On the categorical structure of H^2

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Abstract. The categorical structure of H^2 is shown to be a particular instance of the cokernel of a morphism between symmetric categorical groups.

Introduction

The aim of this note is to show how natural is the use of categorical groups in homological algebra. For this, we give two simple examples concerning classical results on group-extensions. Let us sketch the first example.

Let A and C be two groups, A abelian. A classical theorem in homological algebra asserts that the group of isomorphism classes of extensions of A by C with a fixed operator $\varphi\colon C \to \mathrm{Aut}(A)$ is isomorphic to $H^2_\varphi(C,A)$, the second cohomology group of C with coefficients in A [10]. This theorem has been made more precise by establishing a categorical equivalence between the category of extensions and a certain category whose objects are 2-cocycles [9]. We will show that this category is not an ad hoc construction, but it is the cokernel, performed in the 2-category of symmetric categorical groups, of the usual morphism with codomain the group of 2-cocycles.

Since our aim is not to prove a new theorem, but to explain a classical one, we limit to the simple case of groups, even if similar results can be obtained in more general situations.

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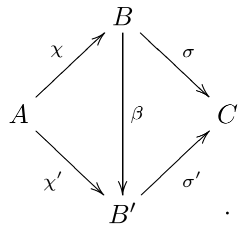
The classical isomorphism

Consider two groups A and C, with A abelian (we use the additive notation for A and the multiplicative notation for C), and fix an operator $\varphi\colon C \to \mathrm{Aut}(A)$. Following [10], recall that an extension of A by C is an exact sequence

$$E\colon\; 0 \longrightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} C \longrightarrow 0 \;\; .$$

Each extension E induces an operator $\varphi_E \colon C \to \mathrm{Aut}(A)$. If E' is another extension, a morphism $\beta \colon E \to E'$ is a group-homomorphism $\beta \colon B \to B'$ such that the following diagram commutes in each part



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These data define the category of extensions of A by C. It is in fact a groupoid because, by the Five Lemma, such a β is an isomorphism. We call

$$OPEXT(C, A, \varphi)$$

the full subcategory of the extensions E such that $\varphi_E = \varphi$. The set

$$\Pi_0(OPEXT(C, A, \varphi))$$

of isomorphism classes of extensions is denoted by $OpExt(C, A, \varphi)$. It is an abelian group under Baer sum, the zero being given by the class of the semi-direct product $A \times_\varphi C$.

Now consider the following sets, which are abelian groups under point-wise sum :

- $\mathcal{C}^2(C, A)$ is the group of maps $g \colon C \to A$ such that $g(1) = 0$
- $\mathcal{Z}^2_\varphi(C, A)$ is the group of maps $f \colon C \times C \to A$ such that for all x, y, z in C we have : $f(x, 1) = 0 = f(1, y)$ and $x \cdot f(y, z) + f(x, yz) = f(x, y) + f(xy, z)$ (where $\cdot$ is the C-action on A induced by φ).

There is a group homomorphism

$$\delta \colon \mathcal{C}^2(C, A) \to \mathcal{Z}^2_\varphi(C, A)$$

defined, for $g \in \mathcal{C}^2(C, A)$ and $x, y \in C$, by $(\delta g)(x, y) = x \cdot g(y) - g(xy) + g(x)$. The cokernel of δ is the second cohomology group $H^2_\varphi(C, A)$ of C with coefficients in A.

