $c_i, y \in I$, $I = [0, 1]$. A non-weighted aggregation operator is the function$Agg : I^n \to I$. A weighted aggregation operator is the function $Agg : V^T \times I^n \to I$, and a normalized weighted aggregation operator is the function $Agg : W^T \times I^n \to I$.

According to [29, 13, 16, 8, 7] , there are some properties for the aggregators:

(1) Boundary conditions: $Agg\left(0, \ldots, 0\right) = 0$ and $Agg\left(1, \ldots, 1\right) = 1$;

(2) Monotonicity: $Agg(x_1,\dots,x_i,\dots,x_n) \geq Agg(x_1,\dots,x_i',\dots,x_n)$ if $x_i \geq x_i'$.
(3) Continuity: A is continuous with respect to each of its variables.
(4) Associativity: $Agg(x_1,x_2,x_3) = Agg(x_1, Agg(x_2,x_3))$
$= Agg(Agg(x_1,x_2),x_3)$.
(5) Symmetry: also known as commutativity or anonymity. For every permutation δ of $\{1,2,\dots,n\}$, the operator satisfies: $Agg\left(x_{\delta(1)},x_{\delta(2)},\dots,x_{\delta(n)}\right) = Agg(x_1,x_2,\dots,x_n)$.
(6) Bisymmetry: $Agg(A(x_{11},x_{12}), Agg(x_{21},x_{22})) =$
$Agg(Agg(x_{11},x_{21}), Agg(x_{12},x_{22}))$
(7) Absorbent Element: $Agg(x_1,\dots,a,\dots,x_n) = a$;
(8) Neutral Element: $Agg^{(n)}(x_1,\dots,e,\dots,x_n) = A^{(n-1)}(x_1,\dots,x_{n-1})$
(9) Idempotence: $Agg(x,x,\dots,x) = x$;
(10) Compensation: $\min_{i=1}^{n}(x_i) \leq Agg(x_1,x_2,\dots,x_n) \leq \max_{i=1}^{n}(x_i)$
(11) Reinforcement: full, downward, and upward reinforcements [29].

Different operators are associated with different choices of the above properties. There are no absolute rules that associate properties to operators. The researchers usually define some properties, and then create their operators.

3. Categories of Aggregation Operators

A non-weighted AO is the special case of a weighted AO such that all weights are equal. This study focuses on discussing the weighted AO. Aggregation operators have been contributed by many researchers. The followings introduce AOs which are frequently used and discussed.

3.1. **Quasi-linear Means.** The general form of quasi-linear means [5, 19, 23] is of the form:

$$qlm(W,C) = h^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\omega_i h(c_i)\right) \quad , \quad c \in I^p. \tag{5}$$

The function $h : I \to \Re$, called the generator of $qlm(w,c)$ is continuous and strictly monotonic. If $h(x) = x^\alpha$, qlm is the weighted root power (wrp) or weighted generalized mean, and other three types are extensions (Table 1).

1. Weighted Root Power $wrp(\alpha;W,C) = \left(\sum\limits_{i=1}^{n} w_i c_i^{\alpha}\right)^{1/\alpha}$	2. Weighted Harmonic mean ($\alpha \to -1$) $whm(W,C) = \frac{1}{\sum_{i=1}^{n}\frac{w_i}{c_i}}$
3. Weighted Geometric mean ($\alpha \to 0$) $wgm(W,C) = \prod\limits_{i=1}^{n} c_i^{w_i}$	4. Weighted Arithmetic mean ($\alpha \to 1$) $wam(W,C) = \sum_{i=1}^{n} w_i c_i$

Table 1. Some Forms of Quasi-linear Means

3.2. **Ordered Weighted Averaging.** OWA [29, 33] is the weighted arithmetic mean (*wam*) in which its weight values are related to the order position of C.

$$owa\,(W,C) = \sum\nolimits_{i=1}^{n} w_i b_j, \tag{6}$$

where b_j is the jth largest of the C, $w_i \in [0,1]$ and $\sum\limits_{i\in\{1,\ldots,n\}} w_i = 1$. w_i can be generated from a regular non-decreasing quantifier Q, which is of the form:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1,\ldots,n, \tag{7}$$

where Q can be defined by $Q\,(\alpha; r) = r^{\alpha}$, $\alpha \geq 0$.

3.3. **Weighted Median.** In weighted median aggregation [30, 23], each element c_i is replaced by two elements:

$$c_i^{+} = (1 - w_i) + w_i \cdot c_i \tag{8}$$

$$c_i^{-} = w_i \cdot c_i. \tag{9}$$

Then the median value is computed by

$$wmed\,(W,C) = Median\left(c_1^{+}, c_1^{-}, \ldots, c_i^{+}, c_i^{-}, \ldots, c_n^{+}, c_n^{-}\right). \tag{10}$$

Alternatively, c_i^{+} and c_i^{-} can be computed by T-conorm and T-norm, denoted as S and T respectively, having the forms as below

$$c_i^{+} = S\,(1 - w_i, c_i) \tag{11}$$

$$c_i^{-} = T\,(w_i, c_i) \tag{12}$$

which S and $T$are also defined in section 3.4.

3.4. **T-norms and T-conorms.** T-norms have the properties in which $T\,(x,1) = x$ and $T\,(x,y) \leq \min\,(x,y)$ whilst T-conorms have the properties in which $S\,(x,0) = x$ and $S\,(x,y) \leq \max\,(x,y)$ [8]. Different kinds of T-norms and T-connorms [8, 23] are shown in table 2.

3.5. **Weighted Gamma Operator.** Zimmermann and Zysno [41] proposed an gamma operator on the unit interval based on T-norms and T-conorms. Calvo and Mesiar [7] modified the equation with a weighted assignment, which is of the form:

$$wgo\,(\alpha; C, W) = \left(\prod\nolimits_{i=1}^{n} c_i^{w_i}\right)^{1-\alpha} \left(1 - \prod\nolimits_{i=1}^{n} (1 - c_i)^{w_i}\right)^{\alpha}. \tag{13}$$

3.6. *OWMAX* **and** *OWMIN***.** Ordered weighted maximum (*owmax*) and ordered weighted minimum operators (*owmin*) were proposed by Dubois el at. [9]. Unlike OWA which deals with weighted arithmetic mean, owmax and owmin apply weighted maximum and minimum [19]. For any weight vector $W = (w_1, \ldots, w_n) \in [0,1]^n$ such that $1 = w_1 \geq \ldots \geq w_n$, *owmax* is of the form:

$$owmax\,(W,C) = \bigvee_{i=1}^{n} \left(w_i \wedge c_{(i)}\right), \quad C \in [0,1]^n. \tag{14}$$

For $W \in [0,1]^n$ such that $w_1 \geq \ldots \geq w_n = 0$, *owmin* is of the form:

$$owmin\,(W,C) = \bigwedge_{i=1}^{n} \left(w_i \vee c_{(i)}\right), C \in [0,1]^n. \tag{15}$$

1. Min-Max	$Tm\left(a,b\right)=\min\left\{a,b\right\}$
	$Sm\left(a,b\right)=\max\left\{a,b\right\}$
2. Lukasiewicz	$Tl\left(a,b\right)=\max\left\{a+b-1,0\right\}$
	$Sl\left(a,b\right)=\min\left\{a+b,1\right\}$
3. Product/	$Tp\left(a,b\right)=ab$
Probabilistic	$Sp\left(a,b\right)=a+b-ab$
4. Dubois & Prade	$Tdp\left(\alpha;a,b\right)=\frac{a\cdot b}{\max\{a,b,\alpha\}},\quad \alpha\in(0,1)$
	$Sdp\left(\alpha;a,b\right)=1-\frac{(1-a)(1-b)}{\max\{(1-a),(1-b),\alpha\}},\alpha\in(0,1)$
5. Yager	$Ty\left(\alpha;a,b\right)=\max\left\{0,1-\left[(1-a)^{\alpha}+(1-b)^{\alpha}\right]^{1/\alpha}\right\}$
	$Sy\left(\alpha;a,b\right)=\min\left\{1,\left(a^{\alpha}+b^{\alpha}\right)^{1/\alpha}\right\},,\alpha>0$
6. Frank	$Tf\left(\alpha;a,b\right)=\log_{\alpha}\left[1+\frac{(\alpha^{a}-1)(\alpha^{b}-1)}{\alpha-1}\right],\alpha>0,\alpha\neq1$
	$Sf\left(\alpha;a,b\right)=1-\log_{\alpha}\left[1+\frac{(\alpha^{1-a}-1)(\alpha^{1-b}-1)}{\alpha-1}\right]$
7. Weber-Sugeno	$Tws\left(\alpha_T;a,b\right)=\max\left\{\frac{a+b-1+\alpha_T\cdot a\cdot b}{1+\alpha_T},0\right\},\alpha_T>-1$
	$Sws\left(\alpha_T;a,b\right)=\min\left\{a+b+\alpha_S\cdot a\cdot b,1\right\},\alpha_S=\frac{\alpha_T}{(1+\alpha_T)}$
8. Schweizer & Sklar	$Tss\left(\alpha;a,b\right)=1-\left[(1-a)^{\alpha}+(1-b)^{\alpha}-(1-a)^{\alpha}(1-b)^{\alpha}\right]^{\frac{1}{\alpha}}$
	$Sss\left(\alpha;a,b\right)=\left[a^{\alpha}+b^{\alpha}-a^{\alpha}b^{\alpha}\right]^{\frac{1}{\alpha}},\quad\alpha>0$

TABLE 2. List of T-norms and T-connorms

3.7. **Leximin Ordering.** Leximin ordering was proposed by Dubois et al. [10]. Yager [31] improved the Lexmin ordering, based on OWA weights. Let Δ denotes a distention threshold between the values being aggregated, the Leximin is of the form:

$$leximin\left(W,C\right)=\sum\nolimits_{i=1}^{n}w_ib_i \tag{16}$$

, where b_i is a sorted $C\in I^n$ in descending order such that $b_1>\ldots>b_n$. In addition,

$$w_j=LexW\left(\Delta,n\right)=\begin{cases}\frac{\Delta^{(n-j)}}{(1+\Delta)^{n-j}}\ ,j=1\\ \frac{\Delta^{(n-j)}}{(1+\Delta)^{n+1-j}},j=2,\ldots,n\end{cases},\quad w_j\in W \tag{17}$$

4. Decision Attitude and Aggregation Operator 1 (DAAO-1, or CSAO-1)

Under uncertainty, different decision makers would have different decision attitudes since they have characteristics of cognitive style or individual difference. The decision attitudes (DAs) can be described by a collection of linguistic terms represented by a collection of DA atomic fuzzy sets, $D=\{d_1,\cdots,d_j,\cdots,d_p\}$, (or the 1^{st} degree DA fuzzy variable) which is further classified as a collection of compound fuzzy sets $HD = \{d_{ij} : i = 1, \ldots, p; j = 1, \ldots, r\}$ with added directional hedge fuzzy sets $H = \{h_1, \ldots, h_r\}$ (The 2^{nd} degree DA fuzzy variable). Details of compound fuzzy variable are shown in section 5.

The range of the membership of a decision attitude fuzzy set is in [0,1] and the aggregated value also belongs to [0,1]. The aggregated value of the membership (or the likelihood) of a decision attitude has the relationship, shown in the following definition.

