Regularity and Symmetry Results for Ground State Solutions of Quasilinear Elliptic Equations

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Abstract

This dissertation deals with the boundary value problem

$$\begin{cases} -\operatorname{div}\Big(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\Big) = \lambda f(|x|,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

$\Omega \subset \mathbb{R}^n$ with $n \geq 1$ is a bounded domain with $C^{1,1}$-boundary, $p > 1$, $\lambda > 0$ and $a : (0,\infty) \to (0,\infty)$ is continuous so that $s \mapsto a(s^p)s^{p-1}$ is strictly increasing with $\lim_{s\to 0} a(s^p)s^{p-1} = 0$.

The main results are

- a study of the radially symmetric solutions and their behaviour, including regularity and information about $\operatorname{sgn}(u')$ and $\operatorname{sgn}(u'')$.
- that if $a(s) \geq c > 0$ is monotone decreasing, $u \mapsto \frac{f(|x|,u)}{u^{\tilde{p}-1}}$ is monotone increasing with $\tilde{p} > p$ and satisfies certain growth conditions, then the boundary value problem has a mountain pass solution. Using Schwarz symmetrization it can be shown that the mountain pass solution has to be radially symmetric. Since Schwarz symmetrization applied to paths will usually not result in admissible paths, this is not an obvious result.
- $C^{1,\alpha}$-regularity of u_λ and norm estimates. The main requirements for this are that $\frac{d}{ds}\left(a(s^p)s^{p-1}\right)$ behaves similar to s^{p-2} close to 0 and $f(|x|,u)$ behaves similar to u^{q-1} close to 0, where $q > p$.

Contents

1 Introduction

The origins of this dissertation are certain questions regarding symmetrical solutions of the prescribed mean curvature equation

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(u) \text{ in } \Omega$$

with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ and generalizations of this type of problem.

The simplest equation of a similar type is the Poisson equation $-\Delta u = \lambda f(u)$ and its natural generalization the p-Laplace $-\Delta_p u := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda f(u)$. The Poisson equation is uniformly elliptic, while the p-Laplace is either degenerate or singular elliptic, however it has a very simple structure enabling scaling arguments that are not possible with any other differential equation. The prescribed mean curvature equation is uniformly elliptic at the origin, which enables certain cutoff arguments. It has an added complication that the growth at infinity often requires considering solutions of bounded variation which will be avoided here.

The class of boundary value problems that will be discussed has the form

$$(P) \qquad \begin{cases} -\operatorname{div}\left(a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

$a, f : \Omega \times [0,\infty) \to [0,\infty)$ are Carathéodory functions, $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ is a bounded domain with $C^{1,1}$-boundary, $p > 1$, $\lambda > 0$ and $s \mapsto a(x, s^p)s^{p-1}$ is strictly increasing with

$$\lim_{s \to 0} a(x, s^p)s^{p-1} = 0.$$

1.1 Overview of the Structure of the Dissertation

Chapter 2 gives a summary of the most important concepts, function spaces and several auxiliary results.

Chapter 3 introduces the formal definition of the boundary value problem and the associated functional J, methods to modify a and f that will be used when proving existence of smooth or nonnegative solutions and two standard regularity theorems which will be used in Chapters 5 and 6.

Chapter 4 investigates radially symmetric solutions of the problem and the corresponding radially symmetric formulation. Using an idea from [8] regularity for radially symmetric weak solutions can be shown under much weaker assumptions than is required for solutions that are not radially symmetric. This approach also enables the study of $\operatorname{sgn}(u')$ and $\operatorname{sgn}(u'')$. This is the foundation for Theorem 4.34 which is a generalization of a result in [8] and shows that if u and the Schwarz symmetrization u^* are critical points of J with $J(u) = J(u^*)$ then $u = u^*$. This allows to prove radial symmetry of minimizers without requiring uniqueness of solutions which can be seen in Chapter 5 and radial symmetry of the mountain pass solution in Theorem 6.3.

Chapter 5 uses the direct method in the calculus of variations to prove existence of local and global minimizers. Methods from the previous chapters are used to prove regularity and how symmetry of ground state solutions can be shown using Lopes-symmetrization and Schwarz symmetrization without uniqueness of solutions.

Chapter 6 contains the main results which originate from an attempt to generalize the result of [40] for the prescribed mean curvature equation to problems that are not uniformly elliptic at the origin. The paper uses an iterative approach to find a minimizer in the Nehari manifold which simultaneously proves existence, regularity and a decay estimate for the norms of solutions as $\lambda \to \infty$.

While modifying this approach did not work, the structure of the functional remained the same in the generalized case. This enabled the construction of explicit paths for the mountain pass theorem that remain admissible after pointwise Schwarz symmetrization and using this Theorem 6.3 shows that the mountain pass solution has to be radially symmetric.

Using similar growth conditions and the additional condition

$$c_e s^{p-2} \leq \tfrac{d}{ds}\Big(a(x, s^p) s^{p-1}\Big) \leq C_e s^{p-2},$$

while dropping some of the other restrictions, it was possible to prove similar regularity and decay estimates for equations that are not uniformly elliptic at the origin in Theorem 6.5. The proof is completely different from the original paper and uses a bootstrap argument to show regularity estimates for the mountain pass solution.

Chapter 7 has examples with numerical simulations. Given a specific boundary value problem, it shows the relationship between radially symmetric weak solutions and associated final value problems.

1.2 Literature

The framework mentioned above contains the Laplace, p-Laplace and the prescribed mean curvature equation, to name only the most prominent examples.

The p-Laplace operator is not uniformly elliptic at the origin, however it has a very particular structure which enables many methods that are not viable if said operator is modified in any way. The following will give an overview of the literature for the prescribed mean curvature equation. It is the origin of this dissertation and despite being uniformly elliptic at the origin, the fact that, unlike the p-Laplace, it has a different behaviour at 0 and ∞ makes it very different from both the Laplace and p-Laplace operators.

[41, 39] give a comprehensive list of results for various right-hand sides f for the prescribed mean curvature equation in dimension $n \geq 2$. Several ideas for energy estimates used in Chapters 5 and 6 have been taken from those papers.

If F oscillates at 0 or ∞ the method of sub- and super-solutions can show the existence of infinitely many positive solutions. The sub/supersolution method can be found in [30] and a proof of the existence of infinitely many solutions in the case of oscillations for the prescribed mean curvature equation can be found in [41]. [28] shows the sub/supersolution method in the case $p > 1$ for a more general problem than what is studied here.

Another multiplicity result for the prescribed mean curvature equation without oscillations in the right-hand side can be found in [38].

Minimizing the energy in the Nehari manifold (see [3] or [13]) can prove existence of positive solutions in situations where otherwise problematic conditions occur, such as a critical exponent in the right-hand side in [3] or a right-hand side that is negative at the origin in [13]. This has been developed further in [40, 36] to show existence of mountain pass solutions where the $W^{2,n+1}$-norm becomes small as $\lambda \to \infty$. In this setting, Theorem 6.3 uses the structure of the Nehari manifold presented in those papers to show symmetry of the ground state solution. Theorem 6.5 originated from the attempt to generalize these papers to $p > 2$ but ultimately a different approach had to be taken.

Existence and multiplicity of solutions can also be shown via topological methods. For example, [11] shows a multiplicity result for the prescribed mean curvature equation using Morse theory and Schauder's fixed point theorem. In [27] Ljusternik-Schnirelmann type theory is used to prove existence for certain eigenvalues in the case $f(x, -u) = f(x, u)$. In [37] nonsmooth critical point theory is used to prove the existence of infinitely many solutions.

There are many one-dimensional results, where, among other things, exact multiplicity results can be obtained for specific right-hand sides, such as [7, 6, 23, 32].

[5] deals with nodal solutions in a similar setting to Theorem 6.5.

[21] shows existence and nonexistence using geometrical arguments in the case $f \leq 0$. [42] gives existence and nonexistence results for a specific right-hand side and in [4] the existence of a unique smooth solution is shown.

[43] shows a general way to prove nonexistence of classical solutions for boundary value problems in a very general setting.

[46, 20] prove symmetry of ground state solutions in $\mathbb{R}^n$ in cases where $f \leq 0$.

[48] has a similar result to Theorem 6.3 for non-smooth functionals that have quadratic growth in $|\nabla u|$.

[50] and [33] show that on an annulus there can be many positive nonradial solutions that are not rotationally equivalent.

2 Preliminaries

This chapter gives an introduction to several standard concepts that will be used throughout, such as Carathéodory functions, Nemytskii mappings, Hölder and Sobolev spaces.

It also includes a detailed presentation of spherical coordinates in n dimensions, for which I could not find a source that includes all relevant statements.

2.1 Notation

$B_R(0) = \{x \in \mathbb{R}^n; |x| < R\}$ is the open ball centered around 0.

$\Omega \subset \mathbb{R}^n$ is a bounded domain, which means that Ω is open and connected and there is an $R > 0$ so that $\Omega \subset B_R(0)$. Ω has a $C^{1,1}$-boundary (see Definition 2.14). The case $n = 1$ is included. The **closure** is denoted by $\overline{\Omega}$.

$\mathbb{N} = \{0, 1, 2, \ldots\}$.

$A \subset B$ is used in the sense $A \subseteq B$.

The **Euclidean norm** for a vector $x \in \mathbb{R}^n$ is written as $|x|$.

If $u \in C^1(\Omega)$, then the gradient is denoted by $\nabla u(x) = \begin{pmatrix} \frac{\partial u(x)}{\partial x_1} \\ \dots \\ \frac{\partial u(x)}{\partial x_k} \end{pmatrix}$.

If $u \in W^{1,p}(\Omega)$ then ∇u is the weak derivative.

$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplace operator.

Constants with small letters are used in lower bounds, while constants with capital letters are used in upper bounds. The index indicates if this is a bound for a or F or related to the Ambrosetti-Rabinowitz condition. C_S will be used as a general constant in the Sobolev embeddings $\|u\|_{L^q} \leq C_S\|\nabla u\|_{L^p}$ and $\|u\|_{C^{0,1-\frac{n}{p}}} \leq C_S\|\nabla u\|_{L^p}$. Norms will always be computed on Ω.

Functions used in estimates will be named similar to constants, such as $d_F(x)$ or $D_a(x)$ and they will be used so they can always be assumed to be nonnegative.

The constants $p \in (1, \infty)$ and $\tilde{p}$ will be related to the growth of A and q, $\hat{q}$ and $\tilde{q}$ will be related to the growth of F.

The scaling factor λ of the right-hand side of the differential equation is always a positive real number.

Almost everywhere or **a.e.** will always refer to the Lebesgue measure.

2.2 Carathéodory functions and Nemytskii mappings

The following statements and proofs are taken from [18].

Definition 2.1. *A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a **Carathéodory** function if*

- ∘ *$x \mapsto f(x,u)$ is measurable for every $u \in \mathbb{R}$,*
- ∘ *$u \mapsto f(x,u)$ is continuous for almost every $x \in \Omega$.*

*A Carathéodory function f is called an **L^p-Carathéodory** function for $p \in [1,\infty]$, if for every $d > 0$ there is a function $g(x) \in L^p(\Omega)$ such that*

$$\forall x \in \Omega \ \forall u \in [-d,d]: \ |f(x,u)| \leq g(x).$$

Theorem 2.2. *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function the composition $x \mapsto f(x,u(x))$ is Lebesgue-measurable for every Lebesgue-measurable function $u : \mathbb{R} \mapsto \mathbb{R}$.*

Proof. See Theorem 2.1 in [18]. □

Definition 2.3. *Let $\mathcal{M}$ be the space of measurable functions on Ω. By the previous theorem for a given Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ the function $u \mapsto N_f(u) := f(x,u(x))$ defines a mapping $N_f : \mathcal{M} \to \mathcal{M}$. The operator N_f is called a **Nemytskii mapping** if f is a Carathéodory function.*

Theorem 2.4. *Suppose that f is a Carathéodory function and there is a constant $c > 0$, a function $b(x) \in L^q(\Omega)$ with $q \in [1,\infty]$ and $r > 0$ such that*

$$|f(x,s)| \leq c|s|^r + b(x), \quad \forall x \in \Omega, \quad \forall s \in \mathbb{R}.$$

Then N_f maps L^{qr} into L^q and it is continuous and maps bounded sets into bounded sets.

Proof. See Theorem 2.3 in [18]. □

Theorem 2.5. *Suppose the Nemytskii mapping N_f maps $L^p(\Omega)$ into $L^q(\Omega)$ for $1 \leq p < \infty$, $1 \leq q < \infty$. Then there is a constant $c > 0$ and $b(x) \in L^q(\Omega)$ such that*

$$|f(x,s)| \leq c|s|^{\frac{p}{q}} + b(x).$$

Proof. See Theorem 2.4 in [18]. □

Theorem 2.6. *Suppose the Nemytskii mapping N_f maps $L^p(\Omega)$ into $L^q(\Omega)$ for $1 \leq p < \infty$, $1 \leq q < \infty$. Then N_f is continuous and maps bounded sets into bounded sets.*

Proof. See Theorem 2.5 in [18]. □

Definition 2.7. *If there is a bounded linear functional $A : X \to \mathbb{R}$ so that*

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{y \in X \setminus \{0\} \\ \|y\| < \varepsilon}} \frac{|J(x+y) - J(x) - Ay|}{\|y\|} = 0,$$

then J is ***Fréchet differentiable*** *at x and $J'(x) := A$ is the* ***Fréchet derivative****.*

Theorem 2.8. *Assume $f(x,s)$ and the partial derivative $f_s(x,s)$ are Carathéodory functions on $\Omega \times \mathbb{R}$. If*

$$|f_s(x,s)| \leq c|s|^m + b(x), \quad \forall s \in \mathbb{R} \quad \forall x \in \Omega$$

where $b(x) \in L^k(\Omega)$, $1 \leq k \leq \infty$, $m > 0$, with $p = nm$, $q = mn/(m+1)$, then $N_f : L^p \to L^q$ and it is continuously Fréchet differentiable with

$$N_f' : L^p \to \mathscr{L}(L^p, L^q)$$

defined by

$$N_f'(u)(v) = N_{f_s}(u)(v), \quad \forall u, v \in L^p.$$

Proof. See Theorem 2.6 in [18]. □

Theorem 2.9. *Let $f(x,s)$ be a Carathéodory function, $F(x,s) = \int_0^t f(x,s)ds$, $p > 1$ and*

$$|f(x,s)| \leq c|s|^{p-1} + b(x), \quad b(x) \in L^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Then $N_f : L^p \to L^{p'}$, $N_F : L^p \to L^1$ and

$$\Phi(u) = \int_\Omega F(x, u(x))dx$$

defines a continuous functional $\Phi : L^p(\Omega) \to \mathbb{R}$ which is continuously Fréchet differentiable.

Proof. See Theorem 2.8 in [18]. □

2.3 Function spaces

2.3.1 Hölder spaces

Definition 2.10. *Let $\alpha \in \mathbb{N}^n$ be a multi-index with $|\alpha| = \sum_{i=1}^n \alpha_i$ and*

$$\partial_\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} u.$$

- *For $\gamma \in [0,1]$ the **Hölder seminorm** is defined as*

$$[u]_\gamma = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\gamma}.$$

- *For $k \in \mathbb{N}$ and $\gamma \in (0,1)$ the* ***Hölder space*** $C^{k,\gamma}(\overline{\Omega})$ *is defined as*

$$C^{k,\gamma}(\overline{\Omega}) = \left\{ u \in C^k(\overline{\Omega})\,;\, \forall\ \alpha \in \mathbb{N}^n, |n| = k : [\partial_\alpha u]_\gamma < \infty \right\}.$$

Lemma 2.11. $C^{k,\gamma}(\overline{\Omega})$ *is a Banach space with the norm*

$$\|u\|_{C^{k,\gamma}} = \|u\|_{C^k} + \sum_{|\alpha|=k} [\partial_\alpha u]_\gamma.$$

Remark 2.12. *In the case $\gamma = 0$ the space $C^{k,\gamma}(\overline{\Omega})$ is the space $C^k(\overline{\Omega})$ and in the case $\gamma = 1$ it is the space of Lipschitz continuous functions on $\overline{\Omega}$.*

Lemma 2.13. *For any $k, l \in \mathbb{N}$ with $k \le l$ and any $0 < \alpha < \beta \le 1$ the space $C^{l,\beta}(\overline{\Omega})$ is compactly embedded in $C^{k,\alpha}(\overline{\Omega})$, written as $C^{l,\beta}(\overline{\Omega}) \subset\subset C^{k,\alpha}(\overline{\Omega})$.*

See 8.6 in [2] for a proof of this lemma.

Definition 2.14. *Ω has a $C^{k,\alpha}$-boundary, with $\alpha \in (0,1]$, $k \in \mathbb{N}$, if for every $b \in \partial\Omega$ there is a neighbourhood B so that, after suitable coordinate transformation, $\partial\Omega \cap B$ is the graph of a $C^{k,\alpha}$-function and $\overline{\Omega} \cap B$ is the intersection of B with the epigraph of that function.*

2.3.2 Sobolev Spaces

This section references several standard concepts and results on Sobolev Spaces. A general overview can be found in [16, 31].

Definition 2.15. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}^n$ functions so that*

$$\int_\Omega u(x) \nabla \varphi(x) dx = - \int_\Omega v(x) \varphi(x) dx$$

for every $\varphi \in C_0^\infty(\Omega)$. Then v is called the ***weak derivative*** *of u on Ω with the notation $\nabla u := v$.*

For $p \in [1, \infty]$ the ***Sobolev space*** *$W^{1,p}(\Omega)$ is defined as*

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) \ ; \ u \text{ has a weak derivative in } L^p \}$$

with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

Theorem 2.16. *$W^{1,p}(\Omega)$ is a Banach space for any $p \in [1, \infty]$.*

$W^{1,p}(\Omega)$ is separable if $p < \infty$ and reflexive if $1 < p < \infty$.

Proof. See Theorem 2 in Chapter 5.2 of [16]. □

2.3.3 Absolute continuity on lines

Definition 2.17. *Let $a < b$, then a function $u : [a, b] \to \mathbb{R}$ is* ***absolutely continuous*** *on $[a, b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=1}^{l} |u(b_k) - u(a_k)| \leq \varepsilon$$

for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, l$ with $[a_k, b_k] \subset [a, b]$ and

$$\sum_{k=1}^{l} (b_k - a_k) \leq \delta.$$

The space of all absolutely continuous functions $u : I \to \mathbb{R}$ is denoted by $AC(I)$.

Theorem 2.18. *A function $u \in L^p(\Omega)$ belongs to $W^{1,p}(\Omega)$ if and only if it has a representative $\overline{u}$ that is absolutely continuous on $\mathscr{L}^{n-1}$ almost every line segments of Ω that are parallel to the coordinate axes and whose first-order (classical) partial derivatives belong to $L^p(\Omega)$. Moreover the (classical) partial derivatives of $\overline{u}$ agree $\mathscr{L}^n$-a.e. with the weak derivatives of u.*

Proof. See Theorem 10.35 in [31]. □

Lemma 2.19 (Stampacchia)**.** *Let $G : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz-continuous function with $G(0) = 0$, then for any $u \in W_0^{1,p}(\Omega)$ the function $G(u)$ is in $W_0^{1,p}(\Omega)$ and*

$$\nabla \left(G(u)\right) = G'(u)\nabla u.$$

Proof. This is Lemma 1.1 in [49]. □

Corollary 2.20. *For $u \in W_0^{1,p}(\Omega)$ the functions*

- $x \mapsto \min\{u(x), 0\}$, *and*
- $x \mapsto \max\{u(x), 0\}$

are also in $W_0^{1,p}(\Omega)$.

Corollary 2.21. *For any function $u \in W^{1,p}(\mathbb{R}^n)$ the function*

$$u(x_1, \ldots, x_{i-1}, |x_i|, x_{i+1}, \ldots, x_k)$$

is also in $W^{1,p}(\mathbb{R}^n)$.

Proof. This follows immediately from the absolute continuity on lines of Sobolev functions because the function obviously remains absolutely continuous in x_i and the L^p-norm and the L^p-norm of the derivative can at most increase by a factor of 2. □

Corollary 2.22. *For $u \in W_0^{1,p}(\Omega)$ the function $|u|$ is also in $W_0^{1,p}(\Omega)$ with $|\nabla |u|| \leq |\nabla u|$.*

Proof. This follows again from the characterization of Sobolev functions via absolute continuity on lines. □

Definition 2.23. *The space $W_0^{1,p}(\Omega)$ is defined as*

$$W_0^{1,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p}(\Omega)}$$

with the norm

$$\|u\|_{W_0^{1,p}} := \|\nabla u\|_{L^p}.$$

By Theorem 2.26 this norm is equivalent to the norm $\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$ on $W_0^{1,p}(\Omega)$.

Lemma 2.24. *$W_0^{1,p}$ is separable if $p \in [1, \infty)$ and weakly closed if $p \in (1, \infty)$. Weakly closed means that bounded sequence have weakly converging subsequences.*

Proof. As a subspace of $W^{1,p}$ it is also separable if $p \in [1, \infty)$. If $p \in (1, \infty)$ it is also a standard result that it is weakly closed and this can be shown in different ways, for example using traces (see Chapter 5.5 in [16]), or using that $W_0^{1,p}$ is a convex subset of $W^{1,p}$ and since it is closed in the norm topology it is also weakly closed. □

2.3.4 Sobolev inequalities

Theorem 2.25 (Gagliardo-Nirenberg-Sobolev inequality). *For $\Omega \subset \mathbb{R}^n$ open and bounded with C^1 boundary, $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$ the function u is in $L^{p^*}(\Omega)$ where $p^* = \frac{np}{n-p}$ with*

$$\|u\|_{L^{p^*}} \leq C_S \|u\|_{W^{1,p}}$$

where C_S depends only on p, n and Ω.

Proof. See Theorem 2 in Chapter 5.6 in [16]. □

Theorem 2.26 (Poincaré inequality). *For $\Omega \subset \mathbb{R}^n$ open and bounded, $1 \leq p < n$ and $q \in \left[1, \frac{np}{n-p}\right]$ there is a $C_S > 0$ so that*

$$\|u\|_{L^q} \leq C_S \|\nabla u\|_{L^p}$$

for any $u \in W_0^{1,p}(\Omega)$. C_S depends only on p, q, n, and Ω.

Proof. See Theorem 3 in Chapter 5.6 in [16]. □

Theorem 2.27 (Morrey's inequality). *If $n < p \leq \infty$, Ω is open and bounded with C^1 boundary, there exists a constant C_S which depends only on p, n and Ω so that for any $u \in W^{1,p}(\Omega)$ there is a representative $u^* \in C^{0,1-\frac{n}{p}}$ of u so that*

$$\|u\|_{C^{0,1-\frac{n}{p}}} \leq C \|u\|_{W^{1,p}}.$$

The constant C only depends on p, n and Ω.

Proof. See Theorem 5 in Chapter 5.6 in [16]. □

Theorem 2.28 (Rellich-Kondrachov). *If $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary and $1 \leq p < n$ then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $1 \leq q < \frac{np}{n-p}$.*

Proof. See Theorem 1 in Chapter 5.7 in [16]. □

2.4 Convex functions

The following lemma is also a standard result that is included here for completeness, as the statements will be used to prove the assumptions in Theorem 3.39.

Lemma 2.29. *For any twice continuously differentiable function $\alpha : \mathbb{R}^n \to \mathbb{R}$ the following statements are equivalent:*

$$\xi^T D^2\alpha(x)\xi \geq 0 \text{ for any } x, \xi \in \mathbb{R}^n \tag{2.1}$$

$$\alpha(y) \geq \alpha(x) + \nabla\alpha(x)(y - x) \text{ for any } x, y \in \mathbb{R}^n \tag{2.2}$$

$$(\nabla\alpha(x) - \nabla\alpha(y))(x - y) \geq 0 \text{ for any } x, y \in \mathbb{R}^n \tag{2.3}$$

$$\alpha(tx + (1 - t)y) \leq t\alpha(x) + (1 - t)\alpha(y) \tag{2.4}$$

for any $t \in [0, 1]$ and $x, y \in \mathbb{R}^n$.

Proof. Let $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$ be arbitrary.

(2.2) $\implies$ (2.1): If (2.1) is wrong, there are $w, \xi \in \mathbb{R}^n$ with $\xi^T D^2\alpha(w)\xi < 0$. By continuity there has to be an $\varepsilon > 0$ so that for any $\tilde{w} \in B_\varepsilon(w)$ the inequality $\xi^T D^2\alpha(\tilde{w})\xi < 0$ still holds. By scaling it can be assumed that $\|\xi\| < \frac{\varepsilon}{2}$.

Using Taylor approximation theorem with the Lagrange version of the remainder, for every $x, y \in \mathbb{R}^n$ there is a $z \in \{sx + (1-s)y\,;\, s \in [0,1]\}$ so that

$$\alpha(y) = \alpha(x) + \nabla\alpha(x)(y-x) + \tfrac{1}{2}(y-x)^T D^2\alpha(z)(y-x). \tag{2.5}$$

For $x = w$ and $y = w + \xi$ this becomes

$$\alpha(w+\xi) = \alpha(w) + \nabla\alpha(w)\xi + \tfrac{1}{2}\xi^T D^2\alpha(z)\xi$$

with

$$z \in \{s(w+\xi) + (1-s)w\,;\, s \in [0,1]\} = \{w + s\xi\,;\, s \in [0,1]\}$$

which is a subset of $B_\varepsilon(w)$. Thus $\frac{1}{2}\xi^T D^2\alpha(z)\xi < 0$ and

$$\alpha(y) = \alpha(w+\xi) > \alpha(w) + \nabla\alpha(w)\xi = \alpha(x) + \nabla\alpha(x)(y-x)$$

which contradicts (2.2).

(2.1) $\implies$ (2.2): Using (2.1) in (2.5) shows (2.2).

(2.2) $\implies$ (2.3): By (2.2) both

$$\begin{aligned} \alpha(y) &\geq \alpha(x) + \nabla\alpha(x)(y-x) \text{ and} \\ \alpha(x) &\geq \alpha(y) + \nabla\alpha(y)(x-y) \end{aligned}$$

hold for all $x, y \in \mathbb{R}^n$ and adding them shows

$$0 \geq (\nabla\alpha(x) - \nabla\alpha(y))(y-x)$$

which proves (2.3).

(2.3) $\implies$ (2.4): Define $\beta(t) = \alpha(tx + (1-t)y)$, then

$$\beta'(t) = \nabla\alpha(tx + (1-t)y)(x-y).$$

With $(x-y) = \frac{1}{t-s}\left(tx + (1-t)y - (sx + (1-s)y)\right)$ (2.3) can be used to show that

$$\begin{aligned}
&\beta'(t) - \beta'(s) \\
&\qquad = \nabla\alpha(tx + (1-t)y)(x-y) \\
&\qquad\qquad - \nabla\alpha(sx + (1-s)y)(x-y)
\end{aligned}$$

is nonnegative which shows that β' is nondecreasing and $\beta : \mathbb{R} \to \mathbb{R}$ is convex. Using the knowledge about convex functions in one dimension the inequality

$$\begin{aligned}
\alpha(x + t(y-x)) &= \beta(t \cdot 1 + (1-t) \cdot 0) \\
&\leq t\beta(1) + (1-t)\phi(0) \\
&= t\alpha(x) + (1-t)\alpha(y)
\end{aligned}$$

follows which proves (2.4).

(2.4) $\implies$ (2.2): Define $\gamma(t) = (1-t)\alpha(x) + t\alpha(y) - \alpha((1-t)x + ty)$. Then $\gamma(0) = 0$ and by assumption $\gamma(t) \geq 0$ on $[0,1]$. Thus $\gamma'(0) \geq 0$ which is equivalent to

$$\alpha(y) - \alpha(x) - \nabla\alpha(x)(y-x) \geq 0.$$

Thus with the following relationship between the statements it can be seen that they are all equal:

$$\begin{tikzcd}
(2.2) \arrow[r, Leftrightarrow] & (2.3) \arrow[r, Rightarrow] & (2.4) \arrow[r, Rightarrow] & (2.5) \arrow[ll, Rightarrow, bend left]
\end{tikzcd}$$
□

2.5 Spherical coordinates in n dimensions

While the spherical coordinates in n dimensions are of course well known, unfortunately I could not find a source where all necessary statements are proved.

Definition 2.30. *Let $n \geq 2$ and $X(r, \phi) : (0, \infty) \times A \to \mathbb{R}^n$ be the function which maps spherical coordinates onto cartesian coordinates with*

$$A := \{0\}^n \cup \big((0, \infty) \times (A_1 \cup A_2)\big) \subset \mathbb{R}^n$$

where

$$A_1 := (0, \pi)^{n-2} \times [0, 2\pi),$$
$$A_2 := \bigcup_{i=1}^{n-2} \Big((0, \pi)^{i-1} \times \{0, \pi\} \times \{0\}^{n-i-1}\Big).$$

X can then be defined as

$$X_j(r, \phi) = r \cos(\phi_j) \prod_{i=1}^{j-1} \sin(\phi_i) \qquad \textit{for } j = 1, \ldots, n-1 \qquad \textit{and}$$
$$X_n(r, \phi) = r \prod_{i=1}^{n-1} \sin(\phi_i).$$

Remark 2.31. *The set A is almost everywhere equal to the set $(0, \infty) \times A_1$ with respect to the n-dimensional Lebesgue measure and $(0, \infty) \times (\{0\}^{n-1} \cup A_2)$ can therefore be ignored when integrating.*

Some trigonometric identities are necessary for the proofs.

Lemma 2.32. *For arbitrary* $\phi_1, \dots, \phi_{n-1} \in \mathbb{R}^n$ *and* $k, n \in \mathbb{N}$ *with* $n > k$

$$\left(\sum_{j=k}^{n-1} \cos^2(\phi_j) \left(\prod_{i=1}^{j-1} \sin^2(\phi_i)\right)\right) + \prod_{i=1}^{n-1} \sin^2(\phi_i) = \prod_{i=1}^{k-1} \sin^2(\phi_i).$$

Proof. Let $k \in \mathbb{N}$ and $n = k+1$. Then

$$\begin{aligned}
&\left(\sum_{j=k}^{k} \cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i)\right) + \prod_{i=1}^{k} \sin^2(\phi_i) \\
&\qquad = \cos^2(\phi_k) \prod_{i=1}^{k-1} \sin^2(\phi_i) + \prod_{i=1}^{k} \sin^2(\phi_i) \\
&\qquad = \prod_{i=1}^{k-1} \sin^2(\phi_i).
\end{aligned}$$

Assume now that the statement holds for a specific n with $n > k$. Then

$$\begin{aligned}
&\left(\sum_{j=k}^{n} \cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i)\right) + \prod_{i=1}^{n} \sin^2(\phi_i) \\
&\quad = \left(\sum_{j=k}^{n-1} \cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i)\right) + \cos^2(\phi_k) \prod_{i=1}^{n-1} \sin^2(\phi_i) \\
&\qquad\qquad + \prod_{i=1}^{n} \sin^2(\phi_i) \\
&\quad = \left(\sum_{j=k}^{n-1} \cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i)\right) + \prod_{i=1}^{n-1} \sin^2(\phi_i) \\
&\quad = \prod_{i=1}^{k-1} \sin^2(\phi_i)
\end{aligned}$$

and the statement holds for $n+1$. By induction it holds for all $n > k$. $\square$

Lemma 2.33. *$X(r, \phi)$ is injective on A.*

Proof. Assume $X(r, \phi) = X(\tilde{r}, \tilde{\phi})$.

By Lemma 2.32 $X_n^2 + \cdots + X_1^2 = r^2 = \tilde{r}^2$ and since $r, \tilde{r} \geq 0$ this implies $r = \tilde{r}$. If $r = 0$ then $\phi = \tilde{\phi} = 0$, so let $r > 0$ and $(r, \phi), (r, \tilde{\phi}) \in (0, \infty) \times (A_1 \bigcup A_2)$.

The function cos is injective on $[0, \pi]$ thus $r\cos(\phi_1) = X_1(r, \phi) = X_1(r, \tilde{\phi}) = r\cos(\tilde{\phi}_1)$ implies $\phi_1 = \tilde{\phi}_1$.

Let now $\phi_i = \tilde{\phi}_i$ for every i with $i < j < n-1$. Then $X_j(r, \phi) = X_j(r, \tilde{\phi})$ implies

$$r\cos(\phi_j)\prod_{i=1}^{j-1}\sin(\phi_i) = r\cos(\tilde{\phi}_j)\prod_{i=1}^{j-1}\sin(\phi_i).$$

If $\prod_{i=1}^{j-1}\sin(\phi_i) = 0$ then $(r, \phi), (r, \tilde{\phi}) \in (0, \infty) \times A_2$ and $\phi_j = \tilde{\phi}_j = \cdots = \phi_{n-1} = \tilde{\phi}_{n-1} = 0$. If it is $\neq 0$ then $\phi_j = \tilde{\phi}_j$ follows again from the injectivity of cos on $[0, \pi]$. By induction $\phi_i = \tilde{\phi}_i$ follows for any $i < n-1$.

$X_{n-1}(r, \phi) = X_{n-1}(r, \tilde{\phi})$ is then equivalent to

$$r\cos(\phi_{n-1})\prod_{i=1}^{n-1}\sin(\phi_i) = r\cos(\tilde{\phi}_{n-1})\prod_{i=1}^{n-1}\sin(\phi_i)$$

which shows $\cos(\phi_{n-1}) = \cos(\tilde{\phi}_{n-1})$ and thus $\phi_{n-1} = \tilde{\phi}_{n-1}$ or $\phi_{n-1} = 2\pi - \tilde{\phi}_{n-1}$. Once again $\prod_{i=1}^{n-1}\sin(\phi_i) = 0$ would imply that $(r, \phi), (r, \tilde{\phi}) \in (0, \infty) \times A_2$ and $\phi_{n-1} = \tilde{\phi}_{n-1} = 0$.

