Journal of Algebra and Related Topics
Vol. 3, No 1, (2015), pp 51-61

A NOTE ON MAXIMAL NON-PRIME IDEALS

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ABSTRACT. The rings considered in this article are commutative with identity $1 \neq 0$. We say that a proper ideal I of a ring R is a maximal non-prime ideal if I is not a prime ideal of R but any proper ideal A of R with $I \subseteq A$ and $I \neq A$ is a prime ideal. That is, among all the proper ideals of R, I is maximal with respect to the property of being not a prime ideal. The concept of maximal non-maximal ideal and maximal non-primary ideal of a ring can be similarly defined. The aim of this article is to characterize ideals I of a ring R such that I is a maximal non-prime (respectively, a maximal non-maximal, a maximal non-primary) ideal of R.

1. INTRODUCTION

The rings considered in this article are nonzero commutative with identity. If R is a subring of a ring T with identity 1, then we assume that $1 \in R$. If a set A is a subset of a set B and $A \neq B$, we denote it symbolically using the notation $A \subset B$. Let P be a property of rings. Let R be a subring of a ring T. Recall from [4] that R is a maximal non-P , if R does not have P, whereas each subring S of T with $R \subset S$ has property P. The concept of maximal non-Noetherian subring of a ring T was investigated in [3]. There are other interesting research articles which appeared in the literature focussing on maximal non-P subring of a ring T (see for example, [2, 4]). Let R be a non-zero commutative ring with identity. A proper ideal I of a ring R is said to be a maximal non-prime ideal of R if the following conditions hold: (i) I is not a

MSC(2010): Primary 13A15; Secondary 13C05.
Keywords: Maximal non-prime ideal, maximal non-maximal ideal, maximal non-primary ideal, maximal non-irreducible ideal.
Received: 12 June 2015, Accepted: 15 July 2015.
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prime ideal of R and (ii) If A is any proper ideal of R such that A contains I properly, then A is a prime ideal of R. Similarly, we can define the concept of a maximal non-maximal (respectively, a maximal non-primary) ideal of R. Motivated by the above mentioned works on maximal non- P subrings, in this article, we focus our attempt on characterizing maximal non-prime (respectively, maximal non-maximal , maximal non-primary) ideals of a ring R. Let I be a proper radical ideal of a ring R. It is proved in Proposition 3.2 that I is a maximal non-primary ideal of R if and only if I is a maximal non-prime ideal of R if and only if I is a maximal non-maximal ideal of R if and only if $I = M_1 \cap M_2$ for some distinct maximal ideals M_1, M_2 of R. Let I be a proper ideal of R such that $I \neq \sqrt{I}$. It is shown in Proposition 4.1 that I is a maximal non-prime ideal of R if and only if I is a maximal non-maximal ideal of R if and only if $\sqrt{I} = M$ is a maximal ideal of R with $M^2 \subseteq I$, and $M = Rx + I$ for any $x \in M \backslash I$. Moreover, it is proved in Proposition 4.2 that I is a maximal non-primary ideal of R if and only if $\sqrt{I} = P$ is a prime ideal of R such that R/I is a quasilocal one-dimensional ring and P/I is a minimal ideal of R/I.

By a quasilocal ring we mean a ring which admits only one maximal ideal. A Noetherian quasilocal ring is referred to as a local ring.. By dimension of a ring R, we mean its Krull dimension and we use the abbreviation $dimR$ to denote the dimension of a ring R. We denote the nilradical of a ring R by $nil(R)$. A ring R is said to be reduced if $nil(R) = (0)$.

2. Some preliminary results

As mentioned in the introduction the rings considered in this article are commutative with identity $1 \neq 0$. We begin with the following lemma.

