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DISCUSSIONS ON CONTINUOUS STOCHASTIC VOLATILITY MODELS

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Abstract. Stochastic volatility (SV) models are substantial for financial markets and decision making because they can capture the effect of time-varying volatility. There are two ways to describe SV; in discrete time setting and continuous time setting. Since the intuitive setting for market trading is normally continuous, it is natural to focus on studying a continuous time setting in a financial environment. In this paper, we review and discuss the most important financial models of continuous stochastic volatility via highlight the advantages and the disadvantages of each one.

1. Introduction: Stochastic Volatility Models in Financial Environment

Stochastic volatility (SV) models are considered the most appropriate approaches to capture an implied volatility smile and fat tailed distribution of asset price return (Kim and Wee, 2014). Such properties can significantly improve the pricing of asset under the Black–Scholes model.

SV models are also substantial for financial markets and decision making because they can capture the effect of time-varying volatility. For this reason, many studies on SV models have been carried out in financial environment such as option pricing, value at risk, risk assessment and portfolio allocation. In addition, SV models also provide alternatives to standard Black-Scholes assumption where observations to volatility do not need to be perfectly correlated with observations of the underlying asset price (Heston, 1993; and Stein and Stein, 1991). These SV type models can offer better information for the joint time-series behavior of option prices and stocks, which could not be captured by using other models.

In a SV model, the constant volatility σ in standard geometric Brownian motion (GBM) model is replaced by a deterministic function of a stochastic process $\sigma(Y_t)$ where Y_t represents the solution of stochastic differential equation (SDE) that is driven by other noise. This implies that SV model has two sources of randomness which can either be correlated or not.

There are two ways to describe SV; in discrete time setting and continuous time setting. Since the intuitive setting for market trading is normally continuous such as derivative pricing (Johnson and Shanno, 1987; Hull and White, 1987;

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Stein and Stein, 1991; Comte and Renault, 1998; and Chronopoulou and Viens, 2012a; 2012b) and portfolio optimization (Pakdel, 2016; and Vierthauer, 2010), it is natural to embark studying a continuous time setting in a financial environment.

2. Classical Stochastic Volatility Models

SV models were first introduced by Taylor (1986) to account for inconsistency in implied volatility values. Taylor recommended modeling the logarithm of volatility as an autoregressive AR(1) process. This model is known as autoregressive stochastic volatility ARSV(1) and is given by

$$x_t = exp\left(\frac{y_t}{2}\right)u_t, \qquad u_t \sim N(0,1)$$
$$y_t = \mu + \phi(y_{t-1} - \mu) + \eta_t, \qquad \eta_t \sim N(0, \sigma^2_{\eta_t}),$$

where x_t denotes the *log* return at time t, where $t = 1, \ldots, T$ and y_t is the *log*–volatility which is assumed to follow a stationary AR(1) process with persistence parameter $|\phi| < 1$. The error terms u_t and η_t are Gaussian white noise sequences. Although the Taylor model is simple and easy to use, it has some drawbacks such as the absence of the mean-reverting part, zero correlation assumption between stock price and volatility, and non-existence of memories in its returns series and volatility.

Subsequently, Johnson and Shanno (1987) used time changing volatility in option pricing where the deterministic function $\sigma(Y_t) = Y_t$ is defined by

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \alpha Y_t dt + \beta Y_t dW_{2t},$$

where α and β are mean and volatility of a volatility of process Y_t, respectively. In this model, Wiener processes W_{1t} and W_{2t} are correlated. The main advantage of this model is that the computational results of option prices are consistent with empirical observations. This model exhibits volatility smile and an increase in value with toward expiry (Mitra, 2011). However, this model only provides numerical method to option pricing instead of in its closed form. The mean-reverting parameter as well as memories of both returns and volatility are also absent in this model.

Scott (1987) later developed the following option pricing model which allows the variance parameter to change randomly of an independent diffusion process,

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \alpha(m - Y_t)dt + \beta dW_{2t},$$

where α, m and β represent mean reverting parameter, mean of volatility and volatility of volatility of process Y_t respectively. The instantaneous volatility parameter for stock prices is assumed to follow Ornstein-Uhlenbeck process. He also noticed that $\sigma(Y_t) = e^{Y_t}$ and the Wiener processes W_{1t} and W_{2t} were not correlated. This model is able to observe marginal improvement in option pricings accuracy as compared to standard Black-Scholes option pricing (Mitra, 2011) and included mean reverting parameter into account. However, its returns series and volatility have no memory and its two sources of randomness are assumed uncorrelated, which is conflicting with the current literature.

