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Sliding Mode Dirichlet Boundary Stabilization of Uncertain Parabolic PDE Systems With Spatially Varying Coefficients

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***Abstract*—We consider the robust boundary stabilization problem of an unstable parabolic partial differential equation (PDE) system with uncertainties entering from both the spatially-dependent parameters and from the boundary conditions. The parabolic PDE is transformed through the Volterra integral into a damped heat equation with uncertainties, which contains the matched part (the boundary disturbance) and the mismatched part (the parameter variations). In this new coordinates, an infinite-dimensional sliding manifold that ensures system stability is constructed. For the sliding mode boundary control law to satisfy the reaching condition, an adaptive switching gain is used to cope with the above uncertainties, whose bound is unknown.**

I. Introduction

Physical systems such as heat conduction processes [1]–[3], chemical tubular reactors, and diffusion-convection-reaction plants are liable to certain degrees of modeling errors and exogenous disturbances. The governing partial differential equations are usually subject to oversimplification due to linearization or non-homogenity from the interior domain, such that the system model requires additional spatially varying or time dependent parameters to manifest some of its critical features [4,5], rather than simply using the common assumption of constant parameters. For distributed parameter systems (DPS) governed by parabolic PDEs with both spatial and time domain parameters, use of finite-dimensional approximation methods(*e.g.*, Galerkin's method) can give rise to the spillover problems due to neglected residue dynamics, and hence will weaken the system performance. It is the main focus of this paper to develop control schemes by studying the intricate PDE model directly.

Boundary control of PDE systems has been well-investigated recently [6]. Most of them are devoted the PDE systems with well-modeled assumption [1,2] or unknown parameters [7,8]. Some of them are dedicated to robust issues of unmodeled dynamics and external boundary disturbances such as [9,10]. In [10], Drakunov *et al.* proposed a sliding mode controller (SMC) for the boundary control problem of a stable heat equation with boundary disturbances. An integral transformation was employed to reformulate this problem into a first-order PDE. A sliding manifold as a function of the distributed states is presented. Unfortunately, this approach is limited to the stable case, and the applied control laws are discontinuous. The chattering problem can not be avoid in reality.

The "chattering phenomenon", regarded as the main drawback of classical SMC, is mainly resulted from the discontinuous control input switching at an infinite frequency [11]. To mitigate chattering, a number of methods (including boundary layer method [12], observer-based solution and higher-order SMC [13]) have been proposed. The boundary layer method is utilized to replace the " signum" function [12]. This approach has effectively reduced the chattering effect with the compromise in robustness of sliding mode and at the cost of steady-state error. On the other hand, the designing goal of high-order sliding modes [13] is focus on $\dot{S}[x(t)] = 0$ rather than $S[x(t)] = 0$, where $S(t)$ is the proposed sliding manifold. By introducing the dynamic auxiliary system, the first derivative of the control signal is formulated as the classical SMC method does, such that the actual control is continuous after the integration.

In this paper, a sliding manifold for boundary control of this infinite-dimensional system will be constructed by using Lyapunov method, which was originally developed for finite-dimensional in [14]. The proposed switching manifold has two features. First, the PDE model is completely adopted, without any model truncated [15]. Last and the most important, the relative degree of this manifold to control input is zero, namely, the switching phenomenon occurs on the derivative of control [16], not on the control [10]. For that reason, the applied sliding mode control input after integration is continuous, such that chattering phenomenon will be mitigated effective. From the viewpoint of technique, the presented approach have the same benefit with that of second-order sliding mode. Furthermore, to compensate to lump system's uncertainties, the sliding gain of control law can be on-line updated.

We consider the boundary control problem for a one-dimensional unstable heat conduction system with spatial varying parameters, as depicted in Fig. 1. The system behavior is governed by the parabolic PDE system

$$U_t(x,t) = \varepsilon U_{xx}(x,t) + g(x)U(x,t) \tag{1}$$

subject to the boundary conditions:

$$U_x(0,t) = 0, \tag{2}$$

$$U(l,t) = Q(t) + d(t), \tag{3}$$

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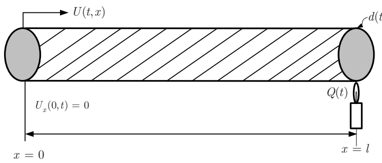

Fig. 1. Heat conduction of a thin rod.