Definition 4.1. An aggregated value y from a normalized aggregation operator Agg of the set of input parameters X belongs to a decision attitude fuzzy set d_j, with the membership value $d_j(y) \in [0,1]$ by the membership function $d_j : y \to I$, $I = [0,1]$.

As the fuzzy set is characterized by the membership function, the same notation d_j is used for a fuzzy set of membership. Usually, the membership function applies a triangular function $\mu_j(a,b,c)$ which is defined by three points.

Different input parameter sets, X's, result in different Effective Aggregation Ranges (EAR) from a collection of the aggregation operators. The effective aggregation range $[y_*, y*]$ is defined as follows.

Definition 4.2. Let the set of the aggregated values from the set of the aggregation operators $\widetilde{Agg}$ be $Y = (y_1, \ldots, y_k, \ldots, y_m)$. The permutation of Y is $\vec{Y} = \{y_{(1)}, \ldots, y_{(k)}, \cdots, y_{(m)}\}$, where $y_{(1)} \le y_{(2)} \cdots \le y_{(m)}$. Thus, the Effective Aggregation Range is $[y_*, y*]$, where $y_* = y_{(1)} = \min(Y)$ is the low-boundary, and $y* = y_{(m)} = \max(Y)$ is the up-boundary.

Lemma 4.3. *The EAR is the proper subset of I, i.e.* $[y_*, y*] \subseteq [0,1]$ *(see Figure 1).*

Proof. As $y = A^{(t)}_{(n)}(\alpha; X) \in I = [0,1]$, $y_* = \min(Y) \ge 0$, and $y* = \max(Y) \le 1$, the lemma holds. □

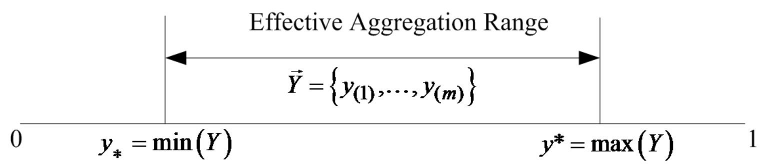

FIGURE 1. Effective Aggregation Range of AOs

Lemma 4.4. *The collection of AOs is the form $\widetilde{Agg} : X \to [y_*, y*]^m$, where m is the dimension (the number) of the output set.*

Proof. This lemma is directly derived from Definition 4.2. □

The CSAO model describes how the cognitive styles of the aggregation operators can be reflected by the decision attitudes. The CSAO can be represented by a collection of the DA fuzzy sets. Thus, the following proposition holds.

Proposition 4.5. *(D_{Agg}): The collection of decision attitude fuzzy sets for an aggregation operator A is of the form:*

$$D_{Agg} = \{\{y, d_1(y)\}, \ldots, \{y, d_j(y)\}, \ldots, \{y, d_p(y)\}\}, y \in [y_*, y*] \tag{18}$$

Proof. Let the collection of decision attitude fuzzy sets be $D = \{d_1, \cdots, d_j, \cdots, d_p\}$, and the discourse universal of D be $[y_*, y*] \subseteq [0, 1]$ (Lemma 4.3). Thus the collection of memberships of the set of decision attitudes D for an aggregation operator is $D_A : [y_*, y*] \to I^p$. As the fuzzy set is generally defined as a collection of pairs, the form is given above. □

Proposition 4.6. **($D_{\widetilde{Agg}}$):** *A collection of the 1st degree DA fuzzy sets $D_{\widetilde{Agg}}$ for a collection of aggregation operators $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$ is of the form:*

$$D_{\widetilde{Agg}} = \begin{pmatrix} \{\{y_{(1)}, d_1\} & ,\ldots, & \{y_{(1)}, d_j\} & ,\ldots, & \{y_{(1)}, d_p\}\} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \{\{y_{(k)}, d_1\} & ,\ldots, & \{y_{(k)}, d_j\} & ,\ldots, & \{y_{(k)}, d_p\}\} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \{\{y_{(m)}, d_1\} & ,\ldots, & \{y_{(m)}, d_j\} & ,\ldots, & \{y_{(m)}, d_p\}\} \end{pmatrix}, \tag{19}$$

where $\{y_{(k)}, d_j\} = \{y_{(k)}, d_j(y_{(k)})\}, \forall k, \forall j,$ *and* $y_{(1)} \leq y_{(2)} \cdots \leq y_{(m)}$.

Proof. It follows from Proposition 4.5 and Lemma 4.4. □

Definition 4.7. The Information Fusion Process $IFP = (\bar{X}, Y, AO*, \{AO\}, SAO)$ is the function which aggregates multiple sources of data granules $\bar{X}$ as a meaningful value Y to represent an object by selecting the most appropriate aggregation operator (AO*) among a set of the AO candidates $\widetilde{AO} = \{AO\}$, i.e. $SAO : \{AO\} \to AO^*$, and $AO^* : \bar{X} \to Y$.

The CSAO model is the ideal function for SAO. Following of the above definition, two definitions are proposed for the selection of AO in $D_{\widetilde{Agg}}$.

Definition 4.8. If an aggregation operator has more than one membership of DAs, the selection of DAs for the AO is of the form:

$$d^*(k) = ArgMax\left(\{\{y_{(k)}, d_1\}, \ldots, \{y_{(k)}, d_j\}, \ldots, \{y_{(k)}, d_p\}\}\right) \tag{20}$$

Definition 4.9. If a DA linguistic term includes more than one aggregation operator, the selection of AOs in a DA linguistic term is of the form:

$$d_j^* = ArgMax\left(\{\{y_{(1)}, d_j\}, \ldots, \{y_{(k)}, d_j\}, \ldots, \{y_{(m)}, d_j\}\}\right) \tag{21}$$

The *DAAO-1* for CSAO is in the following algorithm.

Algorithm 4.10. $DAAO\text{-}1 = CSAO\text{-}1\left(D, \widetilde{Agg}, X\right)$:

Input:

a. A collection of the membership functions of DA fuzzy sets $D = \{d_1, \cdots, d_j, \cdots, d_p\}$;

b. A collection of AOs: $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$;
c. A collection of information granules: $X = (x_1, \ldots, x_i, \ldots, x_n)$;

Process:

Step 1: Compute $\widetilde{Agg}\,(X)$, and then $Y = (y_1, \ldots, y_k, \ldots, y_m)$ is achieved;
Step 2: Get the permutation of Y: $\vec{Y} = \left\{y_{(1)}, \ldots, y_{(k)}, \cdots, y_{(m)}\right\}$;
Step 3: Get $[y_*, y^*] = \left[y_{(1)}, y_{(m)}\right]$;
Step 4: Calculate intervals and modal values for $D$by equally dividing $[y_*, y^*]$;
i. $d_1 = \left(y_*, y_*, y_* + \frac{y^*-y_*}{p-1}\right)$
ii. $d_{j\neq 1,p} = \left(y_* + \frac{y^*-y_*}{p-1}(j-2), y_* + \frac{y^*-y_*}{p-1}(j-1), y_* + \frac{y^*-y_*}{p-1}(j)\right)$
iii. $d_p = \left(y^* - \frac{y^*-y_*}{p-1}, y^*, y^*\right)$
Step 5: Elicit memberships for $D$by interpolation of the three points (a,b,c):
Step 6: Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^*(k)$, $\forall k$.
Step 7: Get $d_j^*, \forall j$.

Output:. $\left\{d_j^*\right\}$. **//END**

This study focuses on discussion of the weighted aggregation operators of which $x_i = \{w_i, c_i\} \in X$is the input.

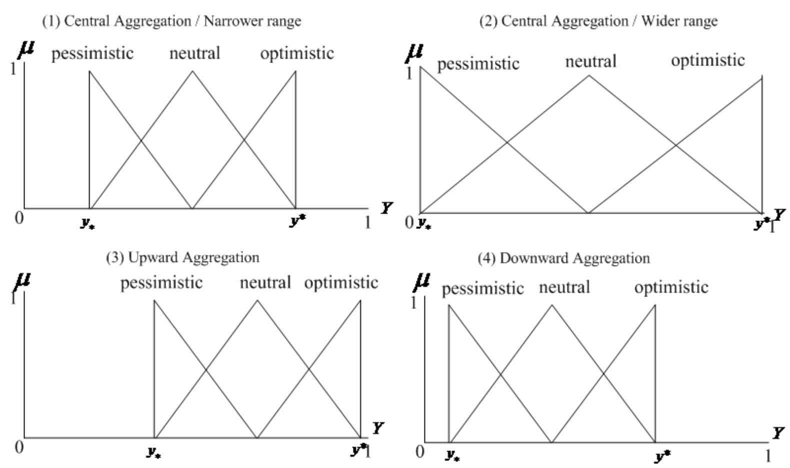

FIGURE 2. Properties of Effective Aggregation Range

To conclude, the CSAO description model is the function $g : X \to I$ or $g = \widetilde{Agg} \circ D = D\left(\widetilde{Agg}\,(X)\right)$. It means that the function g maps the collection of information granules X with the set of the aggregators $\widetilde{Agg}$, to the membership interval [0,1] corresponding to the collection of decision attitude fuzzy sets D.

In most practice, the decision attitudes can be described by three linguistic terms: pessimistic, neutral and optimistic. Figure 2 shows some properties of the DA fuzzy sets.

Properties of EAR can be summarized as followings.

Proposition 4.11. *Let* $y' = mean\,(y_*, y*) = \frac{1}{2}\,(y_* + y*)$, *then*

(1) *Effective aggregation range (EAR) is downward aggregation if* $y' < 0.5$*;.*
(2) *EAR is upward aggregation if* $y' > 0.5$*;*
(3) *EAR is central aggregation if* $y' = 0.5$*;*
(4) *EAR 2 is more upward than EAR 1 if* $y'_1 < y'_2$*. Or EAR 1 is more downward than the EAR 2.*
(5) *EAR 2 is wider than EAR 1 if* $y*_1 - y_{*_1} < y*_2 - y_{*_2}$*. Or EAR 1 is narrower than EAR 2.*

Example 4.12. A numerical example analysis of the algorithm of the CSAO-1 description model is illustrated as follows.

Input

a. Define the collection of decision attitude fuzzy sets:
Let $D = \{d_1, d_2, d_3\}$ represent the set of pessimistic, neural, and optimistic decision attitudes. $d_1 = \mu\,(y_*, y_*, y')$, $d_2 = \mu\,(y_*, y', y*)$, $d_2 = \mu\,(y', y', y*)$, where μ is the triangular membership function.

b. Define a collection of the Aggregation Operators:

$$\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_{17})$$
$$= \left\{ \begin{array}{l} wrp, whm, wgm, wam, owa, ow\max, owmin, \\ Lex\min, wgo, wmed, wmed_l, wmed_{mm}, \\ wmed_{dp}, wmed_y, wmed_f, wmed_{ws}, wmed_{ss} \end{array} \right\}$$

The aggregation operator can be found in section 3. For the notation, $wmed_l$ is $wmed$with Lukasiewicz T-norm and T-connorm. This naming convention is also applied to other $wmed$s, taking different T-norms and T-connorms. In addition, as α affects the aggregation result, different value of α can be regarded as a different operator. This example takes $\alpha = 0.2$, for all parametric operators.

c. Get the collection of information granules:
Let $X = (x_1, \ldots, x_5)$ be weighted criteria; $C = (0.4, 0.5, 0.6, 0.7, 0.9)$, $W = owaW\,(0.6, 5) = (0.3801, 0.1964, 0.1589, 0.1387, 0.1253)$,
and thus $X = ((0.4, 0.1978)\,, \ldots, (0.9, 0.6250))$.