$X_n(r,\phi) = X_n(r,\tilde{\phi})$ is then equivalent to

$$r\sin(\phi_{n-1})\prod_{i=1}^{n-2}\sin(\phi_i) = r\sin(\tilde{\phi}_{n-1})\prod_{i=1}^{n-2}\sin(\phi_i)$$

which implies $\sin(\phi_{n-1}) = \sin(\tilde{\phi}_{n-1})$. The identity

$$\sin(2\pi - \tilde{\phi}_{n-1}) = -\sin(\tilde{\phi}_{n-1})$$

rules out the case $\phi_{n-1} = 2\pi - \tilde{\phi}_{n-1}$ which implies $\phi_{n-1} = \tilde{\phi}_{n-1}$ and therefore X has to be injective. $\square$

Definition 2.34. *Let the function*

$$S : \mathbb{R}^n \to \mathbb{R}^n, \qquad S(x) := (r,\phi), \qquad (r,\phi) \in [0,\infty) \times \mathbb{R}^{n-1}$$

be defined by $r := \sqrt{x_1^2 + \cdots + x_n^2}$,

$$\phi_i := \begin{cases} \arccos\left(\frac{x_i}{\sqrt{x_i^2+\cdots+x_n^2}}\right) & \text{if } x_i^2 + \cdots + x_n^2 \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for $i < n-1$ *and*

$$\phi_{n-1} := \begin{cases} \arccos\left(\frac{x_{n-1}}{\sqrt{x_n^2+x_{n-1}^2}}\right) & \text{if } x_n^2 + x_{n-1}^2 > 0,\ x_n \geq 0, \\ 2\pi - \arccos\left(\frac{x_{n-1}}{\sqrt{x_n^2+x_{n-1}^2}}\right) & \text{if } x_n^2 + x_{n-1}^2 > 0,\ x_n < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.35. $S(\mathbb{R}^n) \subset A$.

Proof. By definition $S(0) = 0$ so it is sufficient to consider $x \neq 0$.

If $x_n^2 + \cdots + x_i^2 = 0$ for a certain $i < n$ let i be the smallest number with that property. $x \neq 0$ implies $i > 1$ and $x_{i-1} \neq 0$. Then $\phi_{i-1} = \arccos\left(\frac{x_{i-1}}{|x_{i-1}|}\right) \in \{0, \pi\}$ and $\phi_i = \cdots = \phi_{n-1} = 0$ by definition.

For $j < i-1$ it follows that $x_j^2 < x_n^2 + \cdots + x_{i-1} + \cdots + x_j^2$ and thus $\frac{x_j}{x_n^2+\cdots+x_j^2} \in (-1,1)$ and $\arccos\left(\frac{x_j}{x_n^2+\cdots+x_j^2}\right) \in (0,\pi)$ which shows $S(x) \in ((0,\infty) \times A_2) \subset A$.

If $x_n^2 + x_{n-1}^2 > 0$ then $\frac{x_i}{\sqrt{x_n^2+\cdots+x_i^2}} \neq \pm 1$ for every $i < n-1$ and therefore $\phi_1, \ldots, \phi_{n-2} \in (0,\pi)$ and by definition $\phi_{n-1} \in [0, 2\pi)$ which proves that $\phi \in A_1$.

Therefore $S(\mathbb{R}^n) \subset A$. $\square$

Lemma 2.36. *$X(S(x)) = x$ for every $x \in \mathbb{R}^n$.*

Proof. Let $x \in \mathbb{R}^n$ and $(r, \phi) := S(x)$.

$x = 0$ implies $(r,\phi) = (0,0)$ and $X((0,0)) = 0 = x$, so assume $x \neq 0$.

If $x_n^2 + \cdots + x_i^2 = 0$ for $i < n$ let i be the smallest number with that property. With $x \neq 0$ this implies $i > 1$ and $x_{i-1} \neq 0$. By definition $\phi_i = \cdots = \phi_{n-1} = 0$ and $\phi_{i-1} \in \{0, \pi\}$. This implies

$$X_n(r,\phi) = r \prod_{i=1}^{n-1} \sin(\phi_i) = 0 \qquad \text{and}$$

$$X_j(r,\phi) = r\cos(\phi_j) \prod_{i=1}^{j-1} \sin(\phi_i) = 0$$

for any $j < n$ with $j \geq i$.

Consider now the case $x_n^2 + \cdots + x_j^2 \neq 0$ with $j < n$. Since arccos maps $[-1, 1]$ onto $[0, \pi]$ and sin is nonnegative on $[0, \pi]$ the identity

$$\begin{aligned}\sin(\phi_j) &= \sin\left(\arccos\left(\tfrac{x_j}{\sqrt{x_j^2+\cdots+x_n^2}}\right)\right) \\ &= \sqrt{1 - \tfrac{x_j^2}{x_j^2+\cdots+x_n^2}} \\ &= \frac{\sqrt{x_n^2 + \cdots + x_{j+1}^2}}{\sqrt{x_n^2 + \cdots + x_j^2}}\end{aligned}$$

holds. Observing that

$$\prod_{i=1}^{j-1} \sin(\phi_i) = \frac{\sqrt{x_n^2 + \cdots + x_j^2}}{\sqrt{x_n^2 + \cdots + x_1^2}}$$

it follows that

$$X_j(r, \phi) = r \cos(\phi_j) \prod_{i=1}^{j-1} \sin(\phi_i) = r \tfrac{x_j}{\sqrt{x_n^2+\cdots+x_j^2}} \tfrac{\sqrt{x_n^2+\cdots+x_j^2}}{\sqrt{x_n^2+\cdots+x_1^2}} = x_j.$$

In the case $x_n^2 + x_{n-1}^2 \neq 0$ the identity $\sin(2\pi - y) = -\sin(y)$ implies

$$\sin(\phi_{n-1}) = \operatorname{sgn}(x_n) \frac{\sqrt{x_n^2}}{\sqrt{x_n^2 + x_{n-1}^2}}.$$

This implies

$$X_n(r, \phi) = r \prod_{i=1}^{n-1} \sin(\phi_i) = \operatorname{sgn}(x_n) \sqrt{x_n^2} = x_n$$

which proves $X(r, \phi) = X(S(x)) = x$. □

Theorem 2.37. *The function* $X : A \to \mathbb{R}^n$ *is bijective and the restriction* $X : (0,\infty) \times A_1 \to X((0,\infty) \times A_1)$ *is a* C^∞*-diffeomorphism where* $\mathbb{R}^n \setminus X((0,\infty) \times A_1)$ *has Lebesgue measure* 0*.*

Proof. The function $X : A \to \mathbb{R}^n$ is obviously in $C^\infty(A)$, the inverse function S is in $C^\infty(X((0,\infty) \times A_1))$.

It remains to show that $X(\{0\}^n \cup (0,\infty) \times A_2)$ has Lebesgue measure 0. If $\phi \in A_2$ then $\phi_{n-1} = 0$ which implies $X_k(r,\phi) \in \mathbb{R}^{n-1} \times \{0\}$ which shows that $X_k((0,\infty) \times A_2) \subset \mathbb{R}^{n-1} \times \{0\}$ and by monotonicity is a set of Lebesgue measure 0. This concludes the proof. □

Remark 2.38. *For any* $(r,\phi) \in (0,\infty) \times A$ *the following statements hold:*

- $\partial_{\phi_k} X_j(r,\phi) = 0 \qquad \text{if } k > j,$
- $\partial_{\phi_k} X_k(r,\phi) = -r \prod_{i=1}^{k} \sin(\phi_i) \qquad \text{if } k < n,$
- $\partial_{\phi_k} X_j(r,\phi) = r \cos(\phi_j) \cos(\phi_k) \prod_{\substack{i=1 \\ i \neq k}}^{j-1} \sin(\phi_i) \qquad \text{if } k < j < n,$
- $\partial_{\phi_k} X_n(r,\phi) = r \cos(\phi_k) \prod_{\substack{i=1 \\ i \neq k}}^{n-1} \sin(\phi_i) \qquad \text{if } k < n.$

Lemma 2.39. *For any* $(r, \phi) \in A$ *and* $k, l \in \{1, \dots, n\}$

$$\partial_r X \cdot \partial_r X = 1, \tag{2.6}$$

$$\partial_r X \cdot \partial_{\phi_k} X = 0, \tag{2.7}$$

$$\partial_{\phi_k} X \cdot \partial_{\phi_k} X = r^2 \prod_{i=1}^{k-1} \sin^2(\phi_i), \tag{2.8}$$

$$\partial_{\phi_k} X \cdot \partial_{\phi_l} X = 0 \quad \text{if } k \neq l, \tag{2.9}$$

which shows that $\partial_r X$ *and* $\partial_{\phi_k} X$ *form an orthogonal basis of* $\mathbb{R}^n$.

Proof. ○ Starting with Eq. (2.6)

$$\partial_r X \cdot \partial_r X = \left(\sum_{j=1}^{n-1} \left(\cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i) \right) \right) + \prod_{i=1}^{n-1} \sin^2(\phi_i) = 1$$

by Lemma 2.32.

○ For Eq. (2.7)

$$\partial_r X \cdot \partial_{\phi_k} X = \partial_r X_k \partial_{\phi_k} X_k + \sum_{j=k+1}^{n-1} (\partial_r X_j \partial_{\phi_k} X_j) + \partial_r X_n \partial_{\phi_k} X_n$$

and if $\sin(\phi_k) \neq 0$, then

$$\begin{aligned} &\sum_{j=k+1}^{n-1} (\partial_r X_j \partial_{\phi_k} X_j) + \partial_r X_n \partial_{\phi_k} X_n \\ &= r \frac{\cos(\phi_k)}{\sin(\phi_k)} \left[\left(\sum_{j=k+1}^{n-1} \cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i) \right) + \prod_{i=1}^{n-1} \sin^2(\phi_i) \right]. \end{aligned}$$

Lemma 2.32 shows that this is equal to $r \frac{\cos(\phi_k)}{\sin(\phi_k)} \prod_{i=1}^{k} \sin^2(\phi_i)$ and with

$$\partial_r X_k \partial_{\phi_k} X_k = r \frac{\cos(\phi_k)}{\sin(\phi_k)} \left[- \prod_{i=1}^{k} \sin^2(\phi_i) \right]$$

it follows that Eq. (2.7) is 0. By continuity this holds for $\sin(\phi_k) = 0$ as well.

- Since

$$\partial_{\phi_k} X \cdot \partial_{\phi_k} X = \partial_{\phi_k} X_k \partial_{\phi_k} X_k + \sum_{j=k+1}^{n-1} (\partial_{\phi_k} X_j \partial_{\phi_k} X_j) + \partial_{\phi_k} X_n \partial_{\phi_k} X_n$$

it follows that

$$\begin{aligned}
&\sum_{j=k+1}^{n-1} (\partial_{\phi_k} X_j \partial_{\phi_k} X_j) + \partial_{\phi_k} X_n \partial_{\phi_k} X_n \\
&\quad = r^2 \frac{\cos^2(\phi_k)}{\sin^2(\phi_k)} \left[\sum_{j=k+1}^{n-1} \left(\cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i) \right) + \prod_{i=1}^{n-1} \sin^2(\phi_i) \right]
\end{aligned}$$

if $\sin(\phi_k) \neq 0$. Using Lemma 2.32 this is equal to

$$r^2 \frac{\cos^2(\phi_k)}{\sin^2(\phi_k)} \prod_{i=1}^{k} \sin^2(\phi_i)$$

and therefore

$$\begin{aligned}
\partial_{\phi_k} X \cdot \partial_{\phi_k} X &= r^2 \prod_{i=1}^{k} \sin^2(\phi_i) + r^2 \frac{\cos^2(\phi_k)}{\sin^2(\phi_k)} \left[\prod_{i=1}^{k} \sin^2(\phi_i) \right] \\
&= r^2 \prod_{i=1}^{k} \sin^2(\phi_i) + r^2 \cos^2(\phi_k) \prod_{i=1}^{k-1} \sin^2(\phi_i) \\
&= r^2 \prod_{i=1}^{k-1} \sin^2(\phi_i).
\end{aligned}$$

By continuity the requirement $\sin(\phi_k) \neq 0$ can be dropped and this proves Eq. (2.8).

- For Eq. (2.9) assume $k < l$ without loss of generality. Then

$$\begin{aligned}\partial_{\phi_k} X \cdot \partial_{\phi_l} X \\ &= \partial\phi_k X_l \partial\phi_l X_l + \left(\sum_{i=l+1}^{n-1} \partial\phi_k X_i \partial\phi_l X_i \right) + \partial\phi_k X_n \partial_{\phi_l} X_n\end{aligned}$$

and similar to before it can be seen that for $\sin(\phi_k), \sin(\phi_l) \neq 0$

$$\begin{aligned}&\left(\sum_{i=l+1}^{n-1} \partial\phi_k X_i \partial\phi_l X_i \right) + \partial\phi_k X_n \partial_{\phi_l} X_n \\ &= r^2 \frac{\cos(\phi_k)\cos(\phi_l)}{\sin(\phi_k)\sin(\phi_l)} \left[\left(\sum_{j=l+1}^{n-1} \cos^2(\phi_j) \prod_{i=1}^{j-1} \sin^2(\phi_i) \right) + \prod_{i=1}^{n-1} \sin^2(\phi_i) \right] \\ &= r^2 \frac{\cos(\phi_k)\cos(\phi_l)}{\sin(\phi_k)\sin(\phi_l)} \left[- \prod_{i=1}^{l} \sin^2(\phi_i) \right]\end{aligned}$$

where Lemma 2.32 was used for the last inequality.

Since

$$\partial\phi_k X_l \partial\phi_l X_l = \frac{\cos(\phi_k)\cos(\phi_l)}{\sin(\phi_k)\sin(\phi_l)} \left[- \prod_{i=1}^{l} \sin^2(\phi_i) \right]$$

it follows that $\partial_{\phi_k} X \cdot \partial_{\phi_l} X = 0$.

Continuity implies this also holds for $\sin(\phi_k) = 0$ and $\sin(\phi_l) = 0$. □

Remark 2.40. *This shows that* $\{\partial_r X, \partial_{\phi_1} X, \ldots, \partial_{\phi_{n-1}} X\}$ *is an orthogonal basis of* $\mathbb{R}^n$*.*

Corollary 2.41. *1.* $X_n^2 + \cdots + X_j^2 = r^2 \prod_{i=1}^{j-1} \sin^2(\phi_i)$ *for any* $j = 1, \ldots, n$ *and any* $(r, \phi) \in A$.

2. $X(r, \phi) = r\partial_r X(r, \phi)$.

3. $X(r, \phi) \in \partial B_r(0)$.

4. $X(1, \phi) = \partial_r X(r, \phi)$.

Lemma 2.42. *For any function* $u(x) \in L^1(B_R(0))$ *the transformation formula holds with*

$$\int_{B_R(0)} u(x)dx = \int_0^R r^{n-1} \int_{A_1} u(X(r, \phi))\Phi(\phi)d\phi dr,$$

where

$$\Phi(\phi) = \prod_{i=1}^{n-2} \sin^{n-1-i}(\phi_i).$$

For any continuous function $u : \partial B_R(0) \to \mathbb{R}$

$$\int_{\partial B_R(0)} u(x)d\sigma = R^{n-1} \int_{A_1} u(X(R, \phi))\Phi(\phi)d\phi.$$

Proof. The matrix $DX(r, \phi) = \left(\partial_r X, \partial_{\phi_1} X, \ldots, \partial_{\phi_{n-1}}\right)$ consists of orthogonal vectors according to Remark 2.40. Using the transformation formula it is sufficient to look at A_1 and ignore A_2 which implies $\sin(\phi_i) \neq 0$ for $i \in \{1, \ldots, n-1\}$. By Lemma 2.39 the matrix

$$M = \left(\partial_r X, \frac{\partial_{\phi_1} X}{r}, \frac{\partial_{\phi_2} X}{r\sin(\phi_1)}, \ldots, \frac{\partial_{\phi_{n-1}} X}{r\prod_{i=1}^{n-2} \sin(\phi_i)}\right)$$

consists of an orthonormal basis of vectors which implies that the matrix is orthogonal and the absolute value of the determinant is

1. Since scaling any row of a matrix simply scales the determinant it now follows that

$$1 = |\det M| = 1 \cdot \frac{1}{r} \cdot \frac{1}{r\sin(\phi_1)} \cdot \dots \cdot \frac{1}{r\prod_{i=1}^{n-2}\sin(\phi_i)} |\det DX(r,\phi)|$$

which proves that

$$|\det DX(r,\phi)| = r^{n-1}\Phi(\phi).$$

The first result then follows from the transformation formula.

For the second result the transformation formula for submanifolds in $\mathbb{R}^n$ is used. Since $|X(r,\phi)| = r$ and the function $X : A \to \mathbb{R}^n$ is bijective, the mapping $\phi \mapsto X(R,\phi)$ is a bijective mapping from A onto $\partial B_R(0)$. The restriction to A_1 is a diffeomorphic mapping to a subset of $\partial B_R(0)$ which differs from $\partial B_R(0)$ only on a set of measure 0.

To use this transformation formula it is necessary to compute $\sqrt{\det\left(\Psi^T\Psi\right)}$ with the matrix

$$\Psi = \left(\partial_{\phi_1}X, \dots, \partial_{\phi_{n-1}}X\right).$$

Noticing that

$$\begin{aligned}
&\Psi^T\Psi \\
&= \begin{pmatrix} \partial_{\phi_1}X \cdot \partial_{\phi_1}X & \partial_{\phi_1}X \cdot \partial_{\phi_2}X & \cdots & \partial_{\phi_1}X \cdot \partial_{\phi_{n-1}}X \\ \vdots & \vdots & \dots & \vdots \\ \partial_{\phi_{n-1}}X \cdot \partial_{\phi_1}X & \partial_{\phi_{n-1}}X \cdot \partial_{\phi_2}X & \cdots & \partial_{\phi_{n-1}}X \cdot \partial_{\phi_{n-1}}X \end{pmatrix} \\
&= \operatorname{diag}\left(R^2, R^2\sin^2(\phi_1), \dots, R^2\prod_{i=1}^{n-2}\sin^2(\phi_i)\right)
\end{aligned}$$

is a diagonal matrix the determinant is

$$R^{2(n-1)} \prod_{i=1}^{n-2} \sin^{2(n-1-i)}(\phi_i)$$

and taking the square root and the transformation formula shows the desired result. $\square$

3 Boundary Value Problem

3.1 Overview

Section 3.2 defines the boundary value problem, the associated functional and the weak formulation.

Section 3.3 states several results concerning the behaviour of solutions, such as nonnegativity of minimizers, Hopfs Lemma and the maximum principle.

Section 3.4 introduces various ways to modify a and f. This is used in Theorems 5.3 and 6.5 in order to be able to include a wider class of differential equations. The modification of f is helpful when proving that there are nonnegative solutions.

Section 3.5 cites the regularity results from the literature that will be used in Chapter 5 and extensively in Chapter 6, Section 6.3.

Section 3.6 introduces the mountain pass theorem and the Palais-Smale compactness condition.

3.2 Weak Solutions

The boundary value problem that will be investigated here in its most general form is

$$\text{(P)} \qquad \begin{cases} -\operatorname{div}\Big(a(x,|\nabla u|^p)|\nabla u|^{p-2}\nabla u\Big) = \lambda f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the following conditions hold:

Assumption 1. *$\Omega \subset \mathbb{R}^n$ with $n \geq 1$ is a bounded domain with $C^{1,1}$ boundary, $p > 1$ and $\lambda > 0$.*

$a(x,s)$ is a Carathéodory function on $\Omega \times (0,\infty)$ and for almost every $x \in \Omega$ the function $s \mapsto a(x,s^p)s^{p-1}$ is strictly increasing in $[0,\infty)$ with $\lim_{s\to 0} a(x,s^p)s^{p-1} = 0$. This implies that the function $a(x,|z|^p)|z|^{p-2}z$ is a Carathéodory function on $\Omega \times \mathbb{R}^n$ when extended to $z = 0$ by 0. Furthermore there is a $C_a > 0$ and a $D_a \in L^{\frac{p}{p-1}}(\Omega)$ so that

$$\Big|a(x,s^p)s^{p-1}\Big| \leq C_a s^{p-1} + D_a(x) \text{ for every } (x,s) \in \Omega \times [0,\infty).$$

$f(x,s)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ and there are $C_f > 0$ and $q > 1$ with $q < \frac{np}{n-p}$ if $p < n$ so that

$$|f(x,s)| \leq C_f|s|^{q-1} + D_f(x) \qquad \text{for every } (x,s) \in \Omega \times \mathbb{R},$$

where $D_f \in L^{\frac{q}{q-1}}$.

Remark 3.1. *It can be argued that the notation of the differential operator as $-\operatorname{div}\Big(a(x,|\nabla u|^p)|\nabla u|^{p-2}\nabla u\Big)$ instead of $-\operatorname{div}(a(x,\nabla u))$ is inconvenient and that there are situations* *where it is unclear which value p should have for a specific differential operator. For example*

$$-\operatorname{div}\left(|\nabla u|\nabla u\right)-\operatorname{div}\left(|\nabla u|^{2}\nabla u\right)=\lambda\left(u^{1.5}+u^{5}\right)$$

would use $p=3$ *in Theorem 5.3 and* $p=4$ *in Theorem 3.46.*

However the specific structure is needed for the main Theorems 6.3 and 6.5 and the benefit of having a consistent notation throughout the thesis seems to outweigh this inconvenience.

Definition 3.2. *A function* $u:\overline{\Omega}\to\mathbb{R}$ *is a* ***classical solution*** *of* (P) *if* $u\in C^2(\Omega)\cap C_0^0(\overline{\Omega})$ *and it satisfies the equation at every point* $x\in\Omega$.

Definition 3.3. *A function* $u\in W_0^{1,p}(\Omega)$ *is a* ***weak solution*** *of* (P) *if*

$$\int_\Omega a(x,|\nabla u|^p)|\nabla u|^{p-2}\nabla u\cdot\nabla v-\lambda f(x,u)v dx=0$$

for every $v\in C_0^\infty(\Omega)$.

Definition 3.4. *Let* $u\in W_0^{1,p}(\Omega)$ *be a nonnegative, nontrivial weak solution of* (P) *so that* $J(u)\leq J(v)$ *for any other nonnegative, nontrivial weak solution v, then u will be called a* ***ground state solution****.*

Definition 3.5. *Let Assumption 1 hold, then the function* $J:W_0^{1,p}(\Omega)\to\mathbb{R}$ *is defined by*

$$J(u)=\int_\Omega\frac{1}{p}A(x,|\nabla u|^p)-\lambda F(x,u)dx \tag{3.1}$$

where

$$A(x,t) = \int_0^t a(x,s)ds \qquad \textit{for } (x,t) \in \Omega \times [0,\infty)$$

and

$$F(x,t) = \int_0^t f(x,s)ds \qquad \textit{for } (x,t) \in \Omega \times \mathbb{R}\,.$$

Define

$$\mathcal{A}(u) := \int_\Omega \frac{1}{p} A(x, |\nabla u|^p)dx \qquad \textit{and} \qquad \mathcal{F}(u) := \int_\Omega F(x,u)dx.$$

Lemma 3.6. *Given Assumption 1 the functional $\mathcal{A} : W_0^{1,p}(\Omega) \to \mathbb{R}$ is well-defined and Frechét differentiable with*

$$\mathcal{A}'(u)(v) = \int_\Omega a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx.$$

Proof. With Assumption 1 and

$$\frac{d}{ds}\frac{1}{p}A(x,s^p) = a(x,s^p)s^{p-1}$$

the functional $\mathcal{A}(u)$ is Fréchet differentiable according to Theorem 2.8. □

Lemma 3.7. *Given Assumption 1 the functional $\mathcal{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$ is well-defined, Frechét differentiable and for any sequence $u_k \in W_0^{1,p}(\Omega)$ that converges weakly to u there is a subsequence u_{k_l} that converges strongly to u in L^q and*

$$\lim_{l\to\infty} \mathcal{F}(u_{k_l}) = \mathcal{F}(u) \qquad \textit{and} \qquad \lim_{l\to\infty} \mathcal{F}'(u_{k_l}) = \mathcal{F}'(u).$$

Proof. The function $\mathcal{F} : L^q(\Omega) \to \mathbb{R}$ is Frechét differentiable with

$$\mathcal{F}'(u)v = \int_\Omega f(x,u)v dx$$

for any $u, v \in L^q(\Omega)$. $\mathcal{F}' : L^q \to (L^q)'$ and is well-defined and continuous. Using the compact embedding $W_0^{1,p}(\Omega) \subset\subset L^q(\Omega)$ the functional can be considered to be defined as $\mathcal{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$ and remains Fréchet differentiable. By Theorem 2.28 a strongly converging subsequence exists and the statements follow from continuity of $\mathcal{F}$ and $\mathcal{F}'$ in L^q. □

Corollary 3.8. *By integration Assumption 1 implies*

$$\tfrac{1}{p} A(x, s^p) \leq \tfrac{C_a}{p} s^p + D_a(x)s$$

and

$$F(x,s) \leq \tfrac{C_f}{q} s^q + D_f(x)s.$$

Lemma 3.9. *The functional J is weakly lower semicontinuous.*

Proof. This is found in [16], 8.2.2, Theorem 1. □

Lemma 3.10.

$$\begin{aligned} &\xi^T \left(D_z^2 \left(\tfrac{1}{p} A(x, |z|^p) \right) \right) \xi \\ &\quad = \Big((p-1)a(x,|z|^p) + p a_s(x,|z|^p)|z|^p \Big) |z|^{p-4} |(z,\xi)|^2 \\ &\qquad + |z|^{p-4} a(x,|z|^p) \left(|z|^2 |\xi|^2 - |(z,\xi)|^2 \right) \end{aligned}$$

and

$$\begin{aligned} &\xi^T \left(D_z^2 \left(\tfrac{1}{p} A(x, |z|^p) \right) \right) \xi \\ &\geq \tfrac{1}{2} |z|^{p-2} |\xi|^2 \min\Big\{ a_s(x,|z|^p) p |z|^p + a(x,|z|^p)(p-1), a(x,|z|^p) \Big\}. \end{aligned}$$

Proof. Either $|(z,\xi)|^2 > \frac{1}{2}|z|^2|\xi|^2$ or $|(z,\xi)|^2 \leq \frac{1}{2}|z|^2|\xi|^2$. In the first case

$$\begin{aligned}\xi^T D_z^2 \left(\tfrac{1}{p}A(x,|z|^p)\right)\xi \\ &\geq \tfrac{1}{2}\Big(a_z(x,|z|^p)p|z|^p + a(x,|z|^p)(p-1)\Big)|z|^{p-2}|\xi|^2.\end{aligned}$$

In the second case

$$\xi^T D_z^2 \left(\tfrac{1}{p}A(x,|z|^p)\right)\xi \geq \tfrac{1}{2}a(x,|z|^p)|z|^{p-2}|\xi|^2.$$

□

The following lemma relates statements between the one-dimensional function $\frac{1}{p}A(x,s^p)$ in $\Omega\times(0,\infty)$ and the function $\frac{1}{p}A(x,|z|^p)$ in $\Omega\times\mathbb{R}^n$.

Lemma 3.11. *1.* $\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p) > 0$ *in* $\Omega\times(0,\infty)$ *if and only if*

$$\xi^T\left(D_z^2\left(\frac{1}{p}A(x,|z|^p)\right)\right)\xi > 0$$

for every $x\in\Omega$ *and* $z,\xi\in\mathbb{R}^n$.

2.
$$\xi^T D_z^2\left(\frac{1}{p}A(x,|z|^p)\right)\xi \geq c_e|z|^{p-2}|\xi|^2$$

holds for all $(x,z)\in\Omega\times\mathbb{R}^n$ *and* $\xi\in\mathbb{R}^n$, *if and only if*

$$\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p) \geq \frac{c_e}{p-1}s^{p-2}$$

for every $(x,s)\in\Omega\times[0,\infty)$.

3. *There is a $C_1 > 0$ so that*

$$\left| \frac{d}{dz_j} \Big(a(x, |z|^p) |z|^{p-2} z \Big)_i \right| \leq C_1 |z|^{p-2} \tag{3.2}$$

for every $(x, z) \in \Omega \times \mathbb{R}^n$ if and only if there is a $C_2 > 0$ so that

$$\frac{d^2}{ds^2} \frac{1}{p} A(x, s^p) \leq C_2 s^{p-2} \tag{3.3}$$

for every $s \in (0, \infty)$.

Proof. 1. If

$$\frac{d^2}{ds^2} \frac{1}{p} A(x, s^p) = \left(a_s(x, s^p) p s^p + a(x, s^p)(p-1) \right) s^{p-2} > 0$$

then by Lemma 3.10 it follows that

$$\xi^T \left(D_z^2 \left(\frac{1}{p} A(x, |z|^p) \right) \right) > 0$$

for any $x \in \Omega$ and $z, \xi \in \mathbb{R}^n \setminus \{0\}$.

If

$$\xi^T \left(D_z^2 \left(\tfrac{1}{p} A(x, |z|^p) \right) \right) > 0$$

for any $x \in \Omega$ and $z, \xi \in \mathbb{R}^n \setminus \{0\}$ then choosing $\xi = \frac{z}{|z|}$ shows that

$$\begin{aligned} 0 &< \xi^T \left(D_z^2 \left(\tfrac{1}{p} A(x, |z|^p) \right) \right) \xi \\ &= \left(a_s(x, s^p) p s^p + a(x, s^p)(p-1) \right) s^p \big|_{s=|z|} \\ &= \left. \frac{d^2}{ds^2} \frac{1}{p} A(x, s^p) \right|_{s=|z|} \end{aligned}$$

and thus $\frac{d^2}{ds^2} \frac{1}{p} A(x, s^p) > 0$ for any $x \in \Omega$ and $s > 0$.

2. If
$$\xi^T D_z^2 \left(\frac{1}{p} A(x, |z|^p)\right) \xi \geq c_e |z|^{p-2} |\xi|^2$$
then choosing $\xi = \frac{z}{|z|}$ shows
$$c_e |z|^{p-2} \leq ((p-1)a(x, |z|^p) + p a_s(x, |z|^p)|z|^p)|z|^{p-2}$$
which in turn shows that $\frac{d^2}{ds^2}\frac{1}{p}A(x, s^p) \geq c_e |z|^{p-2}$.

If
$$\frac{d^2}{ds^2}\frac{1}{p}A(x, s^p) \geq c_e |z|^{p-2}$$
then by integration
$$\frac{c_e}{p-1} s^{p-1} \leq a\left(x, s^p\right) s^{p-1}$$
and using Lemma 3.10 shows that
$$\xi^T \left(D_z^2 \left(\frac{1}{p} A(x, |z|^p) \right) \right) \xi \geq \frac{1}{2} \min \left\{ c_e, \frac{c_e}{p-1} \right\} |z|^{p-2} |\xi|^2 .$$

3. If inequality (3.2) holds then with
$$\begin{aligned} &\left| \frac{d}{dz_j} \left(a(x, |z|^p) |z|^{p-2} z \right)_i \right| \\ &\quad = \left| \sum_{j=1}^{n} \Big[a_z(x, |z|^p) p |z|^{2p-4} z_i z_j \right. \\ &\qquad\qquad \left. + a(x, |z|^p)(p-2)|z|^{p-4} z_i z_j + a(x, |z|^p)|z|^{p-2} \delta_{ij} \Big] \right| \end{aligned}$$

and $z = se_i$ it follows that

$$\begin{aligned}
&\left| \left(\frac{d^2}{ds^2} A(x, s^p) \right)_{s=|z|} \right| \\
&\qquad = \left| |z|^{p-2} \left(a_s(x, |z|^p) p |z|^p + a(x, |z|^p)(p-1) \right) \right| \\
&\qquad = \left| \frac{d}{dz_j} \Big(a(x, |z|^p) |z|^{p-2} z \Big)_i \right|,
\end{aligned}$$

which shows that inequality (3.3) holds with $C_2 := C_1$.

In the other direction assume that

$$\frac{d^2}{ds^2} \frac{1}{p} A(x, s^p) \le C_2 s^{p-2}$$

which implies by integration that $a(x, s^p) s^{p-1} \le \frac{C_2}{p-1} s^{p-1}$ and thus

$$\begin{aligned}
&\left| \frac{d}{dz_j} \Big(a(x, |z|^p) |z|^{p-2} z \Big)_i \right| \\
&= \left| \sum_{j=1}^{n} \Big[a_s(x, |z|^p) p |z|^{2p-4} z_i z_j + a(x, |z|^p)(p-1) |z|^{p-4} z_i z_j \right. \\
&\qquad \left. + a(x, |z|^p) |z|^{p-2} \delta_{ij} - a(x, |z|^p) |z|^{p-4} z_i z_j \Big] \right| \\
&\le n \Big(\big| a_s(x, |z|^p) p |z|^p + a(x, |z|^p)(p-1) \big| |z|^{p-2} \\
&\qquad + \big| a(x, |z|^p) \big| |z|^{p-2} \Big) \\
&\le n \left(C_2 + \frac{C_2}{p-1} \right) s^{p-2} \\
&= n C_2 \frac{p}{p-1} s^{p-2}.
\end{aligned}$$

and thus inequality (3.2) holds with $C_1 = nC_2 \frac{p}{p-1}$.