Lemma 2.1. *Let R be a ring. If P_1, P_2 are incomparable prime ideals of R under inclusion, then $P_1 \cap P_2$ is not a primary ideal of R.*

Proof. Let $I = P_1 \cap P_2$. Since P_1 and P_2 are incomparable under inclusion, there exist $a \in P_1 \backslash P_2$ and $b \in P_2 \backslash P_1$. Note that $ab \in I$. By the choice of a, b, it is clear that $a \notin I$ and no power of $b \in I$. This proves that $I = P_1 \cap P_2$ is not a primary ideal of R. □

Lemma 2.2. *Let R be a reduced ring which is not an integral domain. If every nonzero proper ideal of R is primary, then R has exactly two prime ideals and both of them are maximal ideals of R.*

Proof. Since R is reduced but not an integral domain, it follows that R has at least two minimal prime ideals. Let P_1, P_2 be distinct minimal prime ideals of R. Now we obtain from Lemma 2.1 and the hypotesis that $P_1 \cap P_2 = (0)$. We prove that P_1, P_2 are maximal ideals of R. Let M be a maximal ideal of R such that $P_1 \subseteq M$. We claim that $M = P_1$. Suppose that $P_1 \neq M$. Then $M \nsubseteq P_1 \cup P_2$. Let $a \in M \backslash (P_1 \cup P_2)$ and $b \in P_2 \backslash P_1$. As $ab \notin P_1$, it follows that $ab \neq 0$. Hence Rab is a primary ideal of R. Note that $Rab \subseteq P_2$. Hence it follows from the choice of a that no power of $a \in Rab$. Therefore, $b \in Rab$. This implies that $b = rab$ for some $r \in R$ and so $b(1 - ra) = 0$. As $b \notin P_1$, it follows that $1 - ra \in P_1 \subset M$. From $a \in M$, we obtain that $1 = 1 - ra + ra \in M$. This is a contradiction. Therefore, $P_1 = M$ is a maximal ideal of R. Similarly, it follows that P_2 is a maximal ideal of R. From $P_1 \cap P_2 = (0)$, we get that R has exactly two prime ideals which are P_1 and P_2 and moreover, both are maximal ideals of R. □

Lemma 2.3. *Let R be a ring such that every nonzero proper ideal of R is primary. Then $dim R \leq 1$. Moreover, if R is not a reduced ring, then R is necessarily quasilocal.*

Proof. Suppose that $dim R > 1$. Then there exists a chain of prime ideals $P_1 \subset P_2 \subset P_3$ of R. Let $a \in P_2 \backslash P_1$ and $b \in P_3 \backslash P_2$. Since $ab \notin P_1$, it is clear that $ab \neq 0$ and hence $Rab \neq (0)$. Observe that $Rab \subseteq P_2$. By hypothesis, Rab is a primary ideal of R. From the choice of the element b, it is clear that no power of b can belong to Rab. Hence $a \in Rab$. This implies that $a = rab$ for some $r \in R$ and so $a(1 - rb) = 0$. Since $a \notin P_1$, it follows that $1 - rb \in P_1 \subset P_3$. From $b \in P_3$, we obtain that $1 = 1 - rb + rb \in P_3$. This is a contradiction. Therefore, $dim R \leq 1$.

We next prove the moreover assertion. Suppose that R is not quasilocal. Then there exist at least two distinct maximal ideals M_1, M_2 of R. As we are assuming that R is not a reduced ring, it follows that $M_1 \cap M_2 \neq (0)$. Hence by hypothesis, $M_1 \cap M_2$ is a primary ideal of R. This contradicts Lemma 2.1. Therefore, R is necessarily quasilocal. □

Lemma 2.4. *Let R be a ring which is not reduced. Suppose that (0) is not a primary ideal of R. If every nonzero proper ideal of R is primary, then $nil(R)$ is a minimal prime ideal of R. Indeed, $nil(R)$ is a minimal ideal of R.*