Process

Step 1: Compute$Y$ by$\tilde{A}\,(X)$:

$$Y = \tilde{A}\,(X) = \left\{ \begin{array}{l} 0.5375, 0.5137, 0.5332, 0.5557, 0.6949, 0.3807, 0.4, 0.6939, \\ 0.5019, 0.4619, 0.5127, 0.5, 0.5, 0.5, 0.1193, 0.5199, 0.4868 \end{array} \right\}$$

(k)	*Agg*	$y_{(k)}$	$D\left(y_{(k)}\right)$	$d^*(k)$
1	$wmed_f$	0.1193	{1,0,0}	Pess
2	$ow\max$	0.3807	{0.0917,0.9083,0}	Pess
3	$ow\min$	0.4	{0.0247,0.9753,0}	Ntl
4	$wmed$	0.4619	{0,0.8095,0.1905}	Ntl
5	$wmed_{ss}$	0.4868	{0,0.7231,0.2769}	Ntl
6	$wmed_{mm}$	0.5	{0,0.6773,0.3227}	Ntl
7	$wmed_{dp}$	0.5	{0,0.6773,0.3227}	Ntl
8	$wmed_y$	0.5	{0,0.6773,0.3227}	Ntl
9	wgo	0.5019	{0,0.6707,0.3293}	Ntl
10	$wmed_l$	0.5127	{0,0.6333,0.3667}	Ntl
11	whm	0.5137	{0,0.6298,0.3703}	Ntl
12	$wmed_{ws}$	0.5199	{0,0.6080,0.3920}	Ntl
13	wgm	0.5332	{0,0.5618,0.4382}	Ntl
14	wrp	0.5375	{0,0.5470,0.4530}	Ntl
15	wam	0.5557	{0,0.4838,0.5162}	Opt
16	owa	0.6949	{0,0,1}	Opt
17	Leximin	0.6949	{0,0,1}	Opt

TABLE 3. The Results for $D_{\widetilde{Agg}}$ of 17 AOs

Step 2: Get the $\vec{Y}$:

Get$Ordering\,(Y) = \{14,11,13,15,16,2,3,16,9,4,10,6,6,6,1,12,5\}$, then

$$\vec{Y} = \left\{ \begin{array}{l} 0.1193, 0.3807, 0.4, 0.4619, 0.4868, 0.5, 0.5, 0.5, 0.5019, 0.5127, \\ 0.5137, 0.5199, 0.5332, 0.5375, 0.5557, 0.6939, 0.6949 \end{array} \right\}.$$

Step 3: $[y_*, y*] = \left[y_{(1)}, y_{(m)}\right] = [0.1193, 0.6949]$.

Step 4 and 5: Assign intervals and interpolate memberships for D.

Let $(y_*, y', y*) = [0.1193, 0.4071, 0.6949]$ be substituted by $\mu\,(a, b, c)$ in D. CSAO-1 pattern is shown in Figure 3. It can be observed that the proposed numerical integration is downward integration as$y' = 0.4071 < 0.5$.

Step 6: Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^*(k)$.

Table 3 summarizes the results for$D_{\widetilde{Agg}}$, $\left\{y_{(k)}, D_{Agg}\left(y_{(k)}\right)\right\} \in D_{\widetilde{Agg}}$, $\forall k \in \{1, \cdots, 17\}$.

Step 7 and Output: $\left\{d_j^*\right\} = \{1, 3, 17\}$, which means {$wmed_f$, *owmin*, *owa*/*Leximin*}, where *owa* and *Leximin* produce the same result.

The interpretation of the above example is as follows. The weighted median with other t-connorms and t-norms [23, 30] is likely to produce questionable results. Firstly, t-conform and t-norm are initially designed for aggregation of two fuzzy sets, and are not suitable for weighted criteria, since $wmed\,(W, C)$ has different meanings for $wmed\,(C, W)$. Secondly, the definition of the tuning parameter α

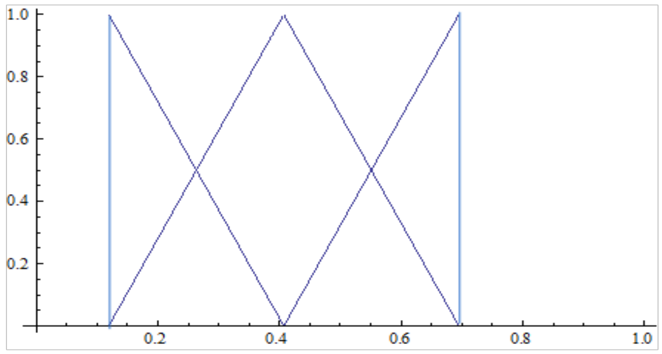

FIGURE 3. Fuzzy Sets in CSAO-1 Pattern

is infinitive since each α represents a new aggregation operator due to different output values. Thirdly, the more criteria are aggregated, the lesser values in W *as* $\sum_{w_i \in W} w_i = 1$ are followed. As t-norms or t-conorms is mainly based on Min and Max of two sets, a misleading result will result.

owmax and *owmin* are not the effective AOs for the decision matrix. The third reason of the above description explains this issue. *Lexmin* and *owa* produce the same result as the weights used by them, and are not defined by their intrinsic functions. If these aggregation operators are removed, the new result is shown in Example 4.13. Further investigation for *owa* is concluded after illustration of Example 4.13.

Example 4.13. Let be $\widetilde{Agg} = \{Agg_1, \ldots, Agg_k, \ldots, Agg_7\}$ ={*wrp,whm,wgm,wam, owa,wgo,wmed*}. Others remain unchanged. The new results for $D_{\widetilde{Agg}}$ are shown in Table 4, and finally, $\{d_j^*\} = \{1, 6, 7\}$, which is {*wmed*, *wam*, *owa*}.

(k)	Agg	$y_{(k)}$	$D_{\widetilde{Agg}}(y_{(k)})$	$d^*(k)$
1	*wmed*	0.4619	{1,0,0}	Pess
2	*wgo*	0.5019	{.6570,0343,0}	Pess
3	*whm*	0.5137	{.5558,04442,0}	Pess
4	*wgm*	0.5332	{.3880,06120,0}	Ntl
5	*wrp*	0.5375	{.3513,06487,0}	Ntl
6	*wam*	0.5557	{.1953,08047,0}	Ntl
7	*owa*	0.6949	{0,0,1}	Opt

TABLE 4. The Results for $D_{\widetilde{Agg}}$ of Seven AOs

Examples 4.12 and 4.13 imply that not all AOs can be applied in DSAO (DAAO). It is similar to that not all the people being suitable for a single job, an interest, or a subject domain as they have different cognitive styles. People who are suitable for a job are pooled and selected accordingly with respect to the senior decision maker. Thus only the suitable AOs can be taken in DSAO, and then classified. The one which mostly reflects the decision maker's cognitive style is selected.

In addition, *owa* seems to produce exaggerate results in the above example. The main reason is that the order of the values of the criteria is sorted in descending order. This action is unnecessary. For one reason, the weight and the criterion are matched; for another reason, the different initial settings of the criteria order are very likely to produce different results. For the third reason, there is no point to mismatch the weight and the criterion pair.

The next section discusses CLOS prior to discuss DSAO-2 which is based on CLOS and CSAO-1.

5. Compound Linguistic Ordinal Scale

Compound Linguistic Ordinal Scale (CLOS) and its application was developed by Yuen [37]. CLOS is a Deductive Rating Strategy (Rs) of the Hedge-Direction-Atom Linguistic Representation Model (HDA-LRM) with a cross reference relationship.

In the HDA-LRM, Compound Linguistic Variable (CLV) $\aleph$, a matrix of a large number of linguistic descriptors is produced by the syntactic rule. The semantic rule "Computing with CLV" maps CLV into representation numbers in matrix $\overline{X}_{\aleph}$ or $\overline{X}$ by Fuzzy Normal Distribution $f_{\overline{X}}(\aleph)$, and produces the numerical results meeting the different requirements of different scenario using few scalable descriptable user-defined parameters.

The Deductive Rating Strategy (Rs) is the ideal rating interface for handing the large scale of CLV. Three key concepts are presented, as follows.

5.1. **Syntactic Rule.** Regarding the syntactic form, CLOS is established on a compound linguistic variable $\alpha \in \aleph_{mn}$ which is comprised of the elements from the linguistic term vectors respectively: hedge vector$\overrightarrow{V_h}$ directional vector $\overrightarrow{V_d} = \left[v_d^-,\ v_d^\theta,\ v_d^+\right]$ and atomic vector $\overrightarrow{V_a} = \left[v_{a_j}\right]$. A matrix of Compound Linguistic Variable (CLV) $\aleph_{mn}$ is built on the syntactic rule algorithm (algorithm 2), $\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$, and has the following form:

$$\begin{bmatrix} \emptyset & v_{hd_1} \oplus v_{a_2} & \cdots & v_{hd_1} \oplus v_{a_{n-1}} & v_{hd_1} \oplus v_{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \emptyset & v_{hd_\eta} \oplus v_{a_2} & \ddots & v_{hd_\eta} \oplus v_{a_{n-1}} & v_{hd_\eta} \oplus v_{a_n} \\ {v_{a_1}}^\theta & {v_{a_2}}^\theta & \ddots & {v_{a_{n-1}}}^\theta & {v_{a_n}}^\theta \\ v_{hd_{\eta+2}} \oplus v_{a_1} & v_{hd_{\eta+2}} \oplus v_{a_2} & \ddots & v_{hd_{\eta+2}} \oplus v_{a_{n-1}} & \emptyset \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{hd_m} \oplus v_{a_1} & v_{hd_m} \oplus v_{a_2} & \cdots & v_{hd_m} \oplus v_{a_{n-1}} & \emptyset \end{bmatrix} \quad (22)$$

, where v_{hd} is the element of the combination of $\overrightarrow{V_h}$ and $\overrightarrow{V_d}$.

