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Sensitivity analysis and model order reduction for random linear dynamical systems

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Abstract

We consider linear dynamical systems defined by differential algebraic equations. The associated input-output behaviour is given by a transfer function in the frequency domain. Physical parameters of the dynamical system are replaced by random variables to quantify uncertainties. We analyse the sensitivity of the transfer function with respect to the random variables. Total sensitivity coefficients are computed by a nonintrusive and by an intrusive method based on the expansions in series of the polynomial chaos. In addition, a reduction of the state space is applied in the intrusive method. Due to the sensitivities, we perform a model order reduction within the random space by changing unessential random variables back to constants. The error of this reduction is analysed. We present numerical simulations of a test example modelling a linear electric network.

Key words: linear dynamical systems, differential algebraic equations, sensitivity analysis, model order reduction, random variables, polynomial chaos, uncertainty quantification, electric circuits.

1 Introduction

Mathematical modelling of technical applications often results in systems of differential algebraic equations (DAEs). Examples are models of electric circuits or multibody dynamics, see [10, 20]. We consider linear dynamical systems, which represent DAEs or ordinary differential equations (ODEs). A Laplace transformation reveals the input-output behaviour in the frequency domain.

The physical parameters of the system may exhibit uncertainties. We substitute the parameters by independent random variables for modelling the uncertainties. The number of parameters is often large, since the technical application involves many components. Thus our aim is to obtain an uncertainty quantification in case of a high-dimensional random space. The large number of random variables makes a numerical simulation by standard methods too costly. Thus we require techniques of model order reduction (MOR) to decrease the complexity.

Appropriate MOR methods to reduce the dimension of the state space for linear dynamical systems already exist based on the transfer function, see [1, 5, 7, 19]. Our idea is to analyse the transfer function also for a reduction of a high-dimensional random space in the stochastic modelling. Concepts for a variance-based sensitivity analysis are available for general functions depending on random variables, see [21].

We compute the required total sensitivity coefficients approximately by the expansions of the generalised polynomial chaos (gPC), following [24]. For this purpose, we investigate nonintrusive approaches based on quadrature as well as intrusive approaches resulting from the stochastic Galerkin method, see [28]. Moreover, an MOR of the state space is considered for the huge systems in the intrusive method. Techniques based on gPC have been applied successfully to nonlinear ODEs in [2, 3] and to linear or nonlinear DAEs in [12, 13, 14, 15, 16, 17].

Sobol [21] suggests to replace random variables with relatively small total sensitivities by constants. We apply this approach to reduce the dimension of the random space observing the sensitivity of the transfer function. While just sparse grids or sampling methods are able to tackle high-dimensional problems, the reduced order model can be resolved by highly accurate tensor-product grids now. We analyse the error of the reduction in the probability space and its interaction between the frequency domain and the time domain. Furthermore, numerical simulations of a test example confirm the efficiency of this MOR in the random space.

The paper is organised as follows. The type of linear problems and the stochastic modelling is introduced in Sect. 2. The gPC expansions and the related numerical

methods are specified in Sect. 3. We outline the sensitivity analysis and construct our MOR approach in Sect. 4. The error of this MOR is estimated stochastically. In Sect. 5, we present results of numerical simulations using the derived strategy. An appendix contains the proofs of the theorems stated in Sect. 4.

2 Dynamical Systems with Random Parameters

In this section, we define the problems to be investigated. Therein, the dependence of linear dynamical systems on physical parameters is substantial.

2.1 Linear dynamical systems

We discuss linear systems of the form

$$\begin{aligned} C(p)x'(t,p) + G(p)x(t,p) &= Bu(t) \\ y(t,p) &= Lx(t,p) \end{aligned} \tag{1}$$

for $t \in I_t$ with $I_t = [0, t_{\text{end}}]$ or $I_t = [0, \infty)$. The matrices $C(p), G(p) \in \mathbb{R}^{N\times N}$ depend on parameters $p \in \Pi \subseteq \mathbb{R}^Q$. Thus the state variables $x : I_t \times \Pi \to \mathbb{R}^N$ are also parameter-dependent. Input signals $u : I_t \to \mathbb{R}^{N_{\text{in}}}$ are introduced via a constant matrix $B \in \mathbb{R}^{N\times N_{\text{in}}}$. We define output signals $y : I_t \times \Pi \to \mathbb{R}^{N_{\text{out}}}$ by the state variables using a constant matrix $L \in \mathbb{R}^{N_{\text{out}}\times N}$.

Often the matrix $C(p)$ is singular for all $p \in \Pi$, i.e., the system (1) represents DAEs. We assume that the associated matrix pencil $C(p) + \lambda G(p)$ is regular for each $p \in \Pi$ to guarantee existence and uniqueness of solutions for initial value problems. Moreover, let the matrix pencil be regular in the limit case $\lambda \to \infty$ to ensure the applicability of time integration techniques, cf. [11]. If $G(p)$ is regular for all $p \in \Pi$, then we obtain existence and uniqueness of stationary solutions for each p.

Without loss of generality, we assume initial values $x(0,p) = 0$, since the transformation $z(t,p) := x(t,p) - x(0,p)$ is applied otherwise. In addition, we consider the case $u(0) = 0$ for the input. Let $X(s,p)$, $Y(s,p)$, $U(s)$ for $s \in \Sigma \subseteq \mathbb{C}$ be the Laplace transforms of the state variables, the output signals and the input signals, respectively. Often just the imaginary axis $s = \mathrm{i}\omega$ for frequencies $\omega \in \mathbb{R}$ is considered. A transformation of the linear dynamical system (1) into the frequency domain yields the input-output relation

$$Y(s,p) = H(s,p)U(s) \tag{2}$$

with the transfer function $H : \Sigma \times \Pi \to \mathbb{C}^{N_{\text{out}} \times N_{\text{in}}}$ defined by

$$H(s,p) := L\left(G(p) + sC(p)\right)^{-1} B \tag{3}$$

for $s \in \Sigma$. The involved inverse matrix can be singular at a finite number of poles. More details are given in [1, 5, 7, 19].

2.2 Stochastic Modelling

The parameters $p \in \Pi$ of the linear dynamical system (1) may exhibit uncertainties. To achieve an uncertainty quantification, we replace the parameters by independent random variables on a random space $(\Omega, \mathcal{A}, \mu)$, i.e.,

$$p : \Omega \to \Pi, \quad p(\omega) = (p_1(\omega), \ldots, p_Q(\omega)).$$

We apply traditional distributions like Gaussian, uniform, beta, etc. to model the uncertainties. Thus a probability density $\rho : \Pi \to \mathbb{R}$ is available. The state variables as well as the output of the system (1) becomes a random process in time domain.

We define associated spaces of integrable functions by

$$L^k(\Pi, \rho) := \{f : \Pi \to \mathbb{K} : \ f \text{ measurable and } \langle f(p)^k \rangle < \infty\}$$

for each integer k and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For a function $f \in L^1(\Pi, \rho)$, we introduce the abbreviation

$$\langle f(p) \rangle := \int_\Omega f(p(\omega)) \, \mathrm{d}\mu(\omega) = \int_\Pi f(p)\rho(p) \, \mathrm{d}p \tag{4}$$

for the expected value. We obtain an inner product

$$\langle f(p)g(p) \rangle = \int_\Pi f(p)g(p)\rho(p) \, \mathrm{d}p \tag{5}$$

on the Hilbert space $L^2(\Pi, \rho)$. We apply the notation (4) also to vector-valued and matrix-valued functions by considering the integration componentwise.