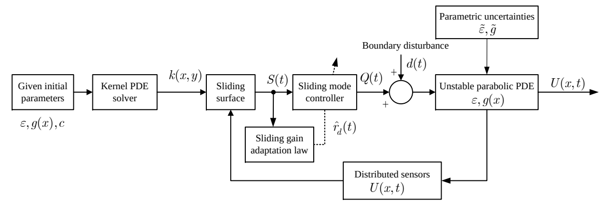

Fig. 2. Configuration of the proposed adaptive sliding mode boundary controller.

where $U(x,t)$ is the temperature of system; $\varepsilon > 0$ is the thermal diffusivity; $g(x) > 0$, $\forall\ 0 \le x \le l$, is analytic function in the domain $D = \{x| 0 < x < l\}$; The boundary condition at zero end, $x = 0$, is Dirichlet type [2]. The Dirichlet boundary actuator $Q(t)$ applied at $x = l$ is corrupted with an exogenous bounded disturbance $d(t) \in C^1([0,\infty)$.

The model (1)-(3) with $d(t) = Q(t) = 0$ can have an arbitrary large numbers of unstable modes for large $g(x)/\varepsilon > 0$. The stabilization problem of this well-known model system (1) with Dirichlet zero boundary condition and (3) has been studied in [2] by using backstepping method. Here, we will further devote to the robust boundary stabilization problem of this unstable infinite-dimensional system (1)-(3) in presence of boundary disturbance $d(\cdot)$ and parametric uncertainties in Section V, by using sliding mode approach.[1]

II. The integral transformation

A. Transformed model

By taking Volterra integral transformation [2,17,18]

$$\omega(x,t) = U(x,t) - \int_0^x k(x,y)U(y,t)dy \tag{4}$$

with the kernel function $k(x,y)$, the system (1)-(3) is converted into a damped heat equation with uncertainties as

$$\omega_t(x,t) = \varepsilon\omega_{xx}(x,t) - c\omega(x,t) \tag{5}$$

$$\omega(0,t) = 0, \tag{6}$$

$$\omega(l,t) = Q(t) + d_\omega(t), \tag{7}$$

The free parameter $c > 0$ is pre-defined for setting desired converge rate of stability. From this point on, we will refer to the PDE system (1)-(3) as the U-system in the U-coordinates and (5)-(7) as the ω-system in the ω-coordinates. The $d_\omega(t)$ is the new uncertainties in corresponding ω-system as

$$d_\omega(t) = d(t) - \int_0^l k(l,y)U(y,t)dy. \tag{8}$$

It is notes that the nominal ω-system is open-loop stable for $c > 0$ and $d_\omega(t) = 0$; however, its closed-loop performance is still deteriorated sharply due to the presence of $d_\omega(t)$. This problem differs greatly from [2], because they merely focus on nominal model of (1)-(3) by using backstepping controller.

[1]In the sequel, to reduce notational overload, the dependence on time will be restrained whenever possible.

B. Existence of kernel function $k(x,y)$

The equivalent relationship between U-system and ω-system is dependent on the existence and well-posed of kernel function $k(x,y)$. This problem is not new one and has been intensively studied in literature [17,19]. Now, to find the solution of $k(x,y)$, by substituting (1)-(3) and (4) into (5)-(7), we have following hyperbolic PDE for solving $k(x,y)$:

$$\varepsilon k_{xx}(x,y) - \varepsilon k_{yy}(x,y) = [g(y) + c]k(x,y) \tag{9}$$

with the boundary conditions

$$k(x,0) = 0, \tag{10}$$

$$k_x(x,x) + k_y(x,x) + \frac{d}{dx}k(x,x) = -[g(x) + c]/\varepsilon, \tag{11}$$

for $(x,y) \in \mathcal{T} = \{x,y : 0 < y < x < l\}$. For this complicated hyperbolic PDE (9)-(11), it is impossible to find the closed-form solution. Following the works of [2] and [18], the numerical solution can be found via variable changes and the method of successive approximation as follows:

