Theory and Applications of Graphs

Volume 7 | Issue 2 Article 4

December 2020

Fault diagnosability of regular graphs

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Recommended Citation

Gu, Mei-Mei; Hao, Rong-Xia; and Cheng, Eddie (2020) "Fault diagnosability of regular graphs," *Theory and Applications of Graphs*: Vol. 7 : Iss. 2 , Article 4.
DOI: 10.20429/tag.2020.070204
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol7/iss2/4

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Cover Page Footnote

Many thanks the anonymous referees for a number of helpful comments and suggestions. This work was supported by Czech Operational Programme Research, Development and Education (OP RDE) International Mobility of Researchers at Charles University project No. CZ.02.2.69/0.0/0.0/16-027/0008495,and China Postdoctoral Science Foundation No. 2018M631322 (Mei-Mei Gu), the National Natural Science Foundation of China Nos. 11971054, 11731002 and the 111 Project of China No. B16002 (Rong-Xia Hao). Part of this work was accomplished while the author Mei-Mei Gu was a doctoral student at Department of Mathematics, Beijing Jiaotong University.

Abstract

An interconnection network's diagnosability is an important measure of its self-diagnostic capability. In 2012, Peng et al. proposed a measure for fault diagnosis of the network, namely, the h-good-neighbor conditional diagnosability, which requires that every fault-free node has at least h fault-free neighbors. There are two well-known diagnostic models, PMC model and MM* model. The *h-good-neighbor diagnosability* under the PMC (resp. MM*) model of a graph G, denoted by $t_h^{PMC}(G)$ (resp. $t_h^{MM^*}(G)$), is the maximum value of t such that G is h-good-neighbor t-diagnosable under the PMC (resp. MM*) model. In this paper, we study the 2-good-neighbor diagnosability of some general k-regular k-connected graphs G under the PMC model and the MM* model. The main result $t_2^{PMC}(G) = t_2^{MM^*}(G) = g(k-1)-1$ with some acceptable conditions is obtained, where g is the girth of G. Furthermore, the following new results under the two models are obtained: $t_2^{PMC}(HS_n) = t_2^{MM^*}(HS_n) = 4n-5$ for the hierarchical star network HS_n, $t_2^{PMC}(S_n^2) = t_2^{MM^*}(S_n^2) = 6n-13$ for the split-star networks S_n^2 and $t_2^{PMC}(\Gamma_n(\Delta)) = t_2^{MM^*}(\Gamma_n(\Delta)) = 6n-16$ for the Cayley graph generated by the 2-tree $\Gamma_n(\Delta)$.

Keywords: 2-good-neighbor diagnosability; PMC model; MM* model; regular graphs; interconnection networks.

2010 Mathematics subject classification: 05C40, 05C25, 68M10,68R10.

1 Introduction

A multiprocessor system is modeled as an undirected simple graph $G = (V, E)$, whose vertices (nodes) represent processors and edges (links) represent communication links.

With the rapid development of multiprocessor systems, processor failure is inevitable along with the number of processors increasing. The process of identifying all the faulty units in a system is called *system-level diagnosis*. For the purpose of self-diagnosis of a system, a number of models have been proposed for diagnosing faulty processors in a network. Among the proposed models, PMC model (that is, Preparata, Metze and Chien's model) [17] and comparison model (MM* model) [16] are widely used. In the PMC model, the diagnosis of the system is achieved through two linked processors testing each other. In the MM* model, to diagnose the system, a processor sends the same task to two of its neighbors, and then compares their responses. The PMC and MM* models have been extensively investigated.

A system is said to be *t-diagnosable* if all faulty units can be identified provided the number of faulty units present does not exceed t. The *diagnosability* is the maximum number of faulty processors which can be correctly identified. The classical diagnosability of a network is quite small owing to the fact that it ignores the unlikelihood of some specific processors failing at the same time. In 2005, Lai et al. [13] introduced a restricted diagnosability of the system called *conditional diagnosability* by assuming that it is almost impossible that all neighbors of one vertex are faulty simultaneously. Inspired by this concept, Peng et al. [18] then proposed the *h-good-neighbor diagnosability*, which requires every fault-free vertex has at least h fault-free neighbors. Furthermore, they evaluated the h-good-neighbor diagnosability of the n-dimensional hypercube Q_n under the PMC model. Yuan et al. [24] and [25] studied the h-good-neighbor diagnosability of the k-ary n-cubes ($k \geq 4$) and 3-ary n-cubes, respectively, under the PMC model and MM* model. Wang et al. [20] and [21] determined the 2-good-neighbor diagnosability of the

Cayley graph generated by transposition trees Γ_n and the alternating graph network AN_n, respectively. More results can be found in [14, 15] etc.

In this paper, we study the 2-good-neighbor diagnosability of some general k-regular k-connected graphs G under the PMC model and the MM* model and obtain the relationship between 2-good-neighbor diagnosability and R^2-connectivity $\kappa^2(G)$. The main result $t_2^{PMC}(G) = t_2^{MM^*}(G) = \kappa^2(G) + g - 1 = g(k-1) - 1$ under the two models with some acceptable conditions is obtained, where g is the girth of G. More precisely, our main result is the following Theorem 1.

Theorem 1. *Let G be a k-regular k-connected graph of order N (that is, with N vertices). Let g be the girth of G, $\ell = cn(G)$ be the maximum number of common neighbors between any two vertices and $\ell' = ca(G)$ be the maximum number of common neighbors between any two adjacent vertices. Suppose further that all of the following conditions hold:*

(1) $k \geq \xi + 2$ *and* $N \geq 2g(k-1) - 1$,

(2) $\ell \leq 2$ *and* $\ell' \leq 1$, *and*

(3) *let F be a 2-good-neighbor faulty set of G, if $|F| \leq g(k-2) - 1$, then $G - F$ is connected.*

Then,

(I) $t_2^{PMC}(G) = g(k-1) - 1$, *and*

(II) $t_2^{MM^*}(G) = g(k-1) - 1$,

where

$$\xi = \begin{cases} \ell & \textit{if } g = 3 \textit{ and } G \textit{ contains no } 5\textit{-cycles}; \\ 3\ell & \textit{if } g = 3 \textit{ and } G \textit{ contains } 5\textit{-cycles}; \\ 2\ell & \textit{if } g = 4 \textit{ and } G \textit{ contains no } 5\textit{-cycles}; \\ 4\ell & \textit{if } g = 4 \textit{ and } G \textit{ contains } 5\textit{-cycles}; \\ 5 & \textit{if } g = 5; \\ 4 & \textit{if } g = 6; \\ 2 & \textit{if } g = 7; \\ 2 & \textit{if } g = 8; \\ 1 & \textit{if } g \geq 9. \end{cases}$$

Furthermore, the following new results about the 2-good-neighbor diagnosability $t_2(G)$ under the PMC model and MM* model are obtained: $t_2^{PMC}(HS_n) = t_2^{MM^*}(HS_n) = 4n-5$ for hierarchical star network HS_n, $t_2^{PMC}(S_n^2) = t_2^{MM^*}(S_n^2) = 6n-13$ for split-star networks S_n^2, and $t_2^{PMC}(\Gamma_n(\Delta)) = t_2^{MM^*}(\Gamma_n(\Delta)) = 6n-16$ for Cayley graph generated by the 2-tree $\Gamma_n(\Delta)$. Especially, the relationship $t_2^{PMC}(G) = t_2^{MM^*}(G) = \kappa^2(G) + g - 1$ of $t_2^{PMC}(G)$ (resp. $t_2^{MM^*}(G)$) and $\kappa^2(G)$ are given. In the literature, most known results about 2-good-neighbor conditional diagnosability of some networks are gotten independently, and some proofs are longwinded. As consequences of our results, some of them can be obtained easily.