Similarly, the Laplace transforms $X(s,p)$, $Y(s,p)$ become random processes in frequency domain. We investigate the transfer function $H(s,p)$ as random process in the following sections, since it describes the input-output behaviour of the stochastic model.

3 Polynomial Chaos Techniques

We introduce the polynomial chaos expansions and associated numerical methods as a tool for the sensitivity analysis in Section 4.2.

3.1 Polynomial Chaos Expansions

The generalised polynomial chaos (gPC) is applicable for a broad class of random distributions, see [26] and the references therein. We use this concept to the state variables and the output of the linear dynamical system (1) as well as the transfer function (3). Assuming finite second moments, the gPC expansion of the state variables reads

$$x(t, p(\omega)) = \sum_{i=0}^{\infty} v_i(t)\Phi_i(p(\omega)) \tag{6}$$

for $t \in I_t$. including a complete set of basis polynomials $(\Phi_i)_{i\in\mathbb{N}}$ with $\Phi_i : \Pi \to \mathbb{R}$. Let the set be orthonormal, i.e., $\langle \Phi_i \Phi_j \rangle = \delta_{ij}$ using the Kronecker-delta, and $\Phi_0 \equiv 1$. The basis polynomials are given a priori by the choice of the random distributions of the parameters, see [26]. The time-dependent coefficient functions $v_i : I_t \to \mathbb{R}^N$ represent inner products (5)

$$v_i(t) = \langle x(t, p)\Phi_i(p)\rangle \quad \text{for each} \;\; i \in \mathbb{N} \tag{7}$$

and each $t \in I_t$. The expansion (6) converges in the norm of the Hilbert space $L^2(\Pi, \rho)$ for each fixed t. We consider the variance of the state variables componentwise by the usual definition $\mathrm{Var}(x_j) := \langle (x_j - \langle x_j \rangle)^2 \rangle$ for each $t \in I_t$ and $j = 1, \ldots, N$. More details can be found in the two monographs [8, 28].

We obtain numerical approximations of a gPC expansion (6) by a truncation to

$$x^M(t, p) = \sum_{i=0}^{M-1} v_i(t)\Phi_i(p). \tag{8}$$

Typically, all polynomials up to some degree D are included in (8), i.e., the number of basis functions becomes $M = (D + Q)!/(D!Q!)$. Now numerical solutions of the finite set of coefficient functions have to be determined.

Likewise, a gPC expansion exists for the output by

$$y(t, p) = \sum_{i=0}^{\infty} w_i(t)\Phi_i(p) \tag{9}$$

for each fixed $t \in I_t$ with coefficients $w_i : I_t \to \mathbb{R}^{N_{\text{out}}}$, since L is a constant matrix. Let the expansion of the transfer function be

$$H(s, p(\omega)) = \sum_{i=0}^{\infty} H_i(s)\Phi_i(p(\omega)) \tag{10}$$

for $s \in \Sigma$ and coefficients $H_i : \Sigma \to \mathbb{C}^{N_{\text{out}} \times N_{\text{in}}}$.

3.2 Nonintrusive Methods

The aim is to compute an approximation of the coefficients in the truncated gPC expansion (8). The exact coefficients represent expected values due to (7). These probabilistic integrals can be approximated by a multidimensional quadrature or by a sampling method. In each case, we obtain an approximation of the form

$$\tilde{v}_i(t) = \sum_{k=1}^{K} \gamma_k \Phi_k(p_k) x(t, p_k) \tag{11}$$

with (deterministic) nodes $p_k \in \Pi$ and weights $\gamma_k \in \mathbb{R}$. Monte-Carlo as well as quasi Monte-Carlo methods exhibit the weights $\gamma_k = \frac{1}{K}$ for all k. This approach is called nonintrusive, since an evaluation of (11) requires to solve K linear dynamical systems (1), which can be done directly. Another name of this type of methods is stochastic collocation, see [27].

In theory, the same fixed quadrature scheme (with constant K) yields approximations of the v_i for each i. In practice, higher-order quadrature techniques have to be used for coefficients v_i associated to basis polynomials Φ_i of a larger degree to obtain sufficiently accurate approximations. Thus the number of nodes K depends on the choice of the maximum degree D.

The above statements for the gPC coefficients of the state variables also apply to the coefficients H_i within the gPC expansion (10) of the transfer function. The approximations read

$$\tilde{H}_i(s) = \sum_{k=1}^{K} \gamma_k \Phi_i(p_k)\, L \left(G(p_k) + sC(p_k)\right)^{-1} B, \tag{12}$$

which requires to solve N_{in} linear systems with an identical matrix for each node $k = 1, \ldots, K$.

3.3 Intrusive Methods

Alternatively, we obtain approximations of the unknown coefficients in the truncated gPC expansion (8) by a stochastic Galerkin method, see [28]. This approach yields a much larger and coupled system

$$\begin{aligned} \hat{C}\hat{v}'(t) + \hat{G}\hat{v}(t) &= \hat{B}u(t) \\ \hat{w}(t) &= \hat{L}\hat{v}(t) \end{aligned} \tag{13}$$

satisfied by approximations $\hat{v} := (\hat{v}_0, \ldots, \hat{v}_{M-1})$ and $\hat{w} := (\hat{w}_0, \ldots, \hat{w}_{M-1})$ of the coefficients for state variables and output signals, respectively, cf. (6) and (9).

The involved matrices $\hat{C}, \hat{G} \in \mathbb{R}^{MN \times MN}$ are defined by the minors

$$\hat{C}_{ij} := \langle C(p)\Phi_i(p)\Phi_j(p)\rangle, \qquad \hat{G}_{ij} := \langle G(p)\Phi_i(p)\Phi_j(p)\rangle \tag{14}$$

for $i, j = 0, 1, \ldots, M-1$. We assume that the components of the matrices $C(p), G(p)$ are polynomials in p. Thus the complete matrices $\hat{C}, \hat{G}$ can be computed exactly by a Gaussian quadrature except for roundoff errors. For example, modified nodal analysis yields matrices with a linear dependence on the parameters in case of linear electric networks, cf. [10]. Otherwise, a redefinition of parameters often enables to meet this assumption (for example, $1/p$ is replaced by $p^* := 1/p$ and a random distribution is considered for p^*). The matrices $\hat{B} \in \mathbb{R}^{MN \times N_{\text{in}}}$ and $\hat{L} \in \mathbb{R}^{MN_{\text{out}} \times MN}$ simply result to

$$\hat{B} := e_1 \otimes B, \qquad \hat{L} := I_M \otimes L \tag{15}$$

using the Kronecker product, the unit vector $e_1 := (1, 0, \ldots, 0)^\top \in \mathbb{R}^M$ and the identity matrix $I_M \in \mathbb{R}^{M \times M}$.