Algorithm 5.1. (Syntactic Rule Algorithm $\aleph_{mn} = G_{\aleph}\left(\overrightarrow{V_h}, \overrightarrow{V_d}, \overrightarrow{V_a}\right)$:

1. Input: Linguistic term sets ($\overrightarrow{V_h}$,$\overrightarrow{V_d}$,$\overrightarrow{V_a}$)

2. Proceed $G_{\overrightarrow{V_{hd}}}\left(\overrightarrow{V_h},\overrightarrow{V_d}\right)=\overrightarrow{V_{hd}}$ by

$$[v_{hd_i}]_{i=1}^{m}=\left[\left(v_{h_\eta}\oplus v_d^-\right),\ \cdots\ ,\left(v_{h_1}\oplus v_d^-\right),\ v_d^\theta\ ,\left(v_{h_1}\oplus v_d^+\right),\ \cdots\ ,\left(v_{h_\eta}\oplus v_d^+\right)\right]$$

3. Proceed $G_{\aleph}\left(\overrightarrow{V_{hd}},\overrightarrow{V_a}\right)$ by

$$G_{\alpha_{ij}}\left(v_{hd_i},v_{a_j}\right)\triangleq\begin{cases}\emptyset & j=1\&i\in\{1,\cdots,((m+1)/2)\}\\ v_{hd_i}\oplus v_{a_j} & j\neq 1,n\&\forall i\\ \emptyset & j=n\&i\in\{((m+1)/2),\cdots,m\}\end{cases},\forall i,j$$

4. Return: $\aleph_{mn}=G_{\aleph}\left(\overrightarrow{V_{hd}},\overrightarrow{V_a}\right)$ **//END**

5.2. **Semantic Rule.** The numerical representation is derived by the semantic rule algorithm or Fuzzy Normal Distribution of the form:

$$\bar{X}_{\aleph}=f_{\overline{X}}(\aleph)=f_{\overline{X}}\left(\left\{\left(\gamma_{\alpha^j},d_{\alpha^j},\tau_{\alpha^j},\{\mu_{\alpha^j\varphi}^{-1}\}^{\varphi}\right)\right\},[X_{\min},X_{\max}],\left(\phi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right) \quad (23)$$

$\bar{X}_{\aleph}$ is the numerical representation of $\aleph$in either fuzzy or crisp value as crisp value is the special case of the fuzzy value. γ_{α^j} is the modal value, d_{α^j} is symmetric distance (by default, $d_{\alpha^1}=d_{\alpha^2}=,\ldots,=d_{\alpha^n}$), τ_{α^j} is tuning parameter of the membership function, μ_{α^j}, of α^j, and $\mu_{\alpha^j\varphi}^{-1}$ is the inversed membership function, which the default setting is the inversed parabola-based membership function $PbMF_{\alpha^j}^{-1}$ is of the form:

$$PbMF_{\alpha^j}^{-1}\left(\mu_{\alpha^j\phi}\right)=\begin{cases}\gamma_{\alpha^j}-d_{\alpha^j}\sqrt{1-\left(\mu_{\alpha^j\phi}\right)^{1/\tau_{\alpha^j}}},\phi='-'\\ \gamma_{\alpha^j},\phi='\theta'\\ \gamma_{\alpha^j}+d_{\alpha^j}\sqrt{1-\left(\mu_{\alpha^j\phi}\right)^{1/\tau_{\alpha^j}}},\phi='+'\end{cases} \quad (24)$$

, where $\varphi='-','\theta''+'$ is determined from $\overrightarrow{V_d}$

$[X_{\min},X_{\max}]$ is the interval of numerical representation of the scale. The 2-tuple input $\left(\phi\left(\overrightarrow{V_h}\right),\lambda_0\right)$ determines the distribution of the $\overrightarrow{V_{hd}}$ in the membership fuzziness process (MFI). Thus $f_{\overline{X}}(\aleph)$ is shown in Algorithm 5.2.

Algorithm 5.2. (Semantic Rule Algorithm / Fuzzy Normal Distribution):

1. Get valid $\left(\left\{\left(\gamma_{\alpha^j},d_{\alpha^j},\tau_{\alpha^j},\{\mu_{\alpha^j\varphi}^{-1}\}^{\varphi}\right)\right\},[X_{\min},X_{\max}],\left(\phi\left(\overrightarrow{V_h}\right),\lambda_0\right)\right)$.
2. Calculate $MCI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)$ and $MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)$ by

$$MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)=MFI\left(\begin{bmatrix}v_{h_1}\\ \vdots\\ v_{h_\eta}\end{bmatrix}\right)$$
$$=\begin{bmatrix}\left[\mu'_{l_i}-\lambda_{\mu_{l_i}}dis\left(v_{h_i}\right),1\right]_{i=1}\\ \left[\left[\mu'_{l_i}-\lambda_{\mu_{l_i}}dis\left(v_{h_i}\right),\mu'_{u_i}+\lambda_{\mu_{l_i}}dis\left(\sigma_i\right)\right]\right]_{i=2}^{\eta-1}\\ \left[0,\mu'_{u_i}+\lambda_{\mu_{l_i}}dis\left(\sigma_i\right)\right]_{i=\eta}\end{bmatrix},$$

where $dis\left(v_{h_i}\right) = \frac{\phi\left(v_{h_i}\right)}{\sum_{\overrightarrow{V_h}} \phi\left(v_{h_i}\right)}$; $\lambda_{\mu_{u_i}} = \lambda_{\mu_{l_i}} = \frac{\lambda_0}{2}$, where$\lambda_{\mu_{u_i}}, \lambda_{\mu_{l_i}} \in [0,1]$ (i.e. $\lambda_0 \in [0,2]$) such that $0 \leq \mu_{u_i} \leq 1$.

3. Calculate $MFI\left(\left[\left[\overrightarrow{V_h}^{+}\right]\right]\right)$ and $MFI\left(\left[\left[\overrightarrow{V_h}^{-}\right]\right]\right)$ by

$$MFI\left(\left[\left[\overrightarrow{V_h}^{-}\right]\right]\right) = vip\left(MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)\right) \equiv \overset{\mu_L^- \ \mu_U^-}{\left\lceil \mu_{l_j}, \mu_{u_j} \right\rceil_{j=1}^{\eta}} = \overset{\mu_L \quad \mu_U}{\left[\left[\mu_{l_j}, \mu_{u_j}\right]\right]_{j=\eta}^{1}}$$

$$MFI\left(\left[\left[\overrightarrow{V_h}^{+}\right]\right]\right) = hrp\left(MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right)\right) \equiv \overset{\mu_L^+ \ \mu_U^+}{\left\lfloor \mu_{l_j}, \mu_{u_j} \right\rfloor_{j=1}^{\eta}} = \overset{\mu_U \quad \mu_L}{\left[\left[\mu_{U_j}, \mu_{l_j}\right]\right]_{j=1}^{\eta}}$$

$$\text{,where } MFI\left(\left[\left[\overrightarrow{V_h}\right]\right]\right) = MFI\left(\left[\begin{array}{c} v_{h_1} \\ \vdots \\ v_{h_\eta} \end{array}\right]\right) = \overset{\mu_L \quad \mu_U}{\left[\left[\mu_{l_j}, \mu_{u_j}\right]\right]_{j=1}^{\eta}}$$

4. Calculate $FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right)$,$\forall j$ by

$$FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right) = \begin{cases} \left[\begin{array}{c} \left[\left[\mu_{\alpha^{j-}}^{-1}\left(\mu_{l_i}{}^{-}\right), \mu_{\alpha^{j-}}^{-1}\left(\mu_{l_i}{}^{-}\right)\right]\right]_{i=1}^{\eta} \\ \left[\left[\mu_{\alpha^j}^{-1}\left(\mu_{l_\theta}\right), \mu_{\alpha^j}^{-1}\left(\mu_{l_\theta}\right)\right]\right]_{\text{i}=\eta+1} \end{array}\right], j=1 \\ \left[\begin{array}{c} \left[\left[\mu_{\alpha^{j-}}^{-1}\left(\mu_{l_i}{}^{-}\right), \mu_{\alpha^{j-}}^{-1}\left(\mu_{l_i}{}^{-}\right)\right]\right]_{i=1}^{\eta} \\ \left[\left[\mu_{\alpha^j}^{-1}\left(\mu_{l_\theta}\right), \mu_{\alpha^j}^{-1}\left(\mu_{l_\theta}\right)\right]\right]_{\text{i}=\eta+1} \\ \left[\left[\mu_{\alpha^{j+}}^{-1}\left(\mu_{l_i}{}^{+}\right), \mu_{\alpha^{j+}}^{-1}\left(\mu_{l_i}{}^{+}\right)\right]\right]_{i=\eta+2}^{m} \end{array}\right], 1 < \text{j} < \text{n} \quad , \forall j \\ \left[\begin{array}{c} \left[\left[\mu_{\alpha^j}^{-1}\left(\mu_{l_\theta}\right), \mu_{\alpha^j}^{-1}\left(\mu_{l_\theta}\right)\right]\right]_{\text{i}=\eta+1} \\ \left[\left[\mu_{\alpha^{j+}}^{-1}\left(\mu_{l_i}{}^{+}\right), \mu_{\alpha^{j+}}^{-1}\left(\mu_{l_i}{}^{+}\right)\right]\right]_{i=\eta+2}^{m} \end{array}\right], j=n \end{cases}$$

5. If $\overline{X}_{\aleph}$ is in fuzzy number, then
$\overline{X} = \left[\left[\left[\left(\bar{x}_{l_{ij}}, \bar{x}_{\pi_{ij}}, \bar{x}_{u_{ij}}\right)\right]\right]_{i=1}^{m}\right]_{j=1}^{n}$,
$\bar{x}_{\pi_{ij}} \in mean\left(FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right)\right)$ and $\left(\bar{x}_{l_{ij}}, \bar{x}_{u_{ij}}\right) \in FI\left(\left[\left[\widehat{\alpha^j}\right]\right]\right)$, $\forall i, j$
6. Return $\overline{X}$;.

//END

5.3. **Deductive Rating Strategy.** It seems incredible that an expert can handle $|\aleph_{7\pm2,7\pm2}| = [21, 73]$ linguistic terms although CLV can produce a large scale of compound linguistic terms. Thus deductive rating strategy is proposed. Algorithm 5.3 shows the rating steps whilst Figure 4 shows an example of the rating interface.

Algorithm 5.3. (Deductive Rating Strategy $\left(\overrightarrow{V_{hdj}}, \overrightarrow{V_a}, Rs\right)$):

1. Observe external information;
2. Understand the problem;
3. Understand the CLOS model;
4. First step rating: choose v_{a_j} in $\overrightarrow{V_a} \equiv \left[v_{a_j}\right]_{j=1}^{n}$;
5. Computer shows second options by

$$\overrightarrow{V_{hdj}} = Rs\left(v_{a_j}\right) = \begin{cases} \left[v_{hd_i}\right]_{i=1}^{\eta} & \text{if } j = 1 \\ \left[v_{hd_i}\right]_{i=1}^{m} & \text{if } j \neq 1, n \\ \left[v_{hd_i}\right]_{i=\eta+2}^{m} & \text{if } j = n \end{cases} .$$

6. Rethink the second option and revise first option;
   6.1 If first option is confirmed, then the rater chooses v_{hd_i} in $\overrightarrow{V_{hdj}}$;
   6.2 Else go to Step 3
7. Return $\alpha_{ij} = \left(v_{hd_i}, v_{a_j}\right)$ **//END**

6. Decision Attitude and Aggregation Operator 2 (DAAO-2, or CSAO-2)

Usually a fuzzy set consists of several AOs. If a decision maker chooses a linguistic term for the decision attitude, although the choices are narrowed, he still needs to choose the right one representing his cognitive style. Thus the DA atomic fuzzy set is further classified by a vector of hedge terms$H = \{h_1, \ldots, h_\eta, \ldots, h_r\}$, which is represented by a vector of the memberships of DA d_j , the following proposition holds.