The remainder of this paper is organized as follows. Section 2 introduces necessary definitions. Our main results are given in Section 3. As applications of our main results, Section 4 concentrates on the applications to some popular interconnection networks. Finally, our conclusions are given in Section 5.

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DOI: 10.20429/tag.2020.070204

2 Preliminaries

Throughout this paper, all graphs are finite, undirected and without loops. We follow [22] for terminologies and notations not defined here.

Let $G = (V(G), E(G))$ be a graph. For a vertex $u \in V(G)$, we use the symbol $N_G(u)$ to denote a set of vertices in G adjacent to u. The cardinality $|N_G(u)|$ represents the degree of u in G, denoted by $d_G(u)$, and $\delta(G)$ is the minimum degree of G. For a vertex set $U \subseteq V(G)$, let $N_G(U) = \bigcup_{v \in U} N_G(v) \setminus U$ and $G[U]$ be the subgraph of G induced by U. If $|N_G(u)| = k$ for every vertex in G, then G is *k-regular*. A subset $S \subseteq V(G)$ is a *vertex-cut* if $G - S$ is disconnected. The *components* of G are its maximal connected subgraphs. The *connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices to be removed from G so that the resulting graph is either disconnected or trivial. Let G be a connected graph, if $G - S$ is connected for every $S \subseteq V(G)$ with $|S| \leq k-1$, then G is *k-connected*.

For two adjacent vertices u and v in G, let $cn(G; u, v)$ denote the number of vertices who are the neighbors of both u and v, that is, $cn(G; u, v) = |N_G(u) \cap N_G(v)|$. Let $cn(G) = \max\{cn(G; u, v) : u, v \in V(G)\}$.

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. For a finite group A and a subset S of A such that $1 \notin S$ (where 1 is the identity element of A) $S = S^{-1}$ (that is, $s \in S$ implies $s^{-1} \in S$), the *Cayley graph* $\mathrm{Cay}(A; S)$ on A with respect to S is defined to have vertex set A and edge set $\{(g, gs) | g \in A, s \in S\}$. (Generally, $S = S^{-1}$ is not required in the definition of a Cayley graph. We impose the condition here so that the corresponding Cayley graph can be treated as undirected.)

A faulty set $F \subseteq V(G)$ is an *h-good-neighbor faulty set* if $|N_G(v) \cap (V(G) \setminus F)| \geq h$ for every vertex $v \in V(G) \setminus F$. An *h-good-neighbor cut* of a graph G is an h-good-neighbor faulty set F such that $G - F$ is disconnected. The minimum cardinality of h-good-neighbor cuts is said to be the *R^h-connectivity* (or *h-good-neighbor connectivity*) of G, denoted by $\kappa^h(G)$. The parameter $\kappa^1(G)$ is equal to *extra connectivity* $\kappa_1(G)$ proposed by Fábrega and Fiol [8], where $\kappa_k(G)$ is the cardinality of a minimum set $S \subseteq V(G)$ such that $G - S$ is disconnected and each component of $G - S$ has at least $k + 1$ vertices. The *symmetric difference* of $F_1 \subseteq V(G)$ and $F_2 \subseteq V(G)$ is defined as the set $F_1 \Delta F_2 = (F_1 - F_2) \cup (F_2 - F_1)$.

The following two lemmas which characterize a graph for h-good-neighbor t-diagnosable under the PMC model and the MM* model, respectively. These lemmas essentially turn the diagnosability problem into a graph theory problem.

Lemma 1. ([24]) *A system $G = (V, E)$ is h-good-neighbor t-diagnosable under the PMC model if and only if there is an edge $(u, v) \in E$ with $u \in V \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ for each distinct pair of h-good-neighbor faulty sets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$.*

The *h-good-neighbor diagnosability* under the PMC model of a graph G, denoted by $t_h^{PMC}(G)$, is the maximum value of t such that G is h-good-neighbor t-diagnosable under the PMC model.

Lemma 2. ([7, 24]) *A system $G = (V, E)$ is h-good-neighbor t-diagnosable under the MM* model if and only if for each distinct pair of h-good-neighbor faulty sets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions.*

(1) *There are two vertices* $u, w \in V \setminus (F_1 \cup F_2)$ *and there is a vertex* $v \in F_1 \Delta F_2$ *such that* $(u, v) \in E$ *and* $(u, w) \in E$.

(2) *There are two vertices* $u, v \in F_1 \setminus F_2$ *and there is a vertex* $w \in V \setminus (F_1 \cup F_2)$ *such that* $(u, w) \in E$ *and* $(v, w) \in E$.

(3) *There are two vertices* $u, v \in F_2 \setminus F_1$ *and there is a vertex* $w \in V \setminus (F_1 \cup F_2)$ *such that* $(u, w) \in E$ *and* $(v, w) \in E$.

The *h-good-neighbor diagnosability* under the MM* model of a graph G, denoted by $t_h^{MM^*}(G)$, is the maximum value of t such that G is h-good-neighbor t-diagnosable under the MM* model.

3 Main result

In this section, we will determine the 2-good-neighbor diagnosability of some general k-regular k-connected graphs G under the PMC model and the MM* model. Before we prove Theorem 1, we would like to comment that a cycle is the most basic connected graph with minimum degree 2. Thus, to find a minimum 2-good-neighbor faulty set, it is natural to find a small cycle in the graph and delete its neighbors. Since the graph is k-regular, one would expect to delete "$g(k-2)$" vertices. However, this assumes that all these $g(k-2)$ neighbors are distinct. This is addressed by the conditions on cn and ca. The conditions $N \geq 2g(k-1)-1$ and $k \geq \xi + 2$ are technical. If the girth is not large enough, there are additional difficulties. Thus the condition on k is much simpler if $g \geq 9$. In fact, if $g \geq 9$, the requirement reduces to $k \geq 3$, which is mild as this excludes only cycles.

Proof of Theorem 1. Let $C = (v_1, v_2, \ldots, v_g, v_1)$ be a shortest cycle in G. The following two claims are useful.

Claim 1. *Let* $S_1 = N_G(C)$, $S_2 = N_G(C) \cup V(C)$. *Then* $|S_1| = g(k-2)$, $|S_2| = g(k-1)$, $\delta(G - S_1) \geq 2$ *and* $\delta(G - S_2) \geq 2$.