Proof. We know from Lemma 2.3 that R is necessarily quasilocal. Let M be the unique maximal ideal of R. Since R is not reduced, $nil(R) \neq (0)$. Hence $nil(R)$ is a primary ideal of R and so it follows from [1, Proposition 4.1] that $\sqrt{nil(R)} = nil(R)$ is a prime ideal of R. Since $nil(R) \subseteq P$ for any prime ideal of R, it follows that $nil(R)$ is a minimal prime ideal of R. As (0) is not a primary ideal of R, it follows from [1, Proposition 4.2] that $\sqrt{(0)}$ is not a maximal ideal of R. Thus $nil(R) \subset M$. We prove that for any nonzero $a \in nil(R)$, $nil(R) = Ra$. First we verify that for any $b \in nil(R)\backslash(0)$ and for any $m \in M\backslash nil(R)$, $bm = 0$. Suppose that $bm \neq 0$. By hypothesis, Rbm is a primary ideal of R. In fact Rbm is a $nil(R)$-primary ideal of R. Since no power of $m \in nil(R)$, we obtain that $b \in Rbm$. This implies that $b = rbm$ for some $r \in R$. Thus $b(1 - rm) = 0$. As $1 - rm$ is a unit in R, it follows that $b = 0$. This is a contradiction. Hence for any nonzero $b \in nil(R)$ and $m \in M\backslash nil(R)$, $bm = 0$. Let $x \in nil(R)$. We assert that $x \in Ra$. This is clear if $x = 0$. If $x \neq 0$, then $xm = 0 \in Ra$. Now Ra is a $nil(R)$-primary ideal of R and no power of $m \in nil(R)$. Hence it follows that $x \in Ra$. This proves that for any nonzero $a \in nil(R)$, $nil(R) = Ra$. This shows that $nil(R)$ is a minimal ideal of R. □

Lemma 2.5. *Let R be a quasilocal ring with M as its unique maximal ideal. Suppose that R is not reduced and $nil(R)$ is a prime ideal of R with $nil(R) \neq M$. If $nil(R)$ is a minimal ideal of R, then (0) is not a primary ideal of R.*

Proof. Let $a \in nil(R), a \neq 0$. Let $b \in M\backslash nil(R)$. Since $nil(R)$ is a simple R-module, it follows that $M(nil(R)) = (0)$ and so $ab = 0$. Now $a \neq 0$ and as $b \notin nil(R)$, it follows that $b^n \neq 0$ for all $n \geq 1$. This proves that (0) is not a primary ideal of R. □

Lemma 2.6. *Let R be a ring which is not reduced. If every nonzero proper ideal of R is a prime ideal of R, then R is quasilocal with $nil(R)$ as its unique maximal ideal and $(nil(R))^2 = (0)$. Moreover, for any $x \in nil(R)\backslash\{0\}$, $nil(R) = Rx$.*

Proof. Since any prime ideal is primary, it follows from Lemma 2.3 that R is necessarily quasilocal. Let M be the unique maximal ideal of R. We prove that $M = nil(R)$. Let $m \in M$. We assert that $m^2 = 0$. Suppose that $m^2 \neq 0$. Then Rm^2 is a prime ideal of R. Therefore, $m \in Rm^2$. This implies that $m = rm^2$ for some $r \in R$ and so $m(1 - rm) = 0$. From $1 - rm$ is a unit in R, it follows that $m = 0$. This is a contradiction. Thus for any $m \in M$, $m^2 = 0$ and so $M = nil(R)$. Hence M is the only prime ideal of R. Let $a, b \in M$. We show that $ab = 0$. This is clear if either $a = 0$ or $b = 0$. Suppose that $a \neq 0$ and $b \neq 0$. Then Ra, Rb are prime ideals of R. Therefore, $Ra = Rb = M$. This implies that $a = ub$ for some unit $u \in R$. It follows from $b^2 = 0$ that $ab = 0$. This proves that $M^2 = (nil(R))^2 = (0)$.

We next prove the moreover part. Let $x \in nil(R)\backslash\{0\}$. Then Rx is a prime ideal of R. From the fact that $nil(R)$ is the only prime ideal of R, it follows that $nil(R) = Rx$. $\square$

3. Radical non-maximal prime ideals

The aim of this section is to determine proper radical ideals I of a ring R such that I is a maximal non-prime ideal. We start with the following lemma.