3.3 Behaviour of Solutions

Lemma 3.12. *Given Assumption 1 and $f(x,s) = 0$ for $s \leq 0$, then any weak solution $u \in W_0^{1,p}(\Omega)$ is nonnegative.*

Proof. Testing with u^- and using $\nabla u \cdot \nabla u^- = |\nabla u^-|^2$ shows that

$$\begin{aligned} 0 &= J'(u)(u^+) \\ &= \int_\Omega a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla u^- - \lambda f(x,u)u^- dx \\ &= \int_\Omega a(x, |\nabla u|^p)|\nabla u|^{p-2}|\nabla u^-|^2 dx \end{aligned}$$

where $f(x,u)u^- \equiv 0$ since u^- is zero if $u \geq 0$ and $f(x,u)$ is zero if $u \leq 0$. Thus $|\nabla u^-| = 0$ almost everywhere and this concludes the proof. □

Lemma 3.13. *Given Assumption 1 and $f(x,s) = 0$ for $s \geq c$ and $s \leq 0$, any critical point $u \in W_0^{1,p}(\Omega)$ of J satisfies $\|u\|_{L^\infty} \leq c$.*

Proof. Testing the differential equation with $(u-c)^+$ and using that

$$\nabla u \cdot \nabla (u-c)^+ = \left|\nabla (u-c)^+\right|^2 \qquad \text{and} \qquad f(x,u)(u-c)^+ = 0$$

shows

$$\begin{aligned} 0 &= J'(u)(u-c)^+ \\ &= \int_\Omega a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla (u-c)^+ - \lambda f(x,u)(u-c)^+ dx \\ &= \int_\Omega a(x, |\nabla u|^p)|\nabla u|^{p-2}\left|\nabla (u-c)^+\right|^2 dx \end{aligned}$$

which implies $(u-c)^+ = 0$ almost everywhere and this proves the statement. $\square$

Lemma 3.14. *Given Assumption 1 and* $-f(x,-s) < f(x,s)$ *for* $s > 0$ *then* $J(|u|) \leq J(u)$ *and* $J(|u|) < J(u)$ *unless* $u(x) \geq 0$.

Proof. For $s > 0$, the inequality

$$\begin{aligned} F(x,-s) &= \int_0^{-s} f(x,t)dt \\ &= -\int_0^{s} f(x,-t)dt \\ &< \int_0^{s} f(x,t)dt = F(x,s) \end{aligned}$$

shows that $-\mathcal{F}(|u|) \leq -\mathcal{F}(u)$. Since $\mathcal{A}(|u|) \leq \mathcal{A}(u)$ this implies $J(|u|) \leq J(u)$ with equality if and only if $u \geq 0$. $\square$

Remark 3.15. *Lemma 3.14 can guarantee nonnegativity of minimizers while it does not directly ensure that all critical points are nonnegative, unlike Lemma 3.12.*

Theorem 3.16 (Pucci & Serrin)**.** *Assume*

$$\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p) > 0 \qquad \textit{for } (x,s) \in \Omega \times (0,\infty)$$

with $a(x,s^p) = a_1(x)a_2(s^p)$, $a_1(x) \geq c > 0$ *and let* u *be a nontrivial weak solution with* $u(x) \geq 0$ *and* $f(x,u(x)) \geq 0$. *Then*

$$u(x) > 0 \textit{ in } \Omega.$$

If Ω satisfies an interior sphere condition and $u \in C^1(\overline{\Omega})$ then $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$ where n is the outer normal to $\partial\Omega$.

Proof. This follows from theorem 8.1 and corollary 8.4 in [44] and the Erratum [45]. □

3.4 Modifying the Functional

Several equations have solutions where the C^1-norm of solutions can be made small in which case the behaviour of $a(x, s)$ and $f(x, s)$ only matters for small values of s.

In this section it is shown that functionals satisfying certain conditions in $\Omega \times [0, \varepsilon]$ can be extended in a way so they satisfy those conditions in $\Omega \times [0, \infty)$ as well.

3.4.1 Extending the differential operator

Let $B : \Omega \times [0, \varepsilon] \to \mathbb{R}$ be a Carathéodory function so that $\frac{d}{ds}B(x, s) =: B_s(x, s)$ and $\frac{d^2}{ds^2}B(x, s) =: B_{ss}(x, s)$ exist and are Carathéodory functions on $\Omega \times (0, \varepsilon]$. Furthermore assume that $B_s(x, s) > 0$ on $\Omega \times (0, \varepsilon]$.

Definition 3.17. *The function $A : \Omega \times [0, \infty) \to \mathbb{R}$ defined by*

$$A(x, s) := \begin{cases} B(x, s) & \text{for } s \in [0, \varepsilon], \\ C_0(x) + C_1(x)s^{\frac{1}{p}} + C_2(x)s & \text{for } s > \varepsilon, \end{cases}$$

with

$$
\begin{aligned}
C_0(x) &:= B(x,\varepsilon) - B_s(x,\varepsilon)\,\varepsilon + B_{ss}(x,\varepsilon)p\,\varepsilon^2,\\
C_1(x) &:= -\frac{p^2}{p-1}\,\varepsilon^{2-\frac{1}{p}}\,B_{ss}(x,\varepsilon),\\
C_2(x) &:= B_s(x,\varepsilon) + \frac{p}{p-1}\,\varepsilon\,B_{ss}(x,\varepsilon),
\end{aligned}
$$

is an extension of B *so that* $A|_{\Omega\times[0,\varepsilon]} = B$.

Define $a(x,s) := \partial_s A(x,s)$.

Corollary 3.18. *The functions* $C_i(x)$ *with* $i \in \{0,1,2\}$ *are measurable on* Ω.

Corollary 3.19. *The following identities hold:*

- *For every* $(x,s) \in \Omega \times (\varepsilon, \infty)$

$$
a(x,s) = A_s(x,s) = C_2(x) + C_1(x)\frac{1}{p}s^{\frac{1}{p}-1},
$$

- *For every* $(x,s) \in \Omega \times (\varepsilon, \infty)$

$$
a_s(x,s) = A_{ss}(x,s) = C_1(x)\frac{1}{p}\left(\frac{1}{p}-1\right)s^{\frac{1}{p}-2},
$$

- *For every* $(x,s) \in \Omega \times \left(\varepsilon^{\frac{1}{p}}, \infty\right)$

$$
\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p) = (p-1)C_2(x)s^{p-2}.
$$

Corollary 3.20. *If* $\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p) > 0$ *in* $\Omega \times (0,\varepsilon)$ *then*

$$
\frac{d^2}{ds^2}\tfrac{1}{p}A(x,s^p) > 0 \text{ in } \Omega \times (0,\infty).
$$

Corollary 3.21. *If $x \mapsto B_{ss}(x,\varepsilon)$ is in $C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in [0,1]$ then $C_1 \in C^{0,\alpha}(\overline{\Omega})$.*

If $x \mapsto B_{ss}(x,\varepsilon)$ and $x \mapsto B_s(x,\varepsilon)$ are in $C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in [0,1]$ then $C_2 \in C^{0,\alpha}(\overline{\Omega})$.

Proof. This follows immediately from the definition. $\square$

Lemma 3.22. *The functions $A(x,s)$, $A_s(x,s)$ and $A_{ss}(x,s)$ are Carathéodory functions on $\Omega \times (0,\infty)$.*

Proof. $s \mapsto A(x,s)$, $s \mapsto A_s(x,s)$ and $s \mapsto A_{ss}(x,s)$ are continuous on $(0,\varepsilon]$ for almost every $x \in \Omega$ by assumption. Continuity on (ε,∞) follows directly from the definition. It remains to be shown that $\lim_{s\downarrow\varepsilon} A(x,s) = A(x,\varepsilon)$ and analogously $\lim_{s\downarrow\varepsilon} A_s(x,s) = A_s(x,\varepsilon)$ and $\lim_{s\downarrow\varepsilon} A_{ss}(x,s) = A_{ss}(x,\varepsilon)$.

Continuity of $s \mapsto A(x,s)$ at $s = \varepsilon$:

$$
\begin{aligned}
&\lim_{s\downarrow\varepsilon} A(x,s) \\
&\qquad = C_0(x) + C_1(x)\,\varepsilon^{\frac{1}{p}} + C_2(x)\,\varepsilon \\
&\qquad = B(x,\varepsilon) - B_s(x,\varepsilon)\,\varepsilon + B_{ss}(x,\varepsilon) p\,\varepsilon^2 \\
&\qquad\qquad + \left(-\frac{p^2}{p-1}\,\varepsilon^{2-\frac{1}{p}}\,B_{ss}(x,\varepsilon)\right)\varepsilon^{\frac{1}{p}} \\
&\qquad\qquad\qquad + \left(B_s(x,\varepsilon) + \frac{p}{p-1}\,\varepsilon\,B_{ss}(x,\varepsilon)\right)\varepsilon \\
&\qquad = B(x,\varepsilon).
\end{aligned}
$$

Continuity of $s \mapsto A_s(x,s)$ at $s = \varepsilon$:

$$\begin{aligned}
\lim_{s\downarrow\varepsilon} A_s(x,s) &= \left(C_1(x)\frac{1}{p}\,\varepsilon^{\frac{1}{p}-1} + C_2(x)\right)\\
&= \left(-\frac{p^2}{p-1}\,\varepsilon^{2-\frac{1}{p}}\,B_{ss}(x,\varepsilon)\right)\frac{1}{p}\,\varepsilon^{\frac{1}{p}-1}\\
&\qquad + \left(B_z(x,\varepsilon) + \frac{p}{p-1}\,\varepsilon\,B_{ss}(x,\varepsilon)\right)\\
&= B_s(x,\varepsilon).
\end{aligned}$$

Continuity of $s \mapsto A_{ss}(x,s)$ at $s = \varepsilon$:

$$\begin{aligned}
\lim_{s\downarrow\varepsilon} A_{ss}(x,s) &= \left(C_1(x)\frac{1}{p}\left(\frac{1}{p}-1\right)\varepsilon^{\frac{1}{p}-2}\right)\\
&= \left(-\frac{p^2}{p-1}\,\varepsilon^{2-\frac{1}{p}}\,B_{ss}(x,\varepsilon)\right)\frac{1-p}{p^2}\,\varepsilon^{\frac{1}{p}-2}\\
&= B_{ss}(x,\varepsilon).
\end{aligned}$$

2. The continuity with respect to s has been shown in the previous segment. Measurability in x for every $s \in (0,\varepsilon]$ follows from the fact that B, B_s and B_{ss} are Carathéodory functions. By definition this also ensures that $C_i(x)$ with $i \in \{1,2,3\}$ are measurable which shows measurability of A in x for $s > \varepsilon$. $\square$

Lemma 3.23. *Let $(x,s) \mapsto B_s(x,s^p)$ be Lipschitz continuous on $\overline{\Omega} \times \left[0,\varepsilon^{\frac{1}{p}}\right]$ and $x \mapsto B_{ss}(x,\varepsilon)$ Lipschitz continuous on $\overline{\Omega}$.*

Then $(x,s) \mapsto a(x,s^p)$ is globally Lipschitz continuous on $\overline{\Omega} \times [0,\infty)$.

Proof. ∘ Let $s, t > \varepsilon^{\frac{1}{p}}$, then

$$\begin{aligned}
&|a(x, s^p) - a(y, t^p)| \\
&\quad = \left| C_2(x) + C_2(x)\frac{1}{p}s^{1-p} - \left(C_2(y) + C_1(y)\frac{1}{p}t^{1-p}\right)\right| \\
&\quad \leq |C_2(x) - C_2(y)| + |C_1(x)|\frac{1}{p}\left|s^{1-p} - t^{1-p}\right| \\
&\qquad + t^{1-p}\frac{1}{p}|C_1(x) - C_1(y)|.
\end{aligned}$$

C_1 and C_2 is Lipschitz continuous and $t^{1-p} \leq \varepsilon^{\frac{1-p}{p}}$ so the first and third term are bound from above by $M|x-y|$.

For the second term the Lipschitz-continuity of $C_1(x)$ implies that $|C_1(x)|$ is bounded and with the mean value theorem and $1-p<0$ there is a ξ between s and t so that

$$\left|s^{1-p} - t^{1-p}\right| = \left|(1-p)\xi^{-p}(s-t)\right| \leq (p-1)\,\varepsilon^{-1}\,|s-t|.$$

- ∘ The case $s, t \leq \varepsilon^{\frac{1}{p}}$ is clear since $B_s(x, s^p)$ is Lipschitz continuous by assumption.
- ∘ Without loss of generality let $s < \varepsilon^{\frac{1}{p}} < t$. Then

$$\begin{aligned}
&|a(y, t^p) - a(x, s^p)| \\
&\quad \leq |a(y, t^p) - a(x, \varepsilon)| + |a(x, \varepsilon) - a(x, s^p)| \\
&\quad \leq C\left(|x-y| + \left|t - \varepsilon^{\frac{1}{p}}\right|\right) + C\left|\varepsilon^{\frac{1}{p}} - s\right| \\
&\quad = C\left(|x-y| + |t-s|\right)
\end{aligned}$$

which shows global Lipschitz continuity on $\overline{\Omega} \times [0, \infty)$.

□

Lemma 3.24. *If there are* $0 < c_e < C_e$ *so that*

$$c_e s^{p-2} \leq \frac{d^2}{ds^2}\frac{1}{p}B(x,s^p) \leq C_e s^{p-2} \qquad \textit{in } \Omega \times [0,\varepsilon]$$

then

$$c_e s^{p-2} \leq \frac{d^2}{ds^2}\frac{1}{p}A(x,s^p) \leq C_e s^{p-2} \qquad \textit{in } \Omega \times [0,\infty).$$

Proof. Since

$$\tfrac{d^2}{ds^2}\tfrac{1}{p}B(x,s^p) = (p-1)C_2(x)s^{p-2}$$

it follows that

$$c_e\, \varepsilon^{p-2} \leq (p-1)C_2(x)\,\varepsilon^{p-2} \leq C_a\, \varepsilon^{p-2}$$

which implies that $\frac{c_e}{p-1} \leq C_2(x) \leq \frac{C_e}{p-1}$ for every $x \in \Omega$ and this proves the statement. □

Lemma 3.25. *If* $B_{ss}(x,s) \leq 0$ *in* $\Omega \times (0,\varepsilon]$ *then* $A_{ss}(x,s) \leq 0$ *in* $\Omega \times (0,\infty)$.

'$\leq$' can be replaced by '$<$', '$\geq$' or '$>$'.

Proof. The statement follows immediately from the definition since

$$A_{ss}(x,s) = \varepsilon^{2-\frac{1}{p}}\, B_{ss}(x,\varepsilon)s^{\frac{1}{p}-2}$$

for $s \geq \varepsilon$. □

3.4.2 Extending the right-hand side of the differential equation

Assume that $G : \Omega \times [0, \varepsilon] \to \mathbb{R}$ is a Carathéodory function on $\Omega \times [0, \varepsilon]$ so that $\frac{d}{ds} G(x, s) =: G_s(x, s)$ exists and is a Carathéodory function on $\Omega \times [0, \varepsilon]$ as well.

Definition 3.26. *The function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by*

$$F(x,s) := \begin{cases} G(x,s) & \textit{for } s \leq \varepsilon \\ G(x,\varepsilon) - G_s(x,\varepsilon)\frac{\varepsilon}{q} + G_s(x,\varepsilon)\frac{s^q}{q\,\varepsilon^{q-1}} & \textit{for } s > \varepsilon \end{cases}$$

is an extension of G so that $F|_{\Omega \times [0,\varepsilon]} = G$.

Define $f(x, s) := \frac{d}{ds} F(x, s)$.

Lemma 3.27. *The functions F and f are Carathéodory functions on $\Omega \times [0, \infty)$.*

Proof. F and f are Carathéodory functions on $\Omega \times [0, \varepsilon]$ by assumption. They are Carathéodory functions on $\Omega \times (\varepsilon, \infty)$ by definition. It thus remains to show that $\lim_{s \downarrow \varepsilon} F(x, s) = F(x, \varepsilon)$ and analogously $\lim_{s \downarrow \varepsilon} f(x, s) = f(x, \varepsilon) = G_s(x, \varepsilon)$.

Therefore

$$\lim_{s \downarrow \varepsilon} F(x,s) = G(x,\varepsilon) - G_s(x,\varepsilon)\frac{\varepsilon}{q} + G_s(x,\varepsilon)\frac{\varepsilon^q}{q\,\varepsilon^{q-1}} = G(x,\varepsilon)$$

and

$$\lim_{s \downarrow \varepsilon} f(x,s) = \lim_{s \downarrow \varepsilon} G_s(x,\varepsilon)\frac{s^{q-1}}{\varepsilon^{q-1}} = G_s(x,\varepsilon)$$

conclude the proof. □

Lemma 3.28 (Growth conditions)**.** *If*

$$c_f s^{q-1} \le G_s(x,s) \le C_f s^{q-1} \qquad \textit{for } (x,s) \in \Omega \times [0,\varepsilon],$$

then

$$c_f s^{q-1} \le f(x,s) \le C_f s^{q-1} \qquad \textit{for } (x,s) \in \Omega \times [0,\infty).$$

Proof. $f(x,u) = F_u(x,u) = G_u(x,\varepsilon)\frac{u^{q-1}}{\varepsilon^{q-1}}$ for $u > \varepsilon$. Then

$$c_f\, \varepsilon^{q-1} \le G_u(x,\varepsilon) \le C_f\, \varepsilon^{q-1}$$

implies

$$c_f u^{q-1} \le G_u(x,\varepsilon)\frac{u^{q-1}}{\varepsilon^{q-1}} = f(x,u) \le C_f u^{q-1}$$

and this proves the result. □

Lemma 3.29. *If $G_s(x,\varepsilon) \ge 0$ and $G(x,s) - \frac{1}{q}G_s(x,s)s \le 0$ in $\Omega \times [0,\varepsilon]$ then*

$$F(x,s) - \vartheta f(x,s) \le 0$$

for any $\vartheta \ge \frac{1}{q}$ and $(x,s) \in \Omega \times [0,\infty)$.

Proof. For $s \ge \varepsilon$

$$\begin{aligned} &F(x,s) - \vartheta f(x,s) \\ &\qquad = \left(G(x,\varepsilon) - G_s(x,\varepsilon)\frac{\varepsilon}{q}\right) + G_s(x,\varepsilon)\frac{1}{\varepsilon^{q-1}}\left(\tfrac{1}{q} - \vartheta\right) s^q \\ &\qquad \le 0 \end{aligned}$$

which implies the result. □

Lemma 3.30. *If there is an $\alpha \leq q-1$ so that G_s satisfies the Nehari condition:*

$$\forall\, 0 < s < t \leq \varepsilon \;\; \forall\, x \in B_R(0) \;\; : \;\; \frac{G_s(x,s)}{s^\alpha} \leq \frac{G_s(x,t)}{t^\alpha}.$$

then

$$\forall\, 0 < s < t \;\; \forall\, x \in B_R(0) \;\; : \;\; \frac{f(x,s)}{s^\alpha} \leq \frac{f(x,t)}{t^\alpha}$$

and thus f satisfies the Nehari condition as well.

Proof.

- The Nehari condition for $0 < s < t \leq \varepsilon$ follows from the assumptions.
- For $\varepsilon \leq s < t$ the derivative is

$$f(x,s) = F_s(x,s) = G_s(x,\varepsilon)\frac{s^{q-1}}{\varepsilon^{q-1}}$$

  and

$$\frac{f(x,s)}{s^\alpha} = G_s(x,\varepsilon)\frac{s^{q-1-\alpha}}{\varepsilon^{q-1}} \leq G_s(x,\varepsilon)\frac{t^{q-1-\alpha}}{\varepsilon^{q-1}} = \frac{f(x,t)}{t^\alpha}$$

  where the inequality follows from $q-1-\alpha \geq 0$.
- The remaining case is $0 < s \leq \varepsilon < t$ and this follows from

$$\frac{f(x,s)}{s^\alpha} \leq \frac{f(x,\varepsilon)}{\varepsilon^\alpha} \leq \frac{f(x,t)}{t^\alpha}.$$

□

3.4.3 Cutting off f for large s

Lemma 3.31. *Let*

$$\chi : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad \chi(s) := \tilde{\chi}(-\tfrac{2}{\delta}s + 1 + \tfrac{2}{\delta}C),$$

where

$$\tilde{\chi}(s) = \frac{1}{\int_{-1}^{1} \varphi(t)dt} \int_{-\infty}^{s} \varphi(t)dt$$

and

$$\varphi(t) := \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & \text{for } t \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\chi \in C^\infty(\mathbb{R}), \qquad \chi(s) = \begin{cases} 1 & \text{for } s \leq C, \\ \in (0,1) & \text{for } s \in (C, C+\delta), \\ 0 & \text{for } s \geq C+\delta. \end{cases}$$

Proof. $\varphi(t)$ is in $C_0^\infty(\mathbb{R})$ by construction and because of the support of φ it follows that

- $\tilde{\chi}(s) = 0$ for any $s \leq -1$,
- $\tilde{\chi}(s) \in (\chi(-1), \chi(1))$ for any $s \in (-1,1)$,
- $\tilde{\chi}(s) = \tilde{\chi}(1) = 1$ for any $s \geq 1$

and the result follows from the construction of $s \mapsto (-\frac{2}{\delta}s + 1 + \frac{2}{\delta}C)$. □

Corollary 3.32. *If $G_s(x,s)$ is an L^∞-Carathéodory function then*

$$f(x,s) = G_s(x,s)\chi(s)$$

is a bounded Carathéodory function so that $f(x,s) = G_s(x,s)$ for $s \leq C$ and $f(x,s) = 0$ for $s \geq C+\delta$.

3.4.4 Modifying f for s smaller than zero

Lemma 3.33. *The function $G_s(x,s)$ can be extended to $s \leq 0$ in one of the following ways:*

- *If $G_s(x,0) = 0$ then*

$$f(x,s) = \begin{cases} G_s(x,s), & s \geq 0 \\ -G_s(x,-s) + s^2, & s < 0 \end{cases}$$

  is a Carathéodory function so that $-f(x,-s) < f(x,s)$ for all $s > 0$.

- *If $G_s(x,0) = 0$ and $G_s(x,s) > 0$ for $s > 0$ then $f(x,s) = 0$ for $s < 0$ also implies $-f(x,-s) < f(x,s)$ for all $s > 0$.*

- *If $G_s(x,0) \geq 0$ with $G_s(x,s) \geq 0$ for $s > 0$ then*

$$f(x,s) = \begin{cases} G_s(x,s), & s \geq 0 \\ G_s(x,0) + s^2, & s < 0 \end{cases}$$

  is a Carathéodory function so that for all $s > 0$ the inequality $-f(x,-s) < f(x,s)$ holds.

3.5 Regularity Theory

The following is theorem 7.1 from chapter 4 of [29]. It is cited here in a slightly simplified version with adjusted terminology:

Theorem 3.34 (Ladyzhenskaya & Ural'tseva)**.** *Let $n \geq m > 1$, $b : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}$ be measurable functions, $q \geq q^* = \frac{nm}{n-m}$ and $u(x)$ be a generalised weak solution in $W^{1,m}(\Omega) \cap L^q(\Omega)$ of*

$$-\operatorname{div}(b(x, \nabla u)) = g(x,u)$$

in $W^{1,m}(\Omega) \cap L^q(\Omega)$ *with* $\operatorname{ess\,sup}_{\partial\Omega} |u| = M < \infty$.

If there is a $\nu > 0$ *so that*

$$b(x,z) \cdot z \geq \nu |z|^m - \varphi_1(x) \qquad \textit{and} \tag{3.4}$$
$$\operatorname{sgn}(u) \cdot g(x,u) \leq (1 + |u|^{\alpha_2}) \varphi_2(x)$$

with

1. $\varphi_1 \in L^{r_1}$, $\varphi_2 \in L^{r_2}$ *with* $r_1, r_2 > \frac{n}{m}$,
2. $0 \leq \alpha_2 < m\frac{n+q}{n} - 1 - \frac{q}{r_2}$,

then $\operatorname{ess\,sup}_{\Omega} |u|$ *is bounded by an expression in terms of* $\|u\|_{L^q}$, M, ν, α_i, $\|\varphi_i\|_{L^{r_i}}$ *and* $\operatorname{meas}\Omega$.

Remark 3.35. *It can be seen that in the case* $m = n$*, the variable* q *can be arbitrary in* $[1, \infty)$. *Let the assumptions hold for* $n = m$ *and such an arbitrary* q.

For any $q \in [1, \infty)$ *the embedding* $W^{1,n}(\Omega) \subset\subset L^q(\Omega)$ *is compact. Thus any solution* $u \in W^{1,n}$ *is in* L^q.

$m < n$ *can be chosen large enough so that* $\frac{mn}{n-m} \geq q$*, the conditions* $r_1, r_2 > \frac{n}{m}$ *as well as* $\alpha_2 < m\frac{n+q}{n} - 1 - \frac{q}{r_2}$ *continue to hold.*

It remains to show Eq. (3.4). *With*

$$\nu_1 |z|^n - \varphi_1(x) \geq \nu_1 \left(|z|^m - 1\right) - \varphi_1(x) = \nu_1 |z|^m - (\varphi_1(x) + \nu_1)$$

and $\varphi_1(x) + \nu_1 \in L^{r_1}$ *this remains true.*

Thus, in case $n = m$ *every* $q \in [1, \infty)$ *is admissible.*

Remark 3.36. *In case $n < m$ Theorem 2.27 shows*

$$\begin{aligned}\nu_1 \|u\|_{C^{0,\alpha}} &\leq C_S \nu_1 \|\nabla u\|_{L^p} \\ &\leq \int_\Omega b(x, \nabla u)\nabla u + \varphi_1(x) dx \\ &= \int_\Omega g(x,u)u + \varphi_1(x) dx \\ &\leq \int_\Omega (1 + |u|^{\alpha_2})|u|\varphi_2(x) + \varphi_1(x) dx\end{aligned}$$

and the last term only depends on L^r norms for suitable r of u, φ_1 and φ_2 which is the same situation as Theorem 3.34.

Remark 3.37. *Translated into the notation of* (P) *the conditions are*

- $a(x, |z|^p)|z|^p \geq \nu_1 |z|^p - \varphi_1$ *with* $\varphi_1 \in L^{r_1}$ *and* $r_1 > \frac{n}{p}$,
- $\operatorname{sgn}(u)\lambda f(x,u) \leq (1 + |u|^{\alpha_2})\varphi_2(x)$ *with* $r_2 > \frac{n}{p}$, $\varphi_2 \in L^{r_2}$,
- $0 \leq \alpha_2 < p\frac{n+q^*}{n} - 1 - q^* \frac{1}{r_2}$ *where* $q^* = \frac{np}{n-p}$ *if* $p < n$ *and* $q^* \geq 1$ *otherwise.*

Corollary 3.38. *If $a(x,s) \geq c_a > 0$ then the theorem also applies to the differential equation*

$$-\operatorname{div}\Big(a\big(x, |\nabla u_0(x)|^p\big)|\nabla u|^{p-2}\nabla u\Big) = \lambda f(x,u)$$

with $u_0 \in W_0^{1,p}(\Omega)$ where the bound does not depend on u_0 in any way.

Theorem 3.39 (Lieberman)**.** *Let $\alpha \in (0,1]$, $\Gamma \geq \gamma > 0$, $k \geq 0$, $M_0 \geq 0$, $m \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ a bounded $C^{1,\alpha}$ domain. Suppose b and g satisfy the structure conditions*

- $\displaystyle\sum_{i,j=1}^{n} \left(\frac{\partial b(x,u,z)}{z_j} \right)_i \xi_i \xi_j \geq \gamma(k + |z|)^m |\xi|^2,$
- $\displaystyle\left| \left(\frac{\partial b(x,u,z)}{z_j} \right)_i \right| \leq \Gamma(k + |z|)^m,$
- $|b(x,u,z) - b(y,v,z)| \leq \Gamma(1 + |z|)^{m+1} \left[|x-y|^\alpha + |u-v|^\alpha \right],$
- $|g(x,u,z)| \leq \Gamma(1 + |z|)^{m+2},$

for any $(x,u,z) \in \partial\Omega \times [-M_0, M_0] \times \mathbb{R}^n$, $(y,w) \in \Omega \times [-M_0, M_0]$ and $\xi \in \mathbb{R}^n$.

If u is a bounded weak solution of the dirichlet problem

$$-\operatorname{div} b(x,u,Du) = g(x,u,Du) \quad \text{in } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega$$

with $|u(x)| \leq M_0$, then there is a positive constant

$$\beta = \beta(\alpha, \Gamma/\gamma, m, n)$$

such that u is in $C^{1,\beta}(\overline{\Omega})$ and

$$|u|_{1+\beta} \leq C(\alpha, \Gamma/\gamma, m, M_0, n, \Phi, \Omega).$$

Proof. This is Theorem 1 from [34]. The names of some constants have been adapted to avoid conflicts with the notation in this dissertation. □

Corollary 3.40. *The boundary value problem* (P) *satisfies the assumptions of Theorem 3.39 if*

$$c_e s^{p-2} \leq \frac{d^2}{ds^2}\frac{1}{p}A(x,s^p) \leq C_e s^{p-2}$$

and

$$|a(x,z) - a(y,z)| \leq \Gamma|x-y|^\alpha$$

and if f is an L^∞-Carathéodory function.

These conditions are not optimal since they only use the case $k = 0$.

Proof. Using Lemma 3.11 this follows directly from Theorem 3.39. □

Lemma 3.41. *Let $(x,s) \mapsto a(x,s^p)$ be Lipschitz-continuous in $\overline{\Omega} \times [0,M]$ for every $M > 0$. Assume that $a(x,s) \geq c_a > 0$ and $u_0 \in C^{1,\alpha}(\overline{\Omega})$, then the boundary value problem*

$$\begin{cases} -\operatorname{div}\Big(a(x,|\nabla u_0|^p)|\nabla u|^{p-2}\nabla u\Big) = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies the conditions of Theorem 3.39. β and $\|u\|_{1+\beta}$ now depend also on the Lipschitz constant of $(x,s) \mapsto a(x,s^p)$ and $\|u_0\|_{C^{1,\alpha}}$, but not on the exact choice of v.

Proof. Let $u_0 \in C^{1,\alpha}$ with $\|u_0\|_{C^{1,\alpha}} \leq M_1$. The structure conditions of Theorem 3.39 now need to be checked for the function $b(x,z) = a(x,|\nabla u_0(x)|^p)|z|^{p-2}z$.

1. The first condition follows directly from

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{\partial b_i}{\partial z_j}\xi_i\xi_j \\ &= \sum_{i,j=1}^{n} a(x, |\nabla u_0(x)|^p) \left[(p-2)|z|^{p-4} z_i z_j + |z|^{p-2}\delta_{i,j}\right] \xi_i\xi_j \\ &= a(x, |\nabla u_0|^p) \left[(p-2)|z|^{p-4}(z,\xi)^2 + |z|^{p-2}|\xi|^2\right].\end{aligned}$$

Either $|(z,\xi)|^2 \le \frac{1}{2}|z|^2|\xi|^2$ or $|(z,\xi)|^2 > \frac{1}{2}|z|^2|\xi|^2$. In the first case

$$\begin{aligned}&a(x, |\nabla u_0|^p) \left[(p-2)|z|^{p-4}(z,\xi)^2 + |z|^{p-2}|\xi|^2\right] \\ &\ge a(x,|\nabla u_0|^p)(p-1)|z|^{p-4}|(z,\xi)|^2 + a(x,|\nabla u_0|^p)\frac{1}{2}|z|^{p-2}|\xi|^2 \\ &\ge \frac{c_a}{2}|z|^{p-2}|\xi|^2\end{aligned}$$

and in the second case either $p-2<0$, in which case a lower bound is given by $c_a|z|^{p-2}|\xi|^2$, or $p-2\ge 0$, in which case

$$\begin{aligned}&a(x, |\nabla u_0|^p) \left[(p-2)|z|^{p-4}(z,\xi)^2 + |z|^{p-2}|\xi|^2\right] \\ &\quad\ge a(x,|\nabla u_0|^p)(p-2)\frac{1}{2}|z|^{p-2}|\xi|^2 + a(x,|\nabla u_0|^p)|z|^{p-2}|\xi|^2 \\ &\quad\ge \left(\frac{1}{2}(p-2)c_a + c_a\right)|z|^{p-2}|\xi|^2.\end{aligned}$$

2. The next condition follows because either $p \ge 2$, in which case

$$\begin{aligned}\left|\frac{\partial b_i}{\partial z_j}\right| &= \left|a(x, |\nabla u_0|^p) \left[(p-2)|z|^{p-4}(z,\xi)^2 + |z|^{p-2}|\xi|^2\right]\right| \\ &\le C_a(p-1)|z|^{p-2}|\xi|^2,\end{aligned}$$

or because otherwise

$$\begin{aligned}&\left|a(x, |\nabla u_0|^p)\left[(p-2)|z|^{p-4}(z,\xi)^2 + |z|^{p-2}|\xi|^2\right]\right|\\&\qquad\le a(x, |\nabla u_0|^p)\left[(2-p)|z|^{p-2}|\xi|^2 + |z|^{p-2}|\xi|^2\right].\end{aligned}$$

Since $p < 2$ implies $3-p \le p-1$ this is bounded from above by the same term $C_a(p-1)|z|^{p-2}|\xi|^2$.