$$k(x,y) = k(\frac{\zeta+\eta}{2}, \frac{\zeta-\eta}{2}) = G(\zeta,\eta) = \sum_{n=0}^{\infty} G_n(\zeta,\eta), \tag{12}$$

with $G_0(\zeta,\eta) = -\frac{1}{4\varepsilon}\int_\eta^\zeta [g(\frac{\tau}{2}) + c]d\tau$ and $G_n(\zeta,\eta) = \frac{1}{4\varepsilon}\int_\eta^\zeta \int_0^\eta [g(\frac{\tau-s}{2}) + c]G_{n-1}(\tau,s)dsd\tau$. Once the solution of $k(x,y)$ is found, the equivalent relationship of U-system and ω-system can be constructed, so does the sequent closed-loop stability.

III. Construction of sliding manifold

This paper aims at developing a sliding mode boundary controller for uncertain parabolic PDE system, which the architecture of proposed methodology is portrayed in Fig.2. The designing procedure is divided into two stages. At the first stage, a novel infinite-dimensional sliding surface is explicity constructed in this section. Next, a continuous variable structure boundary control with switching-gain adaptive law is proposed in Section IV to achieve the control goal $U(x,t) = 0$.

Consider the Lyapunov function that is inspired from the energy-like concept in ω-system as

$$V(t) = \frac{1}{2}\int_0^l \omega^2(x,t)dx > 0 \tag{13}$$

Then, its time derivative along with the system's trajectory yields

$$\begin{array}{rcl}\dot{V}(t) & = & \varepsilon\omega_x(l,t)[Q(t)+d_\omega(l,t)]-\varepsilon\int_0^l \omega_x^2(x,t)dx \\ & & -c\int_0^l \omega^2(x,t)dx\end{array} \tag{14}$$

Choose the switching surface for Dirichlet actuation (7) as

$$S(t)=\omega_x(l,t)=0 \tag{15}$$

Then, on the sliding surface (15), it yields $\dot{V}(t)<0$. We have the following lemma.

Lemma 1: The system (5)-(7) on the sliding surface (15) is exponentially stable in $L_2(0,l)$ norm, with a decay rate $c+\frac{\varepsilon}{2l^2}$.

Proof: According to [20], the Poincaré inequality can be modified as

$$\int_0^l \omega^2(x,t)dx \le 2l\omega^2(0,t)+4l^2\int_0^l \omega_x^2(x,t)dx \tag{16}$$

With the boundary condition (6) and expression (14), we get

$$\dot{V}(t)\le -\frac{\varepsilon+cl^2}{4l^2}\int_0^l \omega^2(x,t)dx=-\left(c+\frac{\varepsilon}{2l^2}\right)V(t)$$

Therefore, $V(t)\le V(0)e^{-(c+\frac{\varepsilon}{2l^2})t}$. On the sliding surface (15), the influence of the control $Q(t)$ and the matched boundary disturbance $d_\omega(t)$ are completely excluded. Thus, this PDE system on sliding surface is exponentially stable. ■

Lemma 1 assures exponential stability of the ω-system when the states lie on the sliding manifold (15), so does the U-system. Rewrite the sliding manifold (15) in terms of U-system, it is

$$S(t)=U_x(l,t)-k(l,l)U(l,t)-\int_0^l k_x(l,y)U(y,t)dy \tag{17}$$

Note that this proposed sliding function requires full state accessibility (measurements throughout the entire domain) in general, where $k(l,y)$ describes the characteristics of the sensor [10]. However, if $g(x)\le 0$, then the PDE (1) is open-loop stable, it yields $S(t)=U_x(l,t)$, with $k(x,y)=0$, because the kernel function $k(x,y)$ is no longer demanded for the stabilization purpose. Thus, for open-loop stable parabolic PDE system, a simple point observation suffices in construction of sliding surface. This results is more simpler then that of [10] under the assumption of full states accessibility.

Remark 1: As observation from (14), the dynamic system on the sliding manifold is dissipative. The input-output operator of $Q(t)$ to $S(t)$ is passive or positive-real, and the corresponding system transfer function is minimum-phase [21].