Proposition 6.1. **($\{d_{ij}\} = f_{\overline{X}}(HD)$):** *The Linguistic Cartesian Product $G_\aleph$ of D and H forms a collection of compound fuzzy sets $HD = \{h_i \oplus d_j : i = 1, \ldots, r; j = 1, \ldots, p\}$, which is of the form.*

$$HD = G_\aleph(H, D) = \begin{bmatrix} \emptyset & h_1 \oplus d_2 & \cdots & h_1 \oplus d_p \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset & h_\eta \oplus d_2 & \ddots & h_\eta \oplus d_p \\ {d_1}^\theta & {d_2}^\theta & \ddots & {d_p}^\theta \\ h_{\eta+2} \oplus d_1 & h_{\eta+2} \oplus d_2 & \ddots & \emptyset \\ \vdots & \vdots & \ddots & \vdots \\ h_r \oplus d_1 & h_r \oplus d_2 & \cdots & \emptyset \end{bmatrix} \tag{25}$$

Let $\{d_{ij}\}$ be the matrix of the fuzzy numbers of HD. $\{d_{ij}\}$ is determined by the semantic rule algorithm $f_{\overline{X}}(HD)$ (algorithm 5.2), which is of the form:

$$\{d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p\} = f_{\overline{X}}(HD)$$

$$= f_{\overline{X}}\left(\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \left\{\mu_{d^j\phi}{}^{-1}\right\}^{\phi}\right)\right\}, [y_*, y^*], \left(\varphi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right) \tag{26}$$

,where $\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \{\mu_{d^j\varphi}^{-1}\}^{\varphi}\right)\right\}$ is the 1st degree DA fuzzy sets which are the symmetric fuzzy set: γ_{d^j} is the modal value, Δ_{d^j} is symmetric distance (by default, $\Delta_{d^1} = \Delta_{d^2} =, \ldots, = \Delta_{d^p}$), τ_{d^j} is the tuning parameter of the membership function, μ_{d^j} is the membership function of d_j or d^j , and $\mu_{d^j\varphi}^{-1}$ is the inverse membership function. The collection of the 1st degree DA fuzzy sets is called the 1st degree DA

fuzzy variable. The parameters of the membership fuzziness process $\left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)$ *determine the distribution of the* 2^{nd} *degree DA fuzzy variable with respect to the corresponding* 1^{st} *degree DA fuzzy sets.*

Proof. It follows from Algorithms 5.1 and 5.2. □

The compound linguistic terms for the decision attitude are used by a deductive rating strategy which is the double step rating process (algorithm 4). The collection of the 2^{nd} degree DA fuzzy sets is shown in the following proposition.

Proposition 6.2. **(** $D"_{\widetilde{Agg}}$**):** *A collection of the* 2^{nd} *degree DA fuzzy sets* $D"_{\widetilde{Agg}}$ *for a collection of aggregation operators* $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$*is of the form:* $D_{\widetilde{Agg}}\left(\vec{Y}\right) =$

$$\begin{bmatrix} \emptyset & \left\{\begin{matrix}\{y_{(1)}, d_{1,2}\}\\ \vdots \\ \{y_{(m)}, d_{1,2}\}\end{matrix}\right\} & \cdots & \left\{\begin{matrix}\{y_{(1)}, d_{1,p}\}\\ \vdots \\ \{y_{(m)}, d_{1,p}\}\end{matrix}\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset & \left\{\begin{matrix}\{y_{(1)}, d_{\eta,2}\}\\ \vdots \\ \{y_{(m)}, d_{\eta,2}\}\end{matrix}\right\} & \ddots & \left\{\begin{matrix}\{y_{(1)}, d_{\eta,p}\}\\ \vdots \\ \{y_{(m)}, d_{\eta,p}\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{y_{(1)}, d_{\eta+1,1}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+1,1}\}\end{matrix}\right\} & \left\{\begin{matrix}\{y_{(1)}, d_{\eta+1,2}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+1,2}\}\end{matrix}\right\} & \ddots & \left\{\begin{matrix}\{y_{(1)}, d_{\eta+1,p}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+1,p}\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{y_{(1)}, d_{\eta+2,1}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+2,1}\}\end{matrix}\right\} & \left\{\begin{matrix}\{y_{(1)}, d_{\eta+2,2}\}\\ \vdots \\ \{y_{(m)}, d_{\eta+2,2}\}\end{matrix}\right\} & \ddots & \emptyset \\ \vdots & \vdots & \ddots & \vdots \\ \left\{\begin{matrix}\{y_{(1)}, d_{r,1}\}\\ \vdots \\ \{y_{(m)}, d_{r,1}\}\end{matrix}\right\} & \left\{\begin{matrix}\{y_{(1)}, d_{r,2}\}\\ \vdots \\ \{y_{(m)}, d_{r,2}\}\end{matrix}\right\} & \cdots & \emptyset \end{bmatrix} \tag{27}$$

Proof. Proposition 4.5 indicates D_{Agg}, which extends further to $D_{\widetilde{Agg}}$ in proposition 4.6. Proposition 6.1 develops the syntactic form and semantic form of the collection of compound fuzzy sets $\{d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p\}$ for the decision attitude. The D_{Agg} can be applied to $f_{\overline{X}}(HD)$. Thus, the form of $D"_{\widetilde{Agg}}$ is derived. □

Regarding the final selection of the representation of 2^{nd} degree DA fuzzy sets and AOs, two definitions are formed.

Definition 6.3. If an aggregation operator has more than one of the 2^{nd} degree DA fuzzy sets, the selection of DAs for the dedicated AO is of the form:

$$d"^{*}(k) = ArgMax\left(\left\{\left\{y_{(k)}, d_{i,j}\right\} : d_{i,j} \neq \emptyset\right\}\right) \tag{28}$$

$d"^{*}(k)$ returns the index of the linguistic label to describe the AO.

Definition 6.4. If the 2^{nd} degree DA fuzzy set d_{ij} includes more than one aggregation operator, the selection of AOs of d_{ij} is of the form:

$$d_{ij}^{*} = ArgMax\left(\left\{\left\{y_{(1)}, d_{ij}\right\}, \ldots, \left\{y_{(k)}, d_{ij}\right\}, \ldots, \left\{y_{(m)}, d_{ij}\right\}\right\}\right) \tag{29}$$

d_{ij}^{*} returns the index in $\vec{Y}$ to represent the linguistic label d_{ij}.

Algorithm 6.5. $DAAO\text{-}2 = CSAO\text{-}2\left(D, \widetilde{Agg}, X, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$ **:**

Input:

a. A collection of the 1^{st} degree DA linguistic variable: $D = \{d_1, \cdots, d_j, \cdots, d_p\}$ is comprised of the membership set $\{\mu_{d^j}\}$ and the corresponding inverse membership set $\left\{\mu^{-1}_{d^j, \varphi='-','\theta','+'}\right\}$ with the tuning factor set $\{\tau_{d^j}\}$;

b. A vector of hedge terms $\overrightarrow{V_h}$ and A vector of directional terms $\overrightarrow{V_d}$;

c. A collection of AOs: $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_m)$;

d. A collection of information granules: $X = (x_1, \ldots, x_i, \ldots, x_n)$;

e. A collection of the parameters of the member fuzziness process: $\left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)$;

Process:

Step 1: Compute $\widetilde{Agg}(X)$, and then $Y = (y_1, \ldots, y_k, \ldots, y_m)$ is achieved;

Step 2: Get the permutation of Y: $\vec{Y} = \left\{y_{(1)}, \ldots, y_{(k)}, \cdots, y_{(m)}\right\}$;

Step 3: Get $[y_*, y^*] = \left[y_{(1)}, y_{(m)}\right]$;

Step 4: Calculate intervals and $\{(\gamma_{d^j}, \Delta_{d^j})\}_{j=1}^{p}$ for $D$by equally diving $[y_*, y^*]$;

i. $d_1 = \left(y_*, y_*, y_* + \frac{y^*-y_*}{p-1}\right) = (\gamma_{d^1}, \gamma_{d^1}, \gamma_{d^1} + \Delta_{d^1})$

ii. $d_{j\neq 1,p} = \left(y_* + \frac{y^*-y_*}{p-1}(j-2), y_* + \frac{y^*-y_*}{p-1}(j-1), y_* + \frac{y^*-y_*}{p-1}(j)\right)$

$$= (\gamma_{d^j} - \Delta_{d^j}, \gamma_{d^j}, \gamma_{d^j} + \Delta_{d^j})$$

iii. $d_p = \left(y^* - \frac{y^*-y_*}{p-1}, y^*, y^*\right) = (\gamma_{d^p} - \Delta_{d^p}, \gamma_{d^p}, \gamma_{d^p})$

Step 5: Elicit memberships μ_{d^j} for $D$by interpolation of (a,b,c).

Step 6: Calculate $D\left(\vec{Y}\right)$, $D_{\widetilde{Agg}}$ and $d^{*}(k)$, $\forall k$.

Step 7: Form HD with rating interface by Algorithm 5.1.

Step 8: Calculate $\{d_{ij} : i = 1, \ldots, r; j = 1, \ldots, p\}$ of HD by $f_{\overline{X}}\left(\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \mu^{-1}_{d^j\varphi}\right)\right\}, [y_*, y^*], \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$ (Algorithm 5.2)

Step 9: Calculate $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$

Step 10: Calculate $d"^{*}(k)$, $\forall k$ in $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$.

Step 11: Calculate d_{ij}^{*}, $i = 1, \ldots, r; j = 1, \ldots, p$

Output: $\left\{d_{ij}^{*}\right\}$ **//END**

Example 6.6. This example is a continuation of Example 4.12. DAAO-2 is illustrated as follows.

Input:

a. $D = \{d_1, d_2, d_3\} = \{\text{P,N,O}\}$;
μ_{d^j} is the symmetric triangular membership, $\forall j \in \{1,2,3\}$;
$\mu^{-1}_{d^j,\varphi='-','\theta','+'}$ is the inversed triangular membership set, $\forall j \in \{1,2,3\}$;
$\tau_{d^j} = 1, \quad \forall j \in \{1,2,3\}$;

b. $\overrightarrow{V_h} = [\text{Little,Quite,Much}]$, and $\overrightarrow{V_d} = [\text{Below,Absolutely,Above}]$

c. A collection of AOs: $\widetilde{Agg} = (Agg_1, \ldots, Agg_k, \ldots, Agg_{17})$;

d. A collection of information granules:
$C = (0.4, 0.5, 0.6, 0.7, 0.9)$,
$W = owaW\,(0.6, 5) = (0.3801, 0.1964, 0.1589, 0.1387, 0.1253)$,
and thus $X = ((0.4, 0.1978), \ldots, (0.9, 0.6250))$;

e. A collection of the parameters of the member fuzziness process:
$\left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right) = (\{1,2,3\}, 0.5)$;

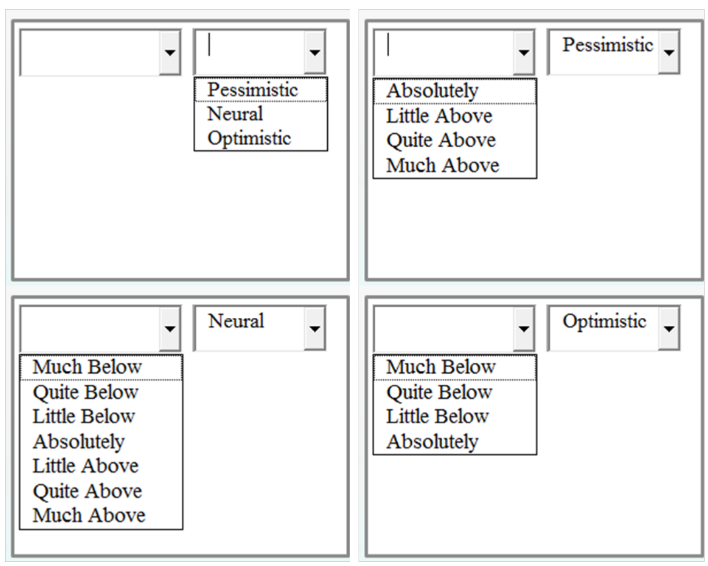

FIGURE 4. Deductive Rating Strategy in the Rating Interface of CLOS

Process:

Step 1-3: $Y = \widetilde{Agg}\,(X) = \left\{ \begin{array}{l} 0.5375, 0.5137, 0.5332, 0.5557, 0.6949, 0.3807, 0.4, 0.6939, \\ 0.5019, 0.4619, 0.5127, 0.5, 0.5, 0.5, 0.1193, 0.5199, 0.4868 \end{array} \right\}$;

$\vec{Y} = \left\{ \begin{array}{l} 0.1193, 0.3807, 0.4, 0.4619, 0.4868, 0.5, 0.5, 0.5, 0.5019, 0.5127, \\ 0.5137, 0.5199, 0.5332, 0.5375, 0.5557, 0.6939, 0.6949 \end{array} \right\}$;

$[y_*, y*] = \left[y_{(1)}, y_{(m)}\right] = [0.1193, 0.6949]$;

Step 4: Calculate intervals and $\{(\gamma_{d^j}, \Delta_{d^j})\}_{j=1}^{3}$ for D:

i. $d_1 = (0.1193, 0.1193, 0.4071)$;

ii. $d_2 = (0.1193, 0.4071, 0.6949)$;

iii. $d_3 = (0.4071, 0.6949, 0.6949)$;

iv. $\{\gamma_{d^j}\} = \{0.1193, 0.4071, 0.6949\}$ and $\Delta_{d^j} = 1, \forall j \in \{1, 2, 3\}$;

Step 5: Elicit memberships μ_{d^j} for D. The results are shown in Figure 3.