Proof of Claim 1. Obviously, $|S_1| = g(k-2)$ and $|S_2| = g(k-1)$. For any vertex x in $G - S_1$, if $x \in V(C)$, then $d_{G-S_1}(x) = 2$. If $x \notin V(C)$, we declare that $|N_G(x) \cap S_1| \leq \xi$. (We remark that the declaration is clearly true if g is sufficiently large; otherwise, this creates a smaller cycle. For graphs with smaller girth, the argument is more technical.)

In fact, note that $C = (v_1, v_2, \ldots, v_g, v_1)$, for $i \in [g]$, $|N_G(x) \cap N_G(v_i)| \leq \ell$. We consider the claim according to the girth of G as follows.

If $g = 3$, then $|N_G(x) \cap S_1| \leq 3\ell$. But if G has no 5-cycles, there exists at most one $i \in \{1, 2, 3\}$ such that $|N_G(x) \cap N_G(v_i)| \leq \ell$.

If $g = 4$, then $|N_G(x) \cap S_1| \leq 4\ell$. But if G has no 5-cycles, there exist at most two $i \in \{1, 2, 3, 4\}$ such that $|N_G(x) \cap N_G(v_i)| \leq 2\ell$.

Note that if $g \geq 5$, then $\ell = cn(G) \leq 1$ (otherwise, there exists a 4-cycle which contradicts the assumption that $g \geq 5$).

If $g = 5$, then $|N_G(x) \cap N_G(v_i)| \leq 1$. Thus, $|N_G(x) \cap S_1| \leq 5$.

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DOI: 10.20429/tag.2020.070204

If $g = 6$, then by Condition (2), $|N_G(x) \cap N_G(v_i)| \le 1$. If $|N_G(x) \cap N_G(v_i)| = 1$, then $|N_G(x) \cap N_G(v_{i+1})| = 0$ and $|N_G(x) \cap N_G(v_{i-1})| = 0$ (where '+' and '−' are the operations with "(mod 6)"). Thus, $|N_G(x) \cap S_1| \le 4$.

If $g = 7$, then by Condition (2), $|N_G(x) \cap N_G(v_i)| \le 1$. If $|N_G(x) \cap N_G(v_i)| = 1$, then for every vertex v in $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$, $|N_G(x) \cap N_G(v)| = 0$, where '+' and '−' are the operations with "(mod 7)". Thus, $|N_G(x) \cap S_1| \le 2$.

If $g = 8$, we can check that $|N_G(x) \cap S_1| \le 2$.

If $g \ge 9$, then by Condition (2), $|N_G(x) \cap N_G(v_i)| \le 1$. If $|N_G(x) \cap N_G(v_1)| = 1$, then $|N_G(x) \cap N_G(v_j)| = 0$ for $j \in [g] \setminus \{1\}$. Thus, $|N_G(x) \cap S_1| \le 1$.

By the above discussion, for any x in $G - S_1$ and $x \notin V(C)$, $|N_G(x) \cap S_1| \le \xi$. This implies that $d_{G-F_1}(x) = k - |N_G(x) \cap S_1| \ge k - \xi \ge 2$ by Condition (1). For any vertex x in $G - S_2$, $|N_G(x) \cap S_2| = |N_G(x) \cap S_1| \le \xi$ as $N_G(x) \cap V(C) = \emptyset$. So $d_{G-S_2}(x) \ge k - |N_G(x) \cap S_2| \ge k - \xi \ge 2$. Thus Claim 1 holds.

Claim 2. *The R^2-connectivity $\kappa^2(G)$ of G is $g(k-2)$.*

Proof of Claim 2. Let $S_1 = N_G(C)$, then $|S_1| = g(k-2)$. Clearly, C is a component of $G - S_1$. If $G - S_1$ is connected, then $N = |V(G)| = |S_1| + |V(C)| = g(k-2) + g = g(k-1) < 2g(k-1) - 1$. The last inequality holds since $1 < g(k-1)$ as $k \ge 2$ and $g \ge 3$. This gives $N < 2g(k-1) - 1$, which contradicts Condition (1). Thus $G - S_1$ is disconnected. By Claim 5, $\delta(G - S_1) \ge 2$. So S_1 is a 2-good-neighbor cut of G, which implies $\kappa^2(G) \le |S_1| = g(k-2)$. On the other hand, by Condition (3), $\kappa^2(G) \ge g(k-2)$. Thus $\kappa^2(G) = g(k-2)$. This establishes Claim 2.

(I) First, we consider $t_2^{PMC}(G)$.

By Claim 1, S_1 and S_2 are both 2-good-neighbor faulty sets of G with $|S_1| = g(k-2)$ and $|S_2| = g(k-1)$. Since $V(C) = S_1 \Delta S_2$ and $N_G(C) = S_1 \subseteq S_2$, there are no edges of G between $V(G) \setminus (S_1 \cup S_2)$ and $S_1 \Delta S_2$. By Lemma 1, G is not 2-good-neighbor $g(k-1)$-diagnosable under the PMC model, this implies that $t_2^{PMC}(G) \le g(k-1) - 1$.

Next we prove $t_2^{PMC}(G) \ge g(k-1) - 1$, that is, G is 2-good-neighbor $[g(k-1)-1]$-diagnosable.

Claim 3. *G is 2-good-neighbor $[g(k-1)-1]$-diagnosable, that is, $t_2^{PMC}(G) \ge g(k-1)-1$.*

Proof of Claim 3. By Lemma 1, it is equivalent to prove: For each distinct pair of 2-good-neighbor faulty sets F_1 and F_2 of G with $|F_1| \le g(k-1)-1$ and $|F_2| \le g(k-1)-1$, there is an edge $(x, y) \in E(G)$ with $x \in V(G) \setminus (F_1 \cup F_2)$ and $y \in F_1 \Delta F_2$.

Suppose, on the contrary, that there are two distinct 2-good-neighbor faulty sets F_1 and F_2 of G with $|F_1| \le g(k-1) - 1$ and $|F_2| \le g(k-1) - 1$ such that there are no edges between $V(G) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$.

Without loss of generality, assume that $F_2 \setminus F_1 \ne \emptyset$. If $V(G) = F_1 \cup F_2$, then $N = |V(G)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le 2g(k-1) - 2$, which contradicts Condition (1). Therefore, $V(G) \ne F_1 \cup F_2$. If $F_1 \cap F_2 = \emptyset$, then the claim is clearly true. Henceforth, we may assume that $F_1 \cap F_2 \ne \emptyset$. Note that since F_1 is a 2-good-neighbor faulty set, $\delta(G - F_1) \ge 2$. Similarly, since F_2 is a 2-good-neighbor faulty set, $\delta(G - F_2) \ge 2$.

Because there are no edges between $V(G) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$, $\delta(G-(F_1 \cup F_2)) \geq 2$ and $\delta(G[F_2 \setminus F_1]) \geq 2$. Similarly, $\delta(G[F_1 \setminus F_2]) \geq 2$ if $F_1 \setminus F_2 \neq \emptyset$. Thus, $F_1 \cap F_2$ is a 2-good-neighbor cut for $F_2 \setminus F_1 \neq \emptyset$ and $G-(F_1 \cup F_2) \neq \emptyset$.