Meanwhile, Hull and White (1987) represented option price by a form of series provided that stochastic volatility is independent of the stock price. They proposed a continuous time diffusion model where $\sigma(Y_t) = \sqrt{Y_t}$ and Y_t obeys *log*-normal process as in the following equations

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \alpha Y_t dt + \beta Y_t dW_{2t},$$

where α and β are mean and volatility of a volatility of process Y_t respectively, with both Wiener processes W_{1t} and W_{2t} are correlated. This model setsthe price of volatility risk to be zero, which is contrary to Heston (1993) who presented a closed–form model with a non–zero price of volatility risk. This model is among the most significant in the literature since it presents closed form solution to European option pricing. Nevertheless, the absence of meanreverting parameter and the nonexistence of both memory of returns and volatility in this model are the flaws.

In the same year, Wiggins (1987) proposed stochastic volatility model under $\sigma(Y_t) = \sqrt{e^{Y_t}}$ and both Wiener processes, i.e W_{1t} and W_{2t} are correlated as given below

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \alpha(m - Y_t)dt + \beta dW_{2t},$$

where α, m and β represent mean reverting parameter, mean of volatility and volatility of volatility of process Y_t, respectively. Although this model has taken into account the mean reverting parameter, it fails including memories in returns and volatility.

In a bid to develop models which can describe the real financial environment better, Stein and Stein (1991) and Schöbel and Zhu (1999) considered stock price distributions that follow diffusion process with a stochastically varying volatility parameter as defined below

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \alpha(m - Y_t)dt + \beta dW_{2t},$$

where α, m and β represent mean reverting parameter, mean of volatility and volatility of volatility of process Y_t, respectively, with assumption of $\sigma(Y_t) = |Y_t|$. The difference between the models of Stein and Stein (1991) and Schöbel and Zhu (1999) is in terms of the correlation between W_{1t} and W_{2t} . It was observed that in the former model, W_{1t} and W_{2t} are not correlated but both parameters are correlated in the latter model. Besides considering mean reverting parameter into account, both models also share the same disadvantages by omitting memory of returns or the memory of volatility.

An attempt to derive a closed-form solution for the pricing of a European call option was made by Heston (1993). In his approach, the deterministic function of volatility is assumed as $\sigma(Y_t) = \sqrt{Y_t}$, provided that Y_t obeys Cox-Ingersoll-Ross (CIR) process as follows

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \theta(\omega - Y_t)dt + \xi\sqrt{Y_t}dW_{2t},$$

where θ, ω and ξ are mean reverting parameter, long variance parameter and volatility of volatility parameter, respectively. The Brownian processes W_{1t} and W_{2t} are correlated. Hestons model stands out from other SV models as it has analytical solution for European options under assumption of correlated Brownian motions. It can also describe the asymmetric smiles by instant correlation between returns series and its volatility. Furthermore, the empirical performance of Hestons model also outperforms other SV models. As a result, this model generates rich mathematical results and enjoys the positivity of the volatility process besides taking into account of mean reverting parameters (Kim and Wee, 2014). However, this model also omits the existence of memories for returns and volatility in which is considered as a drawback of this model.

Hagan et al. (2002) revealed that the market smile dynamics predicted by using local volatility models (i.e. volatility is merely a function of the current asset S_t and of time t) are contrary of observed market behavior. As a treatment of this issue, they proposed an extension of the local volatility model in which the volatility is assumed to be stochastic model and both asset price and volatility are correlated. This extension is called the stochastic alpha-beta-rho (SABR) model as written follows:

$$dS_t = \alpha(Y_t) S_t^{\beta} dW_{1t},$$
$$dY_t = v Y_t dW_{2t},$$

where S_t , $\alpha(Y_t)$ and v are forward value, volatility of forward value and volatility of volatility, respectively. In this case, they assumed $\sigma(Y_t) = |Y_t|$ and Y_t follows a non–mean reverting process. The Wiener processes W_{1t} and W_{2t} are ρ correlated. This is the simplest stochastic volatility model which is homogeneous in S_t and α, which enables to accurately fit the implied volatility curves observed in the marketplace for any single exercise date. This model can also predict the correct dynamics of the implied volatility curves. However, this model also lack of memory in its return or volatility. Furthermore, mean reverting parameter is also not included in this model.