The equations (13) represent a linear dynamical system again. The input-output relation is given by

$$\hat{W}(s) = \hat{H}(s)U(s), \tag{16}$$

where $\hat{W}(s)$ are the Laplace transforms of $\hat{w}(t)$. The associated transfer function $\hat{H} : \Sigma \to \mathbb{C}^{MN_{\text{out}} \times N_{\text{in}}}$ reads

$$\hat{H}(s) := \hat{L}(\hat{G} + s\hat{C})^{-1}\hat{B} \tag{17}$$

for $s \in \Sigma$. This transfer function yields approximations of the gPC coefficients of the original transfer function from (10). It holds that the minor $\hat{H}_i \in \mathbb{C}^{N_{\text{out}} \times N_{\text{in}}}$ (rows with indices $(i-1)N_{\text{out}} + 1, \ldots, iN_{\text{out}}$ in $\hat{H}$) approximates the exact gPC coefficients H_i.

We motivate the approximation $H_i \approx \hat{H}_i$ by two independent interpretations. On the one hand, the time domain function $\hat{w}_i(t)$ represents an approximation of $w_i(t) = \langle y(t,p)\Phi_i(p)\rangle$ and thus the corresponding Laplace transforms should be close to each other. We expect due to (2)

$$\hat{W}_i(s) \approx \langle Y(s,p)\Phi_i(p)\rangle = \langle H(s,p)U(s)\Phi_i(p)\rangle = H_i(s)U(s).$$

Thus it follows $H_i \approx \hat{H}_i$ by a comparison to (16). On the other hand, the behaviour of the involved matrices $\hat{G}, \hat{C}$ often implies that the minors satisfy

$$((\hat{G} + s\hat{C})^{-1})_{ij} \approx \langle (G(p) + sC(p))^{-1}\Phi_i(p)\Phi_j(p)\rangle$$

for $\min\{i, j\} \ll M$. The specific structure of $\hat{B}, \hat{L}$ in (15) causes a simplification into

$$\hat{H}_i(s) = L((\hat{G} + s\hat{C})^{-1})_{i0}B \approx \langle L(G(p) + sC(p))^{-1}B\Phi_i(p)\rangle = H_i(s)$$

due to $\Phi_0 \equiv 1$. Furthermore, we note that conclusions on the regularity of $\hat{G} + s\hat{C}$ using the regularity of $G(p) + sC(p)$ can be done by results for the associated spectra introduced in [22].

4 Model Order Reduction

We derive a reduced order model with respect to the random space based on a sensitivity analysis now.

4.1 Concepts of Model Order Reduction

Several concept of MOR exist for linear and nonlinear dynamical systems, see [1, 5, 19]. We focus on two approaches applied to the linear dynamical system (1):

(i) *Reduction of the state space*
The linear dynamical system (1) is reduced to a smaller system

$$\tilde{C}(p)\tilde{x}'(t,p) + \tilde{G}(p)\tilde{x}(t,p) = \tilde{B}u(t) \tag{18}$$

with solution $\tilde{x} : I_t \times \Pi \to \mathbb{R}^{N_{\text{red}}}$ and $N_{\text{red}} \ll N$. The methods typically yield a matrix $V(p) \in \mathbb{R}^{N \times N_{\text{red}}}$ such that the solution of the reduced system can be projected into the original state space. The aim of the reduction is to still achieve a good approximation of the output, i.e.,

$$\tilde{y}(t,p) := L\, V(p)\tilde{x}(t,p) \approx L\, x(t,p) = y(t,p)$$

should be satisfied as accurate as possible in some error norm.

(ii) *Reduction of the random space*
Based on a sensitivity analysis, the importance of the random parameters is quantified. Let $\mathcal{J} \subset \{1, \ldots, Q\}$ be the indices of the chosen crucial random variables $\tilde{p}$, i.e., $\tilde{p} = \{p_j : j \in \mathcal{J}\}$. We create a mapping $W : \Pi_{\text{red}} \to \Pi$, $\tilde{p} \mapsto p$ componentwise by defining $W_j(p) := p_j$ for $j \in \mathcal{J}$ and $W_j(p) := \bar{p}_j$ for $j \notin \mathcal{J}$ using a constant value $\bar{p}_j$ like the mean $\bar{p}_j := \langle p_j \rangle$, for example. It follows that the linear dynamical system (1) is reduced to

$$C(W(\tilde{p}))\tilde{x}'(t,\tilde{p}) + G(W(\tilde{p}))\tilde{x}(t,\tilde{p}) = Bu(t) \tag{19}$$

with the same dimension of the state space for $\tilde{x}$. Yet the system (19) depends only on $Q_{\text{red}} := |\mathcal{J}|$ random variables. Let $R : \Pi \to \Pi_{\text{red}}$ be the restriction of p onto the variables $\tilde{p}$. It holds that $R \circ W$ is the identity on Π_{red}. The aim is to achieve $Q_{\text{red}} \ll Q$, while still obtaining a good approximation of the output

$$\tilde{y}(t,p) := L\, \tilde{x}(t, R(p)) \approx L\, x(t,p) = y(t,p)$$

in the random space.

Parametric model order reduction (pMOR) yields reduced systems (18), where the parameters p still appear as independent variables, see [4, 6, 25]. Consequently, the computational effort becomes lower in a nonintrusive method for random parameters. However, the application of pMOR is not within the scope of this paper. We will apply a reduction of the state space to the large coupled system (13), since its dimension is large even for small N in case of a high-dimensional random space. In the system (13), the parameters already disappeared due to the probabilistic integration (14).

In the approach (i), reduced order models (18) can be constructed by approximating the associated transfer function (2), see [7]. Thus our idea is to apply the transfer function also within the approach (ii).

4.2 Sensitivity Analysis

The aim is to determine sensitivities, which quantify the influence of a random variable on the output of the system (1). Variance-based sensitivity coefficients are defined via the Sobol decomposition [21] in case of uniform distributions. This concept can be generalised to other random distributions by gPC expansions, see [24].

Let $f \in L^2(\Pi, \rho)$ be a continuous real-valued function depending on the random parameters, whose gPC coefficients are $(f_i)_{i\in\mathbb{N}}$. Thus its variance is given by

$$\mathrm{Var}(f) = \sum_{i=1}^{\infty} f_i^2.$$

We recall that the multivariate polynomials $(\Phi_i)_{i\in\mathbb{N}}$ are just the products of the univariate orthonormal polynomials with respect to each random distribution. The total sensitivity of the jth random parameter reads

$$S_j := \frac{V_j}{\mathrm{Var}(f)} \quad \text{with} \quad V_j := \sum_{i\in\mathcal{I}_j} f_i^2 \quad \text{for} \quad j = 1, \ldots, Q \tag{20}$$

with a set $\mathcal{I}_j$ defined as follows. It holds that $i \in \mathcal{I}_j$ if and only if Φ_i varies with respect to the random variable p_j, i.e., Φ_i includes a nonconstant univariate polynomial in p_j. Hence the bounds $0 \leq S_j \leq 1$ apply for each j. We obtain approximations of the total sensitivities (20) by a truncated expansion

$$V_j^D := \sum_{i\in\mathcal{I}_j^D} f_i^2 \quad \text{with} \quad \mathcal{I}_j^D := \{i \in \mathcal{I}_j : \mathrm{degree}(\Phi_i) \leq D\}. \tag{21}$$

Although the bounds $1 \leq S_1 + \cdots + S_Q \leq Q$ hold, the sum of the total sensitivities is often close to the lower bound. In view of this variability of the sum of

sensitivities, we apply the standardisation

$$S_j^* := S_j \left(\sum_{l=1}^{Q} S_l \right)^{-1} \quad \text{for} \quad j = 1, \ldots, Q \tag{22}$$

to achieve $S_1^* + \cdots + S_Q^* = 1$. This standardisation is useful to compare and to illustrate results.