By Claim 2, $|F_1 \cap F_2| \geq g(k-2)$. Note that $\delta(G[F_2 \setminus F_1]) \geq 2$, so $G[F_2 \setminus F_1]$ has a cycle, say C_1, and the length of C_1 is at least g as the girth of G is g. It now follows that $|F_2 \setminus F_1| \geq g$. Then $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq g + g(k-2) = g(k-1)$, which contradicts $|F_2| \leq g(k-1)-1$. Thus Claim 3 holds.

By the above discussion, $t_2^{PMC}(G) = g(k-1)-1$.

(II) Now we consider $t_2^{MM^*}(G)$.

We first prove $t_2^{MM^*}(G) \leq g(k-1)-1$. By Claim 5, S_1 and S_2 are both 2-good-neighbor faulty sets of G with $|S_1| = g(k-2)$ and $|S_2| = g(k-1)$. Note that $S_1 \Delta S_2 = V(C)$, $S_1 \setminus S_2 = \emptyset$, $S_2 \setminus S_1 = C$, $(V(G) \setminus (S_1 \cup S_2)) \cap V(C) = \emptyset$, and S_1 and S_2 do not satisfy any condition in Lemma 2, so G is not 2-good-neighbor $g(k-1)$-diagnosable. Thus $t_2^{MM^*}(G) \leq g(k-1)-1$.

In the following we prove $t_2^{MM^*}(G) \geq g(k-1)-1$, that is, G is 2-good-neighbor $[g(k-1)-1]$-diagnosable. Suppose, on the contrary that there are two distinct 2-good-neighbor faulty sets F_1 and F_2 of G with $|F_1| \leq g(k-1)-1$ and $|F_2| \leq g(k-1)-1$, but (F_1, F_2) does not satisfy any one of the conditions in Lemma 2. Clearly, $|F_1 \cap F_2| \leq g(k-1)-2$, since $F_1 \neq F_2$, without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. If $V(G) = F_1 \cup F_2$, then $N = |V(G)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2g(k-1)-2$, which contradicts with Condition (1). Therefore, $V(G) \neq F_1 \cup F_2$.

Claim 4. *$G-(F_1 \cup F_2)$ has no isolated vertices.*

Proof of Claim 4. Since F_1 is a 2-good-neighbor faulty set, $|N_{G-F_1}(x)| \geq 2$ for any $x \in V(G) \setminus F_1$. Note that the vertex set pair (F_1, F_2) does not satisfy any one of the conditions in Lemma 2, by the Condition (3) of Lemma 2, for any pair of vertices $u, v \in F_2 \setminus F_1$, there is no vertex $w \in V(G) \setminus (F_1 \cup F_2)$ such that $(u,w) \in E(G)$ and $(v,w) \in E(G)$. Thus, any vertex $x \in V(G) \setminus (F_1 \cup F_2)$ has at most one neighbor in $F_2 \setminus F_1$, $|N_{G-(F_1 \cup F_2)}(x)| \geq 2-1 = 1$, this implies every vertex of $G-(F_1 \cup F_2)$ is not an isolated vertex. The proof of Claim 4 is finished.

If $F_1 \cap F_2 = \emptyset$, then the claim is clearly true. Henceforth, we may assume that $F_1 \cap F_2 \neq \emptyset$. Let $y \in V(G) \setminus (F_1 \cup F_2)$. By Claim 4, y has at least one neighbor in $G-(F_1 \cup F_2)$. Note that the vertex set pair (F_1, F_2) does not satisfy any one of the conditions in Lemma 2, by Condition (3) of Lemma 2, y has no neighbor in $F_1 \Delta F_2$. Since y is arbitrary, there are no edges between $V(G) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$.

Since $F_2 \setminus F_1 \neq \emptyset$, and F_1 is a 2-good-neighbor faulty set, by Condition (3) of Lemma 2, $\delta(G[F_2 \setminus F_1]) \geq 2$. Similarly, $\delta(G[F_1 \setminus F_2]) \geq 2$ if $F_1 \setminus F_2 \neq \emptyset$. Since $V(G) \setminus (F_1 \cup F_2) \neq \emptyset$ and $F_2 \setminus F_1 \neq \emptyset$, $F_1 \cap F_2$ is a 2-good-neighbor cut of G. By Claim 2, $|F_1 \cap F_2| \geq g(k-2)$. Since $\delta(G[F_2 \setminus F_1]) \geq 2$, $G[F_2 \setminus F_1]$ has a cycle C_1 with length at least g as the girth of G is g, it follows that $|F_2 \setminus F_1| \geq g$. Then, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq g + g(k-2) = g(k-1)$, which contradicts $|F_2| \leq g(k-1)-1$. Therefore, G is 2-good-neighbor $[g(k-1)-1]$-diagnosable under the MM* model and $t_2^{MM^*}(G) \geq g(k-1)-1$. The proof is now complete. □

By Theorem 1 and Claim 2, the following Theorem 2 is obtained.

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DOI: 10.20429/tag.2020.070204

Theorem 2. *Let G be a k-regular and k-connected graph and g be the girth of G. If G satisfies all the conditions in Theorem 1, then* $t_2^{PMC}(G) = t_2^{MM^*}(G) = \kappa^2(G) + g - 1 = g(k-1) - 1$.

4 Applications to some networks

As applications of Theorem 1 and Theorem 2, in this section, we determine the 2-good-neighbor diagnosability and the R^2-connectivity for some networks.

4.1 Applications to Hierarchical Star Network HS_n

Definition 1. [1] An n-dimensional star graph, denoted by S_n, is an undirected graph with each vertex representing a distinct permutation of $[n]$ and two vertices are adjacent iff their labels differ only in the first and another position, that is two vertices $u = u_1u_1 \cdots u_n, v = v_1v_2 \cdots v_n$ are adjacent iff $v = u_iu_2u_3 \cdots u_{i-1}u_1u_{i+1} \cdots u_n$ for some $i \in [n] \setminus \{1\}$, where $[n] = \{1, 2, \ldots, n\}$.

Definition 2. ([19]) An *n-dimensional hierarchical star network $HS(n, n)$*, or simply HS_n, is made of $n!$ n-dimensional star graphs S_n, called modules. Each node of HS_n is denoted by a two-tuple address (x, y), where both x and y are arbitrary permutations of n distinct symbols. The first n-bit permutation x identifies the module of x and the second n-bit permutation y identifies the position of y inside its module. Two nodes (x, y) and (x', y') in HS_n are adjacent, if one of the following three conditions holds:

(1) $x = x'$ and $(y, y') \in E(S_n)$; That is, (x, y) is adjacent to (x, y') if $(y, y') \in E(S_n)$.

(2) $x \neq x', x = y$ and $x' = y' = x(1, n)$, where $x(1, n)$ is the permutation by interchanging the nth element with 1st element of x; That is, (x, x) is adjacent to $(x(1, n), x(1, n))$.

(3) $x \neq x', x \neq y$ and $x = y', y = x'$. That is, (x, y) is adjacent to (y, x) if $x \neq y$.