IV. Boundary Controller Design

This section is devoted to develop an adaptive sliding mode boundary stabilizer of the unstable PDE system (1)-(3) to force the system's trajectory will toward the sliding manifold (15) at finite time and then converge to equilibrium $U(x,t)=0$ as $t\to\infty$.

First, to find the relative degree of the sliding manifold (15) respect to the controller input $Q(t)$. With (1), time derivative of $S(t)$ is rewritten as

$$\dot{S}(t)=\omega_{xt}(l,t)=-cS(t)+\varepsilon\omega_{xxx}(l,t), \tag{18}$$

where $\omega_{xxx}(l,t)$ be discontinuous signal, possibly. From the Filippov's sense [22] and the explicit function of "low-pass filter" with a pole at $s=-c$ in (18), the output signal $S(t)$ has explicit solution as $S(t)=S(0)e^{-ct}+\int_0^t \varepsilon\omega_{xxx}(l,\tau t)e^{-c(t-\tau)}d\tau$, which belongs to $C^1[0,\infty)$ continuity. However, the boundary control $Q(t)$ does not appear in (18). Now, we try another way to find their relationship, by integrating both sides of (1) in terms of x from 0 to l, and then taking the time derivative of t, it yields

$$\int_0^l \omega_{tt}(x,t)dx=\varepsilon\dot{S}(t)-\varepsilon\omega_{xt}(0)-\int_0^l c\omega_t(x)dx$$

Since the above results is irrelevant to the spatial variable x, it could be rewritten as

$$\int_0^l \left[\omega_{tt}(x,t)-\frac{\varepsilon}{l}\dot{S}+c\omega_t(x,t)\right]dx\equiv 0$$

For a physical heat conduction system and the mean-value theorem, the above integrand is assumed to be bounded on the interval $[0,l]$. Thus, the term $d_0(t)=\omega_{tt}(l,t)-\frac{\varepsilon}{l}\dot{S}(t)+c\omega_t(x,t)\in\mathcal{H}_\infty$ for $x=l$, where $d_0(t)$ is an unknown but bounded variable. We have

$$\begin{array}{rcl}\dot{S} & = & \frac{l}{\varepsilon}\omega_{tt}(l)+\frac{cl}{\varepsilon}\dot{Q}+\frac{cl}{\varepsilon}\dot{d}_\omega+\frac{l}{\varepsilon}d_0 \\ & = & \frac{cl}{\varepsilon}\dot{Q}(t)+\frac{l}{\varepsilon}\delta_d(t),\end{array} \tag{19}$$

where $\delta_d(t)=\omega_{tt}(l,t)+c\dot{d}_\omega(t)+d_0(t)$ is completely unknown, but could be assumed bounded with its upper bound $\bar{r}_d$, that is, $|\delta_d(\cdot)|<\bar{r}_d$. From (19), it indicates that the relative degree between $Q(t)$ and $S(t)$ is zero, which is quite differ from [10,15] with relative order one. With this advantage, we can develop a sliding mode controller with continuous output signal $Q(t)\in C^1[0,\infty)$ for this system as followings.

A general guideline to control design is to satisfy the reaching condition [23]

$$\dot{S}(t)S(t)<0. \tag{20}$$

Usually, it takes a discontinuous control law to achieve the sliding mode leading to the notorious chattering phenomenon, which greatly degrades the system performance. Moreover, in order to compensate the lumped uncertainties, a sliding mode boundary controller with on-line tuning law of the switching gain is proposed as

$$\dot{Q}(t)=-\hat{r}_d(t)\mathrm{sign}(S(t)), \tag{21}$$

with gain on-line tuning law as

$$\dot{\hat{r}}_d(t)=\frac{l}{a\varepsilon}|S(t)| \tag{22}$$

where $S(t)$ is selected as (15), $\hat{r}_d(t)$ is an estimation of $\bar{r}_d$, and $a > 0$ is a designing parameter for adaptation.

