Avanza. Vol. iv. Fm - IIT, UACJ (2014) 5–13. ISBN: 978-607-9224-52-3

Absence of limit cycles for Kukles-type systems *

Osuna, Osvaldo, Rodríguez-Ceballos, Joel, Villaseñor, Gabriel †

Abstract

In this note, we give a few new criteria for constructing Dulac function related to Kukles-type systems, which allows us to determine the non existence of limit cycles for some generalized Kukles systems. We also present examples in order to illustrate our results.

Keywords: Bendixson-Dulac criterion, Dulac functions, limit cycles, Kukles systems.

1 Introduction

In the now classic work [6], I. Kukles gives necessary and sufficient conditions in order such that the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \sum_{i=0}^{3} h_i(x_1)x_2^i \end{cases} \tag{1}$$

has a center at the origin; in the literature, there are numerous studies on the Kukles equation, see [12], [10] and [9]. In particular, it contains the *Liénard equation* given by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = h_0(x_1) + h_1(x_1)x_2. \end{cases} \tag{2}$$

It is well-known the importance of Liénard systems modelling several oscillatory phenomena, see [1] and [6]. Given the relevance of the equation 1 in the qualitative theory of differential equations is natural to consider some generalizations of this equation. There are some studies in this direction, see [2], [5] and [9].

In this note, we are concerned about limit cycles (isolated periodic orbits) for Kukles-type systems with the following form:

$$\begin{cases} \dot{x}_1 = x_2^k, \\ \dot{x}_2 = h_0(x_1) + h_1(x_1)x_2 + h_2(x_1)x_2^2 + h_n(x_1)x_2^n, \end{cases} \tag{3}$$

*2000, Classifications numbers Ams. 34C25, 34C07, 34A34.

†Instituto de Física y Matemáticas, Universidad Michoacana / Instituto Tecnológico de Morelia.

where $k, n \in \mathbb{N}$ and $n \geq 3$. We obtain some criteria of non-existence of limit cycles for 3. We also present some applications and examples in order to illustrate our results.

Below we review some basic concepts useful in our exposition. Given an open set $\Omega \subset \mathbb{R}^2$, we consider

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2), \quad (x_1,\ x_2) \in \Omega, \end{cases} \tag{4}$$

where f_1, f_2 are real C^1-functions on Ω. Considering the vector field $F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, then the system 4 can be rewritten in the form

$$\dot{x} = F(x), \qquad x = (x_1, x_2) \in \Omega. \tag{5}$$

As usual the divergence of the vector field F is defined by

$$div(F) = div(f_1, f_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}.$$

We consider $C^0(\Omega, \mathbb{R})$ the set of continuous functions and define the sets

$$\mathcal{F}_\Omega^{\pm} := \{f \in C^0(\Omega,\ \mathbb{R}^{\pm} \cup \{0\}) \ :\ \text{vanishes only on a measure zero set}\},$$

and $\mathcal{F}_\Omega := \mathcal{F}_\Omega^- \cup \mathcal{F}_\Omega^+$, along the paper we will use the Lebesgue measure.

Recall that an open subset $\Omega \subset \mathbb{R}^2$ intuitively is said to be l-*connected* if it has l-holes, *i.e.*, if its first fundamental group is a free group with l-generators, we denote $l(\Omega) = l$.

For $h : \Omega \to \mathbb{R}$ a continuous function, let $Z(h) := \{x \in \Omega \ :\ h(x) = 0\}$ be the set of zeros of h.

Following [4] we denote by $l(\Omega, h)$ the sum of the quantities $l(U)$ over all the connected components U of $\Omega \setminus Z(h)$. By a closed oval we mean a subset homeomorphic to the circle $\mathbb{S}^1$; therefore, denote by $co(h)$ the numbers of closed ovals of $Z(h)$ contained in Ω.