3. This condition now depends on $v(x)$ and the value M_1.

$$\begin{aligned}&|b(x,z) - b(y,z)|\\&\quad= |z|^{p-1}\left|a(x, |\nabla u_0(x)|^p) - a(y, |\nabla u_0(y)|^p)\right|\\&\quad\le (1+|z|)^{p-1} L\left(|x-y| + ||\nabla u_0(x)| - |\nabla u_0(y)||\right)\\&\quad\le (1+|z|)^{p-2} L\big(|x-y| + \|u_0\|_{C^{1,\alpha}}|x-y|^\alpha\big).\end{aligned}$$

4. The right side is unaffected so this condition remains true.

□

3.6 Mountain Pass Theorem

3.6.1 Palais-Smale Compactness Condition

Theorem 3.42 (Vitali convergence)**.** *Let μ be a finite positive measure on a measure space $(\Omega, \mathcal{A}, \mu)$. Assume that for $\{f_k\} \subset L^1(\mu)$*

- *the integrals of $|f_k|^p$ are uniformly absolutely continuous, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ so that*

$$\forall E \subset \Omega\ \forall k \in \mathbb{N} : |E| < \delta \quad \Longrightarrow \quad \left|\int_E |f_k|^p d\mu\right| < \varepsilon \ \text{ and}$$

- *f_k converges to f pointwise almost everywhere.*

Then $f \in L^p(\mu)$ and

$$\lim_{k\to\infty} \int_X |f_k - f|^p d\mu.$$

Proof. This is a generalised version of the well-known dominated convergence theorem by Lebesgue, see for example Theorem 1.19 in [2]. □

Definition 3.43. *Given a Banach space X and a functional $J : X \to \mathbb{R}$ which is Frechét differentiable, a sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ is called a **Palais-Smale sequence**, if*

- $\sup\limits_{k\in\mathbb{N}} |J(u_k)| < \infty$
- $\lim\limits_{k\to\infty} \|J'(u_k)\|_{X'} = 0$

*The functional is said to satisfy the (weak) **Palais-Smale compactness condition**, if for a Palais-Smale sequence there exists a $u \in X$ such that*

$$\liminf_{k\to\infty} J(u_k) \le J(u) \le \limsup_{k\to\infty} J(u_k) \qquad \text{and} \qquad J'(u) = 0.$$

The following theorem is a simplified and streamlined version of a much more abstract version given in [10].

Theorem 3.44. *Let Assumption 1 on page 42 hold and assume furthermore that*

- *there are positive constants* $\vartheta > 0$, c_{AR} *and* $C_{AR} > 0$, *and functions* $d_{AR}(x)$, $D_{AR}(x)$ *in* $L^1(\Omega)$. *Let there be an* $\alpha \in [1, p)$ *so that for almost any* $x \in \Omega$ *and any* $s \in \mathbb{R}$

$$\frac{1}{p}A(x, s^p) - \vartheta a(x, s^p)s^p \geq c_{AR}|s|^p - d_{AR}(x), \tag{3.5}$$

$$F(x, s) - \vartheta f(x, s)s \leq C_{AR}|s|^\alpha + D_{AR}(x), \tag{3.6}$$

- *there are* $c_a > 0$, $d_a \in L^{\frac{p}{p-1}}(\Omega)$ *so that*

$$a(x, s^p)s^{p-1} \geq c_a s^{p-1} - d_a(x)$$

for $(x, s) \in \Omega \times [0, \infty)$.

Then J *satisfies the Palais-Smale condition.*

Proof. **Boundedness of the Palais-Smale sequence:** Let $u_k \in W_0^{1,p}(\Omega)$ be a Palais-Smale sequence for the functional J, this means there are constants $C, D > 0$ so that

$$C + D\|\nabla u_k\|_{L^p} \geq J(u_k) - \vartheta J'(u_k)u_k$$

for any $k \in \mathbb{N}$.

Thus, with eq. (3.5) and eq. (3.6),

$$\begin{aligned}
C &+ D\|\nabla u_k\|_{L^p} \\
&\geq \int_\Omega \Big(A(x, |\nabla u_k|^p) - \vartheta a(x, |\nabla u_k|^p)|\nabla u_k|^p \\
&\qquad\qquad - \lambda \left(F(x, u_k) - \vartheta f(x, u_k)u_k\right) \Big) dx \\
&\geq \int_\Omega c_{AR}|\nabla u_k|^p - d_{AR}(x) - \lambda(C_{AR}|u_k|^\alpha + D_{AR}(x))dx \\
&= c_{AR}\|\nabla u_k\|_{L^p}^p - \|d_{AR}\|_{L^1} - \lambda\|D_{AR}\|_{L^1} - \lambda C_{AR}\|u_k\|_{L^\alpha}^\alpha.
\end{aligned}$$

Using the Sobolev inequality this implies

$$C + D\|\nabla u_k\|_{L^p} \geq c_{AR}\|\nabla u_k\|_{L^p}^p - \lambda C_S\|\nabla u_k\|_{L^p}^\alpha - c$$

and this shows that $\|\nabla u_k\|_{L^p}$ is bounded since otherwise $p > \max\{1, \alpha\}$ would lead to a contradiction. As $p > 1$ and $W_0^{1,p}(\Omega)$ is reflexive, there exists $u \in W_0^{1,p}(\Omega)$ and a subsequence u_{k_l} so that

$$u_{k_l} \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega).$$

To simplify notation let the subsequence be denoted by u_k in the following.

Pointwise almost everywhere convergence: The function $s \mapsto A(x, s^p)$ is convex for all $x \in \Omega$ and thus $z \mapsto a(x, |z|^p)|z|^{p-2}z$ is monotone for all $(x, z) \in \Omega \times \mathbb{R}^n$ (see Lemmas 2.29 and 3.11). Therefore

$$\begin{aligned} 0 &\leq \mathcal{A}'(u_k)(u_k - u) - \mathcal{A}'(u)(u_k - u) \\ &= J'(u_k)(u_k - u) - J'(u)(u_k - u) \\ &\qquad + \lambda\mathcal{F}'(u_k)(u_k - u) - \lambda\mathcal{F}(u)(u_k - u) \end{aligned} \tag{3.7}$$

Now $J'(u_k) \overset{k\to\infty}{\longrightarrow} 0$ and $(u_k - u)$ is bounded in $W_0^{1,p}(\Omega)$, so the first term goes to 0 for $k \to \infty$. Since $J'(u) \in \left(W_0^{1,p}(\Omega)\right)'$ and $(u_k - u) \rightharpoonup 0$, the second term goes to 0.

By Theorem 2.28 there is a subsequence (which will still be denoted by u_k) so that u_k converges to u in the L^q-norm. Since $\|f(x, u_k(x))\|_{L^{\frac{q}{q-1}}}$ is bounded it follows that

$$\begin{aligned} \lim_{k\to\infty} |\mathcal{F}'(u_k)(u_k - u)| &\leq \lim_{k\to\infty} \int_\Omega |f(x, u_k)||u_k - u| dx \\ &\leq \lim_{k\to\infty} \|f(x, u_k(x))\|_{L^{\frac{q}{q-1}}} \|u_k - u\|_{L^q} \\ &= 0. \end{aligned}$$

$\lim_{k\to\infty} \mathcal{F}(u)(u_k - u) = 0$ follows since u_k converges to u in L^q. Combining these results with eq. (3.7) shows that

$$\Big(a(x, |\nabla u_k|^p)|\nabla u_k|^{p-2}\nabla u_k - a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u\Big)\,(\nabla u_k - \nabla u) \tag{3.8}$$

converges to 0 in L^1 as $k \to \infty$ since it is nonnegative. After potentially passing to a subsequence again it converges to 0 pointwise almost everywhere (see A 1.10 in [2]). Continuity of $s \mapsto a(x, s^p)s^{p-1}$ and strict monotonicity show that $\nabla u_k \xrightarrow{k\to\infty} 0$ pointwise almost everywhere.

Uniform integrability: Let $\varepsilon > 0$ be arbitrary. It has been shown that (3.8) converges to 0 in the L^1-norm. This implies that there is a $k_0 \in \mathbb{N}$ so that the term (3.8) is smaller than or equal to $\frac{\varepsilon}{2}$ for all $k \geq k_0$.

This obviously holds also when integrating over arbitrary $E \subset \Omega$ instead. Since there are only finitely many $k \in \{0, 1, \ldots, k_0 - 1\}$ there is a $\delta > 0$ so that the integral of (3.8) over E is strictly smaller than $\frac{\varepsilon}{2}$ for every measurable set $E \subset \Omega$ with $|E| < \delta$. The growth conditions on a and the Hölder inequality show that for any k and $|E| < \delta$ this implies

$$
\begin{aligned}
&\int_E a(x, |\nabla u_k|^p)|\nabla u_k|^p dx \\
&\quad < \frac{\varepsilon}{2} + \int_E a(x, |\nabla u_k|^p)|\nabla u_k|^{p-1}|\nabla u| dx \\
&\qquad + \int_E a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u(\nabla u_k - \nabla u) dx \\
&\quad \le \frac{\varepsilon}{2} + \int_E \Big(C_a|\nabla u_k|^{p-1} + D_a(x)\Big)\, |\nabla u| dx \\
&\qquad + \int_E \Big(C_a|\nabla u|^{p-1} + D_a(x)\Big)\, |\nabla u_k - \nabla u| dx \\
&\quad \le \frac{\varepsilon}{2} + C_a\|\nabla u_k\|^{p-1}_{L^p(\Omega)} \left(\int_E |\nabla u|^p dx\right)^{\frac{1}{p}} + \int_E D_a(x)|\nabla u| dx \\
&\qquad + \|\nabla u_k - \nabla u\|_{L^p(\Omega)} \left(\int_E \Big(C_a|\nabla u|^{p-1} + D_a(x)\Big)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

The term $\Big(C_a|\nabla u|^{p-1} + D_a(x)\Big)^{\frac{p}{p-1}}$ can be estimated from above by

$$
2^{\frac{p}{p-1}} \max\left\{C_a^{\frac{p}{p-1}}|\nabla u|^p, |D_a|^{\frac{p}{p-1}}\right\}
$$

which is an L^1-function.

Since $\|\nabla u_k\|^{p-1}_{L^p}$, $\|\nabla u_k - \nabla u\|_{L^p}$ are bounded, by potentially decreasing δ

$$
\int_E a(x, |\nabla u_k|^p)|\nabla u_k|^p dx < \tfrac{3}{4}\,\varepsilon
$$

follows. Now

$$
\begin{aligned}
c_a \int_E |\nabla u_k|^p dx &\le \int_E a(x, |\nabla u_k|^p)|\nabla u_k|^p dx + \int_E d_a(x)\nabla u_k dx \\
&\le \frac{3\,\varepsilon}{4} + \left(\int_E |d_a(x)|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \|\nabla u_k\|_{L^p(\Omega)}
\end{aligned}
$$

by the coercivity condition of a. After potentially decreasing δ again, this is smaller than ε. The Vitali convergence theorem thus shows that $\nabla u_k \to \nabla u$ in L^p and $u_k \to u$ in $W_0^{1,p}(\Omega)$.

Since $J' : W_0^{1,p}(\Omega) \to \left(W_0^{1,p}(\Omega)\right)'$ is continuous in the norm topology on $W_0^{1,p}$ it follows that

$$J'(u) = \lim_{k\to\infty} J'(u_k) = 0,$$

and thus J satisfies the Palais-Smale compactness condition. □

3.6.2 Mountain Pass Theorem

Theorem 3.45. *Let $S \subset W_0^{1,p}(\Omega)$ be closed and let its complement $W_0^{1,p}(\Omega) \setminus S$ have more than one connected component. Let v, w be in distinct connected components of $W_0^{1,p}(\Omega) \setminus S$. Assume J satisfies the Palais-Smale condition and*

$$\inf_{u\in S} J(s) > \max\{J(v), J(w)\}.$$

Then there is a critical point $u \in W_0^{1,p}(\Omega)$ with

$$J(u) = \inf_{\varphi\in\Gamma} \max_{t\in[0,1]} J(\varphi(t)) \geq \inf_{v\in S} J(v),$$

where

$$\Gamma := \left\{\varphi \in C^0([0,1]; W_0^{1,p}(\Omega)) \ ; \ \varphi(0) = v, \ \varphi(1) = w\right\}.$$

This theorem is a classical result in the calculus of variations by Ambrosetti and Rabinowitz and also holds in general Banach spaces and not just the restricted setting given here.

Proof. The theorem is proven by contradiction using either the deformation lemma or Ekelands variational principle, a proof can be found for example in [18], Theorem 5.7. □

3.7 Existence of a Weak Mountain Pass Solution

Theorem 3.46. *Let Assumption 1 hold and assume furthermore that*

$$c_a s^{p-1} - d_a(x) \leq a(x, s^p)s^{p-1},$$

for $c_a > 0$ *and* $d_a \in L^1$*. Assume that there are* $\vartheta > 0$*,* $\alpha < p$ *and positive constants* c_{AR}*,* C_{AR} *and* L^1*-functions* d_{AR} *and* D_{AR} *so that*

$$\frac{1}{p}A(x, s^p) - \vartheta a(x, s^p)s^p \geq c_{AR}s^p - d_{AR}(x) \qquad \text{and}$$
$$F(x, s) - \vartheta f(x, s)s \leq C_{AR}s^\alpha + D_{AR}(x).$$

Let $\Omega_1 \subset \Omega$ *be an open set with positive measure so that*

$$\lim_{s\to\infty} \inf_{x\in\Omega_1} \frac{F(x, s)}{s^p} = \infty.$$

Then there is a $\lambda_0 > 0$ *so that for every* $\lambda \in (0, \lambda_0)$ *there is a mountain pass solution* u_λ*.*

Proof. By the growth conditions for a sufficiently large $d > 0$ it follows that $\inf_{u\in S_d} \mathcal{A}(u) \geq c > 0$ where

$$S_d = \left\{u \in W_0^{1,p}(\Omega)\,;\, \|\nabla u\|_{L^p} = d\right\}$$

and since $\mathcal{F}(u)$ is bounded on S_d it follows that $\inf_{u\in S_d} J(u) > 0$ for any $\lambda \in (0, \lambda_0)$ and a sufficiently small λ. The condition

$$\lim_{s\to\infty} \inf_{x\in\Omega_1} \frac{F(x, s)}{s^p} = \infty$$

implies that for every $d > 0$ there is a function u with $\|\nabla u\|_{L^p} > d$ so that $J(u) < 0$. Combined this shows that the functional has a mountain pass geometry.

By Theorem 3.44 it satisfies the Palais-Smale compactness condition, thus it has a mountain pass solution for any sufficiently small λ. $\square$

4 Radially Symmetric Solutions

4.1 Overview

This chapter lays the groundwork for the radial symmetry of ground state solutions in Chapters 5 and 6.

Section 4.2 introduces the spaces of radially symmetric Sobolev functions and shows the relationship between radially symmetric weak solutions of (P) and the associated radially symmetric functional. This formulation can be used to show existence of radially symmetric solutions when it cannot be shown that solutions of (P) have to be radially symmetric.

Section 4.3 introduces the ordinary differential equation belonging to the radially symmetric formulation of the functional J.

Section 4.4 is based on the choice of test function in the proof of Remark 4.1 in [8]. Using this regularity of radially symmetric solutions and information about $\operatorname{sgn}(u')$ and $\operatorname{sgn}(u'')$ is obtained.

Section 4.5 introduces the well-known symmetrization results. It contains the main result of this section which is that if u and the Schwarz symmetrization u^* are critical points of J with $J(u) = J(u^*)$ then $u = u^*$. A similar result can be obtained from Proposition 2.9 in [8], however the proof is very different. Unlike that result it also allows f to depend on x and requires fewer assumptions on a.

4.2 Radially Symmetric Formulation

Definition 4.1. *A set $\Omega \subset \mathbb{R}^n$ is called **radially symmetric** if*

$$\forall\, x \in \Omega,\, y \in \mathbb{R}^n \;:\; |x| = |y| \implies y \in \Omega.$$

*A function $f : \Omega \to \mathbb{R}$ on a radially symmetric set Ω is called **radially symmetric** if*

$$\forall\, x, y \in \Omega \;:\; |x| = |y| \implies f(x) = f(y).$$

Lemma 4.2. *If Ω is a radially symmetric bounded domain with Lipschitz boundary then Ω is either*

- *an annulus $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ with $R_2 > R_1 > 0$ or*
- *a ball $\Omega = B_R(0)$ with $R > 0$.*

Proof. If $\Omega \subset \mathbb{R}^n$ is radially symmetric, then there is a set $I \subset [0, \infty)$ so that

$$\Omega = \bigcup_{r \in I} \partial B_r(0)$$

where $B_0(0) := \{0\}$ for simplicity of notation.

If Ω is a bounded domain then I has to be a finite interval which implies either $\Omega = B_R(0)$ with $R > 0$, or $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ with $0 \leq R_1 < R_2$.

The Lipschitz boundary rules out the possibility that $\Omega = B_R(0) \setminus \{0\}$ which concludes the proof. $\square$

For a given Ω assume now that I is the corresponding interval, that is, for $\Omega = B_R(0)$ let $I = (0, R)$ and for $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ let $I = (R_1, R_2)$.

4.2.1 Radially Symmetric Sobolev Functions

The statements about radially symmetric Sobolev spaces are a simplified version of what can be found in [19]. That paper only deals with the more interesting case $\Omega = B_R(0)$ since the case of the annulus that is included here is comparatively trivial.

Definition 4.3. *Let Ω be a radially symmetric domain with Lipschitz boundary.*

$$W^{1,p}_{rad}(\Omega) =: \left\{ u \in W^{1,p}(\Omega); u \textit{ is radially symmetric} \right\}$$

is the space of radially symmetric Sobolev functions.

Corollary 4.4. *$W^{1,p}_{rad}(\Omega)$ is complete since any sequence has a pointwise almost everywhere converging subsequence which shows that radial symmetry is preserved in the limit.*

Definition 4.5. *For an open interval I using the norm*

$$\|u\|_{W^{1,p}_{n-1}} := \|u\|_{L^p_{n-1}} + \|u'\|_{L^p_{n-1}}$$

where

$$\|u\|_{L^p_{n-1}} := |\partial B_1(0)| \left(\int_I t^{n-1} |u(t)|^p dt \right)^{\frac{1}{p}}$$

define the weighted Sobolev space

$$\begin{aligned} &W^{1,p}\left(I, t^{n-1}\right) \\ &\quad := \left\{ u : I \to \mathbb{R} \ ; \ u \textit{ has a weak derivative, } \|u\|_{W^{1,p}_{n-1}} < \infty \right\}. \end{aligned}$$

Remark 4.6. *This norm is equivalent to the norm mentioned in* [19]*. It was modified so that it is compatible with the $W^{1,p}$ norm used in this paper, so the embedding is an isometric isomorphism.*

Lemma 4.7. *For any $\tilde{u} \in W^{1,p}(I, t^{n-1})$ there is a function $u \in C^0((0, R])$ or $C^0([R_1, R_2])$ (depending on whether Ω is a ball or an annulus) so that $u = \tilde{u}$ almost everywhere in I.*

Proof. In the case of an annulus this follows directly from Morrey's inequality and in the case of a ball it follows from Remark 2.1 in [19]. □

Lemma 4.8. *For every radially symmetric function $u(x) : \Omega \to \mathbb{R}$ there is a function $\tilde{u}(r)$ so that $u(x) = \tilde{u}(|x|)$ for any $x \in \Omega$. A radially symmetric function $u : \Omega \to \mathbb{R}$ is in $W^{1,p}(\Omega)$ if and only if the function $\tilde{u}$ is in $W^{1,p}(I, t^{n-1})$.*

Thus the spaces are isometrically isomorphic with $\|u\|_{W^{1,p}} = \|\tilde{u}\|_{W^{1,p}_{n-1}}$.

Proof. The case of an annulus is trivial and the case of a ball follows from Theorem 2.3 in [19]. □

Remark 4.9. *For $m > 1$ the spaces $W^{m,p}_{rad}(B_R(0))$ and $W^{m,p}((0, R), t^{n-1})$ can be defined analogously and are generally not isomorphic.*

Definition 4.10. *Let*

$$W^{1,p}_{0,rad}(\Omega) = W^{1,p}_0(\Omega) \cap W^{1,p}_{rad}(\Omega),$$

and

$$W^{1,p}_0((0, R), t^{n-1}) := \left\{ u \in W^{1,p}_0((0, R), t^{n-1}) \; ; \; u(R) = 0 \right\}$$

in case of a ball and

$$W_0^{1,p}((R_1,R_2),t^{n-1}) := \left\{u \in W_0^{1,p}((R_1,R_2),t^{n-1})\ ;\ u(R_1)=u(R_2)=0\right\}$$

in case of an annulus with $0<R_1<R_2$. *In both cases the norm is*

$$\|u\|_{W_{0,n-1}}^{1,p} = |\partial B_1(0)|\left(\int_I t^{n-1}|u'(t)|^p dt\right)^{\frac{1}{p}}.$$

This space is then isometrically isomorphic to $W_0^{1,p}(\Omega)$. Since every function in the space $W_0^{1,p}(I,t^{n-1})$ is continuous with the possible exception of the point 0, the conditions $u(R)=0$ and $u(R_1)=u(R_2)=0$ make sense.

Remark 4.11. *In the case of an annulus* $W_0^{1,p}(I,t^{n-1})$ *is isometrically isomorphic to* $W_0^{1,p}(I)$.

The situation is different in the case of a ball $\Omega = B_R(0)$. *The term* r^{n-1} *allows the function* $u(|x|)$ *in* $W_0^{1,p}(\Omega)$ *to have a singularity at the origin which implies that for any function* $u \in W^{1,p}((0,R))$ *the function* $u(|x|)$ *is in* $W^{1,p}(\Omega)$, *but not vice versa. The function* $v(x)=u(|x|)=|x|^{-2}$ *is in* $W^{1,p}(B_R(0))$ *with* $B_R(0)\subset\mathbb{R}^4$, *but* $u(r)$ *is not in* $W^{1,p}((0,R))$.

Definition 4.12. *Let* $u \in C_0^\infty(\Omega)$, *then* $\hat{u}$ *shall refer to the spherical average defined as*

$$\hat{u}(r) := \fint_{\partial B_r(0)} u(x)d\sigma = \frac{1}{S_{n-1}}\int_{A_1} u(X(r,\phi))\Phi(\phi)d\phi$$

with X *as in Definition 2.30 and*

$$\Phi(\phi) = \prod_{i=1}^{n-2}\sin^{n-1-i}(\phi_i).$$

The following lemma is not a complete characterization of the properties of the spherical average, but rather a simple way to obtain the necessary information needed here.

Lemma 4.13. *If* $u \in C_0^\infty(B_R(0))$ *then* $\hat{u}$ *is in*

$$C_r^1([0,R]) := \Big\{ v \in C^1([0,R]); v'(0) = 0, v(R) = 0 \Big\}$$

Proof. By the bounded convergence theorem the uniform boundedness of $u(X(r,\phi))$ and all its derivatives implies that $\hat{u}$ is differentiable for $r \in (0,R)$ with

$$\hat{u}'(r) = \frac{1}{S_{n-1}} \int_{A_1} \left(\frac{d}{dr} u(X(r,\phi)) \right) \Phi(\phi) d\phi. \tag{4.1}$$

With the derivative $\frac{d}{dr} u(X(r,\phi)) = (\nabla u)(X(r,\phi)) \cdot X(1,\phi)$ and $\frac{d}{dr} X(r,\phi) = \frac{X(r,\phi)}{|X(r,\phi)|} = X(1,\phi) =: \vec{n}$ being the outer unit normal vector, the transformation formula shows that

$$\hat{u}'(r) = \frac{1}{r^{n-1} S_{n-1}} \int_{\partial B_r(0)} \nabla u(x) \cdot \vec{n} d\sigma$$

is continuous in $(0,R]$. Using the theorem of Gauß shows

$$\begin{aligned} |\hat{u}'(r)| &= \left| \frac{1}{r^{n-1} S_{n-1}} \int_{B_r(0)} \Delta u(x) dx \right| \\ &\leq \frac{1}{r^{n-1} S_{n-1}} |B_1(0)| r^n \|\Delta u\|_{C^0} \xrightarrow{r \downarrow 0} 0. \end{aligned}$$

Thus $\hat{u}$ is continuously differentiable on $[0,R]$ with $\hat{u}'(0) = 0$.

Since u has a compact support in Ω, there is a $\tilde{R} < R$ so that $u(x) = 0$ for $|x| > \tilde{R}$. This implies by definition that $\hat{u}(r) = 0$ for $r > \tilde{R}$ and thus $\hat{u}(R) = 0$. $\square$

Lemma 4.14. *The space $C^1_r([0,R])$ is dense in $W^{1,p}_0((0,R),t^{n-1})$.*

Proof. Let $\tilde{u} \in W^{1,p}_0((0,R),t^{n-1})$, then the function $u(x) = \tilde{u}(|x|)$ is in $W^{1,p}_0(B_R(0))$. For any $\varepsilon > 0$ there is a $v \in C^\infty_0(B_R(0))$ with $\|u - v\|_{W^{1,p}} < \varepsilon$. Let $\tilde{v}(r,\phi) = v(X(r,\phi))$ and $\hat{v}(r)$ as in eq. (4.1). Then by Lemma 4.13 $\hat{v} \in C^1_r([0,R])$ and this is obviously a subset of $W^{1,p}_0((0,R),t^{n-1})$.

$$\begin{aligned}
&\frac{1}{S_{n-1}}\|\tilde{u} - \hat{v}\|^p_{L^p_{n-1}} \\
&\quad = \int_0^R r^{n-1}\left|\tilde{u}(r) - \frac{1}{S_{n-1}}\int_{A_1}\tilde{v}(r,\phi)\Phi(\phi)d\phi\right|^p dr \\
&\quad = \int_0^R r^{n-1}\left|\frac{1}{S_{n-1}r^{n-1}}\int_{A_1} r^{n-1}\left(\tilde{u}(r) - \tilde{v}(r,\phi)\right)\Phi(\phi)d\phi\right|^p dr \\
&\quad = \int_0^R r^{n-1}\left|\fint_{\partial B_r(0)}(u(x) - v(x))\,d\sigma\right|^p dr.
\end{aligned}$$

Using the integral formulation of Jensens inequality (see Ü2.9 in [2]) for the convex function $|\cdot|^p$ this is smaller than or equal to

$$\int_0^R r^{n-1}\fint_{\partial B_r(0)}|u(x) - v(x)|^p d\sigma dr$$

which is equal to $\frac{1}{S_{n-1}}\|u - v\|^p_{L^p}$ and thus

$$\|\tilde{u} - \hat{v}\|^p_{L^p_{n-1}} \leq \|u - v\|^p_{L^p}.$$

Repeating this for the derivatives shows

$$\begin{aligned}
&\frac{1}{S_{n-1}}\|\tilde{u}' - \hat{v}'\|_{L^p_{n-1}} \\
&= \int_0^R r^{n-1}\left|\frac{1}{S_{n-1}}\int_{A_1}(\tilde{u}'(r) - \frac{d}{dr}\tilde{v}(r,\phi))\Phi(\phi)d\phi\right|^p dr \\
&= \int_0^R r^{n-1}\left|\frac{1}{S_{n-1}}\int_{A_1}(\nabla u(X(r,\phi)) - \nabla v(X(r,\phi)))\cdot\vec{n}\Phi(\phi)d\phi\right|^p dr \\
&= \int_0^R r^{n-1}\left|\fint_{\partial B_r(0)}(\nabla u(x) - \nabla v(x))\cdot\vec{n}d\sigma\right|^p dr \\
&\leq \int_0^R r^{n-1}\fint_{\partial B_r(0)}|\nabla u(x) - \nabla v(x)|^p d\sigma dr \\
&= \frac{1}{S_{n-1}}\|\nabla u - \nabla v\|_{L^p(B_R(0))}.
\end{aligned}$$

Thus

$$\|\tilde{u} - \hat{v}\|_{W^{1,p}_{n-1}} \leq \|u - v\|_{W^{1,p}} < \varepsilon$$

and this shows that C^1_r is dense in $W^{1,p}_0((0,R),t^{n-1})$. □

4.2.2 Radially Symmetric Functional

Generally, to have radially symmetric solutions the problem has to be radially symmetric. Since the formulation is already symmetric in ∇u the remaining requirement is that $|x| = |y|$ has to imply $A(x,s) = A(y,s)$ (analogously for a, F and f).

To avoid further complicating the notation with the introduction of functions $\tilde{A}(|x|,s) = A(x,s)$ it will be assumed in this chapter that the original problem is reformulated in the form $A(|x|,s)$ and correspondingly for a, F and f.

Lemma 4.15. *Let* $u, v \in W^{1,p}_{0,rad}(\Omega)$*,* $\tilde{u}(|x|) = u(x)$ *and* $\tilde{v}(|x|) = v(x)$*. Define*

$$J_r(\tilde{u}) := S_{n-1} \int_I r^{n-1} \left(\frac{1}{p} A(r, |\tilde{u}'(r)|^p) - \lambda F(r, \tilde{u}(r)) \right) dr. \quad (4.2)$$

Then $J(u) = J_r(\tilde{u})$ *and* $J_r : W^{1,p}_0(I, t^{n-1})$ *is well-defined and Frechét differentiable with*

$$J_r'(\tilde{u})\tilde{v} = S_{n-1} \int_I r^{n-1} \left(a(r, |\tilde{u}'|^p) |\tilde{u}'|^{p-2} \tilde{u}'\tilde{v}' - \lambda f(r, \tilde{u})\tilde{v} \right) dr$$

and $J'(u)(v) = J_r'(\tilde{u})\tilde{v}$*.*

Proof. Using $\nabla u(x) = \tilde{u}'(|x|)\frac{x}{|x|}$ which implies $\nabla u(X(r,\phi)) = \frac{d}{dr}\tilde{u}(r)X(1,\phi)$ and $|\nabla u(X(r,\phi))| = |\tilde{u}'(r)|$ it follows that

$$\begin{aligned}
&J(u) \\
=&\int_\Omega \frac{1}{p} A(|x|, |\nabla u|^p) - \lambda F(|x|, u(x)) dx \\
=&\int_{R_1}^R r^{n-1} \int_{A_1} \left(\frac{1}{p} A(r, |\tilde{u}'(r)|^p) - \lambda F(r, \tilde{u}(r)) \right) \Phi(\phi) d\phi dr \\
=&\left(\int_{R_1}^R r^{n-1} \left(\frac{1}{p} A(r, |\tilde{u}'(r)|^p) - \lambda F(r, \tilde{u}(r)) \right) dr \right) \left(\int_{A_1} \Phi(\phi) d\phi \right) \\
=&|\partial B_1(0)| \int_{R_1}^R r^{n-1} \left(\frac{1}{p} A(r, |\tilde{u}'(r)|^p) - \lambda F(r, \tilde{u}(r)) \right) dr \\
=&J_r(\tilde{u})
\end{aligned}$$

and using $\nabla u(x) \cdot \nabla v(x) = \tilde{u}'(|x|)\tilde{v}'(|x|)$ it follows that

$$\begin{aligned}
&J'(u)(v)\\
&= \int_\Omega a(|x|, |\nabla u|^p)|\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda f(|x|, u(x))dx\\
&= \int_{R_1}^R r^{n-1} \int_{A_1} \Big[a(r, |\tilde{u}'|^p)|\tilde{u}'|^{p-2}\tilde{u}'(r)\tilde{v}'(r) - \lambda f(r, \tilde{u}(r))\tilde{v}(r)\Big] d\phi dr\\
&= J_r'(\tilde{u})(\tilde{v}),
\end{aligned}$$

which proves the theorem. □

Proposition 4.16. *The function $u(x) \in W^{1,p}_{0,rad}(\Omega)$ is a critical point of J if and only if $\tilde{u}$ with $\tilde{u}(|x|) = u(x)$ solves $0 = J_r'(\tilde{u})v$ for any $v \in C^1_R(I)$.*

Proof. If u is a critical point of J, then $J'(u)(v) = 0$ for any function $v \in W^{1,p}_0(\Omega)$. This includes $v \in W^{1,p}_{0,rad}(\Omega)$ which by Lemma 4.15 shows $J_r'(\tilde{u})(\tilde{v}) = J'(u)v = 0$ for any $\tilde{v} \in W^{1,p}_0((R_1, R), t^{n-1})$. Thus $\tilde{u}$ is a critical point of J_r.

On the other hand, if $J_r'(\tilde{u})(\tilde{v}) = 0$ for any $\tilde{v} \in W^{1,p}_0((R_1, R), t^{n-1})$ then $J'(u)(v) = 0$ for any $v \in W^{1,p}_{0,rad}(B_R(0))$ which a priori is not enough to show that u is a critical point of J.

Let now $u \in W^{1,p}_{0,rad}(\Omega)$ and $v \in C^\infty_0(\Omega)$. Then $\nabla u = \tilde{u}'(r)X(1, \phi)$ and $\nabla u \cdot \nabla v = \tilde{u}'(r)\frac{d}{dr}v(X(r, \phi))$ since $\frac{d}{dr}v(X(r, \phi)) = \nabla v \cdot X(1, \phi)$.

This implies

$$
\begin{aligned}
&J'(u)v \\
&= \int_0^R r^{n-1} \Bigg[a(r, |\tilde{u}'(r)|^p) |\tilde{u}'(r)|^{p-2} \tilde{u}'(r) \cdot \left(\int_{A_1} \frac{d}{dr} v(X(r,\phi)) \Phi(\phi) d\phi \right) \\
&\qquad\qquad - \lambda f(r, \tilde{u}(r)) \left(\int_{A_1} v(X(r,\phi)) \Phi(\phi) d\phi \right) \Bigg] dr \\
&= S_{n-1} \int_0^R r^{n-1} \Big[a(r, |\tilde{u}'(r)|^p) |\tilde{u}'(r)|^{p-2} \tilde{u}'(r) \hat{v}'(r) \\
&\qquad\qquad - \lambda f(r, \tilde{u}(r)) \hat{v}(r) \Big] dr \\
&= J_r'(\tilde{u}) \hat{v},
\end{aligned}
$$

where $\hat{v}$ is the symmetrization of v as in Definition 4.12. This implies $\hat{v} \in W_0^{1,p}((R_1, R), t^{n-1})$ and $J'(u)v = J_r'(\tilde{u})\hat{v} = 0$ using Lemma 4.14 which proves the theorem. $\square$

4.3 Ordinary Differential Equation

In Proposition 4.16 it has been shown that a radially symmetric function $u(|x|)$ is a critical point of J if and only if $u(r)$ is a critical point of J_r. Critical points of J are weak solutions of (P) and, similarly, critical points of J_r are weak solutions of an ordinary differential equation.