Select the Lyapunov candidate function

$$V_s(t) = \frac{1}{2}S^2(t) + \frac{1}{2}a\tilde{r}_d^2(t) \tag{23}$$

where $\tilde{r}_d(t) = \hat{r}_d(t) - \bar{r}_d$ is the estimation error. With (19), its time derivative is given by

$$\begin{aligned}\dot{V}_s(t) &= S(t)\dot{S}(t) + a\tilde{r}_d(t)\dot{\hat{r}}_d(t) \\ &= S(t)[\tfrac{cl}{\varepsilon}\dot{Q}(t) + \tfrac{l}{\varepsilon}\delta_d(t)] + a(\hat{r}_d(t) - \bar{r}_d)\dot{\hat{r}}_d(t)\end{aligned}$$

Substituting (21)-(22) into it, we have

$$\begin{aligned}\dot{V}_s(t) &= -\tfrac{l}{\varepsilon}\hat{r}_d(t)\left\|S\right\| + \tfrac{l}{\varepsilon}S\delta_d(t) \\ &< -\tfrac{l}{\varepsilon}\left\|S\right\|(\bar{r}_d - \delta_d(t)) < -\sigma\left\|S\right\|^2\end{aligned}$$

with $\sigma > 0$. Thus, the reachability condition (20) is satisfied and the states of system will approach the switching manifold $S(t) = 0$ within finite time t_s, where t_s is the time that the sliding mode is attained. Once an ideal sliding motion takes place, the closed-loop ω-system behavior will exponential converge to the origin, so does $\lim_{t\to\infty} U(x,t) = 0$.

V. Parametric variations

Consider when the parabolic PDE is subject to not only the boundary disturbance but also to parameter variations. The system model (1) is reformulated as

$$\begin{aligned}U_t(x,t) &= (\varepsilon + \tilde{\varepsilon})U_{xx}(x,t) + (g(x) + \tilde{g})U(x,t) \\ &= \varepsilon U_{xx}(x,t) + g(x)U(x,t) + f(x,t)\end{aligned} \tag{24}$$

where $f(x,t) = \tilde{\varepsilon}U_{xx}(x,t) + \tilde{g}U(x,t) \in C^1([0,l]\times[0,\infty))$ denotes the lumped uncertainties that resulted from the system's parameter variations. The parametric variations $\tilde{\varepsilon}$ and $\tilde{g}$ are assumed to be bounded within constant values, i.e., $|\tilde{\varepsilon}| \le \varepsilon_o$ and $|\tilde{g}| \le g_o$, respectively. The corresponding ω-system counterpart of (24) can be obtained by using the transformation (4) as

$$\omega_t(x,t) = \varepsilon\omega_{xx}(x,t) - c\omega(x,t) + f_\omega(x,t) \tag{25}$$

where $f_\omega(\cdot)$ is the effective variations in the ω-system

$$f_\omega(x,t) = f(x,t) - \int_0^x k(x,y)f(y,t)dy. \tag{26}$$

From (26), the term $f_\omega(\cdot)$ could be further represented in ω-coordinates as

$$\begin{aligned}f_\omega(x) &= \tilde{\varepsilon}U_{xx}(x) + \tilde{g}U(x) \\ &\quad - \textstyle\int_0^x k(x,y)\left[\tilde{\varepsilon}U_{yy}(y) + \tilde{g}U(y)\right]dy \\ &= \tilde{\varepsilon}[U_{xx}(x,t) - \textstyle\int_0^x k(x,y)U_{yy}(y,t)dy] \\ &\quad + \tilde{g}\omega(x,t)\end{aligned} \tag{27}$$

Taking integration by part for (27), with (2) and (9)-(11), the term within the bracket could be further represented as

$$\begin{aligned}&U_{xx}(x,t) - \textstyle\int_0^x k(x,y)U_{yy}(y,t)dy \\ =\ &U_{xx}(x) - k(x,x)U_x(x) + k_y(x,x)U(x) \\ &- \textstyle\int_0^x k_{yy}(x,y)U(y)dy] \\ =\ &\omega_{xx}(x) - \tfrac{c}{\varepsilon}\omega(x) - \tfrac{1}{\varepsilon}[g(x)U(x) \\ &- \textstyle\int_0^x g(y)k(x,y)U(y)dy]\end{aligned} \tag{28}$$