Lemma 3.1. *Let D be an integral domain which is not a field. Then it admits nonzero proper ideals which are not prime ideals.*

Proof. Let $d \in D$ be a nonzero nonunit. Then for any $n \geq 2$, Dd^n is a proper nonzero ideal of D which is not a prime ideal of D. $\square$

Proposition 3.2. *Let R be a ring and I be a proper radical ideal of R. Then the following statements are equivalent:*
(i) I is a maximal non-primary ideal of R.
(ii) $I = M_1 \cap M_2$ for some distinct maximal ideals M_1, M_2 of R.
(iii) I is a maximal non-maximal ideal of R.
(iv) I is a maximal non-prime ideal of R.

Proof. $(i) \Rightarrow (ii)$ Note that R/I is a reduced ring and as I is not primary, it follows that I is not a prime ideal of R and so R/I is not an integral domain. Since I is a maximal non-primary ideal of R, it follows that every nonzero proper ideal of R/I is primary. Hence we obtain from Lemma 2.2 that there exist distinct maximal ideals M_1, M_2 of R such that $I = M_1 \cap M_{2|}$.
$(ii) \Rightarrow (iii)$ We know from Lemma 2.1 that $I = M_1 \cap M_2$ is not a primary ideal and hence it is not a maximal ideal of R. Let A be any proper ideal of R such that $M_1 \cap M_2 \subset A$. Then either $A \nsubseteq M_1$ or $A \nsubseteq M_2$. Without loss of generality we may assume that $A \nsubseteq M_1$. Then $A + M_1 = R$. Hence $1 = a + x$ for some $a \in A$ and $x \in M_1$. Now for any $y \in M_2, y = ay + xy \in A + M_1M_2 = A$. This proves that $M_2 \subseteq A$ and so $A = M_2$. Thus the only proper ideals A of R which contain I properly are M_1 and M_2 and both are maximal ideals of R. Therefore, we obtain that I is a maximal non-maximal ideal of R.
$(iii) \Rightarrow (iv)$ Let A be a proper ideal of R with $I \subset A$. Then by (iii) A is a maximal ideal of R. Hence A is a prime ideal of R. We claim that I is not a prime ideal of R. Suppose that I is a prime ideal of R. Since R/I is not a field, it follows from Lemma 3.1 that R/I admits nonzero proper ideals which are not maximal ideals. This contradicts (iii). Therefore, I is not a prime ideal of R. This shows that I is a maximal non-prime ideal of R.

$(iv) \Rightarrow (i)$ Let A be any proper ideal of R with $I \subset A$. Then by (iv) A is a prime ideal and hence is a primary ideal of R. Since I is a radical ideal of R and is not a prime ideal of R, we get that I is not a primary ideal of R. This proves that I is a maximal non-primary ideal of R. $\square$

4. Non-radical maximal non-prime ideals

The aim of this section is to determine ideals I of a ring R such that $I \neq \sqrt{I}$ and I is a maximal non-prime ideal of R.

Proposition 4.1. *Let I be a proper ideal of a ring R such that $I \neq \sqrt{I}$. Then the following statements are equivalent:*
(i) I is a maximal non-prime ideal of R.
(ii) $\sqrt{I}$ is a maximal ideal of R, $(\sqrt{I})^2 \subseteq I$, and $\sqrt{I} = Rx + I$ for any $x \in \sqrt{I} \backslash I$.
(iii) I is a maximal non-maximal ideal of R.