In summary, there are three main advantages that can be highlighted from the existing models. First, the mean reverting parameter is being taken into account in Scott (1987), Wiggins (1987), Stein and Stein (1991), Schobel and Zhu (1999) and Heston (1993). Second, a closed form of solution is established in Heston (1993) and Hull and White (1987)). Finally, the correlation between the error terms existed in Johnson and Shanno (1987), Hull and White (1978), Wiggins (1987), Hagan (2002), Heston (1993) and Schobel and Zhu (1999).

Based on the previous discussion, it can be deduced that each model mentioned previously has at least one of three main drawbacks. First, the existence of zero correlation between stock price and volatility occurs in models proposed by Taylor (1982), Stein and Stein (1987) and Scott (1987). Second, the absence of incorporating mean-reverting parameter into volatility dynamics is observed in works carried out by Taylor (1982), Johnson and Shanno (1987), Hull and White (1978) and Hagan (2002). Third, it is also noted that all previous models did not consider the existence of memory in neither its returns series nor its volatility component.

Previous literature also suggested that the third drawback pose serious concern in modeling financial asset (Willinger et al., 1999) and (Grau-Carles, 2000). This is strongly supported by the empirical investigations which reveal that volatilities and returns of stock prices habitually show long memory property or long-range dependence (Painter, 1998;and Rejichi and Aloui, 2012). In the following discussion, we will review some of the long memory stochastic volatility (LMSV) models.

3. Stochastic Volatility Models Perturbed by Long Memory

As we mentioned before, empirical studies showed that the volatility of many assets has long memory properties. Thus, taking long memory into account of volatility contribute in providing better understanding for financial transaction and then better forecasting of future risky asset's prices.

Bredit et al. (1998) introduced general case of LMSV model as follows

$$y_t = \sigma_t \xi_t,$$
$$\sigma_t = \sigma\, exp\,\{\frac{v_t}{2}\},$$

where y_t is the return at time t, σ_t is stochastic volatility of return, σ is volatility of volatility, v_t a stationary long memory process, ξ_t is independent and identically distribution, and both $\{v_t\}$ and $\{\xi_t\}$ are independent.

In 1998, Harvey proposed the following equivalent model of LMSV:

$$y_t = \sigma_t \xi_t,$$
$$\sigma_t = \sigma^2\, exp\,\{\frac{\eta_t}{(1-L)^d}\},$$

where y_t is the return at time t, σ_t is stochastic volatility of return, σ is volatility of volatility, L is lag operator, ξ_t is independent identically distribution and η_t a stationary long memory process or η_t has normal independent distribution (nid) (i.e. $\eta_t \sim nid(0, \sigma_\eta)$), with $0 < d < 1$.

These two models share similar advantages in incorporating long memory into their volatility parameter, and both models are also simple in their nature. However, they ignore memory of the returns, mean reverting parameter and the correlation between stock price and volatility.

In the same year, Comte and Renault (1998) introduced long-memory mean reverting volatility processes in the setting of continuous time Hull and White model. They modeled the log of volatility as a fractionally integrated Brownian motion (i.e. $\sigma(Y_t) = e^{Y_t}$ in which Y_t follows fractional Ornstein-Uhlenbeck process for $H > 0.5$) as shown below:

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \alpha(m - Y_t)dt + \beta dB_H(t).$$

According to them, not only this model could empirically capture observed strong smile effect for long maturity times, it also incorporated memory in volatility and considered mean reverting into account. However, this model lacks of memory in its return series in addition to lacks of correlation between stock price and volatility.

Subsequently, Comte et al. (2012) extended Hestons model by influencing long memory to its model based on the fractional integration of a square root volatility process. This approach has been approved by Chronopoulou and Viens (2012a; 2012b) as it succeeded in describing volatilities with strong memory in the long run. They also came up with a new LMSV model as follows

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_{1t},$$
$$dY_t = \alpha Y_t dt + \beta dB_H(t).$$

However, this model fails to consider memory in return series, as well as the assumption of zero correlation between stock price and volatility.

