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Enactment of Implicit Two-Step Obrechhoff-Type Block Method on Unsteady Sedimentation Analysis of Spherical Particles in Newtonian Fluid Media

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Abstract:

Purpose: The analysis of the characteristics of particles motion is considered in this article, where a model which studies a Newtonian fluid media with specific interest on the analysis of unsteady sedimentation of particles is considered. The numerical solution of this first order differential equation model using an Obrechhoff-type block method is presented.

Methodology: The algorithm for the conventional Nyström -type multistep scheme is considered with specific parameter choices in order to obtain the main k-step Obrechhoff-type block method and the required additional method. The unknown coefficients of these methods are obtained by using the concept of Taylor series expansion to obtain the required schemes for the block method which were combined as simultaneous integrators for the solution of the differential equation model.

Findings: The block method gave highly accurate results as compared with the exact solution of the model. Furthermore, at selected values of the physical properties of nanoparticles, the

solutions using the two-step Obrechhoff-type block method was compared with past literatures and the results were seen to be in agreement. The influence of the physical parameters on terminal velocity is also discussed.

Keywords: Block Method; Sedimentation analysis; Tylor series expansion; Terminal velocity.

1. Introduction

The falling and sedimentation of solid particles in gases and liquids occurs in many natural and industrial processes. Sediment transport and deposition in pipe lines [1,2] and alluvial channels [3,4] and chemical and powder processing [5,6] are just a few examples. As a result of its vast areas of application, description and investigation of the motion of immersed bodies in fluids has been of great interest to researchers for a long time and there are numerous investigations of particle settling process in the technical literature. Boillat and Graf [7], Joseph et al. [8] and Cheng [9] are a few examples which investigated spherical particles sedimentation in low and high concentration mixtures. Particle falling in a fluid under the influence of gravity will accelerate until the gravitational force is exactly balanced by the resistance force that includes buoyancy and drag. The constant velocity reached at that stage is called the “terminal velocity” or “settling velocity”. The resistive drag force depends upon the drag coefficient.

During the past decades, a vast body of knowledge has been accumulated on the steady-state motion of spheres in incompressible Newtonian fluids and extensive sets of data were collected which resulted in several theoretical and empirical correlations for the drag coefficient, C_D , in the terms of the Reynolds number, Re . These relationships for spherical particles were presented in treatises and review papers by Clift et al. [10], Khan and Richardson [11] and Chhabra [12], among others. A comparison between a number of these correlations for spheres by Hartman and Yates [13] showed relatively low deviations. In contrast to steady-state motion of particles much less has been reported about the acceleration motion of spherical particles in

incompressible Newtonian fluids. The accelerated motion is relevant to many processes such as particle classification, centrifugal and gravity particle collection and/or separation, where it is often necessary to determine the trajectories of particles accelerating in a fluid [14,15]. Also, for other particular situations, like viscosity measurement using the falling-ball method or rain-drop terminal velocity measurement it is necessary to know the time and distance required for particles to reach their terminal velocities.

It is very difficult to find the exact solutions of heat transfer and fluid flow problems because of their nonlinearity nature. A lot of new approaches are emerging daily to obtain an analytical or numerical solution to these fluid flow problems, especially problems involving the solution of the motion equation of falling objects in fluids. When considering the analytic solution, Joneidi et al [16] mentioned that there are three commonly used approaches to obtain analytic solutions for fluid flow models; Homotopy Perturbation Method (HPM), Differential Transformation Method (DTM) and Homotopy Analysis Method (HAM). Considering the specific model for the motion equation of falling objects in fluids, He's HPM [17] has been adopted as seen in Jalaal and Ganji [18] who considered the motion of objects in inclined tubes filled with incompressible Newtonian fluids, and Jalaal et al. [19] where a solution of the acceleration motion of a vertically falling spherical particle in incompressible Newtonian media was presented. The suitability of the HPM was grounded in the work of Mohyud-Din et al. [20] where it was affirmed that the method has the ability to solve partial differential equation models such as the sedimentation process. The DTM on the other hand, as introduced by Zhou [21] has also been validated to be a useful technique to solve nonlinear differential equations such as the model involving sedimentation analysis. In a recent study by Nouri et al. [22] and Hatami et al. [23], the DTM was adopted to solve the differential equation model of sedimentation analysis of particles and the motion of a particle both in newtonian fluid. HAM was described as a reliable and accurate method, as adopted by Jalaal et al. [24] to solve the

one-dimensional non-linear particle equation. Furthermore, an approach suitable to obtain a semi-exact solution for motion equation of falling objects in fluids known as the Variational Iteration Method (VIM) was implemented by Yaghoobi and Torabi [25] to investigate the acceleration motion of a vertically falling non-spherical particle in incompressible Newtonian media.

Despite the progress made to introduce these analytical solutions, it is observed that numerical solutions are quite important when considering fluid flow models in general [26, 27]. A very common numerical approach is the use of shooting method coupled with Runge-Kutta method. Recent studies utilizing this approach include the works of Hakeem et al. [28], Naveed et al. [29], Ibrahim et al. [30], Raza [31-34].

With specific focus on the motion equation of objects in fluid, the shooting method was employed by Raza et al. [35] to solve the fourth order nonlinear ODEs which described the flow of nanofluids in a semi porous channel with stretching walls, and also for the investigation of critical points for the problem of nanoparticles and micropolar fluid in a channel with changing walls [36]. Bearing in mind that the Runge-Kutta method is suitable for first order differential equations, this implies that the use of this approach require reducing these governing equations to a first-order system. This can be bypassed by the use of block methods, being an approach that can be applied to produce approximate solutions of differential equations at each grid point simultaneously. In addition, block methods have been established to obtain more accurate solutions [37]. Other advantages of the block methods include being self-starting [38, 39], possessing low computational complexity [40], and being less expensive in terms of function evaluations [41]. Also there are some researches in this field that can be useful for readers [42-52].