The extended Bendixson-Dulac criterion is a very useful tool for investigation of limit cycles for planar vector fields which provides bounds for the number of limit cycles, see for instance [3], [4]. The following states a version of extended Bendixson-Dulac criterion:

Proposition 1.1. *([4], Cor. 1) Let $\Omega \subset \mathbb{R}^2$ be an open set with a regular boundary. Suppose that for an analytic function $h : \Omega \to \mathbb{R}$ and a real number s, we have*

$$M_s := f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} + sh\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) = \langle \nabla h, F\rangle + shdiv(F), \tag{6}$$

does not change sign and vanishes only on a measure zero set. Then the limit cycles of system 4 are either totally contained in $Z(h)$, or do not intersect $Z(h)$. Moreover, the number of limit cycles contained in $Z(h)$ is at most $co(h)$ and the number N of limit cycles that do not intersect $Z(h)$ satisfies

$$N \leq \begin{cases} l(\Omega) & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ l(\Omega, h) & \text{if } s < 0. \end{cases} \tag{7}$$

Furthermore, for any $s \neq 0$ the limit cycles of this second type are hyperbolic.

We call a function h in Proposition 1.1 a *Dulac function*. Despite the relevance of this extension of Bendixson-Dulac's criterion it suffers the drawback that there is no general algorithm for finding Dulac's functions. Our results are established with the help of the techniques developed in [8], let us recall the following result:

Proposition 1.2. *([8]) Let $\Omega \subset \mathbb{R}^2$ be an open set with a regular boundary. Suppose that there are both $s \in \mathbb{R}$ and a function $c : \Omega \to \mathbb{R}$ such that*

$$\langle \nabla h, F \rangle + s h\, div(F) = ch, \tag{8}$$

admits an analytic solution h with ch defined on Ω and does not change sign and vanishes only on a null measure subset. Then h is a Dulac function and the conclusions of the Proposition 1.1 are true. In particular, the number of limit cycles contained in $Z(h)$ is at most $co(h)$ and the number N of limit cycles that do not intersect $Z(h)$ satisfies

$$N \leq \begin{cases} l(\Omega) & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ l(\Omega, h) & \text{if } s < 0. \end{cases} \tag{9}$$

Furthermore, for any $s \neq 0$ the limit cycles of this second type are hyperbolic.

2 Results

The next result in [7] provides sufficient conditions in order that a quadratic function does not change sign:

Lemma 2.1. *Given $g_0, g_1, g_2, w : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ continuous functions. If $g_0 \in \mathcal{F}_U$ and $\Delta := g_1(u)^2 - 4g_0(u)g_2(u) \leq 0$, then*

$$Q(u) := g_0(u)w^2 + g_1(u)w + g_2(u) \geq 0 \ (or \ \leq 0).$$

Moreover, if $\Delta \in \mathcal{F}_U^-$, then $Q \in \mathcal{F}_U$.

Let us consider the case $k=1$ and $n=3$, *i.e.*, the Kukles system (1), we get the following:

Proposition 2.2. *If $h_3 \in \mathcal{F}_{\mathbb{R}^2}$ and $h_2^2 - 4h_1h_3 \in \mathcal{F}^-_{\mathbb{R}^2}$, then the Kukles-type system (1) has no limit cycles.*

Proof. The associated equation 8 is

$$x_2 \frac{\partial h}{\partial x_1} + (\sum_{i=0}^{3} h_i(x_1)x_2^i)\frac{\partial h}{\partial x_2} = h[c - sdiv(F)], \tag{10}$$

assuming that $h = h(z)$ depends on $z = z(x_1, x_2)$, and taking

$$x_2 \frac{\partial z}{\partial x_1} + h_0(x_1)x_2 \frac{\partial z}{\partial x_2} = 0,$$

which admits as a solution to $z = -\int^{x_1} h_0(\tau)d\tau + \frac{x_2^2}{2}$, then the above equation becomes

$$[(h_1(x_1) + h_2(x_1)x_2 + h_3(x_1)x_2^2)x_2^2]\frac{dh}{dz} = h(c - s(div(F))), \tag{11}$$