The 3-dimensional hierarchical star HS_3 is shown in Fig. 1.

Remark 1. Each node in HS_n is assigned a label $(x, y) = (x_1x_2 \cdots x_n, y_1y_2 \cdots y_n)$, where $x_1x_2 \cdots x_n$ and $y_1y_2 \cdots y_n$ are permutations of n distinct symbols (not necessarily distinct from each other). The edges of the HS_n are defined by the following n generators:

$$h_1((x, y)) = \begin{cases} (x(1, n), y(1, n)) & \text{if } x = y; \\ (y, x) & \text{if } x \neq y; \end{cases}$$

and

$$h_i((x, y)) = (x, y(1, i)) \text{ for } i \in [n] \setminus \{1\},$$

where $x(1, n)$ is the permutation by interchanging the nth element with 1st element of x.

Let (x, y) be a vertex of HS_n. The *neighbor set* of (x, y) is exactly $\{h_i((x, y)) | i \in I_n\}$. Furthermore, $h_1((x, y))$ is called the *extra neighbor* of (x, y) and $h_i((x, y))$ is called the *internal neighbor* of (x, y) for $2 \leq i \leq n$. Define HS_n^x to be an induced subgraph by the vertex set $\{(x, y) \in V(HS_n) : y \in V(S_n)\}$, which is isomorphic to an n-dimensional star graph S_n identified by x.

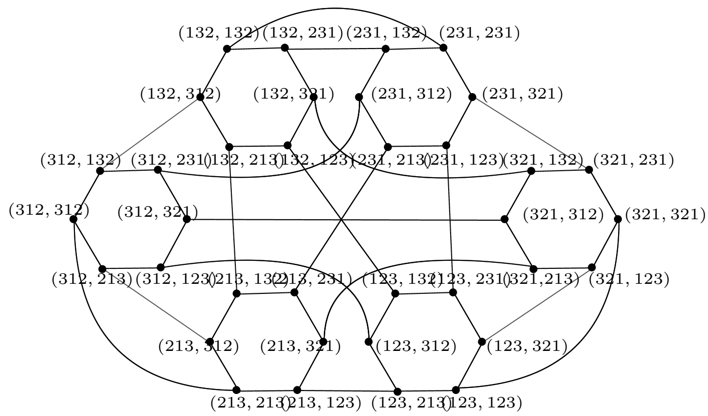

Fig. 1: Hierarchical star network HS_3

Remark 2. Any vertex has exactly one extra neighbor in HS_n, i.e., every vertex (x, y) in HS_n^x is exactly incident to one crossing edge $(x, h_1((x, y)))$. There is one or two crossing edges between any pair of modules. Moreover, for a fixed module HS_n^x, there are two cross edges between HS_n^x and $HS_n^{x(1,n)}$; there is only one cross edge between HS_n^x and HS_n^y, where $y \in \Gamma_n \setminus \{x, y\}$.

Lemma 3. ([19]) *For any integer $n \geq 3$, HS_n is an n-regular n-connected graph, and its girth is 4. Any two vertices have at most two common neighbors in HS_n.*

Recall that HS_n consists of $n!$ modules, each module is isomorphic to the star graph S_n, the known (fault tolerance) properties of S_n are useful.

Lemma 4. *Let U be a subset with $2 \leq |U| \leq 4$ of n-dimensional star graph S_n for $n \geq 5$. The following statements hold.*

(1) ([12,26]) *If $|U| = 2$, then $|N_{S_n}(U)| \geq 2n - 4$.*

(2) ([27]) *If $|U| = 3$, then $|N_{S_n}(U)| \geq 3n - 7$.*

(3) ([26]) *If $|U| = 4$, then $|N_{S_n}(U)| \geq 4n - 10$.*

Lemma 5. *Let F be a faulty subset of n-dimensional star graph S_n for $n \geq 5$. The following statements hold.*

(1) ([23]) *If $|F| \leq 2n - 4$, then $S_n - F$ is connected; or contains two components, one of which is an isolated vertex; or contains two components, one of which is an edge; furthermore, F is the neighborhood of this isolated edge with $|F| = 2n - 4$.*

(2) ([28]) *If $|F| \leq 3n - 8$, then $S_n - F$ is connected; or contains a large component and the union of smaller components which contain at most two vertices in total.*

(3) ([28]) *If $|F| \leq 4n - 11$, then $S_n - F$ is connected; or contains a large component and the union of smaller components which contain at most three vertices in total.*

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DOI: 10.20429/tag.2020.070204

In the following, let F be a faulty subset of n-dimensional hierarchical star network HS_n. For each $\alpha \in \Gamma_n$, let $F_\alpha = F \cap V(HS_n^\alpha)$ and $f_\alpha = |F_\alpha|$. Let $I = \{\alpha : \alpha \in \Gamma_n$ and $HS_n^\alpha - F_\alpha$ is disconnected$\}$, $F_I = \bigcup\limits_{\alpha \in I} F_\alpha$, $f_I = |F_I|$, $\overline{I} = \Gamma_n \setminus I, F_{\overline{I}} = \bigcup\limits_{\alpha \in \overline{I}} F_\alpha$, $f_{\overline{I}} = |F_{\overline{I}}|$ and $HS_n^{\overline{I}} = HS_n[\bigcup\limits_{\alpha \in \overline{I}} V(HS_n^\alpha)]$. These notations will be used throughout the paper. The following Claim holds.

Claim 5. ([9]) *For any $\alpha, \beta \in \Gamma_n$, there exist at least $n!-1$ vertex-disjoint paths connected HS_n^α and HS_n^β.*

Lemma 6. ([9]) *Let F be a faulty subset of $V(HS_n)$ for $n \geq 5$. If $|F| \leq 2n-3$, then $HS_n - F$ either is connected; or contains two components, one of which is an isolated vertex.*

Lemma 7. *Let F be a faulty subset of $V(HS_n)$ for $n \geq 5$. If $|F| \leq 3n-6$, then $HS_n - F$ either is connected; or contains a large component and the union of smaller components which contain at most two vertices in total.*

Proof. Recall that $I = \{\alpha \in \Gamma_n : HS_n^\alpha - F_\alpha$ is disconnected $\}$ and HS_n^α is $(n-1)$-connected, so $f_\alpha \geq n-1$ for any $\alpha \in I$. Since $|F| \leq 3n-6$, $|I| \leq 2$. We claim: $HS_n^{\overline{I}} - F_{\overline{I}}$ is connected. Note that $n!-1 > 3n-6+|I|$ for $n \geq 5$. For any $\alpha', \beta' \in \overline{I}$, by Claim 5, there exists a fault-free path in $HS_n - F$ which connects $HS_n^{\alpha'}$ and $HS_n^{\beta'}$. By the arbitrariness of α' and β', $HS_n^{\overline{I}} - F_{\overline{I}}$ is connected.

Case 1. $|I| = 0$.

In this case, $\overline{I} = \Gamma_n$, $HS_n - F = HS_n^{\overline{I}} - F_{\overline{I}}$ is connected.

Case 2. $|I| = 1$.

Without loss of generality, let $I = \{\alpha\}$. We consider the following three cases.