Given a 2-cocycle, that is an element f of $\mathcal{Z}^2_\varphi(C, A)$, define an extension

$$\epsilon(f) \colon\ 0 \to A \to A \times_f C \to C \to 0$$

in the following way : the underlying set of $A \times_f C$ is the cartesian product of A and C, but the operations are deformed by f, that is $(a, x) + (a_1, y) = (a + x \cdot a_1 + f(x, y), xy)$ and $-(a, x) = (-x^{-1} \cdot a - x^{-1} \cdot f(x, x^{-1}), x^{-1})$ (in particular, if $f = 0$, then $A \times_f C$ is the semi-direct product $A \times_\varphi C$); the morphisms $A \to A \times_f C$ and $A \times_f C \to C$ send a on $(a, 1)$ and (a, x) on x. Passing to isomorphism classes, ϵ

gives rise to a group homomorphism $\mathcal{Z}^2_\varphi(C,A) \to OpExt(C,A,\varphi)$, which factors through the cokernel of δ (because if g is in $\mathcal{C}^2(C,A)$, then

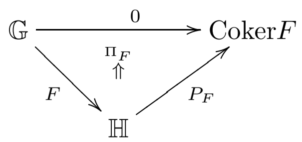
$$\Phi_g \colon A \times_{\delta g} C \to A \times_\varphi C \quad \Phi_g(a,x) = (a + g(x), x)$$

is an isomorphism between $\epsilon(\delta g)$ and the semi-direct product extension. The extension of ϵ to the cokernel of δ gives the isomorphism between $H^2_\varphi(C,A)$ and $OpExt(C,A,\varphi)$.

Categorical groups

Let $\mathbb{C} = (\mathbb{C}, \otimes, I, \dots)$ be a monoidal category. We note by I the unit object and we say that an object X of $\mathbb{C}$ is invertible if there exists an object X^* and two isomorphisms $I \to X \otimes X^*$ and $X^* \otimes X \to I$. We denote by $\Pi_0\mathbb{C}$ the monoid of isomorphism classes of objects of $\mathbb{C}$ (it is commutative if $\mathbb{C}$ is braided), and by $\Pi_1\mathbb{C}$ the commutative monoid $\mathbb{C}(I,I)$ of endomorphisms of I. A cat-group $\mathbb{G}$ is a monoidal groupoid such that each object is invertible [6, 7, 11]. In this case, $\Pi_0\mathbb{G}$ and $\Pi_1\mathbb{G}$ are groups. A morphism $F \colon \mathbb{G} \to \mathbb{H}$ of cat-groups is a monoidal functor ; it is an equivalence iff the induced group homomorphisms $\Pi_0 F$ and $\Pi_1 F$ are isomorphisms. If $\mathbb{G}$ and $\mathbb{H}$ are braided, we say that a morphism F is a γ-morphism if it respects the braiding.

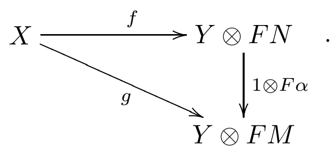
Now fix a γ-morphism $F \colon \mathbb{G} \to \mathbb{H}$ between symmetric cat-groups. The cokernel of F (see [8, 13]) is given by a triple $(\mathrm{Coker}F, P_F, \Pi_F)$ where $\mathrm{Coker}F$ is a symmetric cat-group, P_F a γ-morphism and $\Pi_F \colon F \cdot P_F \to 0$ a monoidal natural transformation, as in the following diagram



(0 is the constant functor which sends each morphism on the identity of the unit object). The cat-group $\mathrm{Coker}F$ can be described in the following way :

- the objects of $\mathrm{Coker}F$ are those of $\mathbb{H}$

- a morphism $[f,N] \colon X \to Y$ in $\mathrm{Coker}F$ is an equivalence class of pairs (f,N) with N an object of G and $f \colon X \to Y \otimes F(N)$ a morphism in $\mathbb{H}$; two such pairs (f,N) and (g,M) are declared equivalent if there exists a morphism $\alpha \colon N \to M$ in $\mathbb{G}$ such that the following diagram commutes