The last two terms in the bracket of (28) can be regarded as feeding $g(x)U(x,t)$ into the linear "time-varying" integral operator (4) whose "impulse response" is

$$h(x,y) = \delta(x-y) - k(x,y),$$

where $\delta(x-y)$ is a Dirac delta function. Therefore, (4) can be rewritten as

$$\omega(x,t) = \int_0^x h(x,y)U(y,t)dy.$$

Similarly, the bracket terms of (28) is $\int_0^x g(y)h(x,y)U(y,t)dy$. This operation can be seen as an inner product of the two functions $g(y)$ and $h(x,y)U(y,t)$ for each $0 \le x \le l$ and $t \ge 0$. By using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}&\left|\textstyle\int_0^x g(y)h(x,y)U(y,t)dy\right| \\ \le\ &\left|\textstyle\int_0^x g(y)dy\right|\cdot\left|\textstyle\int_0^x h(x,y)U(y,t)dy\right| = g_{\max}\left|\omega(x,t)\right|\end{aligned} \tag{29}$$

where $g_{\max} = \left|\int_0^x g(y)dy\right|$. With the above result and (28), the mismatched parametric uncertainties $f_\omega(x,t)$ is bounded as

$$f_\omega(x,t) \le \tilde{\varepsilon}\omega_{xx}(x,t) + [\tilde{g} - \frac{\tilde{\varepsilon}}{\varepsilon}c]\omega(x,t) + \frac{\tilde{\varepsilon}}{\varepsilon}g_{\max}\left|\omega(x,t)\right|$$

According to matching condition [24], the effect of mismatched parametric uncertainties $f(x,t)$ in parabolic PDE system (24) could not be completely rejected within switching manifold. Fortunately, with (25), one can observe that the system (25) could be stabilized via proper setting the value of parameter c, even through there still has a certain range of parametric uncertainties $f_\omega(x,t)$ on it. However, more large of the parameter c, the more large amount of the kernel gain $k(x,y)$ in (9), and so does the corresponding the control effort $Q(t)$. The later numerical result will evidence it. This leads to a trade-off problem. Here, we are going to find the lower bound of c to deal with the problem of mismatched parametric uncertainties as follows.

Theorem 1: The system (24) with boundary conditions (2)-(3) with both parameter variations and boundary disturbance is exponential stable in the sliding mode on the surface (17) if parameter c of (25) is satisfied with

$$c > \tfrac{\varepsilon}{\varepsilon+\varepsilon_0}g_0 + \tfrac{\varepsilon_0}{\varepsilon+\varepsilon_0}g_{\max} \tag{30}$$

Proof: With the Lyapunov function (13), its time derivative yields

$$\begin{aligned}\dot{V}(t) &= \textstyle\int_0^l \omega(x)\left[\varepsilon\omega_{xx}(x) - c\omega(x) + f_\omega(x)\right]dx \\ &\le (\varepsilon + \tilde{\varepsilon})(Q + d_\omega)\omega_x(l) - \varepsilon\textstyle\int_0^l \omega_x^2(x)dx \\ &\quad - c\textstyle\int_0^l \omega^2(x)dx + (\tilde{g} - \tfrac{\tilde{\varepsilon}}{\varepsilon}c)\textstyle\int_0^l \omega^2(x)dx \\ &\quad + \tfrac{\varepsilon_0}{\varepsilon}g_{\max}\textstyle\int_0^l \omega(x)\left|\omega(x)\right|dx\end{aligned}$$

When the system is on the sliding surface (15), it yield

$$\begin{aligned}\dot{V}(t) &\le -\varepsilon\textstyle\int_0^l \omega_x^2(x)dx + [\tfrac{\varepsilon_0}{\varepsilon}g_{\max} + \tilde{g} \\ &\quad - \tfrac{\varepsilon_0}{\varepsilon}c - c]\textstyle\int_0^l \omega^2(x)dx\end{aligned}$$

With Poincaré inequality (16) and condition (30), it yields

$$\dot{V}(t) \leq -2(\tfrac{\varepsilon+\varepsilon_0}{\varepsilon}c - \tilde{g} - \tfrac{\varepsilon_0}{\varepsilon}g_{\max} + \tfrac{\varepsilon}{4l^2})V(t) < 0$$

■

Although the estimation (30) is rather conservative, it can provide a criterion for designing the parameter c.