In the recent work that follows, Mishura and Swishchuk (2010) studied financial markets with stochastic volatilities driven by fractional Brownian motion with Hurst index $H > 0.5$. Firstly, they assumed that stock price S_t satisfies the following stochastic differential equation

$$dS_t = rS_t dt + \sigma(Y_t) S_t dW_t,$$

where r is an interest rate, $\sigma(Y_t)$ is a volatility, and W_t is a standard Brownian motion. Subsequently, they proposed four SV models in which all models incorporated strong memory into its volatility. First, LMSV model driven by fractional Ornstein–Uhlenbeck process where $\sigma(Y_t) = Y_t$ given by

$$dY_t = -aY_t dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is mean–reverting parameter, $\gamma > 0$ is volatility of volatility and B_H is FBM with Hurst index $H > 0.5$, independent of W_t.
Second, LMSV model driven by continuous–time GARCH process where $\sigma(Y_t) = \sqrt{Y_t}$ is expressed as

$$dY_t = a(b - Y_t)dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is meanreverting parameter, b mean–reverting level, $\gamma > 0$ is volatility of volatility, and B_H is FBM with Hurst index $H > 0.5$, independent of W_t.
Third, LMSV model driven by Vasicek processwhere $\sigma(Y_t) = Y_t$ is given by

$$dY_t = a(b - Y_t)dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is mean–reverting speed, b equilibrium level, $\gamma > 0$ is volatility of volatility, and B_H is FBM with Hurst index $H > 0.5$, independent of W_t.
The setbacks of these three models can be abridged into two points. These models ignore the existence of memory in return series and the assumption of zero correlation between stock price and volatility.
Finally, the fourth model is LMSV model driven by GFBM process where $\sigma(Y_t) = \sqrt{Y_t}$ is written as

$$dY_t = aY_t dt + \gamma Y_t dB_H(t),$$

where $a > 0$ is drift, $\gamma > 0$ is volatility of Y_t , and B_H is FBM with Hurst index $H > 0.5$, independent of W_t . This model also has no memory in its return, its mean reverting parameter does not exist, and zero correlation is assumed between stock price and volatility.

Based from previous discussions, the common disadvantage shared in all LMSV models is that they assumed returns series of the stock price is independent, meaning no memory. This is contradictory to most empirical findings conducted by Painter (1998),Willinger et al. (1999), Grau–Carles (2000), and Rejichi, and Aloui (2012), to name only a few. They also suggested GFBM model should be considered as underlying process for financial variables, due to its ability to incorporate long memory in the system under study.

To recap, there are three stages of evolutions for volatility in GBM model. First, GBM model with assumption of constant volatility.Second, GBM model with assumption of stochastic volatility.Third, GBM model with assumption of stochastic volatility influenced by long memory.

Alhagyan, Misiran and Omar (2016 a, 2016 b, 2017) introduced a model of GFBM under the assumption LMSV where the volatility is considered as a fractional Ornstein–Uhlenbeck process where $\sigma(Y_t) = Y_t$ as follows:

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dB_{H_1}(t),$$
$$dY_t = \alpha(m - Y_t)dt + \beta dB_{H_2}(t),$$

where μ is mean of return, Y_t is a stochastic process, $B_{H_1}(t)$ is a fractional Brownian motion (FBM) with Hurst index H_1. While the parameters α, β and m represent mean reverting of volatility, volatility of volatility and mean of volatility, respectively. $B_{H_2}(t)$ is another FBM which is independent from $B_{H_1}(t)$ where both H_1 and H_2 are greater than 0.5 with assumption that this model exhibits long memory. The main disadvantage of this model is the assumption of zero correlation between stock and volatility.

Table 1 below, summarizes all models of continuous stochastic volatility models mentioned under this study, in addition to their main properties.

Table 1. Continuous Stochastic Volatility Models and their main properties.

Scholar	**Mean reverting**	**Corr. error terms**	**Memory of volatility**	**Memory of returns**
Johnson (1979)		x		
Taylor (1986)				
Scott (1987)	x			

Scholar	Mean reverting	Corr. error terms	Memory of volatility	Memory of returns
Hull and White (1987)		x		
Wiggins (1987)	x	x		
Hagan (2002)		x		
Stein and Stein (1991)	x			
Heston (1993)	x	x		
Schöbel and Zhu (1999)	x	x		
Bredit (1998)			x	
Harvey (1998)			x	
Comte and Renault (1998)	x		x	
Mishura and Swishchuk-1(2010)			x	
Mishura and Swishchuk-2(2010)	x		x	
Mishura and Swishchuk-3(2010)			x	
Mishura and Swishchuk-4(2010)	x		x	
Chronopoulou and Viens (2012)			x	
Alhagyan (2017)	x		x	x