Proof. $(i) \Rightarrow (ii)$ Note that R/I is a non-reduced ring in which any non-zero proper ideal is a prime ideal. Hence we obtain from Lemma 2.6 that R/I is a quasilocal ring with $\sqrt{I}/I$ as its unique maximal ideal, $(\sqrt{I}/I)^2 = I/I$, and moreover, $\sqrt{I}/I = R/I(x + I)$ for any $x \in \sqrt{I}\backslash I$. Therefore, $\sqrt{I}$ is a maximal ideal of R, $(\sqrt{I})^2 \subseteq I$, and $\sqrt{I} = Rx + I$ for any $x \in \sqrt{I}\backslash I$.
$(ii) \Rightarrow (iii)$ Since $I \subset \sqrt{I}$, it follows that I is not a maximal ideal of R. Let A be any proper ideal of R such that $I \subset A$. From $(\sqrt{I})^2 \subseteq I \subset A$, it follows that $\sqrt{I} \subseteq \sqrt{A}$. Since $\sqrt{I}$ is a maximal ideal of R, we obtain $\sqrt{I} = \sqrt{A}$. Let $a \in A\backslash I$. Then $a \in \sqrt{I}$. Hence $\sqrt{I} = Ra + I \subseteq A$ and so $A = \sqrt{I}$ is a maximal ideal of R. This proves that I is a maximal non-maximal ideal of R.
$(iii) \Rightarrow (i)$ As $I \subset \sqrt{I}$, it follows that I is not a prime ideal of R. Let A be any proper ideal of R with $I \subset A$. Then A is a maximal ideal and hence is a prime ideal of R. This shows that I is a maximal non-prime ideal of R. $\square$

We next proceed to characterize proper ideals I of a ring R such that $I \neq \sqrt{I}$ and I is a maximal non-primary ideal of R.

Proposition 4.2. *Let I be a proper ideal of a ring R such that $I \neq \sqrt{I}$. Then the following statements are equivalent:*
(i) I is a maximal non-primary ideal of R.
(ii) $\sqrt{I}$ is a prime ideal of R, R/I is quasilocal, $\dim(R/I) = 1$, and $\sqrt{I}/I$ is a simple R/I-module.

Proof. $(i) \Rightarrow (ii)$ As $I \neq \sqrt{I}$ and I is a maximal non-primary ideal of R, it follows that I is not a primary ideal of R, whereas $\sqrt{I}$ is a primary ideal of R. Hence $\sqrt{\sqrt{I}} = \sqrt{I}$ is a prime ideal of R. Let us denote $\sqrt{I}$ by P. Note that R/I is not a reduced ring, the zero-ideal of R/I is not primary but each proper nonzero ideal of R/I is primary. Hence we obtain from Lemma 2.3 that R/I is quasilocal, $dim(R/I) \leq 1$, and moreover, it follows from Lemma 2.4 that P/I is a minimal ideal of R/I (that is, P/I is a simple R/I-module). Let M/I denote the unique maximal ideal of R/I. Since I is not a primary ideal of R, it follows from [1, Proposition 4.2] that $\sqrt{I}$ is not a maximal ideal of R. Therefore, $P/I \subset M/I$ and so $dim(R/I) = 1$.
$(ii) \Rightarrow (i)$ Note that the ring R/I satisfies the hypotheses of Lemma 2.5. Hence it follows from Lemma 2.5 that the zero-ideal of R/I is not a primary ideal. Hence I is not a primary ideal of R. Let A be any proper ideal of R such that $I \subset A$. We consider two cases:
Case(1) $A \subseteq \sqrt{I}$
In this case A/I is a nonzero ideal of R/I and $A/I \subseteq \sqrt{I}/I$. As $\sqrt{I}/I$ is a minimal ideal of R/I, we obtain that $A/I = \sqrt{I}/I$ and so $A = \sqrt{I}$ is a prime ideal of R. Hence A is a primary ideal of R.
Case(2) $A \not\subseteq \sqrt{I}$
Let us denote the unique maximal ideal of R/I by M/I. Note that M is the only prime ideal of R containing A. Hence it follows that $\sqrt{A} = M$. Since M is a maximal ideal of R, we obtain from [1, Proposition 4.2] that A is a primary ideal of R.

This proves that I is a maximal non-primary ideal of R. □

Recall from [1, p.52] that a proper ideal I of a ring R is said to be decomposable if I admits a primary decomposition (that is, I can be expressed as the intersection of a finite number of primary ideals of R). The following proposition characterizes decomposable ideals I of a ring R such that $I \neq \sqrt{I}$ and I is a maximal non-primary ideal.