Subcase 2.1. $n-1 \leq f_\alpha \leq 3n-8$.

Since HS_n^α is isomorphic to S_n, by Lemma 5 (2), $HS_n^\alpha - F_\alpha$ contains a large component, say B, and the union of smaller components which contain at most two vertices in total. Since $(n!-1)-(3n-6)-2-|I| = n!-3n+2 > 0$ for $n \geq 5$, by Claim 5, B is connected to $HS_n^{\overline{I}} - F_{\overline{I}}$. Thus, $HS_n - F$ either is connected; or contains a large component and the union of smaller components which contain at most two vertices in total.

Subcase 2.2. $f_\alpha = 3n-7$.

Since $|F| \leq 3n-6$, $f_{\overline{I}} \leq 1$, by Remark 2, at most one vertex is disconnected with $HS_n^{\overline{I}} - F_{\overline{I}}$ in $HS_n - F$. Thus, $HS_n - F$ either is connected; or contains two components, one of which is an isolated vertex.

Subcase 2.3. $f_\alpha = 3n-6$.

Since $|F| \leq 3n-6$, $f_{\overline{I}} = 0$. By Remark 2, any component of $HS_n^\alpha - F_\alpha$ is connected to $HS_n^{\overline{I}} - F_{\overline{I}}$. Thus, $HS_n - F$ is connected.

Case 3. $|I| = 2$.

Without loss of generality, let $I = \{\alpha, \beta\}$.

Note that HS_n^α (resp. HS_n^β) is isomorphic to S_n and $|F| \leq 3n-6$, so $n-1 \leq f_\alpha, f_\beta \leq 3n-6-(n-1) = 2n-5$. By Lemma 5 (1), $HS_n^\alpha - F_\alpha$ (resp. $HS_n^\beta - F_\beta$) has two components, one of which is an isolated vertex. Let B_α (resp. B_β) be the largest component of $HS_n^\alpha - F_\alpha$ (resp. $HS_n^\beta - F_\beta$). Since $(n!-1)-(3n-6)-2-|I| = n!-3n+1 > 0$ for $n \geq 5$, by Claim 5, B_α (resp. B_β) is connected to $HS_n^{\overline{I}} - F_{\overline{I}}$. The number of vertices which are disconnected to $HS_n^{\overline{I}} - F_{\overline{I}}$ is at most two. Hence, the result holds. □

Lemma 8. *Let F be a faulty subset of n-dimensional hierarchical star network HS_n for $n \geq 5$. If $|F| \leq 4n-9$, then $HS_n - F$ either is connected; or contains a large component and the union of smaller components which contain at most three vertices in total.*

Proof. Recall that $I = \{\alpha : \alpha \in \Gamma_n$ and $HS_n^\alpha - F_\alpha$ is disconnected$\}$ and HS_n^α is $(n-1)$-connected, so $f_\alpha \geq n-1$. Since $|F| \leq 4n-9$, $|I| \leq 3$, we claim: $HS_n^{\overline{I}} - F_{\overline{I}}$ is connected. Note that $n!-1 > 4n-9+|I|$ for $n \geq 5$. For any $\alpha', \beta' \in \overline{I}$, there exists a fault-free path in $HS_n - F$ which connects $HS_n^{\alpha'}$ and $HS_n^{\beta'}$. By the arbitrariness of α' and β', $HS_n^{\overline{I}} - F_{\overline{I}}$ is connected. There are the following four cases.

Case 1. $|I| = 0$.

In this case, $\overline{I} = \Gamma_n$, $HS_n - F = HS_n^{\overline{I}} - F_{\overline{I}}$ is connected.

Case 2. $|I| = 1$.

Without loss of generality, let $I = \{\alpha\}$. We consider the following three subcases.

Subcase 2.1. $n-1 \leq f_\alpha \leq 4n-11$.

Since HS_n^α is isomorphic to S_n, by Lemma 5 (3), $HS_n^\alpha - F_\alpha$ contains a large component, say B, and the union of smaller components which contain at most three vertices in total. Since $(n!-1)-(4n-9)-3-|I| = n!-4n+4 > 0$ for $n \geq 5$, by Claim 5, B is connected to $HS_n^{\overline{I}} - F_{\overline{I}}$. Thus, $HS_n - F$ either is connected; or contains a large component and the union of smaller components which contain at most three vertices in total.

Subcase 2.2. $f_\alpha = 4n-10$.

Since $|F| \leq 4n-9$, $f_{\overline{I}} \leq 1$, by Remark 2, at most one vertex can be disconnected with $HS_n^{\overline{I}} - F_{\overline{I}}$ in $HS_n - F$. Thus, $HS_n - F$ either is connected; or contains two components, one of which is an isolated vertex.

Subcase 2.3. $f_\alpha = 4n-9$.

Since $|F| \leq 4n-9$, $f_{\overline{I}} = 0$. By Remark 2, any component of $HS_n^\alpha - F_\alpha$ is connected to $HS_n^{\overline{I}} - F_{\overline{I}}$. Thus, $HS_n - F$ is connected.

Case 3. $|I| = 2$.

Without loss of generality, let $I = \{\alpha, \beta\}$ and $f_\alpha \geq f_\beta$.

Since $|F| \leq 4n-9$, $n-1 \leq f_\beta \leq 2n-5$ and $n-1 \leq f_\alpha \leq 3n-8$.

By Lemma 5 (1) in HS_n^β, $HS_n^\beta - F_\beta$ contains two components, one of which is an isolated vertex. By Lemma 5 (2) in HS_n^α, $HS_n^\alpha - F_\alpha$ contains a large component and the union of smaller components which contain at most two vertices in total. Let B_α (resp. B_β) be the largest component of $HS_n^\alpha - F_\alpha$ (resp. $HS_n^\beta - F_\beta$). Since $(n!-1)-(4n-9)-3-|I| = n!-4n+3 > 0$ for $n \geq 5$, by Claim 5, B_α (resp. B_β) is connected to $HS_n^{\overline{I}} - F_{\overline{I}}$. The number of vertices which are disconnected to $HS_n^{\overline{I}} - F_{\overline{I}}$ is at most three. Hence, the result holds.

Case 4. $|I| = 3$.

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DOI: 10.20429/tag.2020.070204

Without loss of generality, let $I = \{\alpha, \beta, \gamma\}$.