Block methods can be explicit or implicit. The advantage of adopting implicit block methods over its explicit counterpart is because the order of the implicit method is higher which implies better accuracy. This is because, the higher the order of the numerical method, the better the accuracy [53]. In a recent work by Alkasassbeh et al. [54], an implicit hybrid block method was implemented in a fluid flow model studying the heat transfer of convective fin with temperature-dependent internal heat generation. Although, this approach has impressive performance in terms of accuracy, the presence of off-grid points increases the number of function evaluations and hence more computations. This informs the use of block method without off-grid points in this article. One distinct family of block methods for the numerical approximation of differential equations is the Obrechhoff methods. This family of methods is regarded to be distinct due to the presence of higher derivatives in the method and its maximal order property [55]. Therefore, this article will be considering a maximal order implicit block method to numerically investigate unsteady sedimentation analysis of spherical particles in Newtonian fluid media.

1. Problem Formulation:

The particle sediment phenomenon is modelled using gravity, buoyancy, Drag forces and added mass. According to the Basset-Boussinesq-Ossen (BBO) equation for the unsteady motion of the particle in a fluid, for a dense particle falling in light fluids assuming $\rho \ll \rho_s$ Basset history force is negligible. So, by rewriting force balance for the particle, the equation of motion is gained as follows [19]

$$m \frac{du}{dt} = mg \left(1 - \frac{\rho}{\rho_s} \right) - \frac{1}{8} \pi D^2 \rho C_D u^2 - \frac{1}{12} \pi D^3 \rho \frac{du}{dt} \quad (1)$$

where C_D is the drag coefficient. In the right-hand side of the equation (1), the first term represents the buoyancy effect, the second term corresponds to drag resistance, and the last

term is the added mass effect due to acceleration of fluid around the particle. The main difficulty of solving equation (1) is the non-linear terms due to the non-linearity nature of the drag coefficient C_D . The suggested correlation for C_D of spherical particles which has a good agreement with the experimental data in a wide range of Reynolds number $0 \leq Re \leq 105$ is formulated by

$$C_D = \frac{24}{Re} \left(1 + \frac{1}{48} Re \right) \quad \left. \vphantom{C_D} \right\} \quad (2)$$

$$0 \leq Re \leq 105$$

Jalaal et al. [19] have shown that equation (2) represents a more accurate resistance of the particle in comparison with the pervious equations presented by others. Based on the mass of particle formulated by:

$$m = \frac{1}{6} \pi D^3 \rho_s \quad (3)$$

Equation (1) can be rewritten as

$$a \frac{du}{dt} + bu + cu^2 - d = 0, u(0) = 0 \quad (4)$$

$$a = \left[\frac{1}{12} \pi D^3 (2\rho_s + \rho) \right]$$

$$b = (3\pi D\mu)$$

$$c = \left(\frac{1}{16} \pi D^2 \rho \right) \quad (5)$$

$$d = \frac{1}{6} \pi D^3 g (\rho_s - \rho)$$

In the present study, we choose three different materials for solid particle, Aluminum, Copper and Lead with three different diameters (1, 3, and 5 mm). A schematic of the described problem is shown in Figures 1 and 2. Physical properties of the selected material as well as the resulted

coefficients used in equation (5) are summarized in Tables 1 and 2, respectively. It is needed to note that the more complicated problem addressing this observation has been reported by several researchers. However, the application of new analytical methods has been introduced in this research as a new concept.

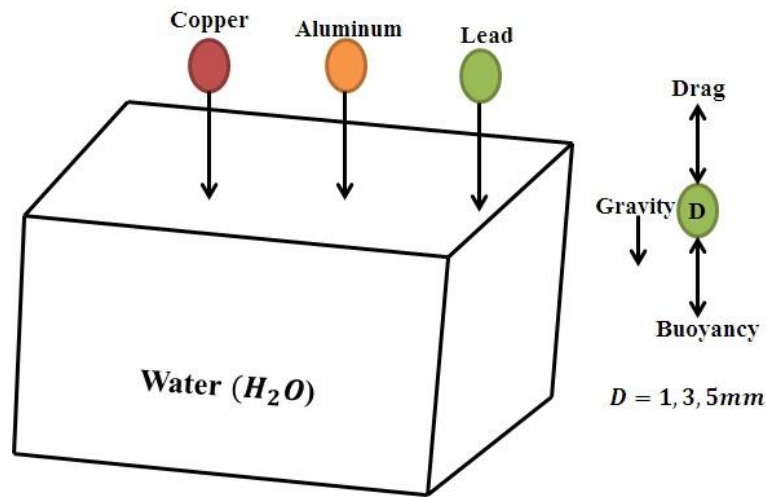


Figure 1: Physical sketch of the proposed problem

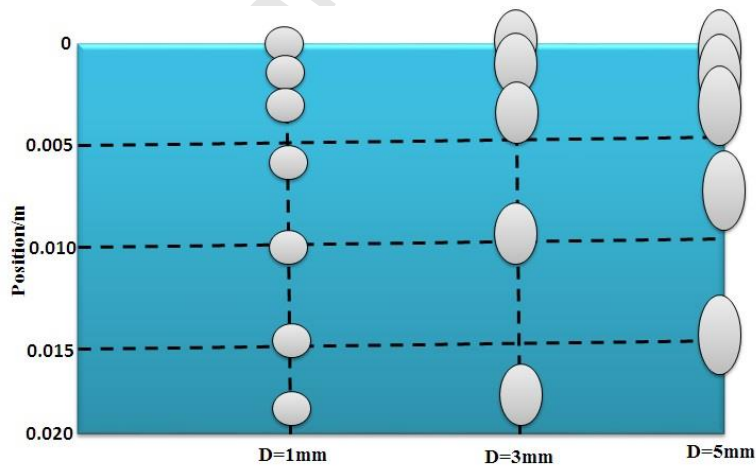


Figure 2: Particles' position for different size of Aluminum particles, time interval $t = 0.02\text{sec}$

2. Development of Implicit Two-Step Obrechhoff-Type Block Method

In order to find the numerical solution of Eq. (4), we employed RK-Fehlberg method and Block method. This article will be adopting an implicit two-step Obrechhoff-type block method to numerically approximate equation (4). The following algorithm is adopted in developing the block method.