The sensitivity analysis tells us, which random variables are crucial or unessential. A parameter-based reduction can be achieved by freezing unessential random variables. We analyse the resulting error briefly. Let $p = (q, r)$ with $q \in \mathbb{R}^{Q_{\text{red}}}$ and $r \in \mathbb{R}^{Q-Q_{\text{red}}}$ be a partition of the random variables. Without loss of generality, the first components q correspond to $\tilde{p}$ in (19). Since we assume independent random variables, associated probability measures μ_q, μ_r are available. We replace the random variables r by constants r_0 within the range of r. The error of this simplification quantified in the space $L^2(\Pi, \rho)$ reads

$$\delta(r_0) := \frac{\sqrt{\langle (f(q,r) - f(q,r_0))^2 \rangle}}{\sqrt{\text{Var}(f)}}, \tag{23}$$

which represents a relative error with respect to the total variance. We assume a positive variance, since an uncertainty quantification is unnecessary otherwise. The following theorem shows that appropriate constants r_0 for freezing unessential variables do exist.

Theorem 1 *For $\varepsilon > 0$, there is a set $B_\varepsilon \subset \mathbb{R}^{Q-Q_{\text{red}}}$ with probability measure $\mu_r(B_\varepsilon) \geq 1 - \varepsilon$ such that*

$$\delta(r_0)^2 < \frac{1 + \varepsilon^{-1}}{\text{Var}(f)} \sum_{i \in \mathcal{I}_r} f_i^2 < (1 + \varepsilon^{-1}) \sum_{j=Q_{\text{red}}+1}^{Q} S_j \tag{24}$$

for all $r_0 \in B_\varepsilon$, where $i \in \mathcal{I}_r$ holds if and only if the basis polynomial Φ_i depends on some random variable in r.

This theorem is mainly a generalisation of Theorem 2 in [21], where uniform distributions are considered. The proof is given in the appendix.

Theorem 1 demonstrates that freezing random variables results in a small error (23) if the associated sensitivity coefficients S_j are relatively small. Using the sharper bound based on $\mathcal{I}_r$ in (24) would be more advantageous. However, all possible partitions $p = (q, r)$ have to be checked in this case, which causes a huge computational effort for large Q (2^Q subsets of $\{p_1, \ldots, p_Q\}$). The total sensitivities allow for just comparing the numbers $S_1, \ldots, S_Q$.

Table 1: Sources of errors in total sensitivities.

nonintrusive technique	intrusive technique
truncation error	truncation error
—	Galerkin method error
quadrature error	—

Often the mean value $r_0 := \langle r \rangle$ achieves a small error, which can be verified by the computation of an approximation to (23). If the mean value causes a bad approximation, then random numbers r_0 are drawn successively until the estimate (24) is satisfied.

We will apply this concept to the componentwise to the matrix-valued transfer function $H(s, p)$ in (2), where the total sensitivities become functions depending on the frequency s. Table 1 summarises the effects causing numerical errors in the computed sensitivity coefficients for the intrusive and the nonintrusive approach. In both methods, a truncation error appears, since a gPC expansion is replaced by a finite sum. The intrusive technique includes the error of the Galerkin method, i.e., the exact solution of the coupled system (13) represents just an approximation of the gPC coefficients. On the one hand, the quadrature error often dominates in the nonintrusive method. On the other hand, a quadrature error does not occur in the intrusive method, since we assume that the matrices $C(p), G(p)$ are polynomials in p, which allow for exact computations. Furthermore, an error appears in the intrusive method if the state space of (13) is reduced.

We briefly discuss the computational effort of the different methods to compute the total sensitivities. Let N_{fr} discrete frequency points be given, where the sensitivity of the transfer function is evaluated. We assume that linear systems are solved by LU-decompositions, which represents the main part of the computational work. In a nonintrusive method with K nodes of the quadrature, we require $N_{\text{fr}}K$ decompositions of systems of dimension N. We recall that K depends on the degree of the polynomials and thus indirectly on M. In the intrusive method, N_{fr} decompositions of the larger dimension MN appear. If we reduce the state space to $N_{\text{red}} \ll MN$ in the stochastic Galerkin method using just a single expansion point $s_0 = \mathrm{i}\omega_0 \in \mathbb{C}$, then we require just a single decomposition of dimension MN and afterwards N_{fr} decompositions of small matrices of dimension N_{red}.

4.3 Reduction of the Random Space

Now we reconsider the MOR approach in the high-dimensional random space from Sect. 4.1. Let $s = \mathrm{i}\omega$ for $\omega \in I_\omega \subseteq \mathbb{R}$, where I_ω is a closed interval. We allow for a single frequency ($I_\omega = \{\omega_0\}$) as well as for an unbounded interval. A threshold $\eta \in (0,1)$, say $\eta = 0.01$, is introduced to control the accuracy of the reduced order model. We restrict ourselfes to single-input-single-output systems, for simplicity, and generalise the strategy later. Let $S_j^*(s)$ for $j = 1, \ldots, Q$ be the sensitivity coefficients (22) for either the real part or the imaginary part of the transfer function $H(s,p)$ associated to the linear dynamical system (1). For fixed s, we reorder these sensitivities to achieve

$$S_{j_1(s)}^*(s) \geq S_{j_2(s)}^*(s) \geq \cdots \geq S_{j_Q(s)}^*(s).$$

If equality holds for two or more coefficients, then a unique ordering follows by assuming ascending indices in such a group. We note that this ordering may vary with s. Let $K(s)$ be the smallest integer such that

$$\sum_{k=1}^{K(s)} S_{j_k(s)}^*(s) > 1 - \eta. \tag{25}$$

In the worst case, it holds $K(s) = Q$. Furthermore, we define the sets of the respective parameters

$$\tilde{p}(s) := \{p_{j_1(s)}, \ldots, p_{j_{K(s)}}\}.$$

In this way, we obtain integers $K_{\mathrm{re}}(s), K_{\mathrm{im}}(s)$ and sets $\tilde{p}_{\mathrm{re}}(s), \tilde{p}_{\mathrm{im}}(s)$ for the real part and the imaginary part, respectively. Now the reduced set of random parameters is defined as

$$\tilde{p} := \bigcup_{\omega \in I_\omega} \left(\tilde{p}_{\mathrm{re}}(\mathrm{i}\omega) \cup \tilde{p}_{\mathrm{im}}(\mathrm{i}\omega)\right). \tag{26}$$

In practice, the interval I_ω is replaced by a finite grid of frequency points. In the worst case, we obtain no reduction, i.e., $\tilde{p} = \{p_1, \ldots, p_Q\}$.