Since $|F| \le 4n-9$, $n-1 \le f_\alpha, f_\beta, f_\gamma \le 4n-9-2(n-1) = 2n-7 < 2n-5$. For $\tau \in I$, note that HS_n^τ is isomorphic to S_n, by Lemma 5 (1), $HS_n^\tau - F_\tau$ contains two components, one of which is an isolated vertex, say x_τ. Since $(n!-1)-(4n-9)-3-|I| = n!-4n+2 > 0$ for $n \ge 5$, by Claim 5, $HS_n^\tau - F_\tau - \{x_\tau\}$ is connected to $HS_n^{\overline{I}} - F_{\overline{I}}$. The number of vertices which are disconnected to $HS_n^{\overline{I}} - F_{\overline{I}}$ is at most three. Hence, the result holds. □

Corollary 1. *Let HS_n be an n-dimensional hierarchical star network for $n \ge 6$. Then $t_2^{PMC}(HS_n) = t_2^{MM^*}(HS_n) = \kappa^2(HS_n) + 3 = 4n-5$.*

Proof. Note that for HS_n, $|V(HS_n)| = n!^{n!}$, $k = n-1$, $cn(HS_n) = 2$, $ca(HS_n) = 0$ and $\xi = 4$, since the girth of HS_n is 4 and HS_n has no 5-cycles. Since $k = n \ge \xi + 2$ for $n \ge 6$ and $|V(HS_n)| = n!^{n!} \ge 8(n-1)-1 = 8n-9$ for $n \ge 6$, Conditions (1) and (2) in Theorem 1 hold. By Lemma 8, Condition (3) in Theorem 1 holds. By Theorem 2, the result holds. □

4.2 Application to the Split-star network S_n^2

Cheng et al. [4] proposed the Split-star networks as alternatives to the star graphs and companion graphs with the alternating group graphs.

Definition 3. *Given two positive integers n and k with $n > k$, note that $[n] = \{1, 2, \ldots, n\}$, and let $\mathcal{P}_n$ be a set of $n!$ permutations on $[n]$. The n-dimensional split-star network, denoted by S_n^2, such that $V(S_n^2) = \mathcal{P}_n$, $E(S_n^2) = \{(p, q)|\ p$ (respectively q) can be obtained from q (respectively p) by either a 2-exchange or a 3-rotation $\}$, where*

(1) *a 2-exchange: interchanges the symbols in the 1st position and the 2nd position, and*

(2) *a 3-rotation: rotates the symbols in the three positions labeled by the 1st position, the 2nd position and the kth position for some $k \in \{3, 4, \ldots, n\}$.*

Let $S_{n,E}^2$ be a subgraph of S_n^2 induced by the set of even permutations, in which the adjacency rule is precisely the 3-rotation. We know that $S_{n,E}^2$ is the alternating group graph AG_n. Let $S_{n,O}^2$ be a subgraph of S_n^2 induced by the set of odd permutations, in which the adjacency rule is precisely the 3-rotation. We have that $S_{n,O}^2$ is also isomorphic to AG_n and $S_{n,O}^2$ is isomorphic to $S_{n,E}^2$ via the 2-exchange $\phi(a_1a_2a_3\cdots a_n) = a_2a_1a_3\cdots a_n$. Hence, there are $\frac{n!}{2}$ independent edges between $S_{n,O}^2$ and $S_{n,E}^2$.

Lemma 9. *Let S_n^2 be the n-dimensional split-star network.*

(1) ([4,5]) *S_n^2 is $(2n-3)$-regular and $\kappa(S_n^2) = 2n-3$ for $n \ge 3$.*

(2) ([6]) *Let x, y be any two vertices of S_n^2, then $cn(S_n^2 : x, y) \le 1$ if x and y are adjacent; $cn(S_n^2 : x, y) \le 2$ otherwise.*

Lemma 10. *Let F be a vertex cut of AG_n for $n \ge 5$.*

(1) ([2]) *If* $|F| \leq 4n-11$, *then* $AG_n - F$ *has two components one of which is a singleton; or* $AG_n - F$ *has two components one of which is an edge,* $|F| = 4n-11$, *and* F *is formed by the neighbors of the edge.*

(2) ([10]) *If* $|F| \leq 6n-19$, *then* $AG_n - F$ *has two components one of which is a singleton, an edge or a 2-path (that is, a path of length 2); or* $AG_n - F$ *has three components, two of which are singletons.*

Lemma 11. *Let* F *be a 2-good neighbor faulty set of* S_n^2 *for* $n \geq 5$. *If* $|F| \leq 6n-16$, *then* $S_n^2 - F$ *is connected.*

Proof. Using the notations established earlier, S_n^2 contains two copies AG_n, say $S_{n,O}^2$ and $S_{n,E}^2$, respectively. Let $F_0 = F \cap V(S_{n,O}^2)$, $F_E = F \cap V(S_{n,E}^2)$. Without loss of generality, assume $|F_0| \geq |F_E|$. Since $2(4n-11) > 6n-16$ for $n \geq 5$, we consider the following two cases.

Case 1. $|F_E| \leq |F_O| \leq 4n-12$.

By Lemma 10 (1), $S_{n,O}^2 - F_O$ (respectively $S_{n,E}^2 - F_E$) is either connected; or has two components, one of which is a singleton. Let B_O (respectively B_E) be the largest component of $S_{n,O}^2 - F_O$ (respectively $S_{n,E}^2 - F_E$). Since $\frac{n!}{2} - (6n-16) - 2 > 0$ for $n \geq 5$, B_O and B_E belong to the same component in $S_n^2 - F$. Note that F is a 2-good-neighbor set of S_n^2, neither of the singleton can remain singleton or for two of them to form an edge in $S_n^2 - F$. This implies that $S_n^2 - F$ is connected.

Case 2. $4n-11 \leq |F_0| \leq 6n-16$.

This implies that $|F_E| \leq (6n-16) - (4n-11) \leq 2n-5$. Note that $S_{n,E}^2$ is isomorphic to AG_n and $\kappa(AG_n) = 2n-4$, so $S_{n,E}^2 - F_E$ is connected. If $S_{n,O}^2 - F_O$ is connected, note that $\frac{n!}{2} - (6n-16) > 0$ for $n \geq 5$, then $S_n^2 - F$ is connected. Thus assume that $S_{n,O}^2 - F_O$ is disconnected in the following.

If $6n-18 \leq |F_0| \leq 6n-16$, then $|F_E| \leq 2$. Note that there are $\frac{n!}{2}$ independent edges between $S_{n,O}^2$ and $S_{n,E}^2$. Thus every component of size at least 3 in $S_{n,O}^2 - F_0$ is part of the component in $S_n^2 - F$ containing $S_{n,E}^2 - F_E$. Thus at most two vertices in $S_{n,O}^2 - F_O$ are not part of this component containing $S_{n,E}^2 - F_E$. This is not possible as F is a 2-good-neighbor set of S_n^2. Thus $S_n^2 - F$ is connected.

Now assume that $4n-11 \leq |F_0| \leq 6n-19$. By Lemma 10, $S_{n,O}^2 - F_O$ has two components, one of which is an edge or a 2-path; or it has three components, two of which are singletons. Let C be the largest component of $S_{n,O}^2 - F_O$. Since $\frac{n!}{2} - (6n-16) - 3 > 0$ for $n \geq 5$, C is part of the component in $S_n^2 - F$ containing $S_{n,E}^2 - F_E$. The smaller components of $S_{n,O}^2 - F_O$ cannot be components of $S_n^2 - F$ since F a 2-good-neighbor set of S_n^2, it must be part of the component in $S_n^2 - F$ containing $S_{n,E}^2 - F_E$. Thus $S_n^2 - F$ is connected. □

The 2-good-neighbor diagnosability of the n-dimensional split-star network and its R^2-connectivity have not been determined so far. By Theorem 2, we immediately get $t_2(S_n^2)$ and $\kappa^2(S_n^2)$ as Theorem 3.