Step 6: Calculate $D\left(\vec{Y}\right) = \left\{D\left(y_{(k)}\right)\right\}$, $D_{\widetilde{Agg}}$ and $d^*(k)$, $\forall k$. The results are shown in Table 3.

Step 7: Form HD with rating interface by Algorithm 5.1.

$\overrightarrow{V_{hd}} = [v_{hd_1}, \cdots, v_{hd_7}] =$ [Much Below,Quite Below, Little Below, Absolutely, Little Above,Quite Above,Much Above], thus

$$HD = G_{\aleph}(H, D) = \begin{bmatrix} \emptyset & MB-N & MB-O \\ \emptyset & QB-N & QB-O \\ \emptyset & LB-N & LB-O \\ A-P & A-N & A-O \\ LA-P & LA-N & \emptyset \\ QA-P & QA-N & \emptyset \\ MA-P & MA-N & \emptyset \end{bmatrix}$$

And the rating interface is shown in Figure 4.

Step 8: Calculate $\{d_{ij}\}$ of HD by

$f_{\overline{X}}\left(\left\{\left(\gamma_{d^j}, \Delta_{d^j}, \tau_{d^j}, \mu^{-1}_{d^{j\varphi}}\right)\right\}, [y_*, y^*], \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$ (Algorithm 5.2). Thus,

$$\{d_{ij}\} = \begin{pmatrix} \emptyset & (0.1193, 0.2092, 0.2992) & (0.4071, 0.4971, 0.5870) \\ \emptyset & (0.2392, 0.3112, 0.3831) & (0.5270, 0.5990, 0.6709) \\ \emptyset & (0.3472, 0.3771, 0.4071) & (0.6350, 0.6649, 0.6949) \\ (0.1193, 0.1193, 0.1193) & (0.4071, 0.4071, 0.4071) & (0.6949, 0.6949, 0.6949) \\ (0.1193, 0.1493, 0.1793) & (0.4071, 0.4371, 0.4671) & \emptyset \\ (0.1433, 0.2152, 0.2872) & (0.4310, 0.5030, 0.5750) & \emptyset \\ (0.2272, 0.3172, 0.4071) & (0.5150, 0.6050, 0.6950) & \emptyset \end{pmatrix}$$

Step 9: Calculate $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$

$$\begin{pmatrix} \emptyset & \left(0^{17}\right) & \begin{pmatrix} 0^3, 0.6095, 0.8860, 0.9672, \\ 0.9672, 0.9672, 0.9463, \\ 0.8265, 0.8152, 0.7455, \\ 0.5978, 0.5503, 0.3482, 0^2 \end{pmatrix} \\ \emptyset & \left(0^1, 0.0332, 0^{16}\right) & \begin{pmatrix} 0^{12}, 0.0860, 0.1455, \\ 0.3981, 0^2 \end{pmatrix} \\ \emptyset & \left(0^1, 0.8799, 0.2372, 0^{14}\right) & \left(0^{17}\right) \\ \left(1, 0^{16}\right) & \left(0^{17}\right) & \left(0^{16}, 1, 1\right) \\ \left(0^{17}\right) & \left(0^3, 0.1716, 0^{13}\right) & \emptyset \\ \left(0^{17}\right) & \begin{pmatrix} 0^3, 0.4285, 0.7742, 0.9576, \\ 0.9576, 0.9576, 0.9838, \\ 0.8665, 0.8523, 0.7652, \\ 0.5806, 0.5212, 0.2686, 0^2 \end{pmatrix} & \emptyset \\ \begin{pmatrix} 0^1, 0.2933, \\ 0.0791, 0^{14} \end{pmatrix} & \begin{pmatrix} 0^{11}, 0.0545, 0.2022, 0.2497, \\ 0.4518, 0^2 \end{pmatrix} & \emptyset \end{pmatrix}$$

"0" means that the membership of AO is equal to zero in this compound linguistic term. The index of "0" means the number of zeros.

Step 10**:** Calculate $d"^{*}(k)$, $\forall k$ in $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$. The results are shown in Table 5.

(k)	Agg	$y_{(k)}$	$D"\left(y_{(k)}\right)$	$d"^{*}(k)$
1	$wmed_f$	0.1193	{A-P(1)}	A-P
2	$ow\max$	0.3807	{MA-P(0.293),QB-N(0.033),LB-N(0.880)}	LB-N
3	$ow\min$	0.4	{MA-P(0.079),QB-N(0.237)}	QB-N
4	$wmed$	0.4619	{LA-N(0.172),QA-N(0.428),MB-P(0.609)}	MB-P
5	$wmed_{ss}$	0.4868	{QA-N(0.774),MB-O(0.886)}	MB-O
6	$wmed_{mm}$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
7	$wmed_{dp}$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
8	$wmed_y$	0.5	{QA-N(0.958),MB-O(0.967)}	MB-O
9	wgo	0.5019	{QA-N(0.984),MB-O(0.946)}	QA-N
10	$wmed_l$	0.5127	{QA-N(0.866),MB-O(0.827)}	QA-N
11	whm	0.5137	{QA-N(0.852),MB-O(0.815)}	QA-N
12	$wmed_{ws}$	0.5199	{QA-N(0.765),MA-N(0.054),MB-O(0.745)}	QA-N
13	wgm	0.5332	{QA-N(0.581),MA-N(0.202),MB-O(0.598),QB-O(0.086)}	MB-O
14	wrp	0.5375	{QA-N(0.521),MA-N(0.250),MB-O(0.550),QB-O(0.145)}	MB-O
15	wam	0.5557	{QA-N(0.269),MA-N(0.452),MB-O(0.348),QB-O(0.398)}	MA-N
16	owa	0.6949	{A-O(1)}	A-O
17	Leximin	0.6949	{A-O(1)}	A-O

TABLE 5. The Results for $D"\left(\vec{Y}\right)$ and $d"^{*}(k)$ of 17 AOs

Step 11 **and Return:** Calculate d^{*}_{ij}, $i = 1, \ldots, r; j = 1, \ldots, p$

$$\left\{d^{*}_{ij}\right\} = d^{*}_{ij}\left(\begin{bmatrix} \emptyset & MB-N & MB-O \\ \emptyset & QB-N & QB-O \\ \emptyset & LB-N & LB-O \\ A-P & A-N & A-O \\ LA-P & LA-N & \emptyset \\ QA-P & QA-N & \emptyset \\ MA-P & MA-N & \emptyset \end{bmatrix}\right) = \begin{bmatrix} \emptyset & 0 & 6,7,8 \\ \emptyset & 2 & 15 \\ \emptyset & 2 & 0 \\ 1 & 0 & 16,17 \\ 0 & 4 & \emptyset \\ 0 & 9 & \emptyset \\ 2 & 15 & \emptyset \end{bmatrix}$$

"0" means no AO is available in this compound linguistic term. Another number means the index in $\vec{Y}$.

If a linguistic term (e.g. MB-O, A-O) includes more than one AOs (e.g. (6,7,8) or (16,17)), either of the AOs can be used since the AOs produce the same result with respect to a compound fuzzy set.

Example 6.7. Using DAAO-2, this example considers only seven AOs used in Example 4.13. Steps 1 and 7 are skipped. The remains of the steps are illustrated as follows:

Step 8**:** Calculate $\{d_{ij}\}$

$$\{d_{ij}\} = \begin{pmatrix} \emptyset & (0.4619, 0.4983, 0.5347) & (0.5784, 0.6148, 0.6512) \\ \emptyset & (0.5105, 0.5396, 0.5687) & (0.6270, 0.6561, 0.6852) \\ \emptyset & (0.5542, 0.5663, 0.5784) & (0.6707, 0.6828, 0.6949) \\ (0.4619, 0.4619, 0.4619) & (0.5784, 0.5784, 0.5784) & (0.6949, 0.6949, 0.6949) \\ (0.4619, 0.4741, 0.4862) & (0.5784, 0.5906, 0.6027) & \emptyset \\ (0.4716, 0.5008, 0.5299) & (0.5881, 0.6173, 0.6464) & \emptyset \\ (0.5056, 0.5402, 0.5784) & (0.6221, 0.6585, 0.6949) & \emptyset \end{pmatrix}$$

Step 9**:** Calculate $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$.

$$D"_{\widetilde{Agg}}\left(\vec{Y}\right) = \begin{pmatrix} \emptyset & \begin{pmatrix} 0^1, 0.9025, 0.5786, \\ 0.0415, 0^3 \end{pmatrix} & (0^7) \\ \emptyset & \begin{pmatrix} 0^2, 0.1101, 0.7813, \\ 0.9282, 0.4477, 0^1 \end{pmatrix} & (0^7) \\ \emptyset & (0^5, 0.1255, 0^1) & (0^7) \\ (1, 0^6) & (0^7) & (0^7, 1) \\ (0^7) & (0^7) & \emptyset \\ (0^2, 0.9615, 0.5566, 0^4) & (0^7) & \emptyset \\ \begin{pmatrix} 0^2, 0.2214, 0.7584, \\ 0.8759, 0.6248, 0^1 \end{pmatrix} & (0^7) & \emptyset \end{pmatrix}$$

"0" means that the membership of AO is equal to zero in this compound linguistic term. The index of "0" means the number of zeros.

Step 10**:** Calculate $d"^*(k)$, $\forall k$ in $D"_{\widetilde{Agg}}\left(\vec{Y}\right)$, which are shown in Table 6.

Step 11**:** and Return : $\left\{d^*_{ij}\right\}$ is shown as follows.

$$\left\{d^*_{ij}\right\} = d^*_{ij}\left(\begin{bmatrix} \emptyset & MB-N & MB-O \\ \emptyset & QB-N & QB-O \\ \emptyset & LB-N & LB-O \\ A-P & A-N & A-O \\ LA-P & LA-N & \emptyset \\ QA-P & QA-N & \emptyset \\ MA-P & MA-N & \emptyset \end{bmatrix}\right) = \begin{bmatrix} \emptyset & 2 & 0 \\ \emptyset & 5 & 0 \\ \emptyset & 6 & 0 \\ 1 & 0 & 7 \\ 0 & 0 & \emptyset \\ 2 & 0 & \emptyset \\ 5 & 0 & \emptyset \end{bmatrix}$$

"0" means no AO is available in this compound linguistic term. Another number means the index in $\vec{Y}$.