Since the transfer function is complex valued, we define the error by

$$e(s, r_0) := \sqrt{\langle |H(s,q,r) - H(s,q,r_0)|^2 \rangle}. \tag{27}$$

It follows that

$$e(s, r_0)^2 := \langle \mathrm{re}^2(H(s,q,r) - H(s,q,r_0)) \rangle + \langle \mathrm{im}^2(H(s,q,r) - H(s,q,r_0)) \rangle.$$

Hence we can analyse the error of the real part and imaginary part separately. However, if the variance of the real part is much smaller than the variance of the

imaginary part or vice versa, then it makes sense to consider the dominating part in the MOR, i.e., the insignificant part is neglected in (26).

The aim is that the error (27) satisfies $e(\mathrm{i}\omega, r_0) < \theta$ with a tolerance $\theta > 0$ for all $\omega \in I_\omega$ and some fixed r_0. Let f be the real part or imaginary part of the transfer function. We require that the error with respect to f is less than $\theta/\sqrt{2}$. It follows that a sufficient condition on η in (25) is

$$\eta \leq \frac{\theta^2}{2} \left((1 + \varepsilon^{-1}) \mathrm{Var}(f) \sum_{j=1}^{Q} S_j \right)^{-1} .$$

with $\varepsilon > 0$ chosen to apply Theorem 1. We note that η is still independent of a choice of the constant values r_0. Theorem 1 tells us that appropriate values r_0 exist for each s for real part as well as for imaginary part. Theoretically, the feasible set of constants r_0 may become tiny in probability if all $\omega \in I_\omega$ are considered. Thus the determination of a suitable r_0 would require to draw many random numbers and to check the error (27). Nevertheless, the set of reasonable constants r_0 often exhibits a relatively high probability due to the correlations between the different frequencies.

In the case of explicit ODEs ($C(p) \equiv I_N$), we obtain an estimate on the error of the output assuming a compact support of the Laplace transform of the input. For simplicity, we consider single-input-single-output systems. The energy of the input is assumed to be bounded.

Theorem 2 *Let $\|u(t)\|_{L^2([0,\infty))} < \infty$ and $U(\mathrm{i}\omega) = 0$ for $\omega \notin I_\omega$. If the system (1) consists of explicit ODEs, then the estimate*

$$\max_{t>0} \|y(t,p) - \tilde{y}(t,p)\|_{L^2(\Pi,\rho)} \leq \tfrac{1}{\sqrt{2\pi}} \cdot \|e(\mathrm{i}\omega, r_0)\|_{L^2(I_\omega)} \cdot \|u(t)\|_{L^2([0,\infty))}$$

holds for constant r_0 with the error (27) provided that the probabilistic integration and the integration in frequency domain can be interchanged.

The proof can be found in the Appendix. For the special case $I_\omega = \mathbb{R}$, parts of the proof coincide with [9], page 2.

If desired, the error (27) can be reformulated to a relative error by division with the variance. For the complex-valued transfer function, the variance is defined as $\mathrm{Var}(H) := \langle |H - \langle H \rangle|^2 \rangle$.

For multiple-input-multiple-output systems, the same concept can be applied in each component of the matrix-valued transfer function. Consequently, the parameter sets (26) for each component have to be united.

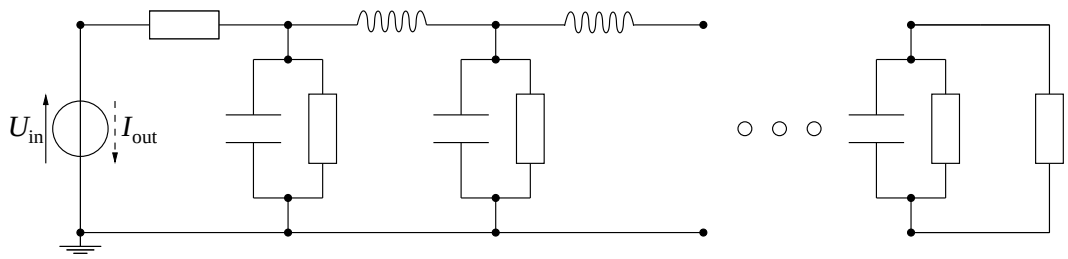

Figure 1: Linear RLC circuit.

5 Test Example: Electric Network

We consider an RLC circuit consisting of a chain of N_{cell} cells depicted in Figure 1. It follows that the network includes N_{cell} capacitances, $N_{\text{cell}} - 1$ inductances and $N_{\text{cell}} + 2$ conductances (reciprocal resistances). A voltage source U_{in} is applied as input, whereas the branch current I_{out} through this source represents the output. This example was introduced in [18].

Modified nodal analysis, see [10], yields a system of DAEs, which can be written in the form (1). The set of random parameters p consists of the capacitances, the inductances and the conductances except for the two conductances at the boundaries. The matrix $C(p)$ includes the capacitances as well as inductances, whereas the conductances are given in $G(p)$. We choose uniform distributions with ranges varying 10% around their mean values. We apply the mean values $\bar{C} = 10^{-9}$ for all capacitances, $\bar{L} = 10^{-6}$ for all inductances and $\bar{G} = 1$ for all conductances except for the two conductances at the boundaries, which are fixed to $G_{\text{bd}} = 2$. The dimensions of the state space and the random space result to $N = 2N_{\text{cell}} + 1$ and $Q = 3N_{\text{cell}} - 1$, respectively.

Due to the uniform distributions, the gPC expansions involve the Legendre polynomials. Sensitivity coefficients are approximated via the truncation (21), where we choose the maximum degree $D = 2$ of the polynomials.

As mentioned in Sect. 4.3, the real part and imaginary part of the transfer function have to be analysed separately. However, the sensitivities of the real part and imaginary part are similar in this example and agree to the sensitivity of the absolute value. Thus we present the total sensitivities only for the absolute value of the transfer function for shortness.

Firstly, we arrange $N_{\text{cell}} = 10$, which yields $N = 21$ state variables and a moderate number of $Q = 29$ random variable. It follows that $M = 465$ basis functions appear in the truncated gPC expansion due to $M = (D+Q)!/(D!Q!)$. We apply three approaches to compute approximations of the gPC coefficients:

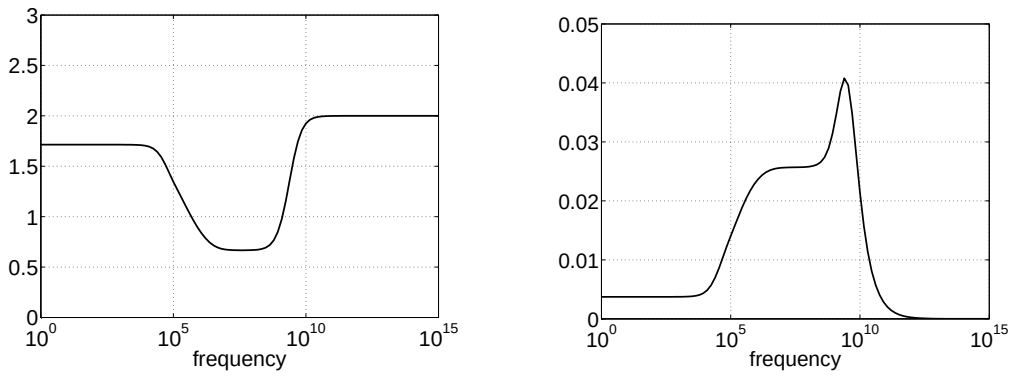

Figure 2: Expected value (left) and standard deviation (right) of absolute value of transfer function computed by the intrusive method ($N_{\text{cell}} = 10$).