VI. Simulation Study

In this section, we present numerical results concerning the solution of kernel function (9)-(11), along with simulation demonstrating of boundary control problem of an unstable heat conduction system with space-dependent coefficient (24) subject to boundary conditions (2)-(3), as shown in Fig.1. The system performances under two different control strategies are compared, including the proposed adaptive SMC methods (21) and other benchmark backstepping boundary controller(BSC),

$$Q(t) = \int_0^l k(l,y)U(y,t)dy, \tag{31}$$

which has proposed in [2,18]. Refer to [18], the system parameters are setup with $\varepsilon = 1$, $g(x) = 14 - 16(x - \frac{1}{2})^2 > 0$, $q = \infty$, $l = 1$ m, and the initial condition $U(x,0) = 1 + \sin(3\pi x/2)$. In this situation, there is one unstable eigenvalue locates in $s = 7.8$ of complex plane, from the characteristic analysis. In order to verify the benefit of chattering diminished of the proposed method (21), the signum function is directly applied without any conventional approximation [12]. All simulations were carried out using finite-difference method by MATLAB software and its PDE toolbox. To demonstrate the robustness of the proposed controller (21), two simulation cases are considered as following:

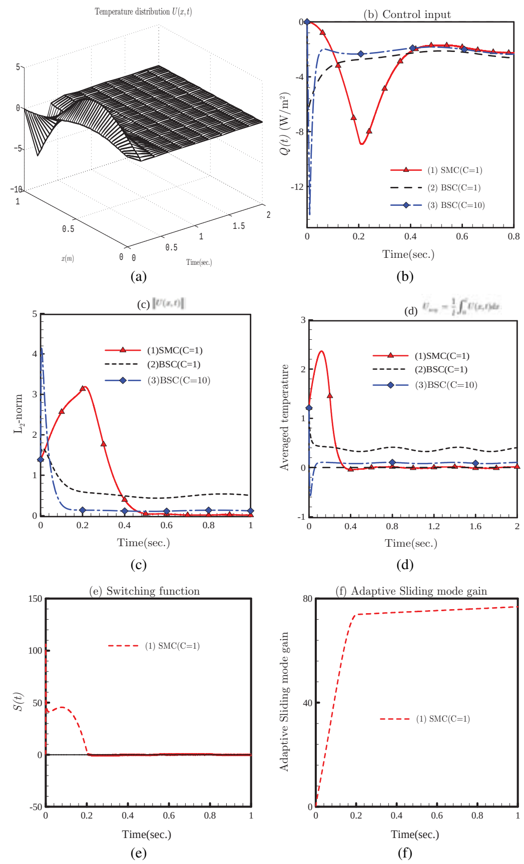

Fig. 3. Closed-loop system responses of case (i): (a) Temperature distribution by adaptive SMC, (b) Applied boundary inputs, (c) Comparisons of the L_2-norm, (d) Comparisons of $U_{avg} = \frac{1}{l}\int_0^l U(x,t)dx$, (e) The evolution of the sliding variable, (f) Adptive sliding gain $\hat{r}_d(t)$ verse time.

A. Case A: System only with boundary disturbance

In this case, the PDE system (24) only subject to boundary disturbance $d(t) = 2 + 0.2\sin(20t)$ at $x = l$. The controller's parameters of (21) are setup with $c = 1$, $a = 0.01$, $r_d(0) = 0$. The simulation results are illustrated in Fig.3. The proposed adaptive SMC method can effectively stabilize this unstable system into the equilibrium $\lim_{t\to\infty} U(x,t) = 0$ within one second, and the L_2-norm of the system's states will converge to zero. On the other hand, the BSC method under two different setting $c = 1$ and $c = 12$ are still failed to the control goal. Although BSC can stabilize the overall states of system into the stable region, the performance is still degraded due to boundary disturbance. This observation is reasonable, because the backstepping control plays a role of feedback stabilizer only for well-known nominal PDE model.