Algorithm 1

START

Step 1: Obtain the coefficients of the implicit Nyström -type multistep scheme

$$u_{n+2} = \sum_{j=0}^{k-2} \alpha_j u_{n+j} + \sum_{i=0}^l h^i \sum_{j=0}^k \beta_{ij} u_{n+j}^{(i)}, \text{ where } k = 2 \text{ is the step number and } l = 2$$

Step 2: Obtain the coefficients of the additional method needed

$$u_{n+1} = \sum_{j=0}^{k-2} \alpha_j u_{n+j} + \sum_{i=0}^l h^i \sum_{j=0}^k \beta_{ij} u_{n+j}^{(i)}$$

Step 3: Combine schemes obtained in *Steps 1* and *2* above to form the desired block method

STOP

Following Algorithm 1 above, the unknown coefficients are obtained using the Taylor series approach [56]. The resultant block method is thus given as

$$\begin{aligned} u_{n+2} &= u_n + \frac{h}{15} \left(7u_n^{(1)} + 16u_{n+1}^{(1)} + 7u_{n+2}^{(1)} \right) + \frac{h^2}{15} \left(u_n^{(2)} - 7u_{n+2}^{(2)} \right) \\ u_{n+1} &= u_n + \frac{h}{240} \left(101u_n^{(1)} + 128u_{n+1}^{(1)} + 11u_{n+2}^{(1)} \right) + \frac{h^2}{240} \left(-13u_n^{(2)} + 40u_{n+1}^{(2)} + 3u_{n+2}^{(2)} \right) \end{aligned} \quad (6)$$

To solve equation (4), the schemes for the block methods were combined as simultaneous integrators for the solution of the ordinary differential equation. Thus, in each iteration, the following steps in (7) compute simultaneously,

$$u_{n+2} = u_n + \frac{h}{15} (7f(t_n, u_n) + 16f(t_{n+1}, u_{n+1}) + 7f(t_{n+2}, u_{n+2})) + \frac{h^2}{15} \left(\frac{df(t_n, u_n)}{dt} - 7 \frac{df(t_{n+2}, u_{n+2})}{dt} \right) \quad (7)$$

$$u_{n+1} = u_n + \frac{h}{240} (101f(t_n, u_n) + 128f(t_{n+1}, u_{n+1}) + 11f(t_{n+2}, u_{n+2})) + \frac{h^2}{240} \left(-13 \frac{df(t_n, u_n)}{dt} + 40 \frac{df(t_{n+1}, u_{n+1})}{dt} + 3 \frac{df(t_{n+2}, u_{n+2})}{dt} \right)$$

2.1 Order and Consistency of the Implicit Two-Step Obrechhoff-Type Block Method

Investigating the order for multistep methods such as the block method under consideration is quite straightforward in comparison to Runge-Kutta methods which can be cumbersome.

To investigate the order of the implicit two-step Obrechhoff-type block method, the linear operator $L_n[u(t)]$, where $u(t)$ is an arbitrarily continuously differentiable function on $[0, b]$

$$L_n[u(t)] = \begin{bmatrix} u_{n+2} - \left(\sum_{j=0}^{k-2} \alpha_j u_{n+j} + \sum_{i=0}^l h^i \sum_{j=0}^k \beta_{ij} u_{n+j}^{(i)} \right) \\ u_{n+1} - \left(\sum_{j=0}^{k-2} \alpha_j u_{n+j} + \sum_{i=0}^l h^i \sum_{j=0}^k \beta_{ij} u_{n+j}^{(i)} \right) \end{bmatrix} \quad (8)$$

Expression (8) is based on equation (6). Recall that $u^{(i)} = f^{(i-1)}(t, u(t))$. Therefore, (8) can be written as:

$$L_n[u(t)] = \begin{bmatrix} u_{n+2} - \left(\sum_{j=0}^{k-2} \alpha_j u_{n+j} + \sum_{i=0}^l h^i \sum_{j=0}^k \beta_{ij} f^{(i-1)}(t_n + jh, u(t_n + jh)) \right) \\ u_{n+1} - \left(\sum_{j=0}^{k-2} \alpha_j u_{n+j} + \sum_{i=0}^l h^i \sum_{j=0}^k \beta_{ij} f^{(i-1)}(t_n + jh, u(t_n + jh)) \right) \end{bmatrix} \quad (9)$$

which becomes, after expanding $u(t_n + jh)$ and $f^{(i-1)}(t_n + jh, u(t_n + jh))$ in a Taylor series about t_n and simplifying we obtain the equation of the following form

$$L_n[u(t)] = \begin{bmatrix} C_{10}u(t_n) + C_{11}hu^{(1)}(t_n) + \dots + C_{1p}h^{(p)}u^{(p)}(t_n) + \dots \\ C_{20}u(t_n) + C_{21}hu^{(1)}(t_n) + \dots + C_{2p}h^{(p)}u^{(p)}(t_n) + \dots \end{bmatrix} \quad (10)$$

The block method is of order p if the local truncation (or discretization) error is $d_n = O(h^p)$,

which is given by:

$$d_n = \frac{L_n[u(t_n)]}{h}$$

So, using the version of the expression given by (10), the block method is of order p if

$$C_{q0} = C_{q1} = \dots = C_{qp} = 0, C_{q(p+1)} \neq 0; q = 1, 2.$$

Combining this result with (10),

$$d_n = C_{q(p+1)} h^p y^{(p+1)}(t_n) + O(h^{p+1}),$$

where $C_{q(p+1)}$ is the error constant of the block method.

Therefore, for the block method (3.1), $C_{q0} = C_{q1} = \dots = C_{q6} = 0; q = 1, 2$ and the error constant

$$\text{is obtained as } [C_{17}, C_{27}] = \left[\frac{1}{4725}, \frac{1}{9450} \right].$$

This implies that the block method is of order $p = 6$.