Proposition 4.17. *If u is a critical point of J_r and $\frac{d^2}{ds^2}\frac{1}{p}A(x, s^p)$ exists, then in any neighbourhood where u is twice continuously differentiable and $r > 0$ it satisfies the second order differential* *equation*

$$
\begin{aligned}
&-\lambda f(r,u)\\
&\quad = u''\Big(a(r,|u'|^p)(p-1)+a_s(r,|u'|^p)p|u'|^p\Big)|u'|^{p-2} \\
&\qquad\qquad + a_r(r,|u'|^p)|u'|^{p-2}u' + \frac{n-1}{r}a(r,|u'|^p)|u'|^{p-2}u'.
\end{aligned}
\tag{4.3}
$$

At points where $\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p)_{s=|v'|} > 0$ *this is equivalent to*

$$
u'' = \frac{\Big(a_r(r,|u'|^p)+\frac{n-1}{r}a(r,|u'|^p)\Big)|u'|^{p-2}(-u')-\lambda f(r,u)}{\Big(a(r,|u'|^p)(p-1)+a_s(r,|u'|^p)p|u'|^p\Big)|u'|^{p-2}}. \tag{4.4}
$$

Proof. Let $I_1 \subset I$ be an interval where v is twice continuously differentiable. Integration by parts shows that for any $v \in C_0^\infty(I)$ $J_r'(u)v = 0$ is equivalent to

$$
0 = \int_{I_1}\left(-\frac{d}{dr}\left(r^{n-1}a(r,|u'|^p)|u'|^{p-2}u'\right) - \lambda r^{n-1}f(r,u)\right)vdr
$$

and using the fundamental lemma of calculus this shows

$$
-\frac{d}{dr}\left(r^{n-1}a(r,|u'|^p)|u'|^{p-2}u'\right) = \lambda r^{n-1}f(r,u).
$$

Thus

$$
\begin{aligned}
&-\lambda f(r,u)\\
&\quad = \frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}a(r,|u'|^p)|u'|^{p-2}u'\right)\\
&\quad = u''\Big(a(r,|u'|^p)(p-1)+a_s(r,|u'|^p)p|u'|^p\Big)|u'|^{p-2}\\
&\qquad\qquad + a_r(r,|u'|^p)|u'|^{p-2}u' + \frac{n-1}{r}a(r,|u'|^p)|u'|^{p-2}u'.
\end{aligned}
$$

If $u'(r) \neq 0$ then

$$\left(a(r,|u'|^p)(p-1)+a_s(r,|u'|^p)p|u'|^p\right)=\frac{d^2}{ds^2}\left(\frac{1}{p}A(x,s^p)\right)\bigg|_{s=|u'|} > 0$$

and rearranging the equation and dividing by that term shows eq. (4.4). □

Corollary 4.18 (Weak solution on an annulus)**.** *If* $[R_1, R_2] \subset (0,\infty)$ *and* $u \in C^2((R_1,R_2)) \cap C_0^0([R_1,R_2])$ *is a nonnegative solution of eq.* (4.3)*, then* u *is a critical point of* J_r*.*

Lemma 4.19 (Weak solution on a ball)**.** *Let* $u \geq 0$ *be a classical solution of eq.* (4.3) *in* $(0,R)$ *with* $u(R) = 0$ *and*

$$\lim_{r\downarrow 0} a\left(|u'(r)|^p\right)|u'(r)|^{p-1}r^{n-1} = 0.$$

Then u *is a critical point of* J_r*.*

Proof. Let $v \in C_r^1([0,R])$, then

$$\begin{aligned}
&J_r'(u)(v)\\
&\quad= \int_0^R r^{n-1}\left(a(|u'|^p)|u'|^{p-2}u'v' - \lambda f(r,u)v\right)dr\\
&\quad= \lim_{s\downarrow 0}\Bigg(\left[r^{n-1}a(|u'|^p)|u'|^{p-2}u'v\right]_{r=s}^R\\
&\qquad\quad + \int_s^R\left(-\frac{d}{dr}\left(r^{n-1}a(|u'|^p)|u'|^{p-2}u'\right)-r^{n-1}\lambda f(r,u)\right)v\,dr\Bigg)\\
&\quad= \lim_{s\downarrow 0}\left(-s^{n-1}a(|u'(s)|^p)|u'(s)|^{p-2}u'(s)v(s)\right)\\
&\quad= 0.
\end{aligned}$$

□

4.4 Regularity, Monotonicity and Curvature

Lemma 4.20. *If $t^{n-1} f(t, u(t))$ is integrable on $[r_0, R]$ then*

$$r \mapsto \frac{1}{r^{n-1}} \int_{r_0}^{r} t^{n-1} f(t, u(t)) dt \tag{4.5}$$

is continuous on $(r_0, R]$. If $f(t, u(t))$ is integrable on $[r_0, R]$ then the mapping given by (4.5) is continuous on $[r_0, R]$.

If $f(r, u(t))$ is continuous on $[r_0, R]$ then the mapping given is continuously differentiable for any $r \geq r_0 \geq 0$.

Proof. Continuity for $r > 0$ follows from standard Lebesgue integration theory. If $r_0 = 0$ and $f(t, v(t))$ is integrable then continuity in $r = 0$ follows from the continuity of $\int_0^r |f(t, u(t))| dt$ in $r = 0$ by

$$\lim_{r \to 0} \left| \frac{1}{r^{n-1}} \int_0^r t^{n-1} f(t, u(t)) dt - 0 \right| \leq \lim_{r \to 0} \int_0^r |f(t, u(t))| dt = 0.$$

Let now $f(r, u(r))$ be continuous. For $r > r_0$ it follows that

$$\frac{d}{ds} \left(\frac{1}{r^{n-1}} \int_{r_0}^{r} t^{n-1} f(t, u(t)) dt \right) = f(r, u(r)) - \frac{n-1}{r^n} \int_{r_0}^{r} t^{n-1} f(t, u(t)) dt.$$

If $r_0 > 0$ the limit for $r \to r_0$ is $f(r_0, u(r_0))$. If $r_0 = 0$ then

$$\begin{aligned}
&\frac{1}{r^n}\int_0^r t^{n-1} f(t,u(t))dt \\
&\quad = \frac{1}{r^n}\int_0^r t^{n-1} f(0,u(0))dt \\
&\qquad\quad + \frac{1}{r^n}\int_0^r t^{n-1}\left(f(t,u(t)) - f(0,u(0))\right)dt \\
&\quad = \frac{1}{n} f(0,u(0)) + \frac{1}{r^n}\int_0^r t^{n-1}\left(f(t,u(t)) - f(0,u(0))\right)dt
\end{aligned}$$

and the absolute value of the integral in the last equation is bounded from above by $\max_{t\in[0,r]} |f(t,u(t)) - f(0,u(0))|$ which goes to 0 for $r \to 0$ because $f(t,u(t))$ is continuous at 0.

Thus

$$\begin{aligned}
\lim_{r\to 0}\frac{d}{dr}\left(\frac{1}{r^{n-1}}\int_0^r t^{n-1} f(t,u(t))dt\right) &= f(0,u(0)) - \tfrac{n-1}{n} f(0,u(0)) \\
&= \tfrac{1}{n} f(0,u(0)).
\end{aligned}$$

□

The following lemma uses the idea from Remark 4.10 in [8] for the test function.

Lemma 4.21. *Let Ω and J be radially symmetric and $u(|x|)$ a radially symmetric weak solution.*

If $\Omega = B_R(0)$ then there is a set N of Lebesgue measure 0 so that

$$-r^{n-1} a(r, |u'(r)|^p)|u'(r)|^{p-2} u'(r) = \lambda \int_0^r t^{n-1} f(t,u(t))dt$$

for every $r \in [0, R] \setminus N$.

If $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ *then*

$$\lambda \int_{r_0}^{r_1} r^{n-1} f(r, u(r)) dr = r_0^{n-1} a(r_0, |u'(r_0)|^p) |u'(r_0)|^{p-2} u'(r_0) - r_1^{n-1} a(r_1, |u'(r_1)|^p) |u'(r_1)|^{p-2} u'(r_1)$$

for every $r_0, r_1 \in [R_1, R_2] \setminus N$.

Proof. Let

$$\varphi_{r_0,\varepsilon}(r) := \begin{cases} 1 & \text{if } r \leq r_0 - \varepsilon, \\ \frac{1}{2\varepsilon}(r_0 + \varepsilon - r) & \text{if } r \in (r_0 - \varepsilon, r_0 + \varepsilon), \\ 0 & \text{if } r \geq r_0 + \varepsilon, \end{cases}$$

with $r_0 \in (0, R)$ and $\varepsilon > 0$ so that $[r_0 - \varepsilon, r_0 + \varepsilon] \subset (0, R)$. $\varphi_{r_0,\varepsilon}$ is absolutely continuous with bounded derivatives and therefore it is in $W^{1,\infty}((0, R))$. u is a weak solution and $\varphi_{r_0,\varepsilon}$ is an admissible test function which implies

$$\int_0^R r^{n-1} a(r, |u'|^p) |u'|^{p-2} u' \varphi'_{r_0,\varepsilon}(r) dr = \lambda \int_0^R r^{n-1} f(r, u) \varphi_{r_0,\varepsilon}(r) dr.$$

With

$$\int_0^R r^{n-1} a(r, |u'|^p) |u'|^{p-2} u' \varphi'_{r_0,\varepsilon}(r) dr = \int_{r_0-\varepsilon}^{r_0+\varepsilon} r^{n-1} a(r, |u'|^p) |u'|^{p-2} u' \left(-\frac{1}{2\varepsilon}\right) dr$$

the Lebesgue differentiation theorem (see [16]) shows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{2\,\varepsilon}\int_{r_0-\varepsilon}^{r_0+\varepsilon} &-r^{n-1}a(r,|u'|^p)|u'|^{p-2}u'dr \\ &= -r_0^{n-1}a(r_0,|u'(r_0)|^p)|u'(r_0)|^{p-2}u'(r_0)\end{aligned} \tag{4.6}$$

for almost every $r_0 \in (0,R)$ and thus

$$-r^{n-1}a(r,|u'(r)|^p)|u'(r)|^{p-2}u'(r) = \lambda\int_0^r t^{n-1}f(t,u(t))dt \tag{4.7}$$

for almost every $r \in [0,R]$. By slightly modifying the test function to

$$\varphi_{r_0,r_1\,\varepsilon}(r) := \begin{cases} 1 & \text{if } r \in [r_0+\varepsilon, r_1-\varepsilon], \\ \frac{r_0-\varepsilon}{2\,\varepsilon} - \frac{r}{2\,\varepsilon} & \text{if } r \in (r_0-\varepsilon, r_0+\varepsilon], \\ \frac{r_1+\varepsilon}{2\,\varepsilon} - \frac{r}{2\,\varepsilon} & \text{if } r \in (r_1-\varepsilon, r_1+\varepsilon], \\ 0 & \text{if } r < r_0-\varepsilon \text{ or } r > r_1+\varepsilon, \end{cases}$$

the same argumentation can be applied to the annulus. There is a set N of measure 0 so that the limit in eq. (4.6) exists for any $r \in [0,R] \setminus N$. For any $r_0, r_1 \in [R_1,R_2] \setminus N$ with $r_0 < r_1$ it follows that

$$\begin{aligned}\lambda\int_{r_0}^{r_1} & r^{n-1}f(r,u(r))dr \\ &= r_0^{n-1}a(r_0,|u'(r_0)|^p)|u'(r_0)|^{p-2}u'(r_0) \\ &\quad - r_1^{n-1}a(r_1,|v'(r_1)|^p)|u'(r_1)|^{p-2}u'(r_1).\end{aligned} \tag{4.8}$$

This is still true if $r_0 \geq r_1$ so eq. (4.8) holds for any $r_0, r_1 \in [R_1,R_2] \setminus N$. ☐

Assumption 2. *Ω is radially symmetric, $a(x,s) = a(s)$ and f has the form $f(|x|,s)$.*

Theorem 4.22. *Given Assumption 1 on page 42 and Assumption 2, any radially symmetric weak solution* $u(|x|) \in W_0^{1,p}(\Omega)$ *is in* $C^1(\overline{\Omega} \setminus \{0\})$.

If $\Omega = B_R(0)$ *and* $f(r, u(r))$ *is integrable on* $[0, R]$ *then* u *is continuously differentiable at the origin with* $u'(0) = 0$.

Proof. The function $s \mapsto a\big(|s|^p\big)|s|^{p-2}s$ is strictly increasing and continuous on $\mathbb{R}$ with $\lim_{s\to 0} a\big(|s|^p\big)|s|^{p-2}s = 0$, therefore there exists a strictly increasing and continuous inverse function $b : [0, \infty) \to [0, \infty)$ with $b(0) = 0$.

In case $\Omega = B_R(0)$ by eq. (4.7) implies that the equation

$$-u'(r) = b\left(\tfrac{1}{r^{n-1}}\lambda \int_0^r t^{n-1} f(t, u(t))dt\right) \tag{4.9}$$

holds almost everywhere in $[0, R]$. The right-hand side of eq. (4.9) is continuous in $(0, R]$ which implies that u' can be modified on a set of measure 0 so that the equation holds everywhere in $(0, R]$. Since u is absolutely continuous and u' is now continuous in $(0, R]$ it follows that u is continuously differentiable in $(0, R]$.

If $f(r, u(r))$ is integrable then it has been shown that the right-hand side is continuous in 0 with limit 0 which implies continuity of u' in $r = 0$ with $u'(0) = 0$.

If $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ then by eq. (4.21) for any $r_0, r \in [R_1, R_2] \setminus N$ it follows that

$$\begin{aligned} -u'(r) = b\Big(&-r_0^{n-1} a\big(|u'(r_0)|^p\big)|u'(r_0)|^{p-2}u'(r_0) \\ &+ \frac{1}{r^{n-1}}\lambda \int_{r_0}^r t^{n-1} f(t, u(t))dt\Big). \end{aligned} \tag{4.10}$$

For a fixed r_0 the right-hand side is continuous for any $r \in [R_1, R_2]$ and thus u' can be modified on a set of measure 0 so the identity holds for every $r \in [R_1, R_2]$. As before this implies implies that u is continuously differentiable on $[R_1, R_2]$. □

Theorem 4.23. *Let Assumption 1 on page 42 and Assumption 2 hold and assume that $f(r,s) > 0$ for $s > 0$ and u is a nonnegative, nontrivial weak solution.*

If $\Omega = B_R(0)$ then $u'(r) < 0$ in $(0, R]$ and $u(r) > 0$ in $[0, R)$.

If $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ then there is a unique point $r_1 \in (R_1, R_2)$ so that $u'(r_1) = 0$ and $u'(r) > 0$ in (R_1, r_1) and $u'(r) > 0$ in (r_1, R_2).

Proof. Let $\Omega = B_R(0)$. Then by eq. (4.9) and $f(r,s) \geq 0$ for $s \geq 0$ it follows that $u'(r) \leq 0$. $\lim_{r \downarrow 0} u(r) \in (0, \infty]$ since u is nontrivial and thus $r^{n-1} f(r, u(r)) > 0$ in a small neighbourhood of 0. This implies that the right-hand side of eq. (4.7) is positive for every $r \in (0, R]$ and thus $u'(r) < 0$ in $(0, R]$ and $u(r) > 0$ in $[0, R)$.

If $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ the function u has to have a maximum $u(r) > 0$ which implies $u'(r) = 0$. Let r_1 be that maximum. Equation (4.8) implies that for any $r_0 \in [R_1, R_2]$

$$r_0^{n-1} a(r_0, |u'(r_0)|^p) |u'(r_0)|^{p-2} u'(r_0) = \lambda \int_{r_0}^{r_1} r^{n-1} f(r, u(r)) dr.$$

Since $u(r_1) > 0$ and $f(r_1, u(r_1)) > 0$ this implies $u'(r) > 0$ for any $r < r_1$ and $u'(r) < 0$ for any $r > r_1$. □

Theorem 4.24. *Let Assumption 1 on page 42 and Assumption 2 hold and assume that* $\frac{d^2}{ds^2}\frac{1}{p}A(s^p) > 0$ *in* $\times(0,\infty)$, $f(r,s) > 0$ *for* $s > 0$ *and that* u *is a nonnegative, nontrivial weak solution. Let* $r_0 \geq 0$ *be in* I*.*

- *If* $u'(r_0) \neq 0$ *then* u *is twice continuously differentiable in a neighbourhood of* r_0*.*
- *If* $u'(r_0) = 0$ *and* $\left(\frac{d^2}{ds^2}\frac{1}{p}A(s^p)\right)\Big|_{s=0} = 0$ *then*
$$\lim_{r\to r_0} u''(r) = -\infty.$$
- *If* $u'(r_0) = 0$ *and* $\left(\frac{d^2}{ds^2}\frac{1}{p}A(s^p)\right)\Big|_{s=0} \in (0,\infty)$ *then* u *is twice continuously differentiable in* r_0 *with* $u''(r_0) \in (0,-\infty)$*.*
- *If* $u'(r_0) = 0$ *and* $\lim_{s\downarrow 0}\left(\frac{d^2}{ds^2}\frac{1}{p}A(s^p)\right) = \infty$ *then* u *is twice continuously differentiable in* r_0 *with* $u''(r_0) = 0$*.*

Proof. $\frac{d^2}{ds^2}\frac{1}{p}A(s^p) > 0$ for $s > 0$ implies that $b(s)$ is continuously differentiable for $s \neq 0$. Thus, if $u'(r) \neq 0$, the argument of b in eq. (4.9) in case of a ball (and eq. (4.10) in case of an annulus) is not zero and thus the right-hand side is continuously differentiable, which implies that u' is continuously differentiable.

Let now $u'(r_0) = 0$. If $r_0 = 0$ then u is bounded and the right-hand side of eq. (4.7) is also continuously differentiable in $r = 0$. Thus the left-hand side is continuously differentiable and the derivative
$$-\frac{d^2}{ds^2}\frac{1}{p}A\left(s^p\right)\Big|_{s=|u'(r)|} u''(r) \tag{4.11}$$
at $r_0 = 0$ has to be equal to $\frac{1}{n}f(0,u(0))$ which is positive. The result now follows directly from the fact that the term (4.11) has

to be positive for $r \to r_0$. The argument for the annulus and the point r_0 with $u'(r_0) = 0$ is identical. $\square$

Proposition 4.25. *Let Assumptions 1 and 2 on page 42 and page 93 hold and assume furthermore that* $f(x, s) > 0$ *for* $s > 0$ *and* $f(R, 0) = 0$.

Let $\Omega = B_R(0)$, $f(R, 0) = 0$ *and* $u(|x|)$ *be a nontrivial, nonnegative and bounded weak solution. Then* $u(r)$ *is concave in a neighbourhood of* 0 *and convex in a neighbourhood of* R.

Let $\Omega = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$, $f(R_2, 0) = 0$ *and* $u(|x|)$ *be a nontrivial, nonnegative and bounded weak solution. Then there is an* $r_0 \in (R_1, R_2)$ *with* $u'(r_0) = 0$ *so that* u *is concave in* $(R_1, r_0]$ *and convex in a neighbourhood of* R_2.

Proof. In case of a ball the derivative of the right-hand side of eq. (4.7) is positive for $r > 0$ sufficiently close to 0 which implies that $u'' < 0$ in a neighbourhood of 0, but possibly not at 0.

Since $u'(R) < 0$ and $f(R, u(R)) = f(R, 0) = 0$ the enumerator in the right hand side of eq. (4.4) is positive which shows that u is convex in a neighbourhood of R.

This also holds in the case of an annulus at R_2. Since $u'(r) > 0$ in (R_1, r_0) the enumerator in the right hand side of eq. (4.4) is negative which shows strict concavity in $[R_1, r_1]$ with $u'(r_1) = 0$. $\square$

4.5 Symmetrization Methods

4.5.1 Lopes symmetrization

The origin of the following method is [35], although uniqueness and not regularity was used to prove symmetry. The modification using regularity that is presented here is certainly not new, however I am not aware of the exact origin.

Theorem 4.26 (Lopes)**.** *Let Ω be a ball or an annulus, let J be radially symmetric ($a(x, s)$ can depend on $|x|$ here) and every critical point of $J(u)$ be continuously differentiable. Then any minimizer $u \in W_0^{1,p}(\Omega)$ has to be radially symmetric.*

Proof. By picking an arbitrary hyperplane T through 0 there are two open and disjoint sets so that $\Omega = \Omega_1 \cup T \cup \Omega_2$.

T has Lebesgue measure 0 so the functional is the sum of the integral on Ω_1 and on Ω_2. Without loss of generality assume that the integral over Ω_1 is less than or equal to the integral over Ω_2. Taking u on Ω_1 and reflecting it at T leads to a new function $\tilde{u} \in W_0^{1,p}(\Omega)$ (see Corollary 2.21) so that $J(\tilde{u}) \leq J(u)$. Since u is already the minimizer this implies equality and $\tilde{u}$ also has to be a critical point. By the assumption on regularity $\tilde{u}$ has to be continuously differentiable as well. Let now $x \in \Omega$ and T be a hyperplane going through 0 and x. The regularity of the mirrored function implies that the directional derivative of u in the normal direction of the hyperplane has to be 0. By rotating the hyperplane around the line through 0 and x it implies that $\nabla u(x)$ is parallel to x.

Consider the great circle that is given by $T \cap \partial B_{|x|}(0)$. Let $\gamma : [0,1] \to T \cap \partial B_{|x|}(0)$ be a parameterization of that great circle so that γ is bijective on $[0,1)$ and $\gamma(0) = \gamma(1)$.

Then $\frac{d}{dt} u(\gamma(t)) = \nabla u(\gamma(t)) \cdot \gamma'(t)$ and because ∇u is normal to $\partial B_{|x|}(0)$ while $\gamma'(t)$ is tangential to it, this has to be 0 everywhere. Thus u is constant on any great circle which proves radial symmetry. $\square$

4.5.2 Moving Plane Method

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary, $\gamma \in \mathbb{R}^n$ be a unit vector and T_α the hyperplane with $\gamma \cdot x = \alpha$.

Starting with α large enough so that $T_\alpha \cap \overline{\Omega} = \emptyset$ the hyperplane is moved until it touches $\partial\Omega$ at α_0. By moving it further and designating $\Sigma(\alpha) \subset \Omega$ the part of Ω that has now been traversed by T_α and $\Sigma'(\alpha)$ the reflection of $\Sigma(\alpha)$ at T_α the smooth boundary of Ω shows that $\Sigma'(\alpha) \subset \Omega$ if T_α is not moved too far.

Let α_1 be the first (and thus largest) α where either T_α is orthogonal to $\partial\Omega$ or $\Sigma'(\alpha)$ is tangent to $\partial\Omega$.

Theorem 4.27 (Gidas, Ni & Nirenberg). *Let $u \in C^2(\overline{\Omega} \cap \{x_1 > \alpha_1\})$ with $u = 0$ on $\partial\Omega \cap \{x_1 > \alpha_1\}$ and $u > 0$ in Ω a solution of*

$$F(x, u, u_1, \dots, u_k, u_{11}, u_{12}, \dots, u_{nn}) = 0.$$

Let $F \in C^1(\overline{\Omega} \times (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^{n\times n})$ with $M \geq m > 0$ so that

$$M|\xi|^2 \geq F_{u_{ij}} \xi_i \xi_j \geq m|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n\,.$$

Let $\gamma = (1, 0, \ldots, 0)$ and assume further that for all $\alpha_1 \leq \alpha < \alpha_0$, $x \in \Sigma(\alpha)$ and x^α the reflection of x at T_α, $u > 0$, $p_1 < 0$ and arbitrary p_α with $\alpha > 1$ and p_{ij} with $i, j \geq 2$ the inequality

$$F(x^\alpha, u, -p_1, p_2, \ldots, p_k, p_{11}, -p_{1\alpha}, \ldots, p_{\beta\gamma}) \geq F(x, u, p_1, p_\alpha, p_{ij})$$

holds.

Let the function $g(x) = F(x, 0, \ldots, 0)$ satisfy either $g(x) \geq 0$ for all x or $g(x) < 0$ for all x on $\partial\Omega \cap \{x_1 > \lambda_1\}$.

For every $\alpha \in (\alpha_1, \alpha_0)$ and $x \in \Sigma(\alpha)$ it follows that $u_{x_1}(x) < 0$ and $u(x) < u(x^\alpha)$. u has to be symmetric in T_{λ_1} if $u_{x_1}(x) = 0$ at a point $x \in \Omega \cap T_{\lambda_1}$.

Proof. See Theorem 2.1' in [22]: □

Theorem 4.28. *Let $a(r, s) = a(s)$, $r \mapsto f(r, u)$ be nonincreasing, u be a bounded weak solution of eq. (4.3) and $\frac{d^2}{ds^2}\frac{1}{p}A(x, s^p) > 0$ in $\overline{\Omega} \times [0, \infty)$. Let $u(r_0) = \max_{r \in (R_1, R_2)} u(r)$.*

For any $h \in (0, u(r_0))$ there are exactly two points $r_1, r_2 \in (R_1, R_2)$ with $r_1 < r_2$ so that $u(r_1) = u(r_2) = h$ and in that case $u'(r_1) \geq |u'(r_2)|$ and $r_0 \leq \frac{r_1 + r_2}{2}$.

If f does not depend on r then $r_0 < \frac{r_1 + r_2}{2}$.

Proof. The one-dimensional differential equation (4.3) on $[R_1, R_2]$ with $[R_1, R_2] \subset (0, \infty)$ can be written as

$$\begin{aligned} 0 \stackrel{!}{=} & F(x, u, u', u'') \\ = & u''\Big(a\big(|u'|^p\big)(p-1) + a_s\big(|u'|^p\big)p|u'|^p\Big)|u'|^{p-2} \\ & + \tfrac{n-1}{r} a\big(|u'|^p\big)|u'|^{p-2}u' + \lambda f(r, u). \end{aligned}$$

Then $F_{u_{ij}} = \frac{d^2}{ds^2}\frac{1}{p}A(x,s^p)$ and since $u \in C^2([R_1,R_2])$ and v' is bounded there are c_e and C_e so that $0 < c_e \le F_{u_{ij}} \le C_e$ for the relevant values.

The term $F(r^\alpha, u, -p_1, p_{11})$ is equal to

$$p_{11}\left(a(|p_1|^p)(p-1) + a_s(|p_1|^p)p|p_1|^p\right)|p_1|^{p-2} \\ + \frac{n-1}{r^\alpha}a(|p_1|^p)|p_1|^{p-2}(-p_1) + \lambda f(r^\alpha, u).$$

Using that

$$\frac{n-1}{r^\alpha}a(|p_1|^p)|p_1|^{p-2}(-p_1) \ge 0 \ge \frac{n-1}{r}a(|p_1|^p)|p_1|^{p-2}p_1$$

since $p_1 < 0$, and that $\lambda f(r^\alpha, u) \ge \lambda f(r,u)$ since $r^\alpha \le r$, it follows that

$$F(r^\alpha, u, -p_1, p_{11}) \ge F(r, u, p_1, p_{11}).$$

Theorem 4.23 shows that there are exactly two points so that $u(r_1) = u(r_2) = h$. In order to use Theorem 4.27 u is supposed to be zero on the boundary points, but no condition is given for $u \mapsto f(r,u)$. Therefore it is possible to replace f with $f(r, u+h)$ and u with $u-h$ and since $u-h > 0$ in (r_1, r_2) the moving plane method is applicable. The hyperplane (which is just a point here) can be moved to $r_3 := \frac{r_1+r_2}{2}$.

As a direct consequence, $u(r_1 + r) \ge u(r_2 - r)$ for any $r \in \left[0, \frac{r_2-r_1}{2}\right]$ and therefore $r_0 \le \frac{r_2+r_1}{2}$. Since $u(r_1) = u(r_2)$ it can be further deduced that

$$\begin{aligned} u'(r_1) &= \lim_{r\downarrow 0}\frac{u(r_1+r)-u(r_1)}{r} \\ &\ge \lim_{r\downarrow 0}\frac{u(r_2-r)-u(r_2)}{r} \\ &= -u'(r_2). \end{aligned}$$

If f does not depend on r and $r_0 = \frac{r_1+r_2}{2}$ then Theorem 4.27 states that u is symmetric about r_0 which implies that $u'(r_2) = -u'(r_1)$ and $u''(r_2) = u''(r_1)$. Looking at Eq. (4.4) this implies that

$$\begin{aligned} 0 &< \frac{n-1}{r_1}\Big(a(|u'(r_1)|)|u'(r_1)|^{p-2}u'(r_1)\Big) \\ &= \frac{n-1}{r_2}\Big(a(|u'(r_2)|)|u'(r_2)|^{p-2}u'(r_2)\Big) \\ &< 0 \end{aligned}$$

which is a contradiction and therefore $r_0 < \frac{r_1+r_2}{2}$. $\square$

There are extensions to the p-Laplace equation and there are extensions to the setting considered here, however it requires knowledge of the set

$$\{x \in \Omega; \nabla u = 0\}$$

which is generally not available. In case of the p-Laplace equation [14] can show that the method works for $u = \phi$ on $\partial\Omega$ provided ϕ has a certain strict monotonicity. [14] shows a moving plane method for the p-Laplace in case $1 < p < 2$. [15] shows this for $p > 2$. Those results also require a priori knowledge that the solutions are C^1 which excludes cases similar to Theorem 5.1 or Theorem 6.3 where none of the regularity theorems can guarantee C^1-regularity for solutions that are not radially symmetric.

4.5.3 Schwarz Symmetrization

An overview of the method of Schwarz symmetrization can be found in [26].

Let $\Omega \subset \mathbb{R}^N$ be bounded and $u : \Omega \subset \mathbb{R}^k \to \mathbb{R}$ a nonnegative function. Let $\Omega^* = B_R(0)$ so that $|\Omega| = |B_R(0)|$. Define E_t to be the superlevel sets of u

$$E_t = \{x \in \Omega; u(x) > t\}\,, \qquad \mu(t) = |E_t|$$

with $|E_t|$ being the Lebesgue measure of E_t.

Definition 4.29. *The* ***Schwarz symmetrization*** *of u, also called* ***spherically symmetric decreasing rearrangement****, is defined by*

$$u^*(x) := \sup\{t \geq 0; \mu(t) > \alpha(n)|x|^n\}$$

with $\alpha(n) = |B_1(0)|$.

Theorem 4.30.

1. *The function u^* is radially symmetric and monotone decreasing in the radial direction and measurable.*
2. *u^* and u are* ***equimeasurable****, meaning their superlevel sets have the same measure: $|E_t| = |E_t^*|$.*
3. *Symmetrization is idempotent: $(u^*)^* = u^*$.*
4. *Symmetrization is invariant under translation:*

$$\forall\, c \in \mathbb{R} \;:\; (u(x+c))^* = u^*(x).$$

5. *Symmetrization is invariant under scaling:*

$$(tu)^*(x) = t(u^*(x)) \quad \textit{for any } t \geq 0.$$

Proof. These are standard results and follow directly from the definition. □

The following theorem follows from Corollary 5.2 in [25] and generalizes the well-known statement that $\int_\Omega f(u(x))dx$ does not change under symmetrization. The equality condition is from Theorem 6.1 in [24].

Theorem 4.31 (Hajaiej)**.** *If $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ so that the conditions in Assumption 1 on page 42 hold and $r \mapsto f(r,s)$ is nonincreasing and nonnegative for almost every $s \geq 0$, then*

$$\int_{B_R(0)} F(|x|, u(x))dx \leq \int_{B_R(0)} F(|x|, u^*(x))dx$$

for any nonnegative $u \in W_0^{1,p}(\Omega)$.

If $r \mapsto f(r,s)$ is strictly decreasing for almost every $s \geq 0$ then equality implies that $u = u^*$.

Theorem 4.32 (Generalised Pólya-Szegő-inequality)**.** *Let $A : [0,\infty) \to [0,\infty)$ be strictly convex with $A(0) = 0$. Then for any nonnegative, weakly differentiable function $u : \mathbb{R}^n \to \mathbb{R}$ with compact support the function u^* is also weakly differentiable and*

$$\int_{\mathbb{R}^n} A(|\nabla u^*|)dx \leq \int_{\mathbb{R}^n} A(|\nabla u|)dx. \tag{4.12}$$

If, furthermore, the set

$$\{x \in \mathbb{R}^n; \nabla u^*(x) = 0 \text{ and } u^*(x) \notin \{0, \|u^*\|_\infty\}\}$$

has Lebesgue measure 0 (with $\|u^*\|_\infty \leq \infty$) then equality in (4.12) implies that u^* is almost everywhere equal to a translate of u.

This theorem is a generalised version of the Pólya-Szegő inequality. The case of equality in the Pólya-Szegő inequality was first studied by Kawohl in [26]. A version for C^2 functions with stricter regularity assumptions in the equality case was shown in [9], while this general form was shown by Cianchi and Fusco in [12], see Theorems 1.4 and 1.5. As a consequence $u \in W_0^{1,p}(B_R(0))$ implies that $u^* \in W_0^{1,p}(B_R(0))$.

Remark 4.33. *It can be easily seen that if u^* is a critical point then it has to be strictly decreasing if $f(x,s) > 0$ for $s > 0$ and u^* is absolutely continuous since $u^* \in W_0^{1,p}$. However this is not enough to show that*

$$\{x \in \mathbb{R}^n; \nabla u^*(x) = 0 \text{ and } u^*(x) \notin \{0, \|u^*\|_\infty\}\}$$

has Lebesgue measure 0.