(i) *nonintrusive method*: quadrature using the Stroud-5 formula [23] in $Q = 29$ dimensions, which results in $K = 1,683$ nodes;

(ii) *intrusive method*: stochastic Galerkin method with a system (13) of dimension $MN = 9,765$;

(iii) *intrusive method with state space reduction*: the system from (ii) is reduced to dimension $N_{\text{red}} = 100$ by a Krylov subspace technique to achieve moment matching in a single point s_0, see [7]. We choose $s_0 = 10^5\text{i}$.

Figure 2 illustrates the expected value and the standard deviation of the transfer function reconstructed by the gPC coefficients of the intrusive method. The resulting sensitivity coefficients (22) are depicted for the three techniques and frequencies $\omega \in [1, 10^{15}]$ in Figure 3, Figure 4 and Figure 5, repectively. We observe a good agreement of the different approaches. Just a single capacitance is dominating for high frequencies, whereas no capacitance is essential for low frequencies. This dominant capacitance is most adjacent to the output current I_{out}. The inductances exhibit a decreasing sequence of sensitivities with respect to their distances to the output current. The sensitivities of the conductances also form a decreasing sequence for high frequencies. However, the conductances are all equally important for low frequencies $\omega < 10^4$. Thus we obtain a promising potential for MOR in random space for high frequencies, whereas just the randomness in the capacitances and inductances can be neglected for low frequencies. For frequencies $\omega > 10^{14}$, the nonintrusive scheme (i) yields physically unreasonable results. However, the variance is tiny for such frequencies, see Figure 2 (right), and thus we are not interested in this region for uncertainty quantification. Furthermore, the approach (iii) causes incorrect values for frequencies $\omega > 10^{12}$, since this domain is far away from the point $\omega_0 = 10^5$ of the moment matching.

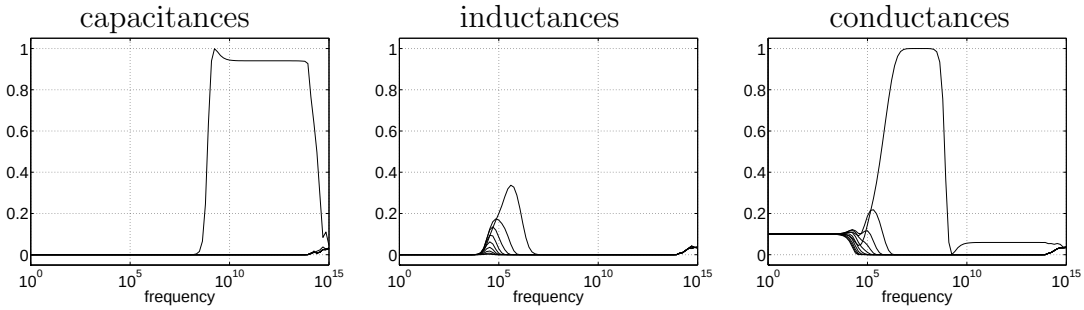

Figure 3: Sensitivities from nonintrusive method ($N_{\text{cell}} = 10$).

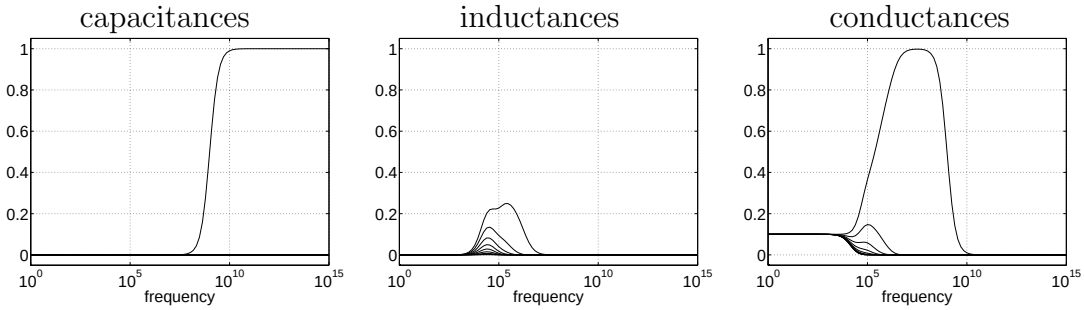

Figure 4: Sensitivities from intrusive method ($N_{\text{cell}} = 10$).

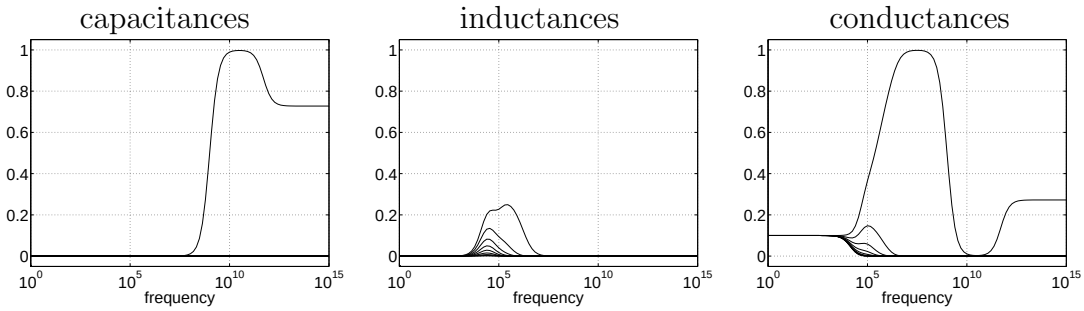

Figure 5: Sensitivities from intrusive method with a reduction of the state space ($N_{\text{cell}} = 10$).

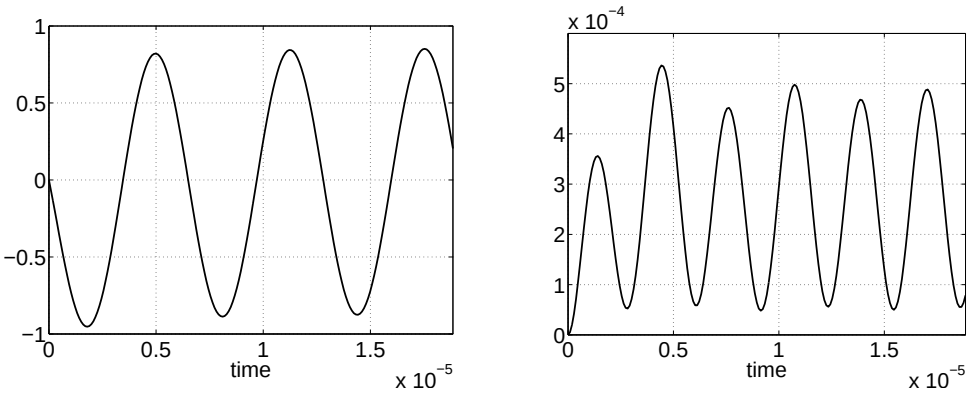

Figure 6: Expected value (left) and variance (right) of output current in transient simulation of the reduced model ($N_{\text{cell}} = 10$).