If $\mathbb{K}$ is a symmetric cat-group, $G\colon \mathbb{H} \to \mathbb{K}$ a γ-morphism and $\varphi\colon F \cdot G \to 0$ a monoidal natural transformation, there exists a unique factorization through the cokernel, that is a unique γ-morphism $G'\colon \mathrm{Coker}F \to \mathbb{K}$ such that $P_F \cdot G' = G$ and $\Pi_F \cdot G' = \varphi$. The functor G' sends an arrow $[f, N]\colon X \to Y$ on the composite

$$(Gf) \cdot (1 \otimes \varphi(N))\colon GX \to G(Y \otimes FN) \simeq GY \otimes G(FN) \to GY \otimes I \simeq GY$$

The cokernel is a bi-limit (see [12] for the notion of bi-limit), so it is characterized, up to equivalences, by its universal property, which is discussed in detail in [13]. Finally, observe that

$$\Pi_0\mathbb{G} \xrightarrow{\Pi_0 F} \Pi_0\mathbb{H} \xrightarrow{\Pi_0 P_F} \Pi_0(\mathrm{Coker}F)$$

is the usual cokernel in the category of abelian groups.

First application

If G is a (abelian) group, we can look at it as a (symmetric) discrete cat-group which we denote by $\underline{D}(G)$. Morphisms between discrete cat-groups are exactly group homomorphisms. The point is that in general the cokernel of a γ-morphism between two discrete symmetric cat-groups is NOT discrete. In fact, its undiscreteness is measured by the kernel, in the usual sense, of the morphism: if $f\colon A \to B$ is a morphism between abelian groups, $\Pi_1(\mathrm{Coker}\underline{D}(f)) = \mathrm{Ker}f$. Moreover, comparing with the general construction of $\mathrm{Coker}F$, we get the following explicit description of $\mathrm{Coker}\underline{D}(f)$:

- the objects of $\mathrm{Coker}\underline{D}(f)$ are the elements of B;
- an arrow between two objects b and b' is an element a of A such that $b = b' + f(a)$.

In other words, the groupoid underlying $\mathrm{Coker}\underline{D}(f)$ is nothing but the Dold-Kan denormalization of the chain complex $\cdots 0 \to 0 \to A \to B$. We apply these facts to the classification of extensions.

First, observe that, under Baer sum, the groupoid $OPEXT(C, A, \varphi)$ is a symmetric cat-group (see [1] for a conceptual proof). Now, the classical theorem recalled above, gives us some informations only on the group $OpExt(C, A, \varphi)$, that is on $\Pi_0(OPEXT(C, A, \varphi))$. To obtain informations on the whole category

$$OPEXT(C, A, \varphi)$$

it suffices to calculate the cokernel of $\delta\colon \mathcal{C}^2(C, A) \to \mathcal{Z}^2_\varphi(C, A)$ in the 2-category of symmetric cat-groups rather than in the category of abelian groups. In fact, with the previous notations, we dispose of two γ-morphisms

$$\underline{D}(\mathcal{C}^2(C, A)) \xrightarrow{\underline{D}(\delta)} \underline{D}(\mathcal{Z}^2_\varphi(C, A)) \xrightarrow{\epsilon} OPEXT(C, A, \varphi)$$

whose composite is naturally isomorphic to the zero-functor. The next proposition qualifies the cokernel of $\underline{D}(\delta)$ as the second cohomology cat-groups of C with coefficients in A.

Proposition : *The γ-morphism*

$$\epsilon' \colon \mathrm{Coker}\underline{D}(\delta) \to OPEXT(C,A,\varphi) \ ,$$

extension of ϵ to the cokernel of $\underline{D}(\delta)$, is an equivalence of cat-groups.

Proof. The fact that ϵ' is an equivalence of cat-groups is equivalent to the fact that $\Pi_0(\epsilon')$ and $\Pi_1(\epsilon')$ are isomorphisms. By the previous description of the cokernel of a discrete morphism, we get :

- $\Pi_0(\epsilon') \colon \Pi_0(\mathrm{Coker}\underline{D}(\delta)) = H^2_\varphi(C,A) \to OpExt(C,A,\varphi)$, which is the classical isomorphism already recalled;
- $\Pi_1(\epsilon') \colon \Pi_1(\mathrm{Coker}\underline{D}(\delta)) = \mathrm{Ker}\delta = \mathcal{Z}^1_\varphi(C,A) \to \Pi_1(OPEXT(C,A,\varphi))$, which is the isomorphism between the group of crossed homomorphisms and the group of automorphisms of $A \times_\varphi C$ inducing the identity on A and C, as in [10] Proposition IV.2.1.