While the set of critical values has to have Lebesgue measure 0 by the theorem of Morse-Sard for any $W_0^{1,p}([0,R])$-function with $p > 1$ (see [17]), the same is not true for the set of critical points. This is obvious for constant functions, but even for strictly decreasing absolutely continuous functions the set of critical points can have positive measure.

[47] *shows the existence of a strictly increasing and absolutely continuous function f whose inverse function is not absolutely continuous by showing that the set of critical points of f has positive Lebesgue measure.*

Theorem 4.34. *Let Assumptions 1 and 2 on page 42 and on page 93 hold and assume that $f(r,s) > 0$ for every $(r,s) \in [0,R] \times (0,\infty)$ and that $r \mapsto f(r,s)$ is nonincreasing for every $s \geq 0$.*

If u and the Schwarz symmetrization u^* are critical points of (P) *with $J(u) = J(u^*)$ and $\Omega = B_R(0)$ then $u = u^*$.*

Proof. By Theorems 4.31 and 4.32 it follows that $J(u^*) \leq J(u)$ by assumption equality implies the equality case in both theorems. By Theorem 4.23 it follows that $\nabla u^* \neq 0$ unless $x = 0$ and $u^* > 0$ in $B_R(0)$. By Theorem 4.32 it follows that $u = u^*$ up to translation but with $u^* > 0$ in $B_R(0)$ and $u = u^* = 0$ on $\partial B_R(0)$ □

Remark 4.35. *This a similar result to Proposition 2.1 in [8] which states that if u and u^* are solutions of* (P) *and*

$$\int_{t_1<u<t_2} a(|\nabla u|^p)|\nabla u|^p dx = \int_{t_1<u^*<t_2} a(|\nabla u|^p)|\nabla u|^p dx$$

with a and f independent of x and several additional condition assumed to hold for a, then u is equal to u^* up to translation.

If u and u^* are critical points then $J'(u)u = 0$ and $J'(u^*)u^* = 0$ and since

$$\int_\Omega f(u)udx = \int_\Omega f(u^*)u^*dx \tag{4.13}$$

it follows that

$$\int_{0<u<\infty} a(|\nabla u|^p)|\nabla u|^p dx = \int_{0<u^*<\infty} a(|\nabla u|^p)|\nabla u|^p dx$$

and thus $u = u^*$.

This does not work if f depends on x since equality in (4.13) *is no longer guaranteed and it seems unclear if the condition can be recovered from $J(u) = J(u^*)$.*

This application seems to be new as usually uniqueness of solutions is used to show radial symmetry via Schwarz symmetrization.

5 Minimizers

In this chapter existence of solutions is shown by using the direct method in the calculus of variations. In all cases considered here the existence of radially symmetric solutions is trivial and those solutions can be found by studying the functional (4.2) directly. A more interesting question is whether the ground state solutions of (P) (i.e. the solutions that minimize the energy among nontrivial and nonnegative solutions) are radially symmetric.

The main results in this chapter will show two different ways to prove radial symmetry of ground state solutions.

In the cases considered here uniqueness of solutions can only occur under additional constraints, such as among positive minimizers or among C^2 ground state solutions. In general there seem to be no uniqueness results of sufficient generality for this framework.

For example, [39] shows the existence of at least 3 solutions in a setting that is compatible with Theorem 5.1. Theorem 3.15 in [41] shows the existence of an infinite sequence of positive solutions in a setting that is compatible with Theorem 5.1.

Example 7.6 demonstrates the existence of two classical solutions in the settings of Theorems 3.46 and 5.3.

This shows that there can be no uniqueness result without significant additional restrictions on the admissible problems.

5.1 Existence and Symmetry for a Coercive Functional

Theorem 5.1. *Let Assumption 1 on* *page 42* *hold and assume there are* $c_A > 0$ *and* $d_A \in L^1(\Omega)$ *so that*

$$c_A s^p - d_A(x) \leq \tfrac{1}{p} A(x, s^p)$$

in $\Omega \times [0, \infty)$*. Assume that for any* $\varepsilon > 0$ *there is a* $D_F \in L^1$ *such that*

$$|F(x, s)| \leq \varepsilon \, |s|^p + D_F(x)$$

in $\Omega \times [0, \infty)$*.*

- *If there is a nonnegative function* $w \in W_0^{1,p}(\Omega)$ *so that* $\int_\Omega F(x, w(x))dx > 0$ *and either* $f(x, 0) = 0$ *or* $f(x, s) > 0$ *for* $s > 0$ *then for any* $\lambda > 0$ *the functional has a ground state solution* $u_\lambda \in W_0^{1,p}(\Omega)$ *with*

$$\liminf_{\lambda \to \infty} \|u_\lambda\|_{L^p} > 0 \qquad \text{and} \qquad \lim_{\lambda \to \infty} J(u_\lambda) = -\infty.$$

- *If additionally* $\Omega = B_R(0)$*,* $a(x, s) = a(s)$*,* $x \mapsto F(x, s)$ *is radially symmetric and nonincreasing in the radial direction and* $f(x, s) > 0$ *in* $\Omega \times (0, \infty)$ *then* u_λ *is radially symmetric with* $u_\lambda^* = u_\lambda$*.*

Proof. ∘ The function f can be modified according to Section 3.4 which implies that any minimizer obtained for the modified functional has to be nonnegative. As a nonnegative critical point it will also be a critical point of the original functional (although no longer necessarily a local or global minimizer). Thus it can be assumed that minimizers are nonnegative.

ε can be chosen so that $c_A - \lambda \varepsilon C_S > 0$ which implies that

$$\begin{aligned} J(u) &\geq c_A \|\nabla u\|_{L^p}^p - \|d_A\|_{L^1} - \lambda \left(\varepsilon \|u\|_{L^p}^p + \|D_F\|_{L^1}\right) \\ &\geq (c_A - \lambda C_S \varepsilon) \|\nabla u\|_{L^p}^p - \|d_A\|_{L^1} - \lambda \|D_F\|_{L^1} \end{aligned}$$

goes to infinity for $\|\nabla u\|_{L^p} \to \infty$. The functional is therefore coercive and has a bounded minimizing sequence $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$. Since $p > 1$ there is a weakly converging subsequence $u_{k_l} \rightharpoonup u_\lambda \in W_0^{1,p}(\Omega)$. The functional is weakly lower semicontinuous which implies

$$J(u_\lambda) \leq \lim_{l\to\infty} J(u_{k_l}) = \inf_{v \in W_0^{1,p}(\Omega)} J(v)$$

and thus u_λ is minimizer of J and a critical point.

No condition has been specified to exclude trivial minimizers for a given λ. Following an argument from [41] it will now be shown that $\lambda \mapsto \mathcal{F}(u_\lambda)$ is nondecreasing which implies that $\liminf_{\lambda\to\infty} \|u_\lambda\|_{L^p} > 0$. First $\lim_{\lambda\to\infty} J(u_\lambda) = -\infty$ since

$$J(u_\lambda) \leq J(w) \leq C_A \|\nabla w\|_{L^p}^p + \|D_A\|_{L^1} - \lambda \int_\Omega F(x, w)dx$$

and this goes to $-\infty$ for $\lambda \to \infty$. Let now $\lambda_1 < \lambda_2$ and u_{λ_1} and u_{λ_2} be the respective global minima. Assume that $\mathcal{F}(u_{\lambda_2}) < \mathcal{F}(u_{\lambda_1})$, the function u_{λ_1} is the minimizer for λ_1 which implies that

$$\mathcal{A}(u_{\lambda_2}) - \lambda_1 \mathcal{F}(u_{\lambda_2}) \geq \mathcal{A}(u_{\lambda_1}) - \lambda_1 \mathcal{F}(u_{\lambda_1}).$$

Using this and the assumption $\mathcal{F}(u_{\lambda_2}) < \mathcal{F}(u_{\lambda_1})$, it follows that

$$\begin{aligned} \mathcal{A}(u_{\lambda_2}) - \lambda_2 \mathcal{F}(u_{\lambda_2}) &= \mathcal{A}(u_{\lambda_2}) - \lambda_1 \mathcal{F}(u_{\lambda_2}) - (\lambda_2 - \lambda_1) \mathcal{F}(u_{\lambda_2}) \\ &> \mathcal{A}(u_{\lambda_1}) - \lambda_1 \mathcal{F}(u_{\lambda_1}) - (\lambda_2 - \lambda_1) \mathcal{F}(u_{\lambda_1}) \\ &= \mathcal{A}(u_{\lambda_1}) - \lambda_2 \mathcal{F}(u_{\lambda_1}) \end{aligned}$$

This contradicts the fact that u_{λ_2} should be the minimizer for λ_2. Thus the assumption is false and $\mathcal{F}(u_\lambda)$ is nondecreasing in λ.

If there were a sequence $\lim_{k\to\infty} \lambda_k = \infty$ with $\lim_{k\to\infty} \|u_{\lambda_k}\|_{L^p} = 0$ it would follow that $\lim_{k\to\infty} \mathcal{F}(u_{\lambda_k}) = 0$ by continuity of $\mathcal{F}$. This has been shown to be false which implies

$$\liminf_{\lambda\to\infty} \|u_\lambda\|_{L^p} > 0.$$

∘ In this case Schwarz symmetrization implies that $J(u_\lambda^*) = J(u_\lambda)$ since u_λ is a global minimizer (of the potentially modified functional) and therefore u_λ^* is also a minimizer and a critical point of the functional.

Theorem 4.23 implies $\nabla u_\lambda^*(x) \neq 0$ unless $x = 0$. By Theorem 4.32 this implies $u_\lambda^* = u_\lambda$ up to translation. The boundary condition and $u_\lambda^* > 0$ in $B_R(0)$ exclude the possibility of a translation and therefore $u_\lambda^* = u_\lambda$. □

Remark 5.2. *If Ω is an annulus then Schwarz symmetrization cannot work, however under additional assumptions for regularity the radial symmetry of the minimizer could be shown using Lopes symmetrization. An example can be seen in the next section in Theorem 5.3.*

5.2 Existence and Symmetry of Smooth Minimizers

Theorem 5.3. *Assume that there are $0 < c_e < C_e$ so that*

$$c_e s^{p-2} \leq \frac{d^2}{ds^2}\frac{1}{p}A(x, s^p) \leq C_e s^{p-2}$$

for $(x,s) \in \Omega \times (0,\varepsilon]$. *Let*

$$\liminf_{s \downarrow 0} \inf_{x \in \Omega} \frac{F(x,s)}{s^p} = \infty$$

and there is a $C_f > 0$ *so that* $|f(x,s)| \leq C_f$ *for any* $(x,s) \in \Omega \times [0,\varepsilon]$.

∘ *Then there is a* $\lambda_0 > 0$ *so that for any* $\lambda \in (0,\lambda_0)$ *the problem has a solution* $u_\lambda \in C^{1,\beta}(\overline{\Omega})$ *with* $\beta \in (0,\alpha)$ *and* $\lim_{\lambda \to 0} \|u_\lambda\|_{C^1} = 0$.

∘ *If* J *is radially symmetric and* Ω *is radially symmetric, then* u_λ *is radially symmetric.*

Proof. ∘ The proof is analogous to Theorem 3.1 in [41]. The function A can be extended to $\Omega \times (0,\infty)$ following Section 3.4. f can be replaced according to Section 3.4.3 so that f remains identical on $\Omega \times \left[0, \frac{\varepsilon}{2}\right]$ and is 0 for $s \geq \varepsilon$. This implies that F is bounded on $\Omega \times [0,\infty)$. f can be modified to be 0 for $s < 0$ to guarantee that the minimizer is nonnegative. The resulting critical point is also a critical point of the original functional if the C^1-norm is sufficiently small and while it may no longer be a minimizer it will be a ground state solution.

By the assumptions on A and the extension it follows that

$$\frac{c_e}{p-1} \leq a(x,s) \leq \frac{C_e}{p-1}$$

and

$$\frac{c_e}{(p-1)} \leq A(x,s) \leq \frac{C_e}{(p-1)}$$

for any $(x,s) \in \Omega \times (0,\infty)$. Thus

$$J(u) \geq \frac{c_e}{p(p-1)} \|\nabla u\|_{L^p}^p - \lambda C$$

which is positive on S_d for a sufficiently large d. The functional is bounded on the set

$$\left\{u \in W_0^{1,p}(\Omega); \|\nabla u\|_{L^p} \leq d\right\}$$

which implies that there is a bounded minimizing sequence u_k. There is a subsequence u_{k_l} which converges weakly against a function u_λ and, because the norm is weakly lower semicontinuous, it follows that $\|\nabla u_\lambda\|_{L^p} \leq d$. Since $\mathcal{A}$ is convex and $\mathcal{F}$ is compact the functional is weakly lower semicontinuous which implies $J(u_\lambda) \leq J(w) < 0 < \inf_{u \in S_d} J(u)$. Therefore $\|\nabla u_\lambda\|_{L^p} < d$ and thus u_λ is a nontrivial critical point of the functional.

Lemma 3.13 implies that $u_\lambda \leq \varepsilon$. The structure conditions in Theorem 3.39 only depend on λ in the upper bound of $\lambda f(x,s)$ where $|s| \leq M := \varepsilon$. This implies that for any λ_0 there are α and M_α so that $\|u_\lambda\|_{C^{1,\alpha}} \leq M_\alpha$ for every $\lambda \in (0, \lambda_0)$.

With $0 = J'(u_\lambda)(u_\lambda)$ it follows that

$$\begin{aligned}\lambda C \geq \lambda \int_\Omega f(x, u_\lambda) u_\lambda dx &= \int_\Omega a(x, |\nabla u_\lambda|^p) |\nabla u_\lambda|^p dx \\ &\geq \frac{c_e}{p-1} \|\nabla u_\lambda\|_{L^p}^p\end{aligned}$$

and thus $\lambda \to 0$ implies that $\|u_\lambda\|_{W^{1p}} \to 0$.

Assuming now there is a sequence $\lambda_k \to 0$ so that $\|\nabla u_k\|_{C^{1,\beta}}$ with $\beta \in (0, \alpha)$ remains bounded away from 0, the compact embedding $C^{1,\beta} \subset\subset C^{1,\alpha}$ implies that there is a subsequence u_{λ_k} which converges in $C^{1,\beta}$. However it converges to 0 in $W_0^{1,p}$ so the limit function can only be 0 which contradicts the assumption of staying bounded away from 0. This proves $\lim_{\lambda \to 0} \|u_\lambda\|_{C^{1,\beta}} = 0$. Thus for sufficiently small λ the function u_λ is also a critical point of the original functional.

Since Ω is open there is an open subset Ω_2 of positive measure so that $\overline{\Omega}_2 \subset \Omega_1$. This implies $\operatorname{dist}(\partial\Omega_1, \Omega_2) > 0$. Thus there exist a nonnegative function $w \in C_0^\infty(\Omega)$ with $w|_{\Omega_2} \equiv 1$ which implies

$$J(sw) \leq \int_\Omega C_a s^p |\nabla w|^p dx - \lambda \int_\Omega F(x, sw) dx.$$

$F(x, s) \geq 0$ for s small enough, therefore

$$\int_\Omega F(x, sw) dx \geq \int_{\Omega_2} F(x, s) dx = s^p \int_{\Omega_2} \frac{F(x, s)}{s^p} dx.$$

By the assumptions s can be made small enough so that

$$\inf_{x \in \Omega_2} \frac{F(x, s)}{s^p} \geq 2 \frac{C_a \|\nabla w\|_{L^p}^p}{\lambda |\Omega|}$$

which implies $J(sw) < 0$ and thus the minimizer is not trivial.

∘ The minimizer of the modified problem is a global minimizer, thus, if Ω and J are radially symmetric, the fact that all critical points are in C^1 implies the symmetry of the minimizer via Lopes symmetrization. □

Remark 5.4. *Since nothing is said about the behaviour of A and F at infinity it is possible that u_λ is not the ground state solution or that the ground state solution is not radially symmetric. However this can only happen if there are solutions that are not in C^1 or if the C^1-norm is large.*

Remark 5.5. *There are many variations of this theorem. The condition on $\frac{d^2}{ds^2} A(x, s^p)$ can be replaced by*

$$c_A s^p - d_A(x) \leq \tfrac{1}{p} A(x, s^p),$$

Assumption 1 on page 42, as well as

$$\liminf_{s \downarrow 0} \inf_{x \in \Omega} \frac{F(x,s)}{s^{\tilde{p}}} = \infty$$

and

$$A(x, s^p) \leq C s^{\tilde{p}}$$

for a $\tilde{p} > 0$. *In this case the functional also has a minimizer for* $\lambda \in (0, \lambda_0)$.

If $\Omega = B_R(0)$ *then radial symmetry can be shown analogously to Theorem 5.1.*

6 Main Results

6.1 Introduction and Main Theorems

The central results in this dissertation were inspired by the paper [40] concerning a regularity and decay estimate for the prescribed mean curvature equation and similar equations. The most general formulation is found in [36]; adapted to the terminology in this dissertation the result of that paper is:

Theorem 6.1 (Lorca & Ubilla)**.** *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded domain with $C^{1,1}$-boundary. If there is a $d > 0$ so that the continuous functions a and f defined on $[0, \infty)$ satisfy*

(f_1) there exist a constant C and $2 < q$ with $q < \frac{2n}{n-2}$ if $n > 2$ so that

$$u^{q-1} \leq f(u) \leq Cu^{q-1} \qquad \text{for all } u \in [0, d].$$

(f_2) There exists a $\delta > 2$ such that

$$\delta F(u) \leq uf(u) \qquad \text{for all } u \in [0, d],$$

where F is defined by $F(u) = \int_0^u f(s)ds$.

(f_3) The function $f(u)/u$ is increasing on the interval $[0, d]$.

(a_1) The function a is nonincreasing such that

$$a(t) \geq a(d) > 0 \qquad \text{for all } t \in [0, d].$$

Then there is a $\Lambda = \Lambda(\Omega, f, a)$ *such that for* $\lambda > \Lambda$ *the boundary value problem*

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^2)\nabla u\right) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

has a nontrivial, nonnegative C^1*-solution* u_λ *and*

$$\|u_\lambda\|_{W^{2,n+1}} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Remark 6.2. *[36] considers the differential equation in the form*

$$-\operatorname{div}\left(a(|\nabla u|)\nabla u\right) = \lambda f(u)$$

which leads to a slightly different a*. However it should be noted that* $a(s^2)$ *is nonincreasing on* $[0, \infty)$ *if and only if* $a(s)$ *is nonincreasing, so the formulation given here is equivalent.*

The idea of the proof is to extend the functions a and f from $[0, d]$ to $[0, \infty)$ in a way that preserves their behaviour and construct an operator T_λ in the following way. Let

$$\mathcal{N} = \{u \in W_0^{1,p}(\Omega) \setminus \{0\}; J'(u)u = 0\}$$

be the Nehari manifold, then for arbitrary $u = u_1 \in \mathcal{N} \cap W^{2,n+1}$ there is a unique solution w_1 of

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^2)\nabla w\right) = \lambda f(u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

It can be shown that there is a unique $t_1 > 0$ so that $t_1 w_1 \in \mathcal{N}$. Using regularity theory for the linear, uniformly elliptic equation (6.2) it follows that $t_1 w_1 \in W^{2,n+1}$. It can then be shown that $J(t_1 w_1) < J(u_1)$ unless u_1 is a solution of the original problem. Repeating this iteration with $u = u_2 := t_1 w_1$ and obtaining $u_3 := t_2 w_2 \in \mathcal{N} \cap W^{2,n+1}$ it can be shown that there is a fixed number of iterations k so that for every $u_1 \in S_\lambda$ where

$$S_\lambda := \Big\{ u \in \mathcal{N} \cap W^{2,n+1} \; ; \; \|u_k\|_{W^{2,n+1}} \leq C_1 \lambda^{-\frac{1}{q-1}} \\ \text{and } J(u_k) \leq C_2 \lambda^{-\frac{2}{q-1}} \Big\}$$

the function u_k will be in S_λ as well. Thus the operator $T_\lambda : S_\lambda \to S_\lambda$, $T(u_1) = u_k$ is well-defined.

S_λ is closed and there is a $u_\lambda \in S_\lambda$ so that $J(u_\lambda) = \inf_{u \in S_\lambda} J(u)$. Because $J(T(u)) \leq J(u)$ with equality if and only if u solves (6.2) it follows that

$$J(u_\lambda) \leq J(T(u_\lambda)) \leq J(u_\lambda)$$

and thus there has to be equality and u_λ is a critical point of (6.1). This also shows that $\|u_\lambda\|_{W^{2,n+1}} \leq C_1 \lambda^{-\frac{1}{q-1}}$.

It should be noted that (6.1) has the form (P) with $p = 2$ and the original idea was to generalize this result to $p > 1$. While almost everything still works for $p > 1$, the modified problem becomes

$$\begin{cases} -\operatorname{div}\Big(a(x, |\nabla u|^2)|\nabla w|^{p-2}\nabla w\Big) = \lambda f(x,u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

and is neither linear, nor uniformly elliptic. It could not be shown that $u \in W^{2,n+1}$ implies $w \in W^{2,n+1}$ and if $u \in C^{1,\alpha}$ it could only be shown that $w \in C^{1,\beta}$ where $\beta < \alpha$ without sufficient control of the norms, preventing the original approach from working in the case $p \neq 2$.

6.1.1 Schwarz Symmetrization

The specific structure of the functional J survives the generalization to $p > 1$ and it can be used to prove radial symmetry of ground state solutions via Schwarz symmetrization. The obvious idea using Schwarz symmetrization for mountain pass solutions is to apply it pointwise to every path. Since $J(\gamma^*(t)) \leq J(\gamma(t))$ it would follow that only considering radially symmetric paths could not lead to an increase in energy. While a non-symmetric mountain pass solution could exist, there would have to be a radially symmetric mountain pass solution with identical energy. The reason why this approach usually does not work is that $t \mapsto \gamma^*(t)$ is generally not a continuous function and therefore no longer an admissible path. The mountain pass theorem can be applied to J_r and this proves existence of a radially symmetric solution. However it is possible that the radially symmetric solution has a higher energy than the general mountain pass solution.

These problems can be prevented given specific assumptions and it can even be shown that ground state solutions must be radially symmetric:

Theorem 6.3. *Let Assumption 1 hold and let* $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 2$, $p > 1$. *Assume that* $|x| = |y|$ *implies* $f(x,s) = f(y,s)$ *for every* $x, y \in B_R(0)$ *and* $s \geq 0$ *and assume that* a *does not depend on* x *and is a continuous function on* $[0, \infty)$. *Assume furthermore that*

($f_{1,s}$) there are positive constants $C_{F,q}$, $C_{F,\tilde{q}}$, $c_{F,\hat{q}}$, *and a function* $d_F(x) \in L^1(\Omega)$ *so that*

$$c_{F,\hat{q}} s^{\hat{q}} - d_F(x) \leq F(x,s) \leq C_{F,\tilde{q}} s^{\tilde{q}} + C_{F,q} s^q$$

for $(x,s) \in \Omega \times (0,\infty)$, *where* $p < \hat{q}$ *and* $p < \tilde{q} < q$. *If* $p < n$, *then let* $q < \frac{np}{n-p}$.

($f_{2,s}$) There is a $\vartheta < \frac{1}{p}$ so that

$$F(x,s) - \vartheta f(x,s)s \leq C_{AR}s^{\alpha} + D_{AR}(x)$$

with $D_{AR}(x) \in L^1(\Omega)$ and $\alpha < p$, $C_{AR} > 0$.

($f_{3,s}$) For almost every $x \in B_R(0)$ the function $s \mapsto \frac{f(x,s)}{s^{p-1}}$ is nonnegative and nondecreasing for $s \geq 0$ and $f(x,s) = 0$ for all $(x,s) \in \Omega \times (-\infty, 0]$.

($f_{4,s}$) For every $s \geq 0$ the function $x \mapsto F(x,s)$ is nonincreasing in the radial direction in the sense that $|x| \leq |y| < R$ implies $F(y,s) \leq F(x,s)$.

($a_{1,s}$) a is continuous in $[0,\infty)$ and nonincreasing with

$$C_a \geq a(s) \geq c_a > 0.$$

($a_{2,s}$) The function $a(s^p)s^{p-1}$ is strictly increasing in $[0,\infty)$ with

$$\lim_{s\to 0} a(s^p)s^{p-1} = 0.$$

Assume furthermore that either $s \mapsto \frac{f(x,s)}{s^{p-1}}$ is strictly increasing or $a(s)$ is strictly decreasing. Then (P) *has a nontrivial ground state solution u_λ for every $\lambda > 0$ and every ground state solution u is in $C^1(\overline{B_R(0)}) \cap C^2(\overline{B_R(0)} \setminus \{0\})$, radially symmetric and decreasing in the radial direction.*

Similar to the original theorem ($f_{3,s}$) and ($a_{1,s}$) are important for the specific structure of the functional, where for every nontrivial and nonnegative $u \in W_0^{1,p}(\Omega)$ there can only be one $t > 0$ so that $tu \in \mathcal{N}$. To show that there is such a t the condition ($f_{1,s}$) is required as well. ($a_{2,s}$) makes the differential equation elliptic and is necessary for the Pólya-Szegő-inequality. ($f_{2,s}$) is necessary for the Palais-Smale condition.

Example 6.4. *The function*

$$F(x,s) = \begin{cases} \frac{3}{2}s^4 & \text{for } s \le 1, \\ \frac{1}{2}s^6\left(1+\sin(s-1)\right) + \frac{1}{2}s^5 & \text{for } s > 1, \end{cases}$$

satisfies the assumptions of Theorem 6.3 with $p = 3$, $\hat{q} = 5$, $\tilde{q} = 4$, $q = 6$, $n = 5$.

This shows that the growth conditions on F *specified in* $(f_{1,s})$ *significantly increase the class of admissible functions compared to* (f_1).

Corollary 6.11 on page 129 shows that (f_3) implies the Ambrosetti-Rabinowitz condition in $(f_{2,s})$ with $\vartheta = \frac{1}{p}$, however $(a_{1,s})$ only implies $\frac{1}{p}A(x,s^p) - \vartheta a(x,s^p)s^p \ge c_{AR}s^p - d_{AR}$ for $\vartheta < \frac{1}{p}$ which is why $(f_{2,s})$ is necessary.

6.1.2 Regularity and Decay Estimate

The second result is

Theorem 6.5. *Assume that* $\Omega \subset \mathbb{R}^n$ *is a bounded domain with* $C^{1,1}$*-boundary,* $n \ge 1$ *and* $p > 1$. *Assume that* $a_s(x,s)$ *and* $f(x,s)$ *are Carathéodory functions and there is an* $\varepsilon > 0$ *so that*

$(f_{1,r})$ *there exist positive constants* c_f *and* C_f *and* $q > p$ *with* $q < \frac{np}{n-p}$ *if* $p < n$ *so that*

$$c_f u^{q-1} \le f(x,u) \le C_f u^{q-1} \qquad \text{for all } u \in [0,\varepsilon].$$

$(f_{2,r})$ *There is a* $\vartheta > 0$ *so that*

$$F(x,s) - \vartheta f(x,s)s \le 0 \qquad \text{for all } s \in [0,\varepsilon].$$

($a_{1,r}$) There is a $c_{AR} > 0$ so that

$$\frac{1}{p}A(x,s^p) - \vartheta a(x,s^p)s^p \geq c_{AR}s^p \text{ for all } (x,s) \in \Omega \times [0,\varepsilon].$$

($a_{2,r}$) There are $0 < c_e < C_e$ so that

$$c_e s^{p-2} \leq \frac{d^2}{ds^2}A(x,s^p) \leq C_e s^{p-2} \qquad \text{for all } (x,s) \in \Omega \times (0,\varepsilon].$$

($a_{3,r}$) The function $a(x,s^p)$ is Lipschitz continuous on $\overline{\Omega} \times [0,\varepsilon]$ and $a_s(x,\varepsilon^p)$ is Lipschitz continuous on $\overline{\Omega}$.

Then there is a $\lambda_0 > 0$ so that the boundary value problem (P) *has a nontrivial, nonnegative solution $u_\lambda \in C^{1,\omega}$ with $\omega \in (0,1)$ and for every $\eta \in \left(0, \frac{1}{q-p}\right)$, there are $M > m > 0$ so that*

$$m\lambda^{-\frac{1}{q-p}} \leq \|u_\lambda\|_{C^{1,\omega}} \leq M\lambda^{-\eta}$$

for every $\lambda \geq \lambda_0$.

This result is very similar to Theorem 6.1, however there are considerable changes in the assumptions and the proof is completely different. The assumption ($a_{2,r}$) replaces assumption (a) and while they are completely independent, the examples given in [36, 40] all satisfy ($a_{2,r}$) while several common differential operators satisfy ($a_{2,r}$) but not (a). Condition ($a_{1,r}$) is implied by (a) and required for the Palais-Smale condition, so it has to be explicitly stated here. ($f_{1,r}$) and ($f_{2,r}$) are necessary for the energy estimates, which prevents ($f_{2,r}$) from being replaced by ($f_{2,s}$) that is used in Theorem 6.3. The condition (f_3) or a variant thereof is not needed here.

The authors of [36] state that: “We point out that though we may apply the Mountain Pass Theorem to obtain a solution of the modified problem $(\hat{P})_\lambda$, we will not have control on the norm of $W^{2,N+1}(\Omega)$ and hence will not be able to return to the original problem." ($(\hat{P})_\lambda$ in that paper is an equivalent formulation of (6.1) in Theorem 6.1)

Indeed no way has been found to show that solutions are in $W^{2,n+1}$ and that this norm can be controlled. However the C^1-norm can be shown to become small and this is enough to allow the return to the original problem.

It should be mentioned that C^2-regularity cannot be expected in general, since in the case of $\Omega = B_R(0)$ it can be seen that solutions cannot be in $C^2(B_R(0))$ for $p > 2$ (see Theorem 4.24).

Solutions u_λ of the model problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda u^{q-1}$$

have the scaling property $u_\lambda = \lambda^{-\frac{1}{q-p}} u_1$, where u_1 is the solution for $\lambda = 1$. This shows that $\eta = \frac{1}{q-p}$ would be the optimal decay constant for the norms of solutions. Examination of the calculations show that $\eta = \frac{1}{q-p}$ can be chosen if $p \geq n$ or if $q - 1 < \frac{p^2}{n-p}$. The inability to choose $\eta = \frac{1}{q-p}$ in other cases is possibly an artefact of the strategy used in the proof and might be overcome by more careful study and adaptation of the regularity results.

Remark 6.6. *While Corollary 6.11 shows that ($f_{3,s}$) implies ($f_{2,r}$) with $\vartheta = \frac{1}{p}$, the reverse is not true. This can be seen by adding a positive peak to f that has almost no area. This does not change F much so that $F(x,s) - \vartheta f(x,s)s < 0$ can remain true, however if the peak is at s_0 and sufficiently large then $\frac{f(x,s_0)}{s_0^{p-1}}$ will be larger than $\frac{f(x,s_0+\varepsilon)}{(s_0+\varepsilon)^{p-1}}$.*

6.2 Proof of the Symmetry Result

Lemma 6.7. *$(a_{1,s})$ implies that for $\vartheta < \frac{1}{p}$ and*

$$c_{AR} := c_a \left(\frac{1}{p} - \vartheta\right) > 0$$

it follows that

$$\frac{1}{p}A(x, s^p) - \vartheta a(x, s^p)s^p \geq c_{AR}s^p \quad \textit{for every } (x, s) \in \Omega \times [0, \infty)$$

and

$$\frac{c_a}{p}s^p \leq \frac{1}{p}A(s^p) \leq \frac{C_a}{p}s^p \quad \textit{for every } (x, s) \in \Omega \times [0, \infty).$$

Proof. Using that $s \mapsto a(x, s)$ is nonincreasing it follows that

$$\begin{aligned}\frac{1}{p}A(x, s^p) - \vartheta a(x, s^p)s^p &= \int_0^s a(x, t^p)t^{p-1}dt - \vartheta a(x, s^p)s^p \\ &\geq a(x, s^p)\int_0^s t^{p-1}dt - \vartheta a(x, s^p)s^p \\ &\geq c_a\left(\frac{1}{p} - \vartheta\right)s^p.\end{aligned}$$

The other inequality follows immediately from $c_a \leq a(s) \leq C_a$. □

6.2.1 Structure of the Functional

The central component in the proof is

Proposition 6.8. *Given $(a_{1,s})$, $(f_{3,s})$ and $(f_{1,s})$, for every nonnegative, nontrivial $u \in W_0^{1,p}(\Omega)$ there is a unique $t > 0$ so that $tu \in \mathcal{N}$ and this is the unique maximum of $t \mapsto J(tu)$ for $t \geq 0$.*

This is similar to Step 2 in the proof of Theorem 1.2 in [40].