In the MOR, we consider the real part and imaginary part separately now. We choose a single frequency $\hat{\omega} := 10^6$ ($I_\omega = \{\hat{\omega}\}$). Three dominating parameters appear for the real part, whereas four essential parameters (including the previous three) are found for the imaginary part. These four elements consist of the two inductances and the two conductances, which are closest to the output current, see Figure 1. Let the sum of their sensitivities (22) be equal to $1 - \eta$. The values η are specified in Table 2. Thus we replace all other random variables by their constant mean values. To verify the reduced order model, a transient simulation is performed in the interval $[0, 3T]$ for $\hat{\omega} = \frac{2\pi}{T}$ with input voltage $u(t) = \sin(\hat{\omega}t)$ and initial values identical to zero. We calculate an approximation of the expected value and the variance of the output current by a Gauss-Legendre quadrature with 4^4 grid points for the four active random variables. The implicit Euler method with constant step sizes yields a numerical solution of the initial value problems of (1), see [11]. Figure 6 illustrates the results. For comparison, we resolve the original unreduced problem by a quasi Monte-Carlo method with 20,000 samples. Table 2 shows the maximum differences in $[0, 3T]$ for the expected value and the variance. The MOR of the random space is successful, since the difference in the variance represents ca. 1% of the magnitude of the variance.

Secondly, we select $N_{\text{cell}} = 35$ to simulate a problem for a high-dimensional random space with $Q = 104$ random variables, while the dimension of the state space becomes $N = 71$. The truncated gPC expansions exhibit $M = 5,565$ basis functions. Now we cannot afford the intrusive approach (ii) or (iii), since at least one linear system of dimension $MN = 395,115$ has to be solved. A direct LU-decomposition becomes too costly. An iterative solution is also critical, since the matrix of the linear system is not symmetric and positive definite. Thus we apply only the nonintrusive method (i) in the numerical simulation, where the scheme Stroud-5 involves $K = 21,633$ nodes in the high-dimensional space. Figure 7

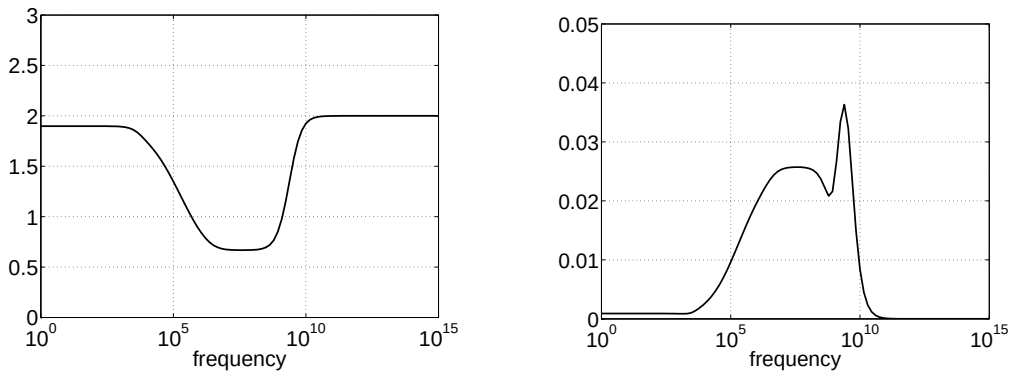

Figure 7: Expected value (left) and standard deviation (right) of absolute value of transfer function computed by the intrusive method ($N_{\text{cell}} = 35$).

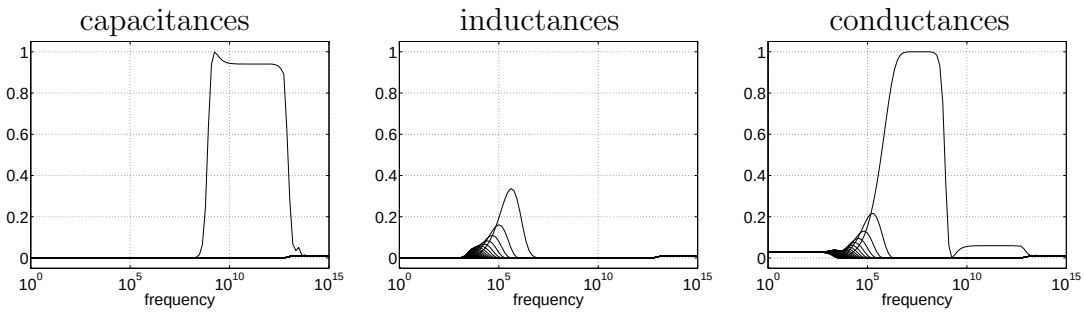

Figure 8: Sensitivities from nonintrusive method ($N_{\text{cell}} = 35$).

depicts the approximations of the expected value and the standard deviation of the transfer function. The reconstructed sensitivities (22) are illustrated in Figure 8. We recognise that the behaviour agrees qualitatively to the previous case of $N_{\text{cell}} = 10$. Again dubious sensitivity coefficients appear in domains with a tiny variance.

At the frequency $\hat{\omega} := 10^9$ ($I_\omega = \{\hat{\omega}\}$), just a single capacitance and a single inductance are dominating. Table 2 characterises the sum of the two sensitivities (22). We freeze all other random variables in our MOR. We recheck our strategy again by a transient simulation using the input voltage $u(t) = \sin(\hat{\omega}t)$. Figure 9 demonstrates the expected value and the variance of the output current, which are computed by a Gauss-Legendre quadrature with 10^2 nodes in the two active random variables. For comparison, a quasi Monte-Carlo simulation with 100,000 samples is applied to the unreduced problem with $Q = 104$ random variables. The maximum differences in the interval $[0, 3T]$ between data of the reduced and the unreduced problem are shown in Table 2. Again the reduced order model represents a good approximation, since the differences for the variance

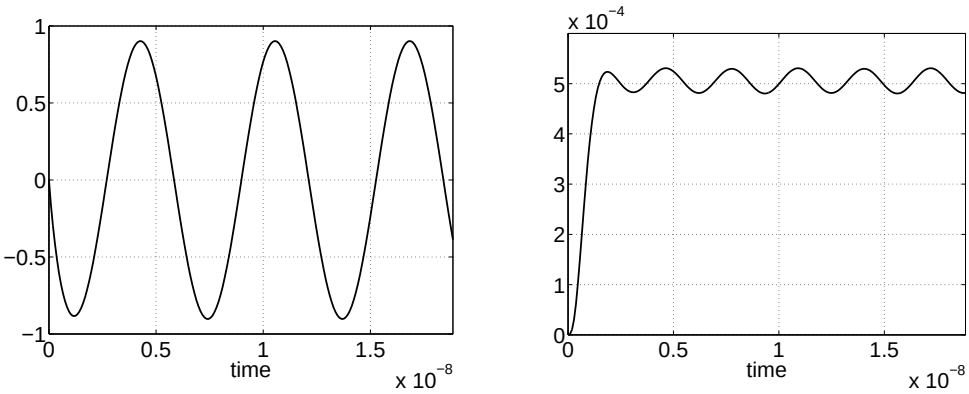

Figure 9: Expected value (left) and variance (right) of output current in transient simulation of the reduced model ($N_{\text{cell}} = 35$).