□

Second application

We consider now another classical way used to compute the group of extensions [10]. Assume that C also is abelian and take φ to be the trivial operator (that is, $x \cdot a = a$ for all x in C and a in A). We consider the subcategory $EXT(C,A)$ of $OPEXT(C,A,\varphi)$ of those extensions

$$0 \to A \to B \to C \to O$$

in which B is abelian ; $Ext(C,A)$ is $\Pi_0(EXT(C,A))$. We can of course repeat the previous argument to explain the equivalence between $EXT(C,A)$ and the cat-group of symmetric 2-cocycles, i.e. 2-cocycles $f \colon C \times C \to A$ such that $f(x,y) = f(y,x)$ for all x,y in C. Otherwise, consider an exact sequence

$$F \colon \ 0 \longrightarrow K \xrightarrow{\ i\ } P \xrightarrow{\ e\ } C \longrightarrow 0$$

of abelian groups. It induces the usual exact sequence

$$\begin{array}{ccccc} 0 \longrightarrow Hom(C,A) & \xrightarrow{\ e_*\ } & Hom(P,A) & \xrightarrow{\ i_*\ } & Hom(K,A) \\ & & & & \downarrow F^* \\ Ext(K,A) & \xleftarrow[\ i^*\ ]{} & Ext(P,A) & \xleftarrow[\ e^*\ ]{} & Ext(C,A) \end{array}$$

(where F^* sends a morphism $\alpha\colon K \to A$ on the extension given by the pushout of i along α) and we have a factorization

$$\overline{F}\colon \mathrm{Coker}(i_*) \to Ext(C,A)$$

of F^* through the cokernel of i_*. Now, if we choose P to be projective, then $Ext(P,A) = 0$, so that the exactness in $Ext(C,A)$ and in $Hom(K,A)$ means that $\overline{F}$ is an isomorphism.
Once again, we have some informations only on the group $Ext(C,A)$, that is on $\Pi_0(EXT(C,A))$. But we have forgotten something : the exactness in $Hom(C,A)$ and in $Hom(P,A)$. What does this exactness mean ? Consider the following γ-morphisms of cat-groups

$$\underline{D}(Hom(P,A)) \xrightarrow{\underline{D}(i_*)} \underline{D}(Hom(K,A)) \xrightarrow{\mathcal{F}} EXT(C,A)$$

where $\mathcal{F}$ is defined as F^*, but without passing to isomorphism classes.

Proposition : *The γ-morphism*

$$\mathcal{F}'\colon \mathrm{Coker}\underline{D}(i_*) \to EXT(C,A)\ ,$$

extension of $\mathcal{F}$ to the cokernel of $\underline{D}(i_*)$, is an equivalence of cat-groups.

Proof. Once again, the fact that $\mathcal{F}'$ is an equivalence corresponds to two classical isomorphisms:

- $\Pi_0(\mathcal{F}')$ is the isomorphism $\overline{F}\colon \mathrm{Coker}(i_*) \to Ext(C,A)$;
- $\Pi_1(\mathrm{Coker}\underline{D}(i_*))$ is the kernel of i_*, which, by exactness in $Hom(P,A)$ and in $Hom(C,A)$, is $Hom(C,A)$. Finally, $\Pi_1(\mathcal{F}')$ is the familiar isomorphism between $Hom(C,A)$ and $\Pi_1(EXT(C,A))$ which sends an element g of $Hom(C,A)$ on the matrix

$$\begin{pmatrix} 1_A & 0_{A,C} \\ g & 1_C \end{pmatrix}.$$

□

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