taking $s = 0$ and $c = (h_1(x_1) + h_2(x_1)x_2 + h_3(x_1)x_2^2)x_2^2$, we need to show that $h_1(x_1) + h_2(x_1)x_2 + h_3(x_1)x_2^2$ belong to $\mathcal{F}_{\mathbb{R}^2}$, but by our hypothesis and Lemma 2.1 this held. Solving 11 we have that $h(x_1, x_2) = \exp(-\int^{x_1} h_0(\tau)d\tau + \frac{x_2^2}{2})$ is a Dulac function. Note that $Z(h) = \emptyset$ contains no ovals, thus $co(h) = 0$. Also as $s = 0$, by Proposition 1.2 the system 1 has no limit cycles, so the result follows. □

Example 2.3. We consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= h_0(x_1) + 2h_2(x_1)x_2 + h_2(x_1)x_2^2 + 2h_2(x_1)x_2^3. \end{aligned}$$

By Proposition 2.2 admits no limit cycles if $h_2 \in \mathcal{F}_{\mathbb{R}^2}$.

Let us consider the case $h_3 = 0$ in the Kukles system (1), we get the following:

Proposition 2.4. *Assume that there exists $\psi(x_1)$ a C^1-function such that the following two conditions hold:*

i).- $\psi'(x_1) + \psi(x_1)h_2(x_1) \in \mathcal{F}_{\mathbb{R}^2}$ and *ii).-* $(h_1(x_1)\psi(x_1))^2 - 4(\psi'(x_1) + \psi(x_1)h_2(x_1))(h_0(x_1)\psi(x_1)) \in \mathcal{F}^-_{\mathbb{R}^2}$,

then the system 1 has no limit cycles.

Proof. The associated equation (8) is written as

$$x_2\frac{\partial h}{\partial x_1} + \left(\sum_{i=0}^{3} h_i(x_1)x_2^i\right)\frac{\partial h}{\partial x_2} = h(c - sdiv(F)),$$

suppose that h has the form $h = \psi(x_1)x_2$ and taking $s = 0$, we get

$$\left(\psi'(x_1) + \psi(x_1)h_2(x_1)\right)x_2^2 + h_1(x_1)\psi(x_1)x_2 + h_0(x_1)\psi(x_1) = ch.$$

Considering the left side as a quadratic function in x_2, its discriminant becomes

$$\Delta = (h_1(x_1)\psi(x_1))^2 - 4(\psi'(x_1) + \psi(x_1)h_2(x_1))(h_0(x_1)\psi(x_1)),$$

by Lemma 2.1 $ch \in \mathcal{F}_{\mathbb{R}^2}$. Note that $Z(h)$ contains no ovals and since $s = 0$, then by Proposition 1.2 the system 1 admits no limit cycles. □

Let us consider the following Kukles-type system:

$$\begin{cases} \dot{x}_1 = x_2^k, \\ \dot{x}_2 = h_0(x_1) + h_1(x_1)x_2 + h_2(x_1)x_2^2 + h_k(x_1)x_2^k, \end{cases} \tag{12}$$

we obtain the next result:

Proposition 2.5. *If $h_2 \in \mathcal{F}_{\mathbb{R}^2}$ and $h_1^2 - 4h_0h_2 \in \mathcal{F}^-_{\mathbb{R}^2}$, then the generalized Kukles system (12) has no limit cycles.*

Proof. The associated equation 8 is

$$x_2^k\frac{\partial h}{\partial x_1} + (h_0(x_1) + h_1(x_1)x_2 + h_2(x_1)x_2^2 + h_k(x_1)x_2^k)\frac{\partial h}{\partial x_2} = h[c - sdiv(F)], \tag{13}$$

assuming that $h = h(z)$ depends on $z = z(x_1, x_2)$ with $x_2^k\frac{\partial z}{\partial x_1} + h_k(x_1)x_2^k\frac{\partial z}{\partial x_2} = 0$, which admits as a solution to $z = -\int^{x_1} h_k(\tau)d\tau + x_2$, then the above equation becomes