The block method is said to be of maximal order if its order p satisfies $p = 2k + 2$ [57]. Hence, the maximal order criterion is satisfied.

In particular, the block method is consistent if it has order $p \geq 1$. Thus, the maximal order block method of order $p = 6$ is consistent.

2.2 Stability of the Implicit Two-Step Obrechhoff-Type Block Method

2.2.1 Stability

The most important stability property, a block method should satisfy is 0-stability to ensure convergence. The term ‘0’ is based on the stability phenomenon in terms of convergence in the limit as step-size $h \rightarrow 0$.

A block method in the following matrix difference equation form

$$A^0 U_{n+k} = A^1 U_{n-k} + h \left[B^0 U_{n+k}^{(1)} + B^1 U_{n-k}^{(1)} \right] + h^2 \left[C^0 U_{n+k}^{(2)} + C^1 U_{n-k}^{(2)} \right] \quad (11)$$

where $U_{n+k}^{(a)} = \left(u_{n+1}^{(a)}, u_{n+2}^{(a)}, \dots, u_{n+k}^{(a)} \right)^T$, $U_{n-k}^{(a)} = \left(u_{n-(k-1)}^{(a)}, u_{n-(k-2)}^{(a)}, \dots, u_n^{(a)} \right)^T$ is 0-stable if the first

characteristic polynomial takes the form

$$\rho(R) = \det(R_{ij}A^0 - A^1); R_{ij} = R^i \delta_{ij} \quad (i=1,2,\dots,k)$$

and the roots of $\rho(R) = 0$ satisfy $|R_j| \leq 1, j=1,\dots,k$.

Following equation (11), the block method (6) is normalized to give the first characteristic polynomial as:

$$\rho(R) = \det(R_{ij}A^0 - A^1) = R(R-1)$$

The roots of $\rho(R) = 0$ satisfy $|R_j| \leq 1, j=1,2$. Hence, the implicit two-step Obrechhoff-type block method is said to be zero-stable.

Theorem [53]: A linear multistep method is convergent iff it is consistent and zero-stable.

Therefore, since the block method is consistent and zero-stable, it is likewise convergent.

2.2.2 Absolute-Stability

To determine the absolute stability of the block method, the resulting expression in the form

$$u_{n+\xi} = u_n + \sum_{i=0}^k \left(\phi_{i\xi} f(t_{n+i}, u_{n+i}) + \tau_{i\xi} \frac{df(t_{n+i}, u_{n+i})}{dt} \right)$$

is applied to the scalar test problem

$$y' = \lambda y, \quad \text{Re}(\lambda) > 0$$

to yield the stability polynomial

$$\pi(q; z) = -(q^\xi) + z \left(\sum_{i=0}^k \phi_{i\xi} q^i \right) + z^2 \left(\sum_{i=0}^k \tau_{i\xi} q^i \right) + 1, \quad z = \lambda h, \xi=1,2$$

The stability region is obtained by plotting the loci of the stability polynomial for $z = e^{i\theta}$ as θ ranges from 0 to 2π .

The stability polynomial of the two-step Obrechhoff-type block method

$$\pi(q; z) = \frac{1}{90} q^3 z^4 - \frac{1}{10} q^3 z^3 + \frac{13}{30} q^3 z^2 - q^3 z + q^3 - \frac{1}{90} r z^4 - \frac{1}{10} r z^3 - \frac{13}{30} r z^2 - r z - r \quad (12)$$

Plotting the roots of the stability polynomial in (12) in boundary locus sense reveals that the block method has a region of absolute stability as shown in the Figure 3.

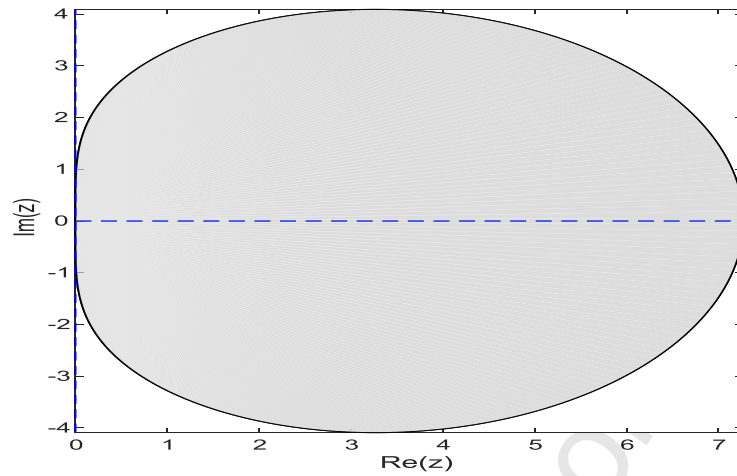


Figure 3. Region of Absolute Stability for the Implicit Two-Step Obrechhoff-Type Block Method

Exact Solution

The model

$$m \frac{du}{dt} = mg \left(1 - \frac{\rho}{\rho_s}\right) - \frac{1}{8} \pi D^2 \rho C_D u^2 - \frac{1}{12} \pi D^3 \rho \frac{du}{dt}$$

rewritten as the linear differential equation (*) with initial condition, follows the steps using

MATLAB

$$\gg \text{ode} = \text{diff}(u,t) == -(1/a)*((c*u^2) + (b*u) - d)$$

$$\gg \text{cond} = u(0) == 0;$$

$$\gg \text{uSol}(t) = \text{dsolve}(\text{ode}, \text{cond})$$

to obtain the exact solution as

$$u(t) = - \frac{\left(b + \tan \left(\frac{t \sqrt{-b^2 - 4cd} - 2a \left(\text{atan} \left(\frac{b}{\sqrt{-b^2 - 4cd}} \right) \right)}{2a} \right) \right) \left(\sqrt{-b^2 - 4cd} \right)}{2c}$$

where

$$a = \left[\frac{1}{12} \pi D^3 (2\rho_s + \rho) \right]$$

$$b = (3\pi D\mu)$$

$$c = \left(\frac{1}{16} \pi D^2 \rho \right)$$

$$d = \frac{1}{6} \pi D^3 g(\rho_s - \rho)$$

The table 1 displays the comparison of the exact solution at $a=b=c=d=1$, with the two-step Obrechhoff-type block method with the help of absolute error.