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References

1. Alhagyan, M., Misiran, M., and Omar, Z.: Estimation of Geometric Fractional Brownian Motion Perturbed by Stochastic Volatility Model, *Far East Journal of Mathematical Sciences* **99 (2)** (2016 a) 221–235.
2. Alhagyan, M., Misiran, M., and Omar, Z.: Geometric Fractional Brownian Motion Perturbed by Fractional Ornstein-Uhlenbeck Process and Application on KLCI Option Pricing, *Open Access Library Journal* **3 (08)** (2016 b) 1.
3. Alhagyan, M., Misiran, M., and Omar, Z.: Surveying the Best Volatility Measurements to Forecast Stock Market, *Applied Mathematical Sciences* **11 (23)** (2017) 1113–1122.
4. Breidt, F. J., Crato, N., and De Lima, P.: The detection and estimation of long memory in stochastic volatility, *Journal of Econometrics* **83 (1)** (1998) 325–348.
5. Chronopoulou, A., and Viens, F. G.: Estimation and pricing under long–memory stochastic volatility, *Annals of Finance* **8 (2)** (2012 a) 379–403.
6. Chronopoulou, A., and Viens, F. G.: Stochastic volatility and option pricing with longmemory in discrete and continuous time, *Quantitative Finance* **12 (4)** (2012 b) 635–649.
7. Comte, F. and Renault, E.: Long memory in continuous–time stochastic volatility models, *Mathematical Finance* **8 (4)** (1998) 291–323.
8. Comte, F., Coutin, L., and Renault, .: Affine fractional stochastic volatility models, *Annals of Finance* **8 (2)** (2012) 337–378.
9. Grau–Carles, P.: Empirical evidence of long–range correlations in stock returns, *Physica A: Statistical Mechanics and its Applications* **287 (3)** (2000) 396–404.
10. Hagan, P. S., Kumar, D., Lesniewski, A. S., and Woodward, D. E.: Managing smile risk, *The Best of Wilmott* **1** (2002) 249–296.
11. Harvey, A. C.: Long memory in stochastic volatility. In J. Knight and S. Satchell (Eds.), *Forecasting Volatility in Financial Markets* (1998) 307–320.
12. Heston, S. L.: A closedform solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* **6 (2)** (1993) 327–343.
13. Hull, J. and White, A.: The pricing of options on assets with stochastic volatilities, *Journal of Finance* **42** (1987) 281–300.
14. Johnson, H., and Shanno, D.: Option pricing when the variance is changing, *Journal of Financial and Quantitative Analysis* (1987) 143–151.
15. Kim, B., and Wee, I. S.: .Pricing of geometric Asian options under Hestons stochastic volatility model, *Quantitative Finance* **14 (10)** (2014) 1795–1809.
16. Mishura, Y., and Swishchuk, A.: Modeling and pricing of variance and volatility swaps for stochastic volatilities driven by fractional Brownian motion, *Applied Statistics, Actuarial and Financial Mathematics* (2010) 52–67.
17. Mitra, S.: A review of volatility and option pricing, *International Journal of Financial Markets and Derivatives* **2 (3)** (2011) 149–179.
18. Painter, S.: Numerical method for conditional simulation of Levy random fields, *Mathematical Geology* **30 (2)** (1998) 163–179.
19. Pakdel, M.: *Essays in financial economics*, Doctoral dissertation, Northern Illinois University, 2016.
20. Rejichi, I. Z., and Aloui, C.: Hurst exponent behavior and assessment of the MENA stock markets efficiency, *Research in International Business and Finance* **26 (3)** (2012) 353–370.
21. Schöbel, R., and Zhu, J.: Stochastic volatility with an Ornstein–Uhlenbeck process: an extension, *European Finance Review* **3 (1)** (1999) 23–46.
22. Scott, L. O.: Option pricing when the variance changes randomly: Theory, estimation, and an application, *Journal of Financial and Quantitative analysis* **22 (4)** (1987) 419–438.
23. Stein, E. M. and Stein, J.: Stock price distributions with stochastic volatility: an analytic approach, *Review of Financial Studies* **4** (1991) 727–752.
24. Taylor, S. J.: *Modelling Financial Time Series*, John Wiley and Sons, New Jersy, 1986.

25. Wiggins, J. B.: Option values under stochastic volatility: Theory and empirical estimates, *Journal of Financial Economics* **19 (2)** (1987) 351–372.
26. Willinger, W., Taqqu, M. S., and Teverovsky, V.: Stock market prices and long–range dependence, *Finance and Stochastics* **3 (1)** (1999) 1–13.

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