$$[h_0(x_1) + h_1(x_1)x_2 + h_2(x_1)x_2^2]\frac{d\ln h}{dz} = c - s(div(F)), \tag{14}$$

taking $s = 0$ and $c = h_0(x_1)+h_1(x_1)x_2+h_2(x_1)x_2^2$, which is a quadratic function respect to x_2, by our hypothesis and Lemma 2.1 we have that $c \in \mathcal{F}_{\mathbb{R}^2}$, thus the equation (14) is written as $\frac{d \ln h}{dz} = 1$, whose solution is

$$h(x_1, x_2) = \exp[\int^{x_1} h_k(\tau) d\tau + x_2].$$

Note that $Z(h) = \emptyset$ contains no ovals. In particular, $co(h) = 0$. Since $s = 0$, then by Proposition 1.2 system 12 has no limit cycles, so the result follows. □

Example 2.6. We consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^5, \\ \dot{x}_2 &= 2 + x_1^2 + x_1 x_2 + (1 + 3x_1^4)x_2^2 + (x_1 - 3x_1 + 8x_1^3)x_2^5. \end{aligned}$$

By Proposition 2.5 we get that the system admits no limit cycles.

Let us consider the following system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = h_0(x_1) + h_1(x_1)x_2 + h_{2n}(x_1)x_2^{2n}, \end{cases} \tag{15}$$

we obtain the next result:

Proposition 2.7. *If any of the following conditions hold:*

a).- $h_0 \in \mathcal{F}^{\pm}_{\mathbb{R}^2}$ and $(\pm)h_{2n} \geq 0$,

b).- $h_{2n} \in \mathcal{F}^{\pm}_{\mathbb{R}^2}$ and $(\pm)h_0 \geq 0$,

then the system 15 has no limit cycles.

Proof. Assume $h_0 \in \mathcal{F}^{+}_{\mathbb{R}^2}$ and $h_{2n} \geq 0$ the other cases are analogous. The associated equation 8 is

$$x_2 \frac{\partial h}{\partial x_1} + (h_0(x_1) + h_1(x_1)x_2 + h_{2n}(x_1)x_2^{2n}) \frac{\partial h}{\partial x_2} = h[c - sdiv(F)], \tag{16}$$

assuming that $h = h(z)$ depends on $z = z(x_1, x_2)$, and taking

$$x_2 \frac{\partial z}{\partial x_1} + h_1(x_1)x_2 \frac{\partial z}{\partial x_2} = 0,$$

which admits as a solution to $z = -\int^{x_1} h_1(\tau)d\tau + x_2$, then the above equation becomes

$$[h_0(x_1) + h_{2n}(x_1)x_2^{2n}] \frac{dh}{dz} = h(c - s(div(F))), \tag{17}$$

taking $s=0$ and $c=h_0(x_1)+h_{2n}(x_1)x_2^{2n}$, which by our hypothesis belong to $\mathcal{F}_{\mathbb{R}^2}$. Solving 17 we have that $h(x_1,x_2)=\exp(-\int^{x_1} h_1(\tau)d\tau+x_2)$ is a Dulac function. Note that $Z(h)=\emptyset$ contains no ovals. In particular, $co(h)=0$. Also as $s=0$, by Proposition 1.2 the system 15 has no limit cycles, so the result follows. □

Example 2.8. We consider the system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= 1+x_1^4+x_1+(1-8x_1+3x_1^5)x_2+(\sin(x_1)+1)x_2^6.\end{aligned}$$

By Proposition 2.7 the above system admits no limit cycles.

Let us consider the following system:

$$\begin{cases}\dot{x}_1 = x_2^{2k+1},\\ \dot{x}_2 = h_0(x_1)+h_1(x_1)x_2+h_{2n+1}(x_1)x_2^{2n+1},\end{cases} \tag{18}$$

we obtain the following:

Proposition 2.9. *If any of the following conditions hold:*

a).- $h_1\in\mathcal{F}^{\pm}_{\mathbb{R}^2}$ *and* $(\pm)h_{2n+1}\geq 0$,

b).- $h_{2n+1}\in\mathcal{F}^{\pm}_{\mathbb{R}^2}$ *and* $(\pm)h_1\geq 0$,

then the system 15 has no limit cycles.