Table 1. the comparison of the exact solution at $a=b=c=d=1$, with the two-step Obrechhoff-type block method

t	Block Method	Exact Solution	Absolute Error (AE)
0	0	0	0
0.2	0.179113346500944130	0.179113346500944240	1.110223e-16
0.4	0.316007091866029990	0.316007091866030040	5.551115e-17
0.6	0.415028297540264160	0.415028297540264270	1.110223e-16
0.8	0.483837459640079630	0.483837459640079740	1.110223e-16
1.0	0.530329756621528260	0.530329756621528150	1.110223e-16
1.2	0.561150740811618860	0.561150740811618980	1.110223e-16
1.4	0.581325694701848580	0.581325694701848810	2.220446e-16
1.6	0.594422666070441360	0.594422666070441700	3.330669e-16
1.8	0.602879032080501420	0.602879032080501980	5.551115e-16

2	0.608320058486297670	0.608320058486298000	3.330669e-16
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$$AE = |\text{exact solution} - \text{Block method solution}|$$

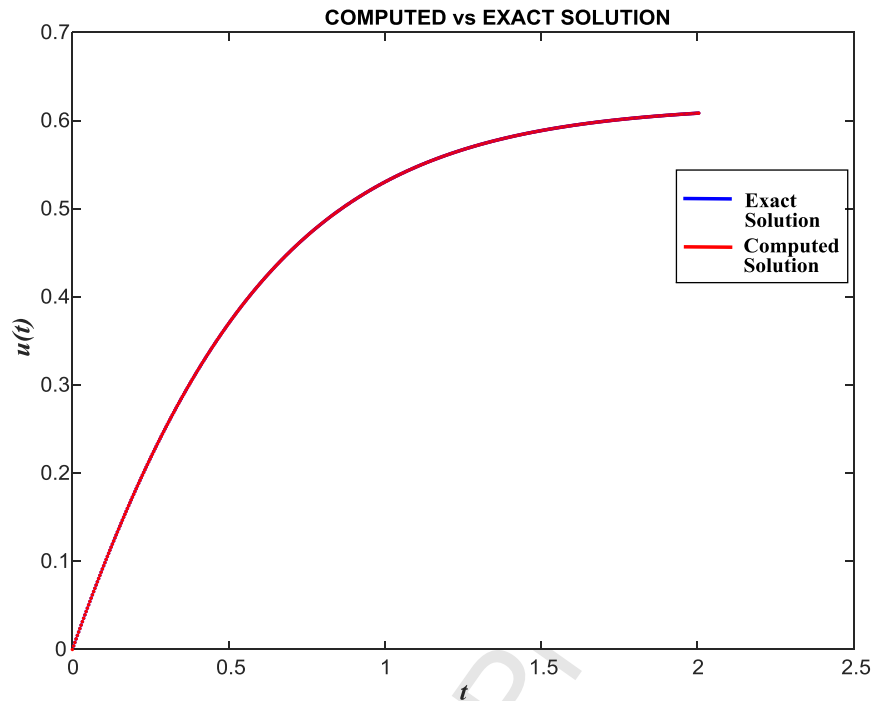


Figure 4: Comparison between Block method with exact solution

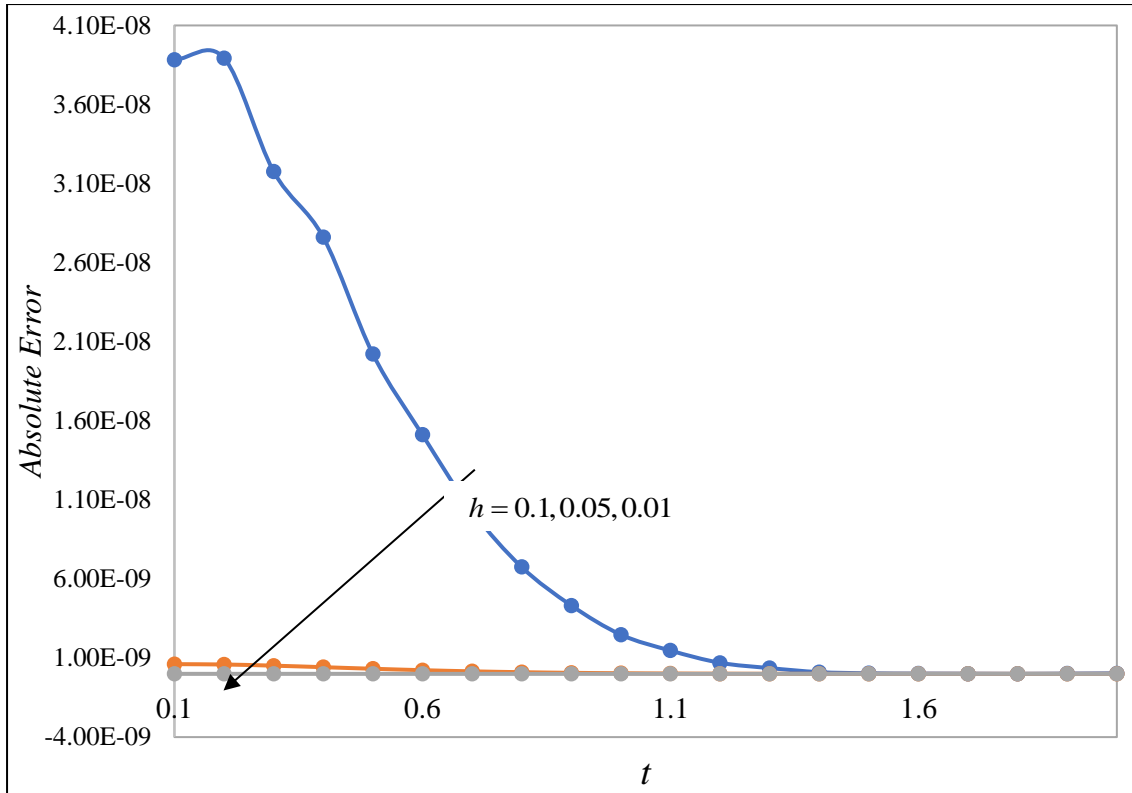


Figure5: Plot of Absolute error over a time (t)