Case B: System with both parametric variations and boundary disturbance

Here, the PDE system subject to the same boundary disturbance $d(t)$ and the parameter variations as $\tilde{\varepsilon} = 0.1$ and $\tilde{g} = 1.4$. With (30), the parameter c is setup with $c = 3 > \frac{\varepsilon}{\varepsilon+\varepsilon_0}g_0 + \frac{\varepsilon_0}{\varepsilon+\varepsilon_0}g_{\max} = 2.5$, with $g_{\max} = 14$. The closed-loop system response is shown in Fig.4. The influence of parametric variation can be effectively restrained, such that the system's states will converge to zero as $t \to 1.2$ sec.

From these two simulations, the presented method has revealed the robustness and performance in the boundary control problems of an uncertain parabolic PDE system with spatially-varying coefficients. Besides, the applied control efforts are smooth and reasonable. With the acid integration, its continuous output signal can reduce the chattering phenomenon in reality.

VII. Conclusions

The problem of boundary stabilization of parabolic PDE systems subject to system uncertainties has been shown to be tractable in this paper. The proposed sliding mode boundary control law has equipped the system with robustness against the spatial dependant coefficients variations and external disturbances.

Use of the Volterra integral has enabled us to study the PDE system in the new coordinates that renders a simple and stable system structure, in which all the uncertainties can be treated as a whole. Moreover, in this same coordinates, the

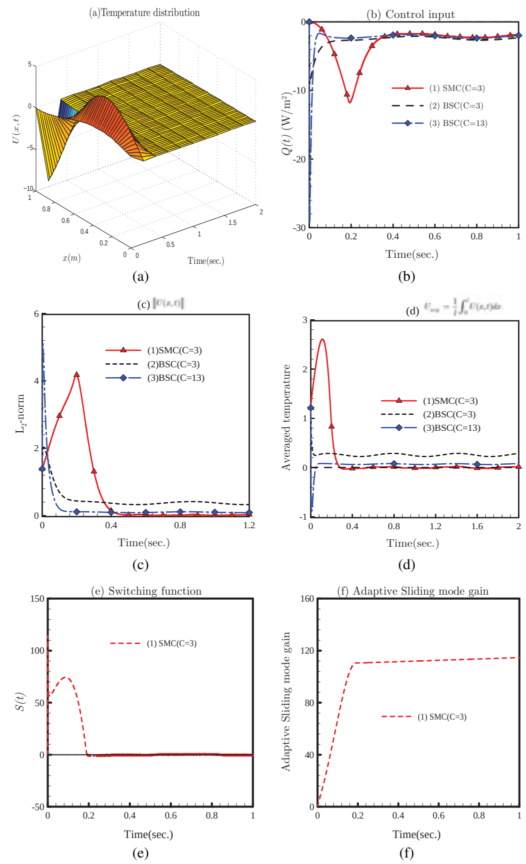

Fig. 4. Closed-loop system responses of case (ii): (a) Temperature distribution by adaptive SMC, (b) Applied boundary inputs, (c) Comparisons of the L_2-norm, (d) Comparisons of $U_{avg} = \frac{1}{l}\int_0^l U(x,t)dx$, (e) The evolution of the sliding variable, (f) Adptive sliding gain $\hat{r}_d(t)$ verse time.

Lyapunov method yields a sliding manifold that explicitly determines the sliding 'vector' in the infinite-dimensional space without the need of a truncated system model. It has a zero-order relative degree property (with respect to the boundary control input) such that a continuous SMC control law can be obtained. The chattering phenomenon that persists in most finite-dimensional control systems is no more an issue here.

The proposed methodology can be easily extended to other benchmark parabolic PDE systems, as long as the solution of kernel function $k(x,y)$ can be found either symbolically or numerically. It is believed that this idea may provides a new avenue to the study of hyperbolic or elliptic equations, or other kinds of higher order PDE control problems. Investigation of these problems from the viewpoints of this paper seem promising.

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