Theorem 3. *Let* S_n^2 *be the* n*-dimensional split-star network. Then* $t_2^{PMC}(S_n^2) = t_2^{MM^*}(S_n^2) = \kappa^2(S_n^2) + 2 = 6n-13$ *for* $n \geq 6$.

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DOI: 10.20429/tag.2020.070204

Proof. Note that for S_n^2, by Lemma 9, $|V(S_n^2)| = n!$, $k = 2n-3$, $cn(S_n^2) = 2$, $ca(S_n^2) = 1$ and $\xi = 6$, since S_n^2 contains 5-cycles. Since $k = 2n - 3 \geq \xi + 2$ for $n \geq 6$ and $|V(S_n^2)| = n! \geq 6(2n - 3 - 1) - 2 = 12n - 26$ for $n \geq 6$, Conditions (1) and (2) in Theorem 1 hold. By Lemma 11, Condition (3) in Theorem 1 holds. By Theorem 2, the result follows. □

4.3 Application to the Cayley graph generated by 2-tree $\Gamma_n(\Delta)$

Definition 4. Let Γ be the alternating group, the set of even permutations on $\{1, 2, \ldots, n\}$, and the generating set Δ be a set of 3-cycles. The corresponding Cayley graph $\mathrm{Cay}(\Gamma, \Delta)$ is denoted by $\Gamma_n(\Delta)$. To get an undirected Cayley graph, we will assume that whenever a 3-cycle (abc) is in Δ, so is its inverse (acb). We can depict Δ via a graph H with vertex set $[n]$, where a triangle K_3 on vertices a, b, and c corresponds to each pair of a 3-cycle (abc) and its inverse in Δ, where a hyperedge of size 3 corresponds to each pair of a 3-cycle and its inverse in Δ. We consider a simpler case when H has a tree-like structure. Such a graph is built by the following procedure. We start from K_3, then repeatedly add a new vertex, joining it to exactly two adjacent vertices of the previous graph. Any graph obtained by this procedure is called a 2-tree. If v is a vertex of a 2-tree H with the property that H can be generated in such a way that v is the last vertex added, then v is called a leaf of the 2-tree. If Δ is the set of 3-cycles via a 2-tree H, then $\Gamma_n(\Delta)$ is called the *Cayley graph generated by 2-trees* Δ.

The *alternating group graph* AG_n [11], can be viewed as the Cayley graph generated by the graph having a tree-like (in fact, star-like) structure of triangles.

Lemma 12. ([2]) *Let* $G = \Gamma_n(\Delta)$ *be a Cayley graph generated by the* 2*-tree* Δ *for* $n \geq 4$*. Then*

(1) G *is* $(2n-4)$*-regular and* $(2n-4)$*-connected.*

(2) $\Gamma_n(\Delta)$ *does not contain* $K_4 - e$*, that is,* K_4 *with an edge deleted, and* $K_{2,3}$ *as a subgraph. For any two vertices* u *and* v*,* $cn(G : u, v) = 1$ *if* u *and* v *are adjacent,* $cn(G : u, v) \leq 2$ *otherwise.*

Lemma 13. ([3]) *Let* $\Gamma_n(\Delta)$ *be a Cayley graph generated by the* 2*-tree* Δ*. Then* $\kappa^2(\Gamma_n(\Delta)) = 6n - 18$ *for* $n \geq 6$*.*

The 2-good neighbor diagnosability of the Cayley graph generated by the 2-tree $\Gamma_n(\Delta)$ has not been determined so far. By Theorem 2, we immediately get $\kappa^2(\Gamma_n(\Delta))$.

Note that for $\Gamma_n(\Delta)$, by Lemma 12, $|V(\Gamma_n(\Delta))| = n!/2$, $k = 2n - 4$, $cn(\Gamma_n(\Delta)) = 2$, $ca(\Gamma_n(\Delta)) = 1$ and $\xi = 6$, since $\Gamma_n(\Delta)$ contains 5-cycles. Since $k = 2n - 4 \geq t + 2$ for $n \geq 6$ and $|V(\Gamma_n(\Delta))| = n!/2 \geq 6(2n - 4 - 1) - 2 = 12n - 32$ for $n \geq 6$, Conditions (1) and (2) in Theorem 1 hold. Condition (3) in Theorem 1 holds by Lemma 13. Thus, by Theorem 2, the following result holds.

Theorem 4. *Let* $\Gamma_n(\Delta)$ *be a Cayley graph generated by the* 2*-tree* Δ*. Then* $t_2^{PMC}(\Gamma_n(\Delta)) = t_2^{MM^*}(\Gamma_n(\Delta)) = \kappa^2(\Gamma_n(\Delta)) + 2 = 6n - 16$ *for* $n \geq 6$*.*

5 Concluding remarks

In this paper, the 2-good-neighbor diagnosability of a general regular graph G is studied. The main result $t_2^{PMC}(G) = t_2^{MM^*}(G) = \kappa^2(G)+g-1 = g(k-1)-1$ under some conditions is obtained. As consequences of our results, the 2-good-neighbor diagnosability and R^2-connectivity $\kappa^2(G)$ of many networks including some known results can be obtained. The following new results are obtained: $t_2^{PMC}(HS_n) = t_2^{MM^*}(HS_n) = 4n-5$ for hierarchical star network HS_n, $t_2^{PMC}(S_n^2) = t_2^{MM^*}(S_n^2) = 6n-13$ for split-star networks S_n^2, and $t_2^{PMC}(\Gamma_n(\Delta)) = t_2^{MM^*}(\Gamma_n(\Delta)) = 6n-16$ for Cayley graph generated by the 2-tree $\Gamma_n(\Delta)$.

In the literature, most known results about 2-good-neighbor conditional diagnosability of some networks are obtained via ad-hoc methods under various techniques. In this paper, we unified these approaches to obtain general results. As consequences of these results, some of them can be obtained easily.

Observing that Wang et al. [20] and [21] obtained 2-good-neighbor diagnosability of the Cayley graphs generated by transposition trees and the alternating group networks, respectively, under the PMC model and MM* model. We can deduce these results by Theorem 1 and Theorem 2 as directive corollaries. The details are omitted.

Furthermore, the h-good-neighbor diagnosability of a general regular graph G for $h \geq 3$ is a challenging work which need to be studied in the future. Since we have established a relationship between the R^2-connectivity and the 2-good-neighbor diagnosability of regular graphs (under certain conditions), any R^2-connectivity result (even an upper bound result) for such an interconnection network will "automatically" give a 2-good-neighbor diagnosability result (respectively an upper bound result) for this network by Theorem 2. This advances the study of 2-good-neighbor diagnosability as one can now leverage on such existing results rather than applying ad-hoc methods. It would also be interesting to see whether we can apply Theorem 2 in "reverse," that is, find a network in the literature where its 2-good-neighbor diagnosability was obtained via an ad-hoc method but its R^2-connectivity has not been evaluated.

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DOI: 10.20429/tag.2020.070204