Table 1: Physical properties of nanoparticles

Material	Density ρ (kg/m^3)	Viscosity ($kg/(m.s)$)
Aluminum	2702	--
Copper	8940	--
Lead	11340	--
Water	996.51	0.001

Table 2: Coefficient in Eqn. 4

Particle material	Diameter /mm	a	b	c	d

Aluminum	1	0.0000016756496	0.00000942477796	0.000195664281	0.000008760256
	3	0.0000452425392	0.00002827433389	0.001760978529	0.000236526917
	5	0.0002094562001	0.00004712388981	0.004891607024	0.001095032024
Copper	1	0.0000049418587	0.00000942477796	0.000195664281	0.000040801768
	3	0.0001334301866	0.00002827433389	0.001760978529	0.001101647738
	5	0.0006177323456	0.00004712388981	0.004891607024	0.00510022101
Lead	1	0.0000061984958	0.00000942477796	0.000195664281	0.000053129377
	3	0.0001673593872	0.00002827433389	0.001760978529	0.001434493196
	5	0.0007748119783	0.00004712388981	0.004891607024	0.006641172207

Results & Discussions:

Figure 3 traces out the curve in the complex plane indicating the boundary between unstable and stable solutions when adopting the implicit two-Step Obrechhoff-type block method. The grey shaded region gives the values of $z = \lambda h$ where the scheme will be stable while the unshaded region shows the unstable region. Hence, the choice of h -values will be within the stable region. However, it is worth noting that the most important stability property a good numerical method should satisfy is 0-stability. This is because this is the stability property that ensures convergence as discussed in earlier part of this article. Therefore, as the implicit two-Step Obrechhoff-type block method is 0-stable, it could also be referred to as being simply stable, where the key term '0' is based on the stability phenomenon in terms of convergence in the limit as step-size (h) tends to zero. Fig. 4 further depicts the convergence in solution of the

two-step Obrechhoff-type block method to the exact solution obtained. The computed solution is observed overlapping the exact solution, therefore the two-step Obrechhoff-type block method possess high accuracy with strong convergence properties. The convergence is depicted in the satisfaction of the 0-stability property shown in the Fig.4 showing the absolute error for selected iteration points as step-size $h \rightarrow 0$.

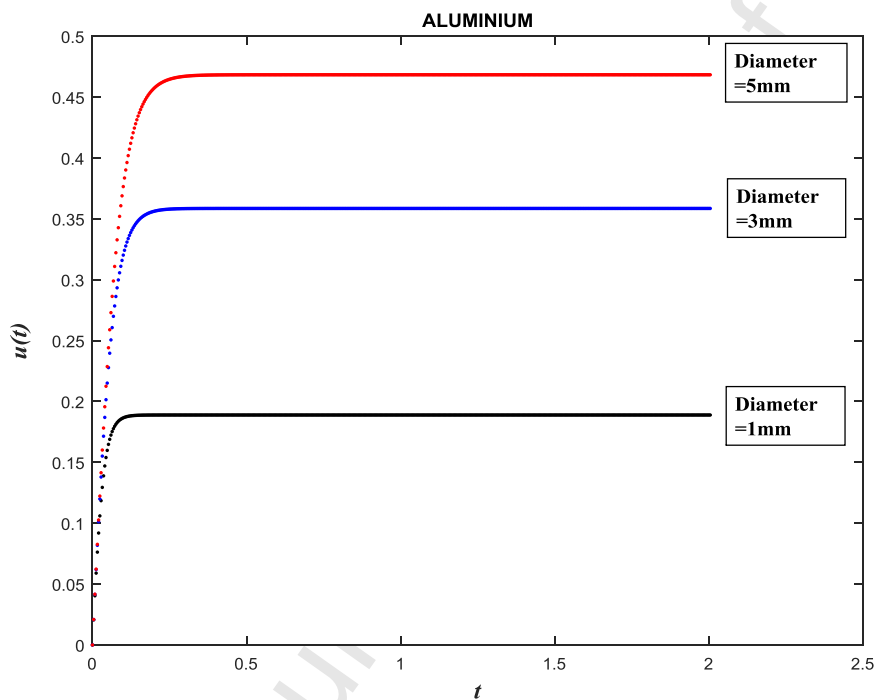


Figure 6: Velocity variation for different particle diameter (Aluminium)