Proposition 4.3. *Let I be a proper ideal of a ring R such that $I \neq \sqrt{I}$ and I is decomposable. The following statements are equivalent:*
(i) I is a maximal non-primary ideal of R.
(ii) $\sqrt{I}$ is a prime ideal of R, $(R/I, M/I)$ is quasilocal, $dim(R/I) = 1$, $I = \sqrt{I} \cap q$, where q is a M-primary ideal of R, $q \neq M$, and $\sqrt{I}/I$ is a simple R/I-module.

Proof. $(i) \Rightarrow (ii)$ It follows from $(i) \Rightarrow (ii)$ of Proposition 4.2 that $\sqrt{I}$ is a prime ideal of R, R/I is quasilocal, $\dim(R/I) = 1$, and $\sqrt{I}/I$ is a simple R/I-module. Let M/I denote the unique maximal ideal of R/I.

We are assuming that I is decomposable. Let $I = q_1 \cap \cdots \cap q_n$ be an irredundant primary decomposition of I in R with q_i is a P_i-primary ideal of R for each $i \in \{1, \ldots, n\}$. Since I is not a primary ideal of R, it follows that $n \geq 2$. Note that $\sqrt{I} = \cap_{i=1}^{n} P_i$. As $\sqrt{I}$ is a prime ideal of R, it follows that $\sqrt{I} = P_i$ for some $i \in \{1, 2, \ldots, n\}$. Without loss of generality we may assume that $\sqrt{I} = P_1$. Since $P_i \neq P_j$ for all distinct $i, j \in \{1, 2, \ldots, n\}$, it follows that $P_1 \subset P_j$ for all $j \in \{2, \ldots, n\}$. As P_1/I and M/I are the only prime ideals of R/I, it follows that $n = 2$ and $P_2 = M$. Note that $I \subseteq P_1 \cap q_2$. We assert that $I = P_1 \cap q_2$. Since $q_1 \not\subseteq q_2$, it follows that $P_1 \not\subseteq q_2$. Let $x \in P_1 \backslash q_2$ and let $y \in q_2 \backslash P_1$. Observe that $xy \in P_1 \cap q_2$ but no power of y belongs to $P_1 \cap q_2$ and $x \notin P_1 \cap q_2$. Hence $P_1 \cap q_2$ is not a primary ideal of R. As we are assuming that I is a maximal non-primary ideal of R, it follows that $I = P_1 \cap q_2$. Since $I \neq \sqrt{I}$, it follows that $q_2 \neq M$.
$(ii) \Rightarrow (i)$ This follows immediately from $(ii) \Rightarrow (i)$ of Proposition 4.2. $\square$

Example 4.4. Let $R = K[[X, Y]]$ be the power series ring in two variables X, Y over a field K. It is well-known that R is a local ring with $M = RX + RY$ as its unique maximal ideal. Let $I = RX^2 + RXY$. Observe that $I = RX \cap M^2$. Note that $\sqrt{I} = RX$ is a prime ideal of R, $M^2 \neq M$ is a M-primary ideal of R, $dim(R/I) = 1$, and RX/I is a simple R/I-module. Hence it follows from $(ii) \Rightarrow (i)$ of Proposition 4.3 that I is a maximal non-primary ideal of R.

5. Maximal non-irreducible ideals

Recall that an ideal I of a ring R is irreducible, if I is not the intersection of any ideals I_1, I_2 of R with $I \subset I_i$ for each $i \in \{1, 2\}$. The aim of this section is to determine proper ideals I of a ring R such that I is a maximal non-irreducible ideal of R. We first characterize proper radical ideals I of R such that I is a maximal non-irreducible ideal of R.

Proposition 5.1. *Let I be a proper radical ideal of a ring R. Then the following statements are equivalent:*
(i) I is a maximal non-irreducible ideal of R.
(ii) $I = M_1 \cap M_2$ for some distinct maximal ideals M_1, M_2 of R.