One can purely use DAAO-1, or DAAO-2. However, the selection function by $ArgMax$ is excessively straightforward in DAAO-1 in many AO candidates for one DA linguistic term d_j , whilst DAAO-2 contains no AOs for some linguistic terms if insufficient AO candidates for the relatively large scale of the compound linguistic terms. Regarding the number of AO candidates, the selection strategy to combine DAAO-1 and DAAO-2 is of the following algorithm.

(k)	*Agg*	$y_{(k)}$	$D"\left(y_{(k)}\right)$	$d"^{*}(k)$
1	*wmed*	0.4619	{A-P(1)}	A-P
2	*wgo*	0.5019	{QA-P(0.962),MB-N(0.903)}	QA-P
3	*whm*	0.5137	{QA-P(0.557),MA-P(0.221),MB-N(0.579),QB-N(0.110)}	MB-N
4	*wgm*	0.5332	{MA-P(0.758),MB-N(0.042),QB-N(0.781)}	QB-N
5	*wrp*	0.5375	{MA-P(0.876),QB-N(0.928)}	QB-N
6	*wam*	0.5557	{MA-P(0.625),QB-N(0.448),LB-N(0.126)}	MA-P
7	*owa*	0.6949	{A-O(1)}	A-O

TABLE 6. The Results for $D"\left(\vec{Y}\right)$ and $d"^{*}(k)$ of Seven AOs

Algorithm 6.8. (Selection Strategy, $SAO\left((d_j, h_i), \left(D_{\widetilde{Agg}}, d"^{*}(k)\right)\right)$):
Input: $D_{\widetilde{Agg}}$ of DAAO-1, and $d"^{*}(k)$ of DAAO-2.

Selection Process:

Step 1: Select an atomic term of DA d_j.

Step 2: Check if no AO return for the d_j in $D_{\widetilde{Agg}}$,
True: Return empty message and go to Step 1.
False: Go to Step 3.

Step 3: Check if only one AO return for the d_j in $D_{\widetilde{Agg}}$,
True: Return $Agg_{(k)}$.
False: Go to Step 4.

Step 4: Select the directional hedge term h_i.

Step 5: Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $d"^{*}(k)$,
True: Return empty message and go to Step 4 or 1.
False: Return $Agg_{(k)} = d"^{*}(k)$.

Return: $Agg_{(k)}$. **//End**

Example 6.9. Consider Examples 4.13 and 6.7. The rating interface can be referred to Figure 4. Three cases are illustrated.

Case 1: d_3= 'Opt'.
Input: $D_{\widetilde{Agg}}$ of DAAO-1 in table 4 and $d"^{*}(k)$ of DAAO-2 in Table 6.
Selection Process
Step 1: Select an atomic term of DA: d_3="Opt".
Step 2: *owa* return for the d_j in $D_{\widetilde{Agg}}$,
Step 3: Only one AO return for the d_j in $D_{\widetilde{Agg}}$,
Return: $Agg_{(7)} = owa$

Case 2: d_2="Ntl".
Input: $D_{\widetilde{Agg}}$ of DAAO-1 in table 4 and $d"^{*}(k)$ of DAAO-2 in Table 6.

Selection Process:

Step 1: Select an atomic term of DA: d_2="Ntl".

Step 2 and 3: wgm, wrp, and wam return for the d_j.

Step 4: Select the directional hedge term h_i.

Step 5: Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $d"^* (k)$,

True: Return empty message and go to Step 4 (As d_j="Ntl" is assumed,
Step 1 is skipped).
False: Return $Agg_{(k)} = d"^* (k)$.

Return: $Agg_{(k)} = wgo, wrp, wam$ depends on which valid h_i is firstly selected.

Case 3: which d_1="Pes", is similar to Case 2. $Agg_{(k)} = wmed, wgo, wrp$ depends on which valid h_i is firstly selected.

7. CSAO in Decision Matrix

In a decision matrix, more than one alternative is considered. This means different input value sets$X$'s possibly produce different $\{d^* (k)\}$, $\left\{d_j^*\right\}$, and $\left\{d_{ij}^*\right\}$. To address this issue, three definitions are created as follows:

Definition 7.1. In the decision matrix, the linguistic presentation of the style of the decision attitude for the AOs is computed by the form:

$$\left\{d_\beta^* (k)\right\}^* = Max \left(Mode \left(Join \left(\left\{d_\beta^* (k)\right\}\right)\right)\right), \tag{30}$$

where β is the index of the alternative of the decision matrix. $Join$ is the function which combines the matrices, and $Mode$ is the value that occurs the most frequently in an entry of $Join\left(\left\{\left\{d_{ij}^*\right\}_\beta\right\}\right)$.

Definition 7.2. In a decision matrix, the AO of the style of the decision attitude for the linguistic terms is computed as:

$$\left\{\left\{d_j^*\right\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\left\{d_j^*\right\}_\beta\right\}\right)\right)\right) \tag{31}$$

Definition 7.3. let $\left\{d_{ij}^*\right\}_\beta$ be the DAAO-2 pattern of the alternativeβ. Then, the pattern of the decision matrix is of the form:

$$\left\{\left\{d_{ij}^*\right\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\left\{d_{ij}^*\right\}_\beta\right\}\right)\right)\right) \tag{32}$$

If more than one AO index is returned in the entry, the index number with the highest value is chosen since it is likely to produce higher value for each alternative of the decision matrix. Thus the Max is taken. Also Max can eliminate "0" values. The Selection Strategy in Decision matrix is illustrated in Algorithm 7.4.

Algorithm 7.4. $(Agg_{(k)} = \overline{CSAO}\left((h_i, d_j), \{X\}, \widetilde{Agg}, D, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$:

Input: $(h_i, d_j), D, \widetilde{Agg}, X, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)$

Process:

Step 1: Calculated $d_\beta^*(k)$ in $CSAO1\left(D, \widetilde{Agg}, X_\beta\right) \quad \forall\beta \in \{1,\ldots,|\{X\}|\}$ (Algorithm 4.10)

Step 2: $\{d_j^*\}_\beta = CSAO1\left(D, \widetilde{Agg}, X_\beta\right)$, $\forall\beta \in \{1,\ldots,|\{X\}|\}$ (Algorithm 4.10)

Step 3: $\{d_{ij}^*\}_\beta = CSAO2\left(D, \widetilde{Agg}, X_\beta, \left(\overrightarrow{V_h}, \overrightarrow{V_d}\right), \left(\phi\left(\overrightarrow{V_h}\right), \lambda_0\right)\right)$, $\forall\beta \in \{1,\ldots,|\{X\}|\}$ (Algorithm 5.1)

Step 4: $\left\{d_\beta^*(k)\right\}^* = Max\left(Mode\left(Join\left(\left\{d_\beta^*(k)\right\}\right)\right)\right)$

Step 5: $\left\{\{d_j^*\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\{d_j^*\}_\beta\right\}\right)\right)\right)$

Step 6: $\left\{\{d_{ij}^*\}_\beta\right\}^* = Max\left(Mode\left(Join\left(\left\{\{d_{ij}^*\}_\beta\right\}\right)\right)\right)$

Step 7: Check if no AO return for the d_j in $\left\{d_\beta^*(k)\right\}^*$,

True: Return Empty message and go to Input to request another d_j.
False: Go to Step 4.

Step 8: Check the numbers of AO's return for the d_j in $\left\{d_\beta^*(k)\right\}^*$,

1: Return $Agg_{(k)} = \left\{d_\beta^*(k)\right\}^*$ without considering h_i.
2-3: Return $Agg_{(k)} = \left\{\{d_j^*\}_\beta\right\}^*$ without considering h_i.
≥ 4 : Go to Step 9.

Step 9: Check if no AO return for the $d_{ij} = h_i \oplus d_j$ in $\left\{\{d_{ij}^*\}_\beta\right\}^*$,

True: Return empty message and go to **Input** with new (h_i, d_j).
False: Return $Agg_{(k)} = \left\{\{d_{ij}^*\}_\beta\right\}^*$.

Return: $Agg_{(k)}$. **//End**

The use of this algorithm is shown in the section 7.3. The next section performs the numerical analyses for the proposed DSAO model to validate its usability and validity.

8. Numerical Analyses

Three major analyses are performed and discussed as follows.

8.1. Scenario.

Consider a decision matrix as follows,

$$\begin{array}{ccc} & W & (w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5) \\ & C & (c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5) \\ \bar{O} = & \begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{array} & \begin{pmatrix} 0.5 & 0.5 & 0.6 & 0.7 & 0.9 \\ 0.5 & 0.7 & 0.9 & 0.8 & 0.5 \\ 0.6 & 0.9 & 0.5 & 0.7 & 0.5 \\ 0.4 & 0.5 & 0.6 & 0.8 & 0.9 \\ 0.5 & 0.9 & 0.5 & 0.7 & 0.5 \end{pmatrix} \end{array},$$

where $W = owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$, which is shown in Table 7.

In this section, firstly, ten different decision matrices of the above form are created with 10 weight sets (Table 7). The matrices are further aggregated by 10 aggregation operators defined as: $\widetilde{Agg} = (whm, wgm, wam, wmed, wrp01, wrp05, wrp20, wgo01, wgo05, wgo09)$, where 01 means $\alpha = 0.1$, and so on.

Secondly, regarding discussion of the research values, the decision matrix with $\alpha = 0.9$ is selected for the application of DSAO-2.

δ	w_1	w_2	w_3	w_4	w_5
0.1	0.851	0.061	0.038	0.028	0.022
0.2	0.725	0.108	0.070	0.053	0.044
0.3	0.617	0.143	0.098	0.077	0.065
0.4	0.525	0.168	0.122	0.099	0.085
0.5	0.447	0.185	0.142	0.120	0.106
0.6	0.381	0.196	0.159	0.139	0.125
0.7	0.324	0.202	0.173	0.156	0.145
0.8	0.276	0.205	0.184	0.172	0.163
0.9	0.235	0.203	0.193	0.187	0.182
1	0.200	0.200	0.200	0.200	0.200

TABLE 7. W Generated by $owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$

8.2. **Properties of Individual AOs.** Ten decision matrices of the variation of weight sets are aggregated by ten AOs. The weight sets are generated by $owaW(\delta), \delta \in \{0.1, 0.2, \ldots, 1\}$ and are shown in Table 7. The larger δ means the less gap among the individual weights. When $\delta = 1$, all weights are of equal values. The data are plotted in Figures 5 and 6 .

Figures 5 and 6 show that different AOs behave differently for different decision matrices. This means that each AO has a different style. *wrp and wgo* with different α produce different results and likely different ranks. This means that a different AO with different α can have its own style.

Although $w_1 > w_2 > \ldots > w_5$ except for $\delta = 1$, the distribution among the weights are narrowed whilst δ increases. The sensitivity of each AO for the changes of weight is different. When the difference among the weights get less (e.g. increase of δ), the outputs of *wgo05 and wgo09* decrease while the outputs of others AOs increase. In addition, *wmed* has relative sensitivity of the change of the values of weights.