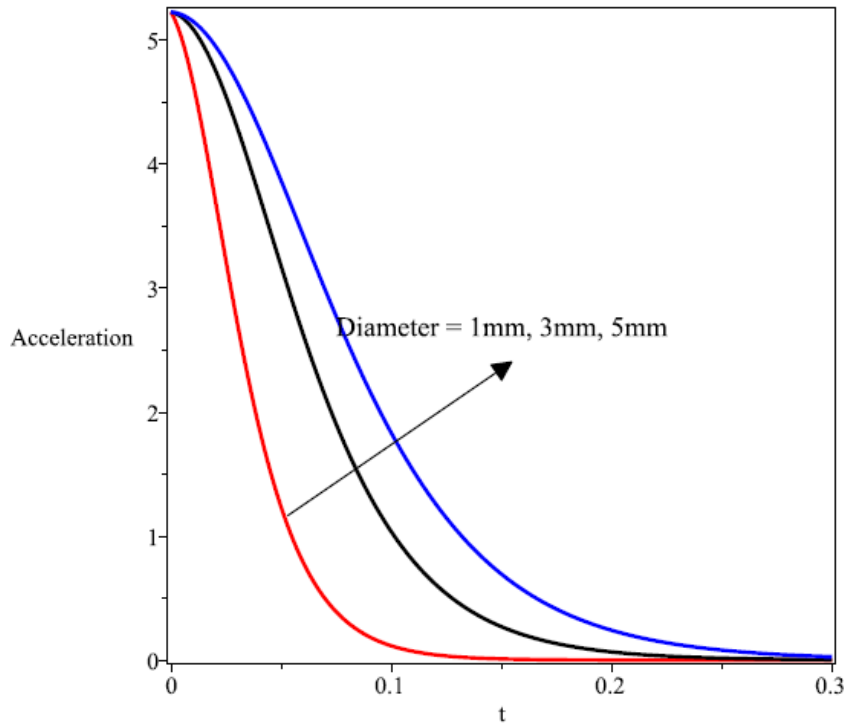


Figure 7: Acceleration for different particle diameter (Aluminum)

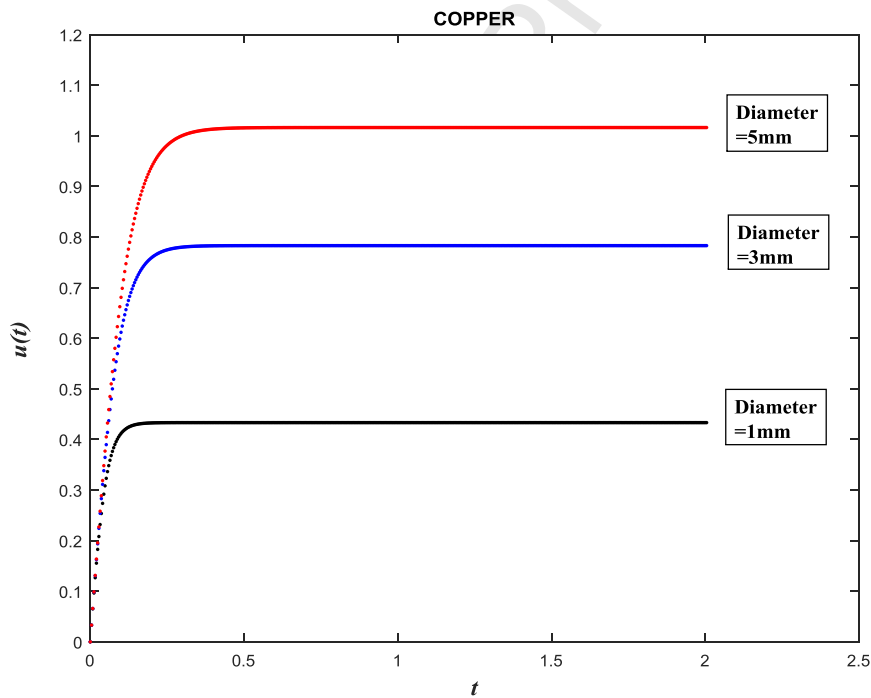


Figure 8: Velocity variation for different particle diameter (Copper)

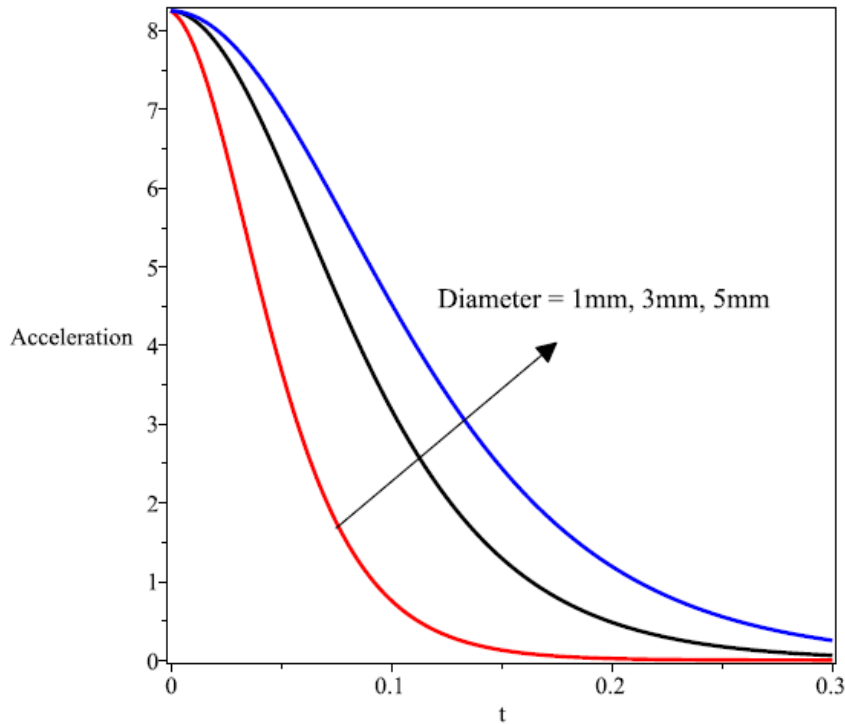


Figure 9: Acceleration for different particle diameter (Copper)

Fig.5 represents the graph of the absolute error over a time with changing step length. It is observed from this profile that the graph of the absolute error is strictly monotonically decreasing as step length of the developed method reduces. This is due to the fact that the developed method is going to be well behaved if the step length of the method reduces. Impact of the Aluminium type nanoparticle's diameter on velocity profile and acceleration profile is presented in Fig.6 and Fig.7 respectively. As the diameter of the particles increases velocity profile increases rapidly, correspondingly, trend of the acceleration also behaves like same. Figure 8 and 9 showed the impact of different diameter size of copper type nanoparticles on velocity and acceleration profiles. As the diameter of the nanoparticles increase velocity and acceleration profile increase monotonically. In a same vein, Lead type nanoparticle's diameter effects are exposed in Figure 10 and 11 for velocity and acceleration profile respectively. Afterwards, to present some rational examples, Aluminum, Copper and Lead are chosen in different sizes sub converged in water. Physical properties of the materials and determined coefficients for Eq. (4) in these real models are recorded in Tables 2.