Proof. By $(f_{1,s})$ and Lemma 6.7

$$\begin{aligned}
\frac{c_a}{p} t^p \|\nabla u\|_{L^p}^p - \lambda \left(C_{F,q} t^q C_S^q \|\nabla u\|_{L^p}^q + C_{F,\tilde{q}} t^{\tilde{q}} C_S^{\tilde{q}} \|\nabla u\|_{L^p}^{\tilde{q}} \right)& \\
\leq \frac{c_a}{p} t^p \|\nabla u\|_{L^p}^p - \lambda \left(C_{F,q} t^q \|u\|_{L^q}^q + t^{\tilde{q}} C_{F,\tilde{q}} \|u\|_{L^{\tilde{q}}}^{\tilde{q}} \right)& \\
\leq J(tu)& \\
\leq \frac{C_a}{p} t^p \|\nabla u\|_{L^p}^p - \lambda \left(c_F t^{\hat{q}} \|u\|_{L^{\hat{q}}}^{\hat{q}} + \|d_F\|_{L^1} \right).&
\end{aligned}$$

This is positive for small enough t, negative for large t and 0 for $t = 0$. Thus $t \mapsto J(tu)$ needs to have at least one critical point which is a global maximum. As a side note it shows that

$$J(u) \geq \frac{c_a}{p} t^p d^p - \lambda \left(C_{F,q} t^q C_S^q d^q + C_{F,\tilde{q}} t^{\tilde{q}} C_S^{\tilde{q}} d^{\tilde{q}} \right) \tag{6.3}$$

for $\|\nabla u\|_{L^p} = d$ which shows that $J(u) \geq c > 0$ on

$$S_d = \left\{ u \in W_0^{1,p}(\Omega)\,; \|\nabla u\|_{L^p} = d \right\}$$

for sufficiently small $d > 0$.

A critical point of $t \mapsto J(tu)$ implies that

$$0 = J'(tu)u = \tfrac{1}{t} J'(tu)(tu).$$

Assume $J(tu)$ has two critical points $0 < s < t$. The function a is decreasing and with $J'(su)(su) = 0$ and $J'(tw)(tw) = 0$ the chain of equations

$$\begin{aligned}
\left(\frac{s}{t}\right)^p \int_\Omega \lambda f(x,tu)tudx &= \left(\frac{s}{t}\right)^p \int_\Omega a\big(t^p|\nabla u|^p\big)|\nabla tu|^p dx \\
&\le \left(\frac{s}{t}\right)^p \int_\Omega a\big(s^p|\nabla u|^p\big)|\nabla tu|^p dx \\
&= \int_\Omega a\big(s^p|\nabla u|^p\big)|\nabla su|^p dx \\
&= \int_\Omega \lambda f(x,su)sudx
\end{aligned}$$

follows.

At a point x so that $u(x) \in (0,\infty)$ condition $(f_{3,s})$ implies

$$\begin{aligned}
f(x,su(x))su(x) &= \frac{f(x,su(x))}{(su(x))^{p-1}} s^p u^p(x) \\
&\le \frac{f(x,tu(x))}{(tu(x))^{p-1}} s^p u^p(x) \\
&= \left(\frac{s}{t}\right)^p f(x,tu(x))tu(x).
\end{aligned}$$

If $u(x) = 0$, the outer inequality

$$f(x,su(x))su(x) \le (\tfrac{s}{t})^p f(x,tu(x))tu(x)$$

is trivially true.

This can now be combined with the previous computation to obtain

$$\begin{aligned}
\left(\frac{s}{t}\right)^p \int_\Omega \lambda f(x,tu)tudx &\le \int_\Omega \lambda f(x,su)sudx \\
&\le \left(\frac{s}{t}\right)^p \int_\Omega \lambda f(x,tu)tudx.
\end{aligned}$$

By assumption, either a is strictly decreasing or $\frac{f(x,s)}{s^{p-1}}$ is strictly increasing. Since $f(x, u(x))$ is not zero everywhere one of the inequalities has to be strict and it follows that

$$\left(\frac{s}{t}\right)^p \int_\Omega \lambda f(x, tu)tudx < \left(\frac{s}{t}\right)^p \int_\Omega \lambda f(x, tu)tudx$$

which is a contradiction.

Therefore there is a unique positive critical point of $t \mapsto J(tu)$ in $(0, \infty)$ and it is a maximum. □

6.2.2 Constructing Special Paths for the Mountain Pass Theorem

The idea that enables the Schwarz symmetrization is constructing a very specific path that does not encounter the problem discussed in [1] which is that the pointwise Schwarz symmetrization of an admissible path in the mountain pass theorem is generally not continuous. Otherwise this could be used to show $J(\gamma^*(t)) \leq J(\gamma(t))$ and symmetry of mountain pass solutions would easily follow in most situations.

What will be done instead for arbitrary u is constructing a path from 0 to a u_0 through u in three separate steps:

1. Use $t \mapsto tu$ with $t \in [0, c_w]$
2. Connect $c_w u$ and $c_w u_0$ using the straight line $t(c_w u) + (1 - t)(c_w u_0)$
3. Connect $c_w u_0$ and u_0 using $t \mapsto tu_0$

If c is chosen large enough, $J(\gamma(t))$ is positive only on the first part and combining this with Proposition 6.8 enables Schwarz symmetrization.

Lemma 6.9. *Let $u_0 \in W_0^{1,p}(\Omega)$ be a nonnegative radially symmetric function that is strictly decreasing in the radial direction so that $J(tu_0) < 0$ for $t > 1$. Then for every nonnegative, nontrivial $u \in W_0^{1,p}(\Omega)$ there is a $c_w > 1$ so that*

$$\gamma(t) = \begin{cases} tu, & t \in [0, c_w], \\ (t-c_w)c_w u_0 + (1-(t-c_w))c_w u, & t \in (c_w, c_w+1], \\ c_w \frac{c_w+1}{t} u_0, & t \in (c_w+1, c_w(c_w+1)]. \end{cases}$$

is a continuous path connecting 0 *with* u_0 *so that* $J(\gamma(t)) < 0$ *for* $t \in (c_w, c_w(c_w+1)]$.

Proof. If u and u_0 are linearly dependant then $\gamma(t)$ for $t > c_w$ simply repeats parts of the path $\gamma_{|[0,c_w]}$ and $J(\gamma(t)) < 0$ follows trivially if c_w is chosen large enough. Therefore it can be assumed that u and u_0 are not linearly dependant. Thus $\inf_{t\in[0,1]} \|v(t)\|_{L^{\hat{q}}}^{\hat{q}} =: c_v > 0$ for $v(t) = tu + (1-t)u_0$. As a consequence the term

$$\frac{\|\nabla v(t)\|_{L^p}^p}{\|v(t)\|_{L^{\hat{q}}}^{\hat{q}}}$$

is well-defined for $t \in [0,1]$ and has a maximum $m > 0$ at a point $t_0 \in [0,1]$.

Since

$$\begin{aligned} J(c_w v(t)) &\leq C_a c_w^p \|\nabla v(t)\|_{L^p}^p - \lambda c_{F,\hat{q}} c^{\hat{q}} \|v(t)\|_{L^{\hat{q}}}^{\hat{q}} + \lambda \|d_F\|_{L^1} \\ &\leq c_w^p \|v(t)\|_{L^{\hat{q}}}^{\hat{q}} \left(C_a m - \lambda c_{F,\hat{q}} c_w^{\hat{q}-p} \right) + \lambda \|d_F\|_{L^1} \end{aligned}$$

and $\|v(t)\|_{L^{\hat{q}}}^{\hat{q}} \geq c_v > 0$ the constant $c_w > 1$ can be chosen large enough so that $J(c_w v(t))$ is negative for every $t \in [0,1]$.

With a c_w chosen this way it follows that $J(\gamma(t)) < 0$ for $t \in (c_w, c_w+1]$ and since $c_w > 1$ it follows that $J(\gamma(t)) < 0$ for $t \in (c_w+1, c_w(c_w+1)]$. $\square$

Lemma 6.10. *$(f_{3,s})$ implies that*

$$F(x,t) - F(x,s) \geq \frac{1}{p} f(x,s) s^{1-p}(t^{p-1} - s^{p-1}) \tag{6.4}$$

for every $s > 0$, $t \geq 0$ and almost all $x \in \Omega$.

This lemma is a generalization of a result that can be found in Step 3 of the proof of Theorem 1.2 in [36].

Proof. Let $0 < s \leq t$, then

$$\frac{d}{ds}\left(F\left(x, s^{\frac{1}{p}}\right)\right) = \frac{1}{p} \frac{f\left(x, s^{\frac{1}{p}}\right)}{\left(s^{\frac{1}{p}}\right)^{p-1}} \leq \frac{1}{p} \frac{f\left(x, t^{\frac{1}{p}}\right)}{\left(t^{\frac{1}{p}}\right)^{p-1}} = \frac{d}{dt}\left(F\left(x, t^{\frac{1}{p}}\right)\right)$$

follows from the Nehari condition $(f_{3,s})$.

It has been shown that the derivative is increasing, therefore the function $s \mapsto F(x, s^{\frac{1}{p}})$ is convex and continuously differentiable on $(0, \infty)$. Thus

$$F\left(x, t^{\frac{1}{p}}\right) - F\left(x, s^{\frac{1}{p}}\right) \geq \frac{1}{p} f\left(x, s^{\frac{1}{p}}\right) s^{\frac{1}{p}-1}(t-s)$$

for all $s, t > 0$ and equivalently

$$F(x,t) - F(x,s) \geq \frac{1}{p} f(x,s) s^{1-p}(t^p - s^p)$$

for all $s, t > 0$ and almost all $x \in \Omega$. Since $t \mapsto F(x,t)$ and t^{p-1} are continuous in $[0, \infty)$, the inequality extends to $t = 0$. $\square$

Corollary 6.11. *For $t = 0$ in inequality (6.4) it follows that*

$$-F(x,s) \geq -\frac{1}{p} f(x,s)s$$

and thus

$$F(x,s) - \frac{1}{p} f(x,s)s \leq 0$$

for all $s \geq 0$ and almost all $x \in \Omega$.

Proposition 6.12. *Given $(f_{1,s})$, $(f_{2,s})$, $(f_{3,s})$, $(a_{1,s})$ and $(a_{2,s})$ there is a mountain pass solution $u_\lambda \in W_0^{1,p}(\Omega)$ so that*

$$J(u_\lambda) = \inf_{u \in \mathcal{N}} J(u).$$

Proof. By Lemma 6.7, Corollary 6.11, $(f_{1,s})$ and $(f_{2,s})$ as well as $a(s^p)s^p \geq c_a s^p$ the functional satisfies the Palais-Smale compactness condition following Theorem 3.44. By the inequality (6.3) it follows that $\inf_{u \in S_d} J(u) \geq c_1 > 0$ if d is chosen small enough. Let u be an arbitrary, nonnegative, nontrivial and radially symmetric function which decreases in the radial direction. Then by the growth conditions there is a $t_0 > 0$ so that for $u_0 := tu$ it follows that $\|t\nabla u_0\|_{L^p} > d$ and $J(tu_0) < 0$ for $t \geq 1$.

Since $J(0) = 0$ this shows the functional has a mountain pass geometry and therefore there is a mountain pass solution $u_\lambda \in W_0^{1,p}(B_R(0))$ for every $\lambda > 0$.

The mountain pass solution minimizes the energy in the Nehari manifold. Otherwise there would be a $u \in \mathcal{N}$ with $J(u) < J(u_\lambda)$ and using Proposition 6.8 and Lemma 6.9 there would be an admissible path γ_0 with

$$J(\gamma_0(t)) \leq J(u) < J(u_\lambda) = \inf_{\gamma \in \Gamma} \max_t J(\gamma(t))$$

which is a contradiction. □

6.2.3 The Modified Problem

Now a detour is necessary in order to be able to show that the Schwarz symmetrization of u_λ is also a solution of the problem. This uses the method of the modified problem from [36, 40].

For a given fixed $u \in \mathcal{N}$ consider

$$\begin{cases} -\operatorname{div}\left(a(x, |\nabla u|^p)|\nabla w|^{p-2}\nabla w\right) = \lambda f(x,u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

This is the Euler-Lagrange equation of the functional

$$J_u(w) = \int_\Omega \frac{1}{p} a(x, |\nabla u|^p)|\nabla w|^p - \lambda f(x,u) w dx$$

with

$$J'_u(w)v = \int_\Omega a(x, |\nabla u|^p)|\nabla w|^{p-2}\nabla w \cdot \nabla v - \lambda f(x,u) v dx.$$

Lemma 6.13. *Given* $(a_{1,s})$ *and* $(f_{1,s})$*, the problem* (6.5) *has a unique weak solution*

$$w \in W_0^{1,p}(\Omega) \setminus \{0\}$$

for every $u \in \mathcal{N}$*.*

This is essentially a simple application of the direct method in the calculus of variations.

Proof.

$$J_u(tw) = t^p \left(\int_\Omega \frac{1}{p} a(x, |\nabla u|^p)|\nabla w|^p dx \right) - t \left(\lambda \int_\Omega f(x,u) w dx \right)$$

Since both integrals are positive and $p > 1$, the functional is negative for a fixed $w \not\equiv 0$ and a sufficiently small $t > 0$. By the Hölder inequality and the Sobolev embedding the functional can be estimated from below by

$$J_u(w) \geq \frac{c_a}{p} \|\nabla w\|_{L^p}^p - \lambda \left(C_{F,q} \|u\|_{L^q}^{q-1} + C_{F,\tilde{q}} \|u\|_{L^{\tilde{q}}}^{\tilde{q}-1} \right) \|\nabla w\|_{L^p}.$$

This shows that the functional is coercive. By strict convexity of the first term and linearity of the second term, the functional is strictly convex and thus weakly lower semicontinuous. Coercivity implies that the minimizing sequence $w_k \in W_0^{1,p}(\Omega)$ with

$$\lim_{k \to \infty} J(w_k) = \inf_{v \in W_0^{1,p}(\Omega)} J(v) \in (-\infty, 0)$$

is bounded in $W_0^{1,p}(\Omega)$. Because $W_0^{1,p}(\Omega)$ is reflexive there is a weakly convergent subsequence $w_{k_l} \to w \in W_0^{1,p}(\Omega)$.

Lower semicontinuity shows that

$$J(w) \leq \lim_{l \to \infty} J(w_{k_l}) = \inf_{v \in W_0^{1,p}(\Omega)} J(v)$$

and thus w is the minimizer. If $\tilde{w}$ is a different minimizer then by the strict convexity

$$J\left(\frac{w + \tilde{w}}{2}\right) < \frac{J(w) + J(\tilde{w})}{2} = \inf_{v \in W_0^{1,p}(\Omega)} J(v)$$

which is a contradiction, so the minimizer is unique. □

Lemma 6.14. *For $u \in \mathcal{N}$ let $w \in W_0^{1,p}(\Omega)$ be the solution of the modified problem* (6.5). *Then*

$$\int_\Omega f(x, u) u dx \leq \int_\Omega f(x, u) w dx.$$

This is similar to a calculation in Step 3 in the proof of Theorem 1.2 in [40].

Proof. $u \in \mathcal{N}$ is equivalent to

$$\int_\Omega a(x, |\nabla u|^p)|\nabla u|^p dx = \lambda \int_\Omega f(x,u)udx$$

and because w solves (6.5) weakly, $J_u'(w)v = 0$ for all $v \in W_0^{1,p}(\Omega)$, including $v = u$ and $v = w$, which implies

$$\int_\Omega a(x, |\nabla u|^p)|\nabla w|^{p-2}\nabla w \nabla u dx = \lambda \int_\Omega f(x,u)udx$$

and

$$\int_\Omega a(x, |\nabla u|^p)|\nabla w|^p dx = \lambda \int_\Omega f(x,u)wdx.$$

Using those equations it follows that

$$\begin{aligned}
\lambda &\int_\Omega f(x,u)udx \\
&= \int_\Omega a(x, |\nabla u|^p)|\nabla w|^{p-2}\nabla w \cdot \nabla u dx \\
&\leq \left(\int_\Omega a(x, |\nabla u|^p)|\nabla w|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x, |\nabla u|^p)|\nabla u|^p dx\right)^{\frac{1}{p}} \\
&= \left(\lambda \int_\Omega f(x,u)w\right)^{\frac{p-1}{p}} \left(\lambda \int_\Omega f(x,u)udx\right)^{\frac{1}{p}}.
\end{aligned}$$

Dividing by $\left(\int_\Omega f(x,u)udx\right)^{\frac{1}{p}}$ shows that

$$\int_\Omega f(x,u)udx \leq \int_\Omega f(x,u)wdx.$$

$\square$

Proposition 6.15. *Given* (f_3)*,* (a_s) *and* $u \in \mathcal{N}$*, let* w *be the solution of* (6.5) *with* $t > 0$ *so that* $tw \in \mathcal{N}$*. Then*

$$J(tw) \leq J(u)$$

with equality if and only if u *is a solution of the original boundary value problem.*

This is essentially the generalization of Step 3 in the proof of Theorem 1.2 in [40].

Proof. Using the concavity of $A(s)$ and Lemma 6.10

$$\begin{aligned}
&J(tw) - J(u) \\
&\quad = \int_\Omega \frac{1}{p} A(x, |\nabla tw|^p) - \frac{1}{p} A(x, |\nabla u|^p) \\
&\qquad\qquad - \lambda(F(x, tw) - F(x, u)) dx \\
&\quad \leq \int_\Omega \frac{1}{p} a(x, |\nabla u|^p) \big(|\nabla tw|^p - |\nabla u|^p\big) \\
&\qquad\qquad - \frac{1}{p} \lambda\big(f(x, u) u^{1-p} \big((tw)^p - u^p\big)\big) dx \\
&\quad = \frac{1}{p} \left(\int_\Omega a(x, |\nabla u|^p) |t \nabla w|^p - \lambda f(x, u) u^{1-p} (tw)^p dx \right) \\
&\qquad\qquad - \frac{1}{p} \left(\int_\Omega a(x, |\nabla u|^p) |\nabla u|^p - \lambda f(x, u) u dx \right) \\
&\quad = \frac{t}{p} \left(\int_\Omega a(x, |\nabla u|^p) |\nabla w|^p - \lambda f(x, u) u^{1-p} w^p dx \right)
\end{aligned}$$

where the last equality follows from $u \in \mathcal{N}$. Now w is a solution of (6.5), so the term involving a in the last line can be replaced with f to obtain

$$J(tw) - J(u) \leq \lambda \frac{t}{p} \left(\int_\Omega f(x,u)w - f(x,u)u^{1-p} w^p dx \right). \quad (6.6)$$

With the Hölder inequality and Lemma 6.14 it follows that

$$\begin{aligned}
\int_\Omega f(x,u)wdx &= \int_\Omega \left(\frac{f(x,u)}{u^{p-1}} \right) u^{p-1} wdx \\
&= \int_\Omega \left(\frac{f(x,u)}{u^{p-1}} \right)^{\frac{p-1}{p}} (u^p)^{\frac{p-1}{p}} \left(\frac{f(x,u)}{u^{p-1}} \right)^{\frac{1}{p}} (w^p)^{\frac{1}{p}} dx \\
&\leq \left(\int_\Omega \frac{f(x,u)}{u^{p-1}} u^p dx \right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{f(x,u)}{u^{p-1}} w^p dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega f(x,u)wdx \right)^{\frac{p-1}{p}} \left(\int_\Omega f(x,u)u^{1-p} w^p dx \right)^{\frac{1}{p}}.
\end{aligned}$$

Dividing by $(\int_\Omega f(x,u)wdx)^{\frac{p}{p-1}}$ implies

$$\int_\Omega f(x,u)wdx \leq \int_\Omega f(x,u)u^{1-p} w^p dx.$$

Combined with inequality (6.6) this proves $J(tw) - J(u) \leq 0$.

Equality holds if and only if all inequalities are equalities. Because of the Hölder inequality in the last set of inequalities this means that u^p has to be a constant multiple of w^p and therefore u has to be a constant multiple of w. Since both functions are nonnegative and nontrivial this implies that there is an $s > 0$ so that $u = sw$. But $u = sw \in \mathcal{N}$ and $tw \in \mathcal{N}$ and by Proposition 6.8 this implies $s = t$ and therefore $u = tw$.

Using this fact in (6.5) and choosing $v = tw$ gives

$$\begin{aligned} 0 = J'_u(w)v &= \int_\Omega a(x, |\nabla u|^p)|\nabla w|^{p-2}\nabla w \cdot \nabla v - \lambda f(x,u)v dx \\ &= \int_\Omega a(x, |t\nabla w|^p)|t\nabla w|^p t^{1-p} - \lambda f(x,tw)tw dx \\ &= t^{1-p}\int_\Omega \lambda f(x,tw)tw dx - \int_\Omega \lambda f(x,tw)tw dx \end{aligned}$$

and since $p > 1$ it follows that $t = 1$. Thus $w = u$ and therefore (6.5) shows that u has to be a solution of the original differential equation. □

Now the theory is in place to complete the proof of Theorem 6.3.

6.2.4 Proof

Proof of Theorem 6.3. By Proposition 6.12 there is a mountain pass solution u_λ so that $J(tu_\lambda) < J(u_\lambda)$ for $t > 0$ and $t \neq 1$.

Since $(tu)^* = t(u^*)$ it follows from Theorems 4.31 and 4.32 that

$$J(tu_\lambda^*) \leq J(tu_\lambda) \leq J(u_\lambda) \text{ for every } t > 0$$

where the last inequality is strict unless $t = 1$. Since there is a $t > 0$ so that $tu_\lambda^* \in \mathcal{N}$ and u_λ minimizes the energy in the Nehari manifold it follows that $t \mapsto J(tu_\lambda^*)$ has its maximum at $t = 1$ with $J(u_\lambda^*) = J(u_\lambda)$.

Let now w be the solution of (6.5) for $u = u_\lambda^*$ and $t_0 > 0$ so that $t_0 w \in \mathcal{N}$. Then by Proposition 6.15

$$J(t_0 w) \leq J(u_\lambda^*) = J(u_\lambda)$$

and again using that u_λ minimizes the energy in the Nehari manifold there has to be equality in the inequality. Proposition 6.15 then implies that u_λ^* is a solution of (P).

By Theorem 4.34 it follows that $u_\lambda^* = u_\lambda$ which concludes the proof. □

6.3 Proof of the Regularity Result

By Definitions 3.17 and 3.26 and Section 3.4.4 the functions can be extended so that all of the assumptions of Theorem 6.5 hold in $\Omega \times (0, \infty)$ for A and $\Omega \times \mathbb{R}$ for F with $f(x, s) = 0$ for $s \leq 0$ which implies nonnegativity of critical points by Lemma 3.12. If the statement of the theorem holds for this modified problem, then by choosing λ large enough solutions will also be solutions of the original problem. Assume therefore in the remaining chapter that the functions are extended in this way and that the assumptions hold in $\Omega \times (0, \infty)$ and $\Omega \times \mathbb{R}$ respectively.

The existence of a $W_0^{1,p}(\Omega)$-solution to the problem follows from the mountain pass theorem. Growth estimates can be used to show that there is a constant $C > 0$ so that $\|u_\lambda\|_{L^p} \leq C\lambda^{-\frac{1}{q-p}}$.

The main idea is to use a frozen differential equation similar to (6.5) with Theorem 3.34 and the growth estimate stated above. Doing this multiple times leads to a sequence of estimates

$$\|u_\lambda\|_{L^\infty} \leq M_k \lambda^{-\gamma_k}.$$

It can be shown that $\lim_{k\to\infty} \gamma_k = \frac{1}{q-p}$ which implies that there is a $k \in \mathbb{N}$ so that $\gamma_k > \frac{1}{q-1}$ and for such a k

$$|\lambda f(x, u_\lambda(x))| \leq M_k^{q-1} C_f \lambda^{1-\gamma_k(q-1)} \leq M_k^{q-1} C_f \lambda_0^{1-\gamma_k(q-1)}.$$

Theorem 3.39 then shows existence of $\omega \in (0,1)$ and $M > 0$ so that $\|u_\lambda\|_{C^{1,\omega}} \leq M$. This step can now be repeated for a frozen differential equation in order to obtain the stated decay estimate for the solutions u_λ if γ_k is large enough.

6.3.1 Growth Estimates

Corollary 6.16. *$(a_{2,r})$ implies that for $c_a = \frac{c_e}{p-1}$ and $C_a = \frac{C_e}{p-1}$ it follows that*

$$c_a \leq a(x,s) \leq C_a \quad \text{for all } (x,s) \in [0,\infty)$$

and

$$c_a \leq A(x,s) \leq C_a.$$

Lemma 6.17. *By Corollary 6.16 and $(f_{1,r})$ functions in the Nehari manifold are uniformly bounded away from zero:*

$$u \in \mathcal{N} \quad \Longrightarrow \quad \|\nabla u\|_{L^p} \geq \left(\frac{c_a}{C_f C_s}\right)^{\frac{1}{q-p}} \lambda^{-\frac{1}{q-p}}.$$

This is a generalization of estimates found in Step 4 of the proof of Theorem 1.2 in [40].

Proof. $u \in \mathcal{N}$ is equivalent to

$$\int_\Omega a(x, |\nabla u|^p)|\nabla u|^p dx = \int_\Omega \lambda f(x,u)u dx.$$

Estimating the left side from below and the right side from above via the growth estimates gives

$$c_a \|\nabla u\|_{L^p}^p \leq \lambda C_f \|u\|_{L^q}^q.$$

The Sobolev inequality $\|u\|_{L^q}^q \leq C_S^q \|\nabla u\|_{L^p}^q$ implies

$$1 \leq \lambda \frac{C_f C_S^q}{c_a} \|\nabla u\|_{L^p}^{q-p}$$

and this proves the statement. $\square$

Remark 6.18. *If $u \in C^1(\overline{\Omega})$, the estimate*

$$\left(\frac{c_a}{C_f C_s}\right)^{\frac{1}{q-p}} \lambda^{-\frac{1}{q-p}} \leq \|\nabla u\|_{L^p} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}} \leq \|\nabla u\|_\infty |\Omega|^{\frac{1}{p}}$$

shows a lower bound for the C^1-norm.

Lemma 6.19. *By ($a_{1,r}$) and ($f_{2,r}$) for every $u \in \mathcal{N}$ the energy $J(u)$ is bounded from below by*

$$J(u) \geq c_{AR} \|\nabla u\|_{L^p}^p.$$

This is again a generalization of estimates found in Step 4 of the proof of Theorem 1.2 in [40].

Proof.

$$\begin{aligned} J(u) &= J(u) - \vartheta J'(u)(u) \\ &= \int_\Omega \frac{1}{p} A(|\nabla u|^p) - \vartheta a(|\nabla u|^p)|\nabla u|^p \\ &\qquad\qquad - \lambda \left(F(x,u) - \vartheta f(x,u)u\right) dx \\ &\geq c_{AR} \|\nabla u\|_{L^p}^p. \end{aligned}$$

$\square$

Proposition 6.20. *Given the assumptions for Theorem 6.5 there is a mountain pass solution for every $\lambda > 0$ and for such a solution u_λ the estimate*

$$J(u_\lambda) \leq C_1 \lambda^{-\frac{p}{q-p}}$$

holds with

$$C_1 = \left(\frac{1}{p} - \frac{1}{q}\right) \left(\frac{C_a^q}{c_f^p}\right)^{\frac{1}{q-p}} \tilde{C}_S^{-\frac{qp}{q-p}}$$

where $\tilde{C}_S = \sup_{u \in W_0^{1,p} \setminus \{0\}} \frac{\|u\|_{L^q}}{\|\nabla u\|_{L^p}}$.

Proof. Let $u \in W_0^{1,p}(\Omega)$ with $\|\nabla u\|_{L^p} = d$. Then by the growth conditions

$$\begin{aligned} J(u) &\geq c_a \|\nabla u\|_{L^p}^p - \lambda C_f \|u\|_{L^q}^q \\ &\geq c_a \|\nabla u\|_{L^p}^p - \lambda C_f C_S^q \|\nabla u\|_{L^p}^q \\ &= c_a d^p - \lambda C_f C_S^q d^q \end{aligned}$$

which is positive for sufficiently small $d > 0$ since $q > p$.

Then, for a nontrivial, nonnegative $u \in W_0^{1,p}(\Omega)$,

$$J(tu) \leq C_a t^p \|\nabla u\|_{L^p}^p - \lambda c_f t^q \|u\|_{L^q}^q$$

and thus $J(tu) < 0$ for large enough t. Let $\lambda_0 \geq 0$ be fixed, $u_0 := t_0 u$ with t_0 so that $C_a t^p \|\nabla u\|_{L^p}^p - \lambda_0 c_f t^q \|u\|_{L^q}^q < 0$ for every $t \geq t_0$. Then J has a mountain pass geometry and since it satisfies the Palais-Smale compactness condition it has a mountain pass solution for every $\lambda \geq \lambda_0$.

Let $u \in W_0^{1,p}$ be a nonnegative, nontrivial function. If u and u_0 are linearly dependant, then tu with $t \geq 0$ is an admissible path for the mountain pass theorem.

If they are not linearly dependant, then they can be connected by a path which can be constructed as follows:

With $v(t) = tu_0 + (1-t)u$ the term

$$\frac{\|\nabla v(t)\|_{L^p}^p}{\|v(t)\|_{L^q}^q}$$

is well-defined for $t \in [0,1]$ and has a maximum $m > 0$ at a point $t_1 \in [0,1]$. Let $c = 2\max\left\{t_2, 1, \left(\frac{C_a m}{\lambda_0 c_f}\right)^{\frac{1}{q-p}}\right\}$ where t_2 is chosen so that $J(tu) < 0$ for $t \geq t_2$. Then

$$\begin{aligned}
J(cv(t)) &\leq C_a c^p \|\nabla v(t)\|_{L^p}^p - \lambda_0 c_f c^q \|v(t)\|_{L^q}^q \\
&\leq \left(C_a \frac{\|\nabla v(t)\|_{L^p}^p}{\|v(t)\|_{L^q}^q} - \lambda_0 c_f c^{q-p} \right) c^p \|v(t)\|_{L^q}^q \\
&\leq \left(\frac{C_a m}{\lambda_0 c_f} - c^{q-p} \right) \lambda_0 c_f c^p \|v(t)\|_{L^q}^q \\
&< 0
\end{aligned}$$

for every $t \in [0,1]$. Thus $J(cv(t-c)) < 0$ for $t \in [c, c+1]$ and

$$\gamma(t) = \begin{cases} tu, & \text{if } \ t \in [0,c], \\ (t-c)cu_0 + (1-(t-c))cu, & \text{if } \ t \in (c, c+1], \\ c\frac{c+1}{t} u_0, & \text{if } \ t \in (c+1, c(c+1)], \end{cases}$$

with $t \in [0, c(c+1)]$ is an admissible path connecting 0 and u_0 via u. By construction of this path $J(\gamma(t)) < 0$ for $t \in (c, c(c+1)]$ and therefore

$$\max_{t \in [0, c(c+1)]} J(\gamma(t)) = \sup_{t > 0} J(tu).$$

Since there may be paths with a lower maximum this shows that for the mountain pass solution u_λ

$$J(u_\lambda) = \inf_{\gamma\in\Gamma} \max_s J(\gamma(s)) \le \inf_{\substack{u\in W_0^{1,p}\setminus\{0\} \\ u\ge 0}} \sup_{t>0} J(tu).$$

Thus, an upper estimate for $J(u_\lambda)$ can be computed by looking at

$$J(tu) \le C_a t^p \|\nabla u\|_{L^p}^p - \lambda c_f t^q \|u\|_{L^q}^q =: \alpha(t).$$

The function α has a global positive maximum $\alpha(t_3)$ since $\alpha(t) > 0$ for a small t and $\lim_{t\to\infty} \alpha(t) = -\infty$. α is continuously differentiable, so at this point the derivative of α has to be zero, thus

$$0 \overset{!}{=} \alpha'(t_3) = C_A \|\nabla u\|_{L^p}^p t_3^{p-1} - \lambda c_f \|u\|_{L^q}^q t_3^{q-1}$$

which is equivalent to $t_3 = 0$ or

$$t_3 = \left(\frac{C_A \|\nabla u\|_{L^p}^p}{\lambda c_f \|u\|_{L^q}^q} \right)^{\frac{1}{q-p}}.$$

This shows that

$$\sup_{t>0} J(tu) \le \frac{1}{p} C_A t^p \|\nabla u\|_{L^p}^p - \lambda \frac{c_f}{q} t^q \|u\|_{L^q}^q = \alpha(t) \le \alpha(t_3)$$

and

$$\begin{aligned}
\alpha(t_3) &= t_3^p \left(\frac{C_A}{p} \|\nabla u\|_{L^p}^p - \lambda \frac{c_f}{q} \|u\|_{L^q}^q \frac{C_A \|\nabla u\|_{L^p}^p}{\lambda c_f \|u\|_{L^q}^q} \right) \\
&= \left(\frac{C_A \|\nabla u\|_{L^p}^p}{\lambda c_f \|u\|_{L^q}^q} \right)^{\frac{p}{q-p}} \left(\frac{C_A}{p} \|\nabla u\|_{L^p}^p - \frac{1}{q} C_A \|\nabla u\|_{L^p}^p \right) \\
&= \left(\frac{1}{p} - \frac{1}{q} \right) \left(\frac{C_A^q}{\lambda^p c_f^p} \right)^{\frac{1}{q-p}} \left(\frac{\|\nabla u\|_{L^p}}{\|u\|_{L^q}} \right)^{\frac{qp}{q-p}}.
\end{aligned}$$

Taking the infimum over all nontrivial $u \in W_0^{1,p}(\Omega)$ with $u \geq 0$ means $\frac{\|\nabla u\|_{L^p}}{\|u\|_{L^q}}$ will be the inverse of the optimal Sobolev constant $\tilde{C}_S$ and thus

$$J(u_\lambda) \leq \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{C_A^q}{c_f^p}\right)^{\frac{1}{q-p}} \tilde{C}_S{}^{-\frac{qp}{q-p}} \lambda^{-\frac{p}{q-p}}.$$

$\square$

Corollary 6.21. *By Lemma 6.19 and Proposition 6.20 there is a $C_2 > 0$ so that*

$$\|\nabla u_\lambda\|_{L^p} \leq C_2 \lambda^{-\frac{1}{q-p}}$$

for every mountain pass solution u_λ and $\lambda > 0$.

6.3.2 Iterative Regularization

Central for the proof is the repeated use of Theorem 3.34 with appropriately chosen constants α, β and r leading to a sequence γ_k.