Proof. $(i) \Rightarrow (ii)$ Since I is a proper radical ideal of R, it follows from [1, Proposition 1.14] that I is the intersection of all the prime ideals P of R such that $P \supseteq I$. Let C be the collection of all prime ideals P of R such that P is minimal over I. Observe that we obtain from [5, Theorem 10] that I is the intersection of all members of C. Since I is not irreducible

and any prime ideal is irreducible, we get that C contains at least two elements. Let $P_1, P_2 \in C$ be distinct. We assert that $C = \{P_1, P_2\}$. Suppose that there exists $P_3 \in C$ such that $P_3 \notin \{P_1, P_2\}$. Then it is clear that $I \subset P_2 \cap P_3$ and $P_2 \cap P_3$ is non-irreducible. This is in contradiction to the assumption that I is a maximal non-irreducible ideal of R. Therefore, $C = \{P_1, P_2\}$ and so $I = P_1 \cap P_2$. We next show that P_1 and P_2 are maximal ideals of R. Towards showing it, we first prove that $P_1 + P_2 = R$. Suppose that $P_1 + P_2 \neq R$. Let M be a maximal ideal of R such that $P_1 + P_2 \subseteq M$. Since P_1 and P_2 are not comparable under the inclusion relation, there exist $a \in P_1 \backslash P_2$ and $b \in P_2 \backslash P_1$. Consider the ideals $J_1 = I + Ra + Rb^2$ and $J_2 = I + Ra^2 + Rb$ of R. It is clear that $I \subseteq J_1 \cap J_2$. As $a^2 \in (J_1 \cap J_2) \backslash I$, it follows that $I \subset J_1 \cap J_2$. Since I is a maximal non-irreducible ideal of R, we obtain that $J_1 \cap J_2$ is irreducible. Therefore, either $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$. If $J_1 \subseteq J_2$, then $a = x + ra^2 + sb$ for some $x \in I = P_1 \cap P_2$ and $r, s \in R$. This implies that $a(1 - ra) = x + sb \in P_2$. As $a \notin P_2$, we obtain that $1 - ra \in P_2$. Therefore, $1 = ra + 1 - ra \in P_1 + P_2 \subseteq M$. This is a contradiction. Observe that we get a similar contradiction if $J_2 \subseteq J_1$. Hence $P_1 + P_2 = R$. Let M_1 be a maximal ideal of R such that $P_1 \subseteq M_1$. Since $P_1 + P_2 = R$, it follows that the ideal $M_1 \cap P_2$ is not irreducible. As $I \subseteq M_1 \cap P_2$, we obtain that $I = P_1 \cap P_2 = M_1 \cap P_2$. Since $P_1 \not\supseteq P_2$, it follows that $P_1 \supseteq M_1$ and so $P_1 = M_1$ is a maximal ideal of R. Similarly it can be shown that P_2 is a maximal ideal of R. Thus $I = M_1 \cap M_2$ for some distinct maximal ideals M_1, M_2 of R.
$(ii) \Rightarrow (i)$ If $I = M_1 \cap M_2$ for some distinct maximal ideals M_1, M_2 of R, then it is clear that I is not irreducible. It is verified in the proof of $(ii) \Rightarrow (iii)$ of Proposition 3.2 that M_1 and M_2 are the only proper ideals J of R such that $I \subset J$. Since M_1 and M_2 are both irreducible, we obtain that I is a maximal non-irreducible ideal of R. □

Let I be a proper ideal of a ring R such that $I \neq \sqrt{I}$. We next attempt to characterize such ideals I in order that I is a maximal non-irreducible ideal of R. We do not know the precise characterization of such ideals. However, we have the following partial results.