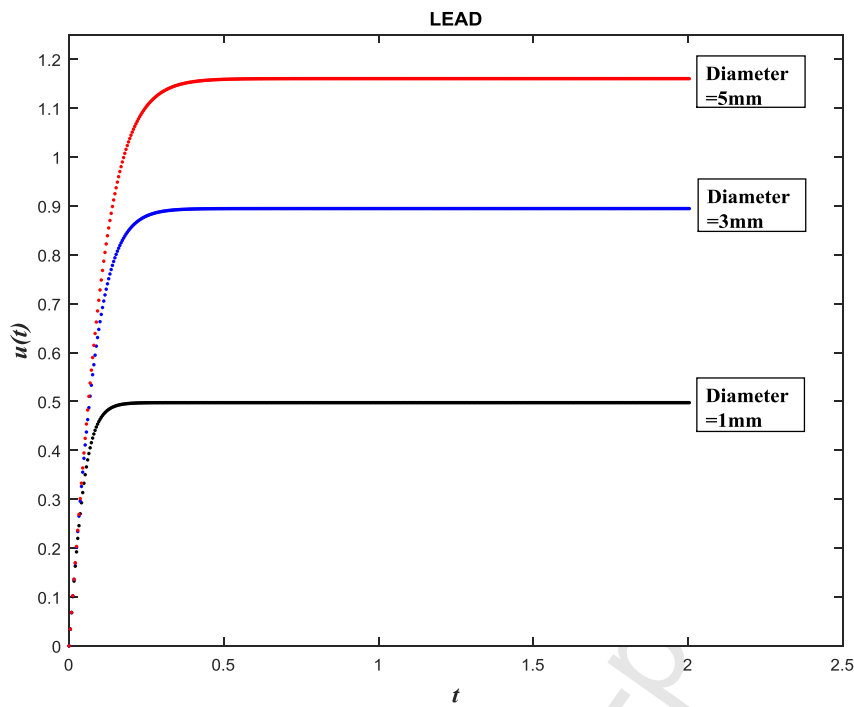


Figure 10: Velocity variation for different particle diameter (Lead)

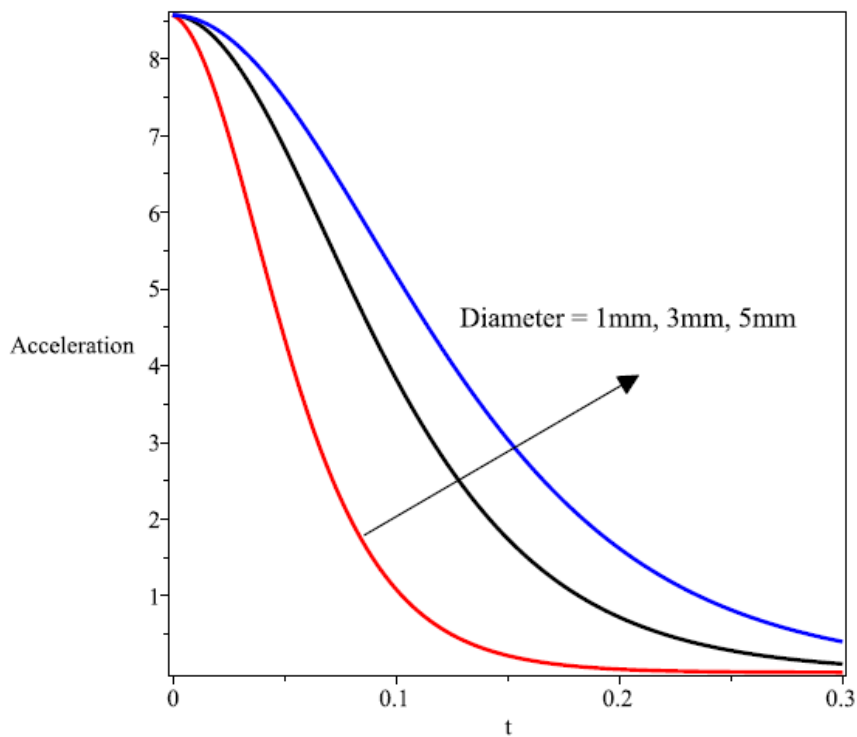


Figure 11: Acceleration for different particle diameter (Lead)

Conclusion:

The introduction of block methods as a numerical approach to approximate solution of differential equations has helped with the computational rigor problems attributed to past approaches. This is because block methods are self-starting and require no predictors or starting values. Likewise, Obrechhoff-type methods have been confirmed to possess impressive accuracy due to the presence of higher derivative in the method. Hence, the motivation in this article to implement a two-step Obrechhoff-type block method to investigate unsteady sedimentation analysis of spherical particles in Newtonian fluid media. The derivation of the two-step Obrechhoff-type block method involved a modification of the conventional Nyström-type multistep scheme with certain parameter values. The required coefficients for block method were obtained using Taylor series expansion approach of obtaining coefficients for linear multistep methods. A further investigation into the properties of the block method showed its consistency, zero-stability, and hence convergence, which is also displayed in the numerical results obtained. The self-starting property of the two-step Obrechhoff-type block method reduced the computational rigor involved in adopting the method to solve the differential equation model showing the unsteady sedimentation of particles. In addition, as a result of the presence of higher derivative in the block method, its impressive accuracy is observed in the solution comparison to the exact solution. The graphs and tables further display the usability of the two-step Obrechhoff-type block method. Hence, the two-step Obrechhoff-type block method is recommended as an approach to obtain approximate solutions to first order differential equation models.

Conflict of interest

The author do not have any conflict of interest in this work.

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Highlight

- The two-step Obrechhoff-type block method reduced the computational.
- The accuracy of the block method is observed in comparison with the exact solution.
- This method is recommended as approximation for first order differential equation.

Journal Pre-proof