Table 2: Values η from the sum of sensitivities $1 - \eta$ of selected random variables and maximum differences for $t \in [0, 3T]$ in expected value and variance of output current between reduced and unreduced problem for the two test cases.

N_{cell}	Q	$\hat{\omega}$	η for real part	η for imag. part	max. diff. expected v.	max. diff. variance
10	29	10^6	$2 \cdot 10^{-3}$	$8 \cdot 10^{-3}$	$1.2 \cdot 10^{-4}$	$3.5 \cdot 10^{-6}$
35	104	10^9	$4 \cdot 10^{-7}$	$7 \cdot 10^{-7}$	$9 \cdot 10^{-5}$	$4 \cdot 10^{-7}$

come up to just ca. 0.1% of the magnitude of the variance.

6 Conclusions

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Acknowledgements

The authors would like to thank Dr. Michael Striebel (ZF Lenksysteme GmbH, Schwäbisch Gmünd, Germany) for providing the MATLAB code modelling the linear dynamical system in our test example.

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Appendix

Proof of Theorem 1:

Since we assume independent traditional random distributions, densitities ρ_q, ρ_r are available corresponding to the measures μ_q, μ_r. We define the functions

$$u(q) := \int_{\Pi_r} f(q,r)\rho_r(r)\,\mathrm{d}r - f_0, \qquad v(r) := \int_{\Pi_q} f(q,r)\rho_q(q)\,\mathrm{d}q - f_0.$$

The gPC expansion of f yields

$$u(q) = \sum_{i\in\mathcal{I}_u, i\neq 0} f_i\Phi_i(q,r), \qquad v(r) = \sum_{i\in\mathcal{I}_v, i\neq 0} f_i\Phi_i(q,r).$$

It holds that $i \in \mathcal{I}_u$ if and only if Φ_i does not depend on some variable in r and $i \in \mathcal{I}_v$ if and only if Φ_i does not depend on some variable in q. Furthermore, let

$$w(q,r) := \sum_{i\in\mathcal{I}_w} f_i\Phi_i(q,r)$$

with $i \in \mathcal{I}_w$ if and only if Φ_i depends on at least one variable in q and one variable in r. Thus $i = 0$ is not within $\mathcal{I}_w$. It holds that $f(q,r) = f_0 + u(q) + v(r) + w(q,r)$. A substitution of the random variables r by constants r_0 yields the difference

$$f(q,r) - f(q,r_0) = v(r) + w(q,r) - v(r_0) - w(q,r_0).$$

Mimicing the steps for the proof of Theorem 2 in [21], it follows that a set B with $\mu_r(B) \geq 1-\varepsilon$ exists such that

$$\delta(r_0)^2 < \frac{1+\varepsilon^{-1}}{\mathrm{Var}(f)} \sum_{i\in\mathcal{I}_v\cup\mathcal{I}_w, i\neq 0} f_i^2$$

for all $r_0 \in B$. The definitions of the index sets implies $(\mathcal{I}_v\backslash\{0\}) \cup \mathcal{I}_w = \mathcal{I}_r$. We recall that $\mathcal{I}_j$ for $j = 1,\ldots,Q$ is defined by $i \in \mathcal{I}_j$ if and only if Φ_i depends on the random variable p_j. If $i \in \mathcal{I}_r$, then there is a variable p_j in r ($j > Q_{\mathrm{red}}$) such that $i \in \mathcal{I}_j$. The definition (20) yields

$$\frac{1}{\mathrm{Var}(f)} \sum_{i\in\mathcal{I}_r} f_i^2 \leq \frac{1}{\mathrm{Var}(f)} \sum_{j=Q_{\mathrm{red}}+1}^{Q} \sum_{i\in\mathcal{I}_j} f_i^2 = \sum_{j=Q_{\mathrm{red}}+1}^{Q} S_j,$$

which completes the proof. □

Proof of Theorem 2:

Let $\tilde{H}(s,p) := H(s,q,r_0)$ be the approximation of the exact transfer function $H(s,p)$. We obtain by using the Cauchy-Schwarz inequality in $L^2(I_\omega)$

$$\begin{aligned}
\max_{t>0} |y(t,p) - \tilde{y}(t,p)| &\leq \max_{t>0} \left| \frac{1}{2\pi} \int_{\mathbb{R}} (Y(\mathrm{i}\omega,p) - \tilde{Y}(\mathrm{i}\omega,p)) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega \right| \\
&\leq \max_{t>0} \frac{1}{2\pi} \int_{\mathbb{R}} |Y(\mathrm{i}\omega,p) - \tilde{Y}(\mathrm{i}\omega,p)| \cdot \left| \mathrm{e}^{\mathrm{i}\omega t} \right| \, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |H(\mathrm{i}\omega,p) - \tilde{H}(\mathrm{i}\omega,p)| \cdot |U(\mathrm{i}\omega)| \, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{I_\omega} |H(\mathrm{i}\omega,p) - \tilde{H}(\mathrm{i}\omega,p)| \cdot |U(\mathrm{i}\omega)| \, \mathrm{d}\omega \\
&\leq \frac{1}{2\pi} \left(\int_{I_\omega} |H(\mathrm{i}\omega,p) - \tilde{H}(\mathrm{i}\omega,p)|^2 \, \mathrm{d}\omega \right)^{\frac{1}{2}} \left(\int_{I_\omega} |U(\mathrm{i}\omega)|^2 \, \mathrm{d}\omega \right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{2\pi} \int_{I_\omega} |H(\mathrm{i}\omega,p) - \tilde{H}(\mathrm{i}\omega,p)|^2 \, \mathrm{d}\omega \right)^{\frac{1}{2}} \left(\int_0^\infty |u(t)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}}
\end{aligned}$$

pointwise for $p \in \Pi$. For $I_\omega = \mathbb{R}$, this estimate is already given in [9], page 2. It follows that

$$\begin{aligned}
\langle |y(t,p) - \tilde{y}(t,p)|^2 \rangle &\leq \left\langle \frac{1}{2\pi} \int_{I_\omega} |H(\mathrm{i}\omega,p) - \tilde{H}(\mathrm{i}\omega,p)|^2 \, \mathrm{d}\omega \right\rangle \|u(t)\|^2_{L^2([0,\infty))} \\
&= \left(\frac{1}{2\pi} \int_{I_\omega} \langle |H(\mathrm{i}\omega,p) - \tilde{H}(\mathrm{i}\omega,p)|^2 \rangle \, \mathrm{d}\omega \right) \|u(t)\|^2_{L^2([0,\infty))} \\
&= \left(\frac{1}{2\pi} \int_{I_\omega} e(\mathrm{i}\omega,r_0)^2 \, \mathrm{d}\omega \right) \|u(t)\|^2_{L^2([0,\infty))}
\end{aligned}$$

for each $t > 0$ using the definition (27) of the error e. Taking the square root yields the desired formula. $\square$

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