Lemma 5.2. *Let I be a proper ideal of a ring R such that $I \neq \sqrt{I}$. If I is a maximal non-irreducible ideal of R, then $\sqrt{I}$ is a prime ideal of R and moreover, R/I is quasilocal.*

Proof. Let C be the colletion of all prime ideals P of R such that P is minimal over I. We assert that C is singleton. Let $P, Q \in C$. Since $I \neq \sqrt{I}$, it is clear that $I \subset P \cap Q$. As I is a maximal non-irreducible ideal of R, it follows that $P \cap Q$ is irreducible. Hence either $P \subseteq Q$ or

$Q \subseteq P$. Therefore, $P = Q$. This shows that there is only one prime ideal P of R such that P is minimal over I. Thus $\sqrt{I} = P$ is a prime ideal of R.

We next show that R/I is quasilocal. Let M, N be maximal ideals of R such that $I \subseteq M \cap N$. Since $I \neq \sqrt{I}$, it follows that $I \subset M \cap N$. As $M \cap N$ is irreducible, we obtain that either $M \subseteq N$ or $N \subseteq M$. Hence $M = N$. This shows that R/I is quasilocal. □

Lemma 5.3. *Let* (T, N) *be a quasilocal ring such that* $(0) \neq \sqrt{(0)}$ *and* (0) *is a maximal non-irreducible ideal of* T. *Then* $dim_{T/N}(N/N^2) \leq 2$.

Proof. Suppose that $dim_{T/N}(N/N^2) \geq 3$. Let $\{a, b, c\} \subseteq N$ be such that $\{a+N^2, b+N^2, c+N^2\}$ is linearly independent over T/N. Consider the ideals $J_1 = Ta + Tc$ and $J_2 = Tb + Tc$. By the choice of a, b, c, it is clear that $J_1 \not\subseteq J_2$, $J_2 \not\subseteq J_1$ and so $J_1 \cap J_2$ is not an irreducible ideal of T. Moreover, as $c \in J_1 \cap J_2$, it follows that $J_1 \cap J_2 \neq (0)$. This contradicts the hypothesis that (0) is a maximal non-irreducible ideal of T. Therefore, $dim_{T/N}(N/N^2) \leq 2$. □

Lemma 5.4. *Let* (T, N) *be a quasilocal ring such that* $(0) \neq \sqrt{(0)}$ *and* $dim_{T/N}(N/N^2) = 2$. *Then the following statements are equivalent:*
(i) (0) *is a maximal non-irreucible ideal of* T.
(ii) $N^2 = (0)$.

Proof. By hypothesis, $dim_{T/N}(N/N^2) = 2$. Let $\{a, b\} \subseteq N$ be such that $\{a + N^2, b + N^2\}$ is a basis of N/N^2 as a vector space over T/N.
$(i) \Rightarrow (ii)$ Consider the ideals $J_1 = N^2 + Ta$ and $J_2 = N^2 + Tb$. By the choice of the elements a, b, it is clear that $J_1 \not\subseteq J_2$ and $J_2 \not\subseteq J_1$. Hence the ideal $J_1 \cap J_2$ is not irreducible. Since (0) is a maximal non-irreducible ideal of T, it follows that $J_1 \cap J_2 = (0)$. As $N^2 \subseteq J_1 \cap J_2$, we obtain that $N^2 = (0)$.
$(ii) \Rightarrow (i)$ It follows from $N^2 = (0)$ and from the choice of the elements a, b that $Ta \not\subseteq Tb$, $Tb \not\subseteq Ta$, and $Ta \cap Tb = (0)$. This implies that (0) is not an irreducible ideal of T. Let J be any nonzero proper ideal of T. Then either $dim_{T/N}(J) = 1$ or 2. If $dim_{T/N}(J) = 2$, then $J = N$ is irreducible. Suppose that $dim_{T/N}(J) = 1$. Let A, B be proper ideals of T such that $J = A \cap B$. If $J \neq A$ and $J \neq B$, then we get that $A = B = N$ and so $J = N$. This is a contradiction. Hence either $J = A$ or $J = B$. This shows that J is irreucible. Hence (0) is a maximal non-irreducible ideal of T. □

Acknowledgements

We are very much thankful to the referee for a very careful reading of this paper and valuable suggestions. We are also very much thankful to Professor H. Ansari-Toroghy for his support.

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