Proof. Assume $h_1\in\mathcal{F}^{+}_{\mathbb{R}^2}$ and $h_{2n+1}\geq 0$. We seek a Dulac function h depending on z such that

$$x_2^{2k+1}\frac{\partial z}{\partial x_1}+h_0(x_1)\frac{\partial z}{\partial x_2}=0,$$

and we take $s=0$. The rest proof is similar to that of Proposition 2.7. □

Example 2.10. We consider the system

$$\begin{aligned}\dot{x}_1 &= x_2^5,\\ \dot{x}_2 &= 3-x_1^4+x_1+(2+x_1^2+4x_1^{10})x_2+(x_1^5+3x_1^2-7)^2x_2^7.\end{aligned}$$

By Proposition 2.9 the above system admits no limit cycles.

Acknowledgements. The authors want to thank the comments from anonymous referee.

Bibliography

[1] T. Carletti, G. Villari, "A note on the existence and uniqueness of limit cycles for Liénard systems", *J. Math. Anal. Appl.*, 307, (2005), 763-773.

[2] J. Chavarriga, I. García, E. Sáez, I. Szántó, "Limit cycles in Kukles system of arbitrary degree with invariant ellipse", *Nonlinear Analysis: theory, methods and appl.*, 67 (4), (2007), 1005-1014.

[3] L. A. Cherkas, "Dulac functions for polynomial autonomous systems on a plane", *Diff. eqs.*, 33, (1997), 692-701.

[4] A. Gasull, H. Giacomini, "Some applications of the extended Bendixson-Dulac Theorem", *Progress and challenges in Dyn. syst. Springer*, 54, (2013), 233-252.

[5] J. Llibre, A. Mereu, "Limit cycles for generalized Kukles polynomial diff. systems", *Nonlinear Analysis*, 75 (4), (2011), 1261-1271.

[6] I. Kukles, "On some cases of the discrimination between focus and center", *Dokl. Akad. Nauk. SSSR*, 42, (1944), 208-211.

[7] M. Lahyane, O. Osuna, J. Rodríguez-Ceballos, G. Villaseñor-Aguilar, "A note on limit cycles for generalized Liénard systems", Preprint, (2014).

[8] O. Osuna, G. Villaseñor-Aguilar, "On the Dulac functions for multiply connected domains", *E. J. of Qualitative Theory of Differential Equations*, 61, (2013), 1-11.

[9] R. Rabanal, "On the limit cycles of a class of Kukles type diff. systems", *Nonlinear Analysis*, 95, (2014), 676-690.

[10] E. Sáez, I. Szántó, "Coexistence of algebraic and nonalgebraic limit cycles in Kukles system", *Per. Math. Hung.*, 56 (1), (2008), 137-142.

[11] G. Sansone, "Sopra lequazione di Liénard delle oscillazioni di rilassamento", *Ann. Math. Pura Appl.*, 28 (4), (1949), 153-181.

[12] H. Zang, T. Zhang, Y. Tian, M. Tadé, "Limit cycles for the Kukles system", *J. of Dyn. and control syst.*, 14 (2), (2008), 283-298.

Osuna, Osvaldo *(osvaldo@ifm.umich.mx)*
Instituto de Física y Matemáticas, Universidad Michoacana.
Edif. C-3, Ciudad. Universitaria, C.P. 58040. Morelia, Michoacán, México.

Rodríguez-Ceballos, Joel *(joel@ifm.umich.mx)*
Instituto Tecnológico de Morelia, Depto. de Ciencias Básicas.
Edif. AD, Morelia, Michoacán, México.

Villaseñor-Aguilar, Gabriel *(gabriel@ifm.umich.mx)*
Instituto Tecnológico de Morelia, Depto. de Ciencias Básicas.
Edif. AD, Morelia, Michoacán, México.