Regarding the patterns of the AO population in the figures, the figures show that the lines of AOs are closer while δ decreases. When δ increases, which means the gap of the weights of the criteria is reduced, the lines get farther apart. The main reason is that the criteria in a high index becomes more significant, and the values of the criteria in a higher index are more than the values of the criteria in a lower index.

Regarding the patterns of CSAO, the number of the AOs in Opt should be more than the number of the AOs in Pes. The main reason is that more lines are located

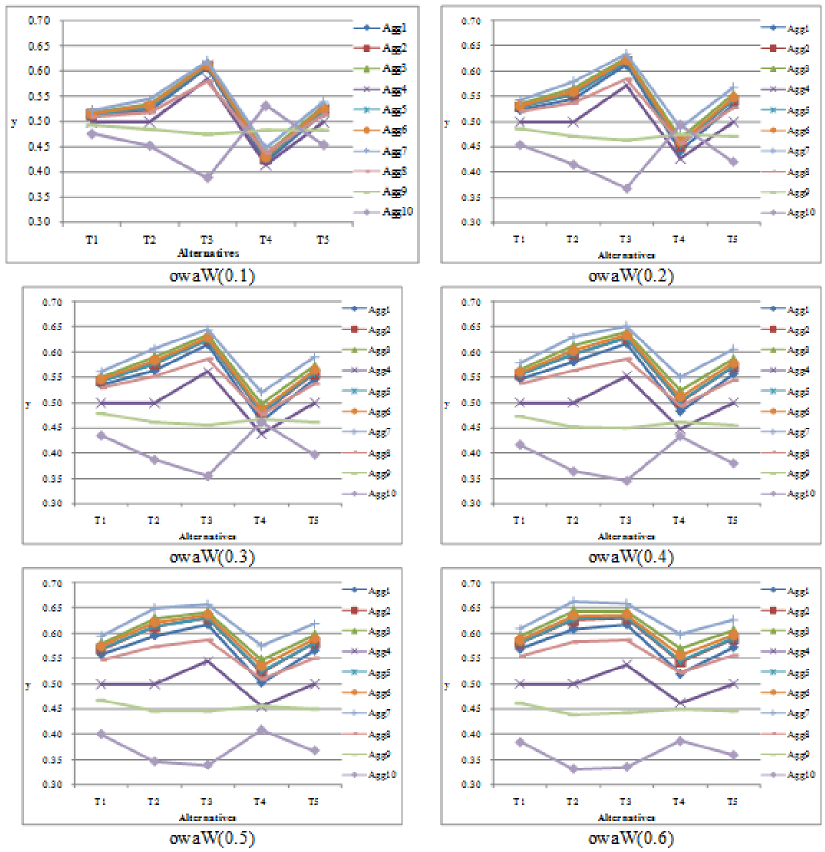

FIGURE 5. Results of Individual Aggregation Operators (Part I)

in upper position of the y-axis. In the next sub-section this issue is investigated in depth.

8.3. **Selection of AO by CSAO.** What a decision maker finally feels of interest is not the properties of the aggregation operators, but which AO is the most suitable. In fact, there is likely no absolute answer. In the real world, no decision maker can always guarantee an absolutely accurate answer (except for those who are arrogant), but the best and the most appropriate answer which he think it is correct (but others may not agree). Similarly, why they make different decisions when the objective situation and background are the same? One of the explanations is that they have different cognitive styles or individual differences. Some make clever decisions whilst some do not. In the mathematician's view, how they make decision can be modeled by equations. In the CSAO model, each AO reflects a different cognitive style. CSAO is used to classify the cognitive styles. This research proposes that

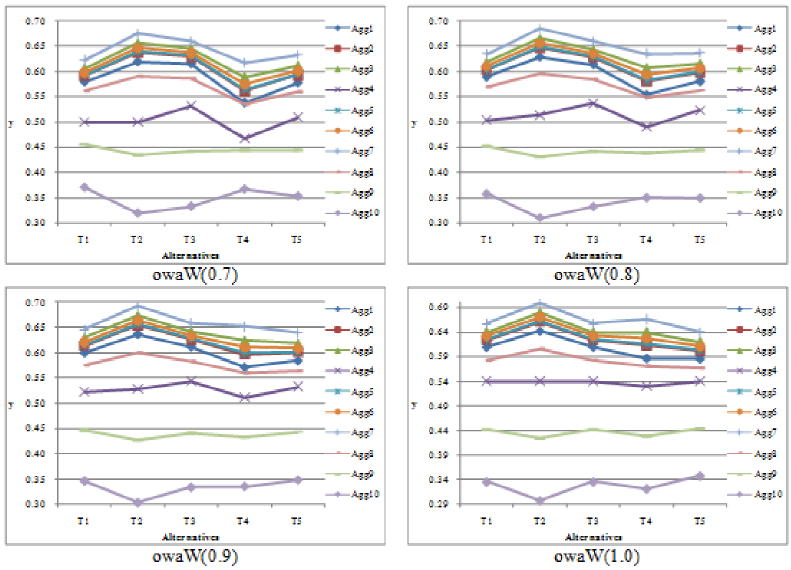

FIGURE 6. Results of Individual Aggregation Operators (Part II)

CSAO is represented by DAAO-1 and DAAO-2. The rating interface can be referred to Figure 4.

Tables 8 and 9 show the $\left\{d_{\beta}^{*}(k)\right\}^{*}$ and $\left\{\{d_{j}^{*}\}_{\beta}\right\}^{*}$ of DAAO-1 of the proposed decision matrix where $W = owaW\,(0.9)$. Interestingly, no matter which alternative input set of the decision matrix is used, the order of the AOs (k) is preserved to be the same.

(k)	Agg	$d_1^*(k)$	$d_2^*(k)$	$d_3^*(k)$	$d_4^*(k)$	$d_5^*(k)$	$\left\{d_{\beta}^{*}(k)\right\}^{*}$
1	$wrp01$	Pes	Pes	Pes	Pes	Pes	Pes
2	$wrp05$	Ntl	Ntl	Ntl	Ntl	Ntl	Ntl
3	$wgo05$	Ntl	Ntl	Ntl	Ntl	Ntl	Ntl
4	wam	Opt	Opt	Opt	Ntl	Ntl	Opt
5	$wrp20$	Opt	Opt	Opt	Ntl	Opt	Opt
6	$wgo01$	Opt	Opt	Opt	Opt	Opt	Opt
7	$wgo09$	Opt	Opt	Opt	Opt	Opt	Opt
8	$wmed$	Opt	Opt	Opt	Opt	Opt	Opt
9	wgm	Opt	Opt	Opt	Opt	Opt	Opt
10	whm	Opt	Opt	Opt	Opt	Opt	Opt

TABLE 8. The Linguistic Presentation of the Style of the Decision Attitude for the AOs of the Decision Matrix $\{d^{*}(k)\}^{*}$

j	d	$\{d_j^*\}_1$	$\{d_j^*\}_2$	$\{d_j^*\}_3$	$\{d_j^*\}_4$	$\{d_j^*\}_5$	$\left\{\{d_j^*\}_\beta\right\}^*$
1	Pes	1	1	1	1	1	1
2	Ntl	3	3	3	3	3	3
3	Opt	10	10	10	10	10	10

TABLE 9. The AO of the Style of the Decision Attitude for the Linguistic Terms of the Decision Matrix $\left\{\{d_j^*\}_\beta\right\}^*$

If the decision maker chooses *Opt*, there are seven options to represent the Optimistic AO. It is too subjective to use ArgMax in equation 21, thus DAAO-2 is needed. From DAAO-2 (Algorithm 5.1), $\{d_{ij}^*\}_1$, $\{d_{ij}^*\}_2$,$\{d_{ij}^*\}_3$,$\{d_{ij}^*\}_4$,$\{d_{ij}^*\}_5$ are as below respectively:

$$\begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 0 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 2 & 4 \\ \emptyset & 2 & 7 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 4 & \emptyset \\ 2 & 7 & \emptyset \end{bmatrix}, \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 0 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 6 & \emptyset \end{bmatrix}$$

From equation 32 in Definition 7.3, then

$$\{d_{ij}^*\}^* = Max\left(Mode\left(Join\left(\left\{\{d_{ij}^*\}_\beta\right\}\right)\right)\right) = \begin{bmatrix} \emptyset & 0 & 3 \\ \emptyset & 2 & 5 \\ \emptyset & 0 & 9 \\ 1 & 0 & 10 \\ 0 & 3 & \emptyset \\ 0 & 3 & \emptyset \\ 2 & 5 & \emptyset \end{bmatrix}$$

If a decision maker chooses “Pes” for the AO in the decision system, in the first rating step, there is only one choice, $wrp01$, as it is indicated in Table 8. The second rating category in V_{hd} is unnecessary.

If “Ntl” is chosen, for the representation of AO, $wrp05$ and $wgo05$ are the candidates, by using equation 7, where $wgo05$ is for “Ntl”.

When “Opt” is chosen, there are seven candidates. It is too straightforward to use ArgMax (equation 21). Thus the second rating category in V_{hd} is needed. The index of the AO can be found in $\{d_{ij}^*\}^*$. $wgo05$,$wrp20$,$wgo09$ and wgm are the options with respect to the choice of the second rating linguistic term.

9. **Conclusions**

As different aggregation operators produce different results, these results can be described by the possibility likelihoods of the cognitive styles. The selection of the aggregation operators is related to the likelihoods of the cognitive styles of the operators. To achieve the proposal, the Cognitive Style and Aggregation Operator (CSAO) model is proposed to analyze the mapping relationship between aggregation operators and cognitive style on the basis of fuzzy set theory. The CSAO model has two types of Decision Attitude and Aggregation Operator (DAAO) model: DAAO-1, DAAO-2. The difference is that DAAO-1 applies classical single dimension linguistic terms whilst DAAO-2 applies the compound linguistic terms. Three Algorithms for AO selection are developed.

The appropriate operators will be chosen according to the linguistic terms of the decision attitudes in the CSAO model. The cognitive style is characterized by the decision attitude. The CSAO model is useful for measuring the distribution of the AOs.

Examples 4.12 and 6.6 test 17 AOs. On the basis of the result pattern, Examples 4.13 and 6.7 select only 7 AOs. From the numerical examples, it can be concluded that the weighted median with other t-connorms and t-norms, *owmax*, *owmin*, and *owa* is not appropriate for the aggregation of the decision matrix. The reasons are stated after the numerical Examples 4.12 and 4.13.

In the section of numerical analyses, 10 AOs are tested for 10 decision matrices. The best practices of AO selection are illustrated using the combination of DAAO-1 and DAAO-2.

Limitation of the CSAO model is that the CSAO relies on the definitions of the candidates. If some candidates are abnormal, the CSAO pattern will be abnormal too. Usually the abnormal operators produce excessively optimistic or excessively positive results. In this case, the expert can remove the abnormal AO by his perception, and then recalculate the patterns again. After several refinements of the patterns, the appropriate CSAO model can be developed.

The CSAO is devoted to a proposal as how to map a collection of aggregation operators into a collection of decision attitudes by the CSAO model. This model is typically useful for those unsolved issues in the selection of aggregation operators. The OA candidates are determined by the decision maker with respect to the cognitive styles, which are characterized by decision attitudes. Thus the CSAO model is useful for the decision making applications with consideration of the cognitive styles (or decision attitudes) of the decision makers.

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