Lemma 6.22. *Let $p \in (1, n)$ and $q \in \left(p, \frac{np}{n-p}\right)$. Define*

$$r := \max\left\{\frac{n}{p} + \varepsilon, \frac{q}{q-1}\right\} \quad \textit{with} \quad \varepsilon := \frac{1}{2}\left(\frac{q}{q-p} - \frac{n}{p}\right),$$

$$\alpha := q - 1 - \beta \quad \textit{with} \quad \beta := \min\left\{\frac{np}{n-p}\frac{1}{r}, q-1\right\}$$

$$\textit{and} \quad \tau := \frac{1}{p-1}\left(\frac{\beta}{q-p} - 1\right).$$

Then

- $r \geq \frac{1}{2}\left(\frac{q}{q-p} + \frac{n}{p}\right) > \frac{n}{p}$,
- $1 < \beta r \leq \frac{np}{n-p}$,
- $0 \leq \alpha < \frac{np}{n-p} - 1 - \frac{np}{n-p}\frac{1}{r}$ *and*
- $0 < \tau \leq \frac{1}{q-p}$.

Proof.

- Since $nq - pq = (n-p)q < (n-p)\frac{np}{n-p} = np$ it follows that $n(q-p) = nq - np < pq$ and thus $\frac{n}{p} < \frac{q}{q-p}$ which shows that $\varepsilon > 0$ and

$$\frac{n}{p} < \frac{n}{p} + \varepsilon = \frac{1}{2}\left(\frac{n}{p} + \frac{q}{q-p}\right).$$

- The inequality $\beta r \leq \frac{np}{n-p}$ follows immediately from the definition. If $\beta = \frac{np}{n-p}\frac{1}{r}$ then

$$\beta r = \frac{np}{n-p} > 1$$

and if $\beta = q - 1$ then

$$\beta r \geq (q-1)\frac{q}{q-1} = q > p > 1.$$

Thus, in either case $1 < \beta r \leq \frac{np}{n-p}$.

- If $\beta = q - 1$ then $\alpha = q - 1 - \beta = 0$. If $\beta = \frac{np}{n-p}\frac{1}{r}$ then

$$\alpha = q - 1 - \beta < \frac{np}{n-p} - 1 - \frac{np}{n-p}\frac{1}{r}.$$

- It remains to calculate τ for the cases $\beta = q - 1$ and $\beta = \frac{np}{n-p}\frac{1}{r}$ with

$$r = \frac{q}{q-1} \qquad \text{or} \qquad r = \frac{1}{2}\left(\frac{q}{q-p} + \frac{n}{p}\right).$$

In case $\beta = q - 1$ it follows that

$$\tau = \frac{1}{p-1}\left(\frac{q-1}{q-p} - 1\right) = \frac{1}{q-p}.$$

If $\beta = \frac{np}{n-p}\frac{1}{r}$ and $r = \frac{1}{2}\left(\frac{q}{q-p} + \frac{n}{p}\right)$ it follows that

$$\begin{aligned}(p-1)\tau &= \left(\frac{np}{n-p}\right)\left(\frac{1}{q-p}\right)\frac{1}{\frac{1}{2}\left(\frac{q}{q-p} + \frac{n}{p}\right)} - 1\\ &= \frac{2np}{(n-p)(q+(q-p)\frac{n}{p})} - 1\end{aligned}$$

and since $q < \frac{np}{n-p}$ this is strictly larger than

$$\begin{aligned}\frac{2np}{(n-p)\left(\frac{np}{n-p} + (\frac{np}{n-p} - p)\frac{n}{p}\right)} - 1\\ &= \frac{2np}{np + np\frac{n}{p} - (n-p)n} - 1\\ &= \frac{2np}{2np} - 1\\ &= 0.\end{aligned}$$

If $\beta = \frac{np}{n-p}\frac{1}{r}$ and $r = \frac{q}{q-1}$ then

$$(p-1)\tau = \frac{1}{q-p}\frac{np}{n-p}\frac{q-1}{q} - 1 = \frac{np}{n-p}\frac{1}{q}\frac{q-1}{q-p} - 1$$

and since $\frac{q-1}{q-p} > 1$ and $q < \frac{np}{n-p}$ this term is positive as well.

As β is the minimum of those choices it follows that in either case

$$0 < \tau \leq \frac{1}{q-p}.$$

□

Lemma 6.23. *Let* $\gamma_0 = 0$ *and* $\gamma_{k+1} = \gamma_k + \tau(1-(q-p)\gamma_k)$. *Then* $\gamma_1 = \tau$ *and either* $\tau = \frac{1}{q-p}$ *and* $\gamma_k = \frac{1}{q-p}$ *for all* $k \in \mathbb{N}\setminus\{0\}$, *or* $\tau < \frac{1}{q-p}$ *and the sequence* γ_k *is strictly increasing with* $\lim_{k\to\infty}\gamma_k = \frac{1}{q-p}$.

Proof. By construction $\gamma_1 = \tau \in \left(0, \frac{1}{q-p}\right]$.

If $\tau = \frac{1}{q-p}$ and $\gamma_k = \tau$ then $\gamma_{k+1} = \tau + \tau(1-1) = \tau$.

If $\tau < \frac{1}{q-p}$ and $\gamma_k \in \left(0, \frac{1}{q-p}\right)$ then $1-(q-p)\gamma_k > 0$. This implies

$$\gamma_{k+1} = \gamma_k + \tau\,(1-(q-p)\gamma_k) > \gamma_k$$

and thus γ_k is strictly increasing. On the other hand $\tau < \frac{1}{q-p}$ implies $1-(q-p)\tau > 0$ and

$$\gamma_{k+1} = \tau + \gamma_k\,(1-(q-p)\tau) < \tau + \frac{1}{q-p}\,(1-(q-p)\tau) = \frac{1}{q-p}.$$

Therefore γ_k is bounded and as a strictly increasing sequence it converges to a γ. Taking the limit on both sides of the definition shows

$$\gamma = \lim_{k\to\infty}\gamma_{k+1} = \lim_{k\to\infty}\Big(\gamma_k + \tau(1-(q-p)\gamma_k)\Big) = \gamma + \tau(1-(q-p)\gamma)$$

and this has the unique solution $\gamma = \frac{1}{q-p}$ which concludes the proof. □

Proposition 6.24. *Let the assumptions in Theorem 6.5 with $p \leq n$ hold and $\lambda_0 > 0$ be arbitrary. Then for every $\gamma \in \left(0, \frac{1}{q-p}\right)$ there is an $M > 0$ so that*

$$\|u_\lambda\|_{L^\infty} \leq M\lambda^{-\gamma}$$

for every $\lambda \geq \lambda_0$ and every mountain pass solution u_λ.

Proof. If $p < n$ let r, α, β and τ be as in Lemma 6.22 and if $p = n$ choose $\alpha = 0$, $\beta = q - 1$, $\tau = \frac{1}{q-p}$ and $r = \frac{q}{q-1}$. This implies $r = \frac{q}{q-1} > 1 = \frac{n}{p}$ and $\beta r = q \in (1, \infty)$ and thus the choice of constants is admissible for the regularity theorem following Remark 3.35.

The function $u = u_\lambda$ solves the differential equation

$$-\operatorname{div}\left(a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = u^\alpha \left(\lambda f(x, u_\lambda(x))u_\lambda^{-\alpha}(x)\right).$$

It should be noted that the function $f(x, u_\lambda(x))u_\lambda^{-\alpha}(x)$ does not have a singularity at points where $u_\lambda(x) = 0$ due to the growth conditions for f and $\alpha \leq q - 1$. The right-hand side is equal to $u^\alpha\phi_\lambda(x)$ with

$$\phi_\lambda(x) := \lambda f(x, u_\lambda(x))u_\lambda^{-\alpha}(x).$$

Since $a(x, |z|^p)|z|^p \geq c_e|z|^p$ Theorem 3.34 states that $\operatorname{ess\,sup}_\Omega |u_\lambda|$ exists and only depends on α, r, $\|u_\lambda\|_{L^{\frac{np}{n-p}}}$ and $\|\phi_\lambda\|_{L^r}$. With

$$\|\phi_\lambda\|_{L^r} \leq \lambda C_f \left(\int_\Omega |u_\lambda|^{(q-1-\alpha)r} dx\right)^{\frac{1}{r}} = C_f\lambda\|u_\lambda\|_{L^{\beta r}}^\beta$$

and $1 < \beta r \leq \frac{np}{n-p}$ it follows that

$$\|u_\lambda\|_{L^{\beta r}} \leq C_S\|\nabla u\|_{L^p} \leq C_S C_2 \lambda^{-\frac{1}{q-p}}$$

by Corollary 6.21 and Theorem 2.26. It is assumed that C_S is large enough to encompass both Sobolev embeddings.

Since $1 - \frac{\beta}{q-p} = -(p-1)\tau$ it follows that for every $\lambda \geq \lambda_0$

$$\|\phi_\lambda\|_{L^r} \leq C_f C_S^\beta C_2^\beta \lambda^{-(p-1)\tau} \leq C_f C_S^\beta C_2^\beta \lambda_0^{-(p-1)\tau}.$$

Thus there is an upper bound for $\|\phi_\lambda\|_{L^r}$ which is independent of λ for $\lambda \geq \lambda_0$. Corollary 6.21 and Theorem 2.26 again show that

$$\|u_\lambda\|_{L^{\frac{np}{n-p}}} \leq C_S \|\nabla u\|_{L^p} \leq C_S C_2 \lambda_0^{-\frac{1}{q-p}}$$

and the upper bound on $\|u_\lambda\|_{L^{\frac{np}{n-p}}}$ also does not depend on λ.

Thus the upper bound on $\operatorname{ess\,sup}_\Omega |u_\lambda|$ is independent of λ and there is an M_0 so that

$$\|u_\lambda\|_{L^\infty} \leq M_0 \lambda^{-\gamma_0}$$

for all $\lambda \geq \lambda_0$ with $\gamma_0 = 0$.

Assume now that $\|u_\lambda\|_{L^\infty} \leq M_k \lambda^{-\gamma_k}$ with $k \geq 0$ (see Lemma 6.23 for the definition of γ_k).

The function $u = \lambda^{\gamma_{k+1}} u_\lambda$ solves the differential equation

$$-\operatorname{div}\Big(a(x, |\nabla u_\lambda|^p)|\nabla u|^{p-2}\nabla u\Big) = \lambda^{(p-1)\gamma_{k+1}+1} f(x, u_\lambda(x)). \quad (6.7)$$

The left-hand side satisfies the condition $a(x, |\nabla u_\lambda(x)|)|z|^p \geq c_e |z|^p$ and

$$\begin{aligned}\|\lambda^{\gamma_{k+1}} u_\lambda\|_{L^{\frac{np}{n-p}}} &\leq \lambda^{\gamma_{k+1}} C_S \|\nabla u_\lambda\|_{L^p} \\ &\leq C_S C_2 \lambda^{\gamma_{k+1} - \frac{1}{q-p}} \\ &\leq C_S C_2 \lambda_0^{\gamma_{k+1} - \frac{1}{q-p}}\end{aligned}$$

since $\gamma_{k+1} \le \frac{1}{q-p}$, so this term does not depend on λ.

The right-hand side of eq. (6.7) is bound by $(1+|u|^0)\tilde{\phi}_\lambda(x)$ with

$$\tilde{\phi}_\lambda(x) = \lambda^{(p-1)\gamma_{k+1}+1} f(x, u_\lambda)$$

so once again it remains to control the norm of $\tilde{\phi}_\lambda$ to control the upper bound on $\|\lambda^{\gamma_{k+1}} u_\lambda\|_{L^\infty}$.

$$\begin{aligned}
\left\|\tilde{\phi}_\lambda\right\|_{L^r} &= \lambda^{(p-1)\gamma_{k+1}+1}\left(\int_\Omega \left|u_\lambda^\alpha f(x,u_\lambda) u_\lambda^{-\alpha}\right|^r dx\right)^{\frac{1}{r}} \\
&\le \lambda^{(p-1)\gamma_{k+1}+1} C_f \left(\int_\Omega |u_\lambda|^{(q-1-\alpha)r} dx\right)^{\frac{1}{r}} \|u_\lambda\|_{L^\infty}^\alpha \\
&= \lambda^{(p-1)\gamma_{k+1}+1} C_f \|u_\lambda\|_{L^{\beta r}}^\beta \|u_\lambda\|_{L^\infty}^\alpha \\
&\le C_f C_2^\beta M_n^\alpha C_S^\beta \lambda^{(p-1)\gamma_{k+1}+1-\frac{\beta}{q-p}-\alpha\gamma_k}.
\end{aligned}$$

The exponent of λ is $(p-1)\gamma_{k+1}+1-\frac{\beta}{q-p}-\alpha\gamma_k$ and it will now be shown that this is 0. With

$$\gamma_{k+1} = \gamma_k + \tau(1-(q-p)\gamma_k) = \tau + \gamma_k\left(1-(q-p)\tau\right)$$

and $1-\frac{\beta}{q-p} = -(p-1)\tau$ as well as

$$\alpha = q-1-\beta = p-1+q-p-\beta = (p-1)\left(1-(q-p)\tau\right)$$

it follows that

$$\begin{aligned}
&(p-1)\gamma_{k+1}+1-\frac{\beta}{q-p}-\alpha\gamma_k \\
&= (p-1)\left(\tau+\gamma_k\left(1-(q-p)\tau\right)-\tau+\left(1-(q-p)\tau\right)\gamma_k\right) = 0.
\end{aligned}$$

Thus there is a constant $M_{k+1} > 0$ so that the essential supremum of the solution of (6.7) is bound by M_{k+1} for every $\lambda \ge \lambda_0$. Since

$\lambda^{\gamma_{k+1}} u_\lambda$ solves the equation this implies $\|u_\lambda\|_{L^\infty} \leq M_{k+1}\lambda^{-\gamma_{k+1}}$. By induction there is a sequence M_k so that for all $k \in \mathbb{N}$

$$\|u_\lambda\|_{L^\infty} \leq M_k \lambda^{-\gamma_k}$$

with $\lim_{k\to\infty} \gamma_k = \frac{1}{q-p}$.

Hence for every $\gamma \in \left(0, \frac{1}{q-p}\right)$ there is a $k \in \mathbb{N}$ so that $\gamma_k > \gamma$ which concludes the proof. □

Corollary 6.25. *If $p > n$ then*

$$\|u\|_{C^{0,1-\frac{n}{p}}} \leq C_S \|\nabla u\|_{L^p} \leq C_S C_2 \lambda^{-\frac{1}{q-p}}$$

by Theorem 2.27 and Corollary 6.21. In this case the statement of the previous theorem follows trivially.

It has now been shown that $\|u_\lambda\|_{L^\infty}$ decays fast enough in all cases which is the critical component in showing that $|\lambda f(x, u_\lambda(x))|$ remains bounded even as $\lambda \to \infty$. This is the key to applying Theorem 3.39.

6.3.3 Proof

Proof of Theorem 6.5. It has now been shown that for every $\lambda_0 > 0$ and $\gamma \in \left(0, \frac{1}{q-p}\right)$ there is an $M > 0$ so that

$$\|\nabla u_\lambda\|_{L^p} \leq C\lambda^{-\frac{1}{q-p}} \qquad \text{and} \qquad \|u_\lambda\|_{L^\infty} \leq M\lambda^{-\gamma}$$

for every $\lambda \geq \lambda_0$ and every mountain pass solution u_λ.

This implies

$$|\lambda f(x, u_\lambda(x))| \leq C_f \lambda \lambda^{-(q-1)\gamma} \leq C_f \lambda_0^{1-(q-1)\gamma}.$$

Theorem 3.34 can now be used for the boundary value problem

$$\begin{cases} -\operatorname{div}\left(a(x, |\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda f(x, u_\lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It implies that there is an $\omega_2 \in (0,1)$ and M_{ω_2} so that $\|u_\lambda\|_{C^{1,\omega_2}} \leq M$ for every $\lambda \geq \lambda_0$ since all terms which ω_2 and M_{ω_2} depend on are independent of λ.

The function

$$\lambda^{\gamma\frac{q-1}{p-1}-\frac{1}{p-1}} u_\lambda$$

solves the differential equation

$$\begin{cases} -\operatorname{div}\left(a(x, |\nabla u_\lambda|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda^{\gamma(q-1)-1}\lambda f(x, u_\lambda) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Since $(x, s) \mapsto a(x, s^p)$ is globally Lipschitz continuous the function $a(x, |\nabla u_\lambda|^p)$ is in C^{0,ω_2} with the norm being bound by the Lipschitz constant and M_{ω_2} and therefore it is independent of λ.

Using this with the computation

$$\begin{aligned} \left|\lambda^{(\gamma(q-1)-1)+1} f(x, u_\lambda)\right| &\leq M_{\omega_2}^{q-1} C_f \lambda^{(\gamma(q-1)-1)+1-(q-1)\gamma} \\ &= C_f M_\alpha^{q-1} \end{aligned}$$

shows that there is an $\omega \in (0, \omega_2)$ and $M_\omega > 0$ so that for every $\lambda \geq \lambda_0$

$$\left\|\lambda^{\gamma\frac{q-1}{p-1}-\frac{1}{p-1}} u_\lambda\right\|_{C^{1,\omega}} \leq M_\omega$$

or, equivalently,

$$\|u_\lambda\|_{C^{1,\omega}} \leq M_\omega \lambda^{\frac{1}{p-1}-\gamma\frac{q-1}{p-1}}.$$

Note that $\gamma > \frac{1}{q-1}$ implies

$$\frac{1}{p-1} - \gamma \frac{q-1}{p-1} < 0.$$

This gives a decay estimate for the $C^{1,\omega}$-norm of u_λ. In the limit $\gamma = \frac{1}{q-p}$ it follows that

$$\begin{aligned}
\frac{1}{p-1} - \gamma \frac{q-1}{p-1} &= \frac{1}{p-1} - \frac{1}{q-p} \frac{q-1}{p-1} \\
&= \frac{q-p-(q-1)}{(p-1)(q-p)} \\
&= -\frac{1}{q-p}.
\end{aligned}$$

Thus for every $\eta \in \left(0, \frac{1}{q-p}\right)$ it is possible to choose γ close enough to $\frac{1}{q-p}$ so that

$$\frac{1}{p-1} - \gamma \frac{q-1}{p-1} < -\eta$$

and thus

$$\|u_\lambda\|_{C^{1,\omega}} \leq M_\omega \lambda^{-\eta}$$

for every $\lambda \geq \lambda_0$ and every mountain pass solution u_λ.

Remark 6.18 shows the lower bound for the $C^{1,\omega}$-norm. $\square$

7 Examples

There have been 4 main existence theorems shown so far:

- Theorem 5.1 shows the existence of a minimizer using the behaviour of A and F at ∞.
- Theorem 5.3 shows the existence of a smooth minimizer using the behaviour of $\frac{d^2}{ds^2}A(x, s^p)$ and the absence of singularities of $f(x, s)$ at the origin.
- Theorem 3.46 shows the existence of a mountain pass solution using the behaviour of A and F at ∞.
- Theorem 6.5 shows the existence of a smooth mountain pass solution using the behaviour of $\frac{d^2}{ds^2}A(x, s^p)$ and $f(x, s)$ at the origin.

The specific mountain pass geometry in Theorems 3.46 and 6.5 automatically implies the existence of a local minimizer, however in the case of Theorem 6.5 the local minimizer has to be 0, since for any sufficiently small $d > 0$ the functional J is positive on S_d.

The growth conditions of Theorem 5.1 and Theorem 3.46 at ∞ are incompatible, but Theorem 5.3 and Theorem 3.46 can be used to show existence of two positive solutions which will be demonstrated in Example 7.6.

Proposition 4.16 shows that any radially symmetric weak solution of (P) is a critical point of J_r. In case of a ball those solutions are in $C^2((0,R])$ with $u'(r) < 0$ in $(0,R]$. In case of an annulus those solutions are in $C^2([R_1,R] \setminus \{r_1\})$ with $u'(r) < 0$ in $(r_1,R]$ where the continuity of u'' in r_1 depends on $\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p)$. Under suitable assumptions regarding a and f the initial value problem (4.4) with $u(R) = 0$, $u'(R) = -c < 0$ and $u(r) > 0$ for $r < R$ has a unique solution with a maximal existence interval. Thus weak solutions are associated with solutions of this initial value problem (up until the point r_1 in case of an annulus).

The following theorem gives a precise characterization of the solutions of this initial value problem for a specific example. It can be shown for more general formulations including cases that are singular elliptic at the origin.

Proposition 7.1. *Let*

$$\frac{1}{p}A(x,s^p) = \frac{1}{2}s^2 + \frac{1}{3}s^3$$

and $f(s)$ be locally Lipschitz continuous in $[0,\infty)$ and positive in $(0,\infty)$. Then the final value problem for critical points of Eq. (4.2) *can be stated as*

$$\begin{aligned} u''(r) &= -\frac{\frac{n-1}{r}(u' + |u'|u') + \lambda f(u)}{1+2|u'|} && \text{for } r < R,\\ u(r) &> 0 && \text{for } r < R,\\ u(R) &= 0,\\ u'(R) &= -c \end{aligned}$$

and for any $c > 0$ it has a unique solution u in an interval $(R-\varepsilon, R) \subseteq (0,R)$.

There is a unique minimal r_0 so that u exists on (r_0,R) and one of the following statements is true:

- $r_0 = 0$ *and* $u'(0) = 0$, *or*
- $r_0 = 0$, $\lim_{r\to 0} u(r) = \infty$ *and* $\lim_{r\to 0} u'(r) = -\infty$, *or*
- $r_0 > 0$ *and* $\lim_{r\to r_0} u(r) = 0$.

Proof. This problem can be seen as a (backwards) initial value problem $y' = F(x, y)$ with $y \in \mathbb{R}^2$. By the standard Picard-Lindelöf theorem the solution exists locally in a neighbourhood $[R - \varepsilon, R] \times [0, a] \times [c - \delta, c + \delta]$ of $(R, u(R), u'(R))$. By the standard extension theorem there is a maximal interval (r_0, R) so that the differential equation cannot exist on a larger interval $(r_1, R) \supsetneq (r_0, R)$.

If the interval is maximal then either $r_0 > 0$ or $r_0 = 0$. If $r_0 > 0$ then the inability to extend the solution implies that either $u(r_0) = 0$ or that $\liminf_{r\downarrow r_0} |u'(r)| = \infty$ (since $\lim_{r\downarrow r_0} u(r) = \infty$ also implies $\lim_{r\downarrow r_0} u'(r) = -\infty$).

If $r_0 = 0$ then either u is bounded with $u'(0) = 0$ or u' is unbounded, because again u being unbounded implies that u' is unbounded.

Using the comparison principle for ordinary differential equations the possibility that u' is unbounded and $r_0 > 0$ can be excluded:

- If $u'(r) < 0$ then

  $$u''(r) > \frac{\frac{n-1}{r}\frac{1}{2}|u'|(1 + 2|u'|) - \lambda f(u(0))}{1 + 2|u'|}.$$

  Thus, if $u'(r_2)$ is sufficiently negative so that the right-hand side is positive, u will be convex and $u'(r) < 0$ for all $r < r_2$.

The differential equation $v''(r) = -\frac{n-1}{r}v'$ has the family of solutions $v'(r) = Cr^{-(n-1)}$ and it can be seen that v' can only be unbounded when $r \to 0$. Using the estimate

$$u''(r) < -\frac{\frac{n-1}{r}u'(1+2|u'|) + \lambda f(u)}{1+2|u'|} < -\frac{n-1}{r}u'$$

and choosing C so that $u'(r_1) = v'(r_1)$ it follows that $u''(r) > v''(r)$ in (r_0, r_1) and thus $0 > u'(r) > v'(r)$ for $r < r_1$ which implies that u' cannot go to $-\infty$ at $r_0 > 0$.

- If $u'(r) > 0$ then $u''(r) < 0$ and thus u' being unbounded implies $\lim_{r\downarrow r_0} u'(r) = \infty$ and $\lim_{r\downarrow r_0} u''(r) = -\infty$. Since $u > 0$ it has to be bounded and $f(u)$ can be assumed to be bounded.

  Analogously this shows

  $$0 > u''(r) = -\frac{\frac{n-1}{r}(u' + |u'|u') - \lambda f(u)}{1+2|u'|} > -\frac{n-1}{r}u' - D$$

  with $D > 0$.

  The differential equation

  $$v''(r) = -\frac{n-1}{r}v' - D$$

  has the family of solutions

  $$v'(r) = Cr^{-(n-1)} - \frac{D}{n}r$$

  which are bounded on any interval $[r_0, R] \subset (0, R]$. Choosing a point r_1 and C so that $v'(r_1) = u'(r_1)$ it follows that $u''(r) > v''(r)$ and $0 < u'(r) < v'(r)$ for $r < r_1$. This shows that $u'(r)$ is bounded since v' is bounded.

Thus the possibility that either u or u' is unbounded on an interval $[\varepsilon, r] \subset (0, R)$ can be excluded and therefore $u(r_0) = 0$ if $r_0 > 0$.

If $r_0 = 0$ and u is bounded on $(0, R)$ then by Lemma 4.19 it is a weak solution which by Theorem 4.22 and Theorem 4.24 implies that $u \in C^2([0, R])$ with $u'(0) = 0$ and $u''(0) < 0$.

If u is unbounded then

$$\lim_{r \to 0} u'(r) = -\infty \text{ and } \lim_{r \to 0} u''(r) = \infty.$$

This concludes the proof.

$\square$

Corollary 7.2. *If $r_0 > 0$ then the function corresponds to a radially symmetric solution of* (P) *on the annulus with $R_1 = r_0$.*

If $r_0 = 0$ and $u'(0) = 0$ then the function corresponds to a radially symmetric solution of (P) *on the ball.*

If Assumption 1 on page 42 is satisfied then any weak solution has to be bounded by Theorem 3.34, which implies that unbounded solutions of the initial value problem are not weak solutions in that case.

Solving this initial value problem for different $c > 0$ is therefore a good way to get an idea of the possible solutions and their behaviour on the ball $B_R(0)$ and annuli with $R_2 = R$ and $R_1 \in (0, R)$.

In more general cases if

$$\lim_{s \to 0} \frac{d^2}{ds^2} \frac{1}{p} A(x, s^p) = \infty$$

then conventional numerical solvers lack stability and accuracy for initial value problems and if

$$\lim_{s\to 0}\frac{d^2}{ds^2}\frac{1}{p}A(x,s^p)=0$$

they usually fail once the solution reaches $u'(r)=0$. This makes them unable to show solutions on an annulus, although it is possible to get solutions on the ball even if the ellipticity degenerates at the origin.

If computing solutions in the degenerate or singular case are the primary goal then other methods should be chosen such as finite element methods or wavelet methods. Those are the correct approaches for weak solutions and able to deal with the lack of regularity and unstable behaviour that can happen in those cases. However their complexity puts them out of the scope of simple examples here and the approach via the ordinary differential equation is interesting because (in the somewhat restricted setting of problems that are uniformly elliptic at the origin) this can be used to visualize all possible candidates of solutions.

For that reason the examples will be visualized with equations that are uniformly elliptic at the origin. Uniform ellipticity at the origin is not necessary for the analytical results with the exception of Example 7.3, where it is used to prove that the mountain pass solution on the ball is radially symmetric. It is merely used to exclude problems with the numerical simulations.

7.1 Numerical Solver

The numerical solutions were computed using the solve_ivp function from the Python package SciPy 1.3.2. The method used is the implicit Radau IIA Runge-Kutta method of order 5 for ordinary differential equations. A semi-manual shooting method was used to obtain a suitable initial value where an arbitrary endpoint $r_0 > 0$ was not desirable.

7.2 Examples

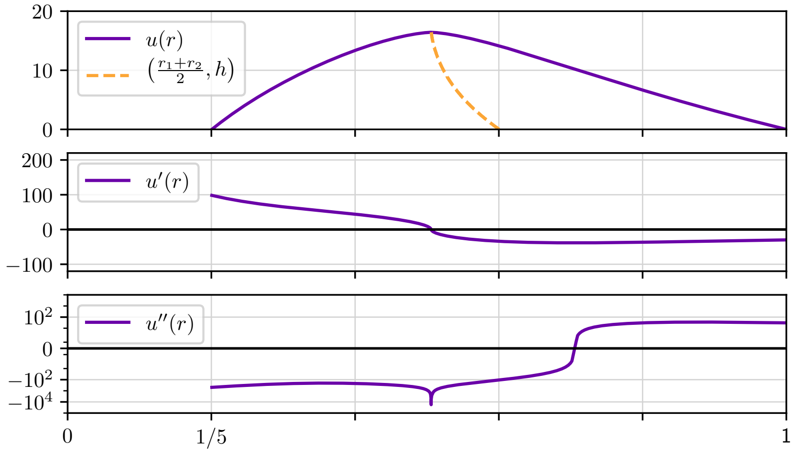

Figure 7.1: $-\operatorname{div}(\nabla u + |\nabla u|\nabla u) = 4u^3$, $n = 3$

Example 7.3. *Let* $\frac{1}{p}A(x, s^p) = \frac{1}{2}s^2 + \frac{1}{3}s^3$, $n = 3$, $F(x, s) = s^4$.

Following Theorem 3.46 the functional has a mountain pass solution u_λ*. Theorem 3.34 shows that it has to be bounded and Theorem 3.39 shows that it has to be in* $\in C^{1,\alpha}$*. This does not follow directly from anything that was proved in this thesis as this is the case where* $k > 0$*, but it is easy to check. The boundary value problem is uniformly elliptic for this solution* u_λ *and thus higher regularity for* u_λ *follows. This can be used to argue that* *every solution on the ball has to be radially symmetric using the moving plane method Theorem 4.27. This is the only argument and example that relies on uniform ellipticity at the origin to make an analytical argument.*

In case of an annulus there is a radially symmetric mountain pass solution, however it does not necessarily have to be a ground state solution.

The local minimizer associated with the mountain pass geometry is the trivial solution.

Figure 7.1 shows the solution u on the annulus $B_1(0) \setminus \overline{B_{1/5}(0)}$. By Theorem 4.28 it follows that $r_0 < \frac{r_1+r_2}{2}$ for any $r_1 < r_2$ with $u(r_1) = u(r_2) = h$. The curve $\left(\frac{r_1+r_2}{2}, h\right)$ is shown in Fig. 7.1 together with the solution u.

By differentiating eq. (4.10) *it can be seen that $u''(r) < 0$ in $(R_1, r_0 + \varepsilon)$ and $u''(R_2) > 0$. The numerical simulation shows that there is only a single point in $(1/5, 1)$ where $u''(r) = 0$, but it is not entirely clear if this can be guaranteed.*

Example 7.4. *Figure 7.2 shows the numerical simulation of the solution of $\frac{1}{p}A(x, s^p) = \frac{1}{2}s^2$, $\lambda = 1$ and the right-hand side $f(s) = 50s^{1.2}\,(1.2 + \sin(5000s))$ on the annulus $B_1(0) \setminus \overline{B_{1/5}(0)}$ in $\mathbb{R}^5$. This shows that despite the fact that $f(x, s) = f(s)$, $f \in C^1(\mathbb{R})$, $f(0) = 0$ and $f(s) > 0$ for $s > 0$ the possibility of multiple points where u changes curvature cannot be excluded.*

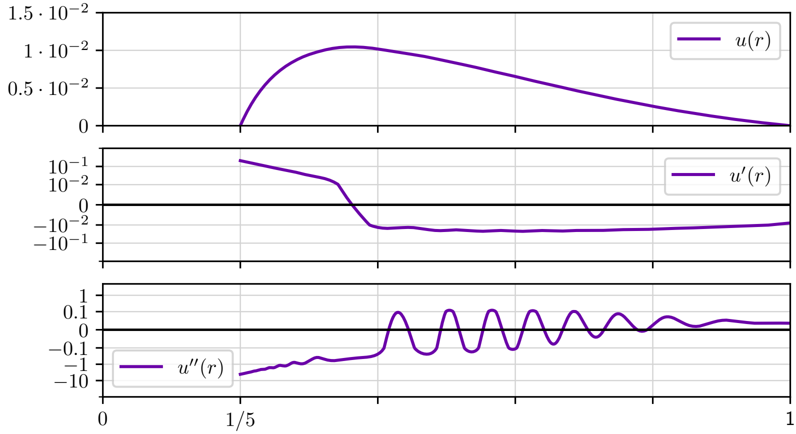

Figure 7.2: The radially symmetric solution to

$$-\Delta u = 50|s|^{1.2}(1.2 + \sin(5000s)) \text{ in } B_1(0) \setminus \overline{B_{1/5}(0)} \subset \mathbb{R}^5 .$$

Example 7.5. *If $\frac{1}{p}A(x, s^p) = \frac{1}{2}s^2 + \frac{1}{3}s^3$, $n = 5$ and $F(x, s) = s^{15}$ Theorem 6.5 is not admissible since $\frac{2n}{n-2} = \frac{10}{3} < 15$ and since $\frac{3n}{n-3} = \frac{15}{2} < 15$ there is no direct way to use the mountain pass theorem because depending on the choice of space $W_0^{1,p}(\Omega)$ the functional is either not well-defined or does not satisfy the Palais-Smale condition.*

It is unknown whether there is a radially symmetric solution on a ball, however using $W^{1,1}((R_1, R_2)) \subset L^q((R_1, R_2))$ for any $q \geq 1$ shows that the mountain pass theorem can be used on J_r and there is a radially symmetric solution on an annulus.

Looking at the numerical solutions of the ordinary differential equation in Fig. 7.3 suggests that every initial value $u'(R) < 0$

leads to a solution on some annulus (R_1, R) *and that no radially symmetric solution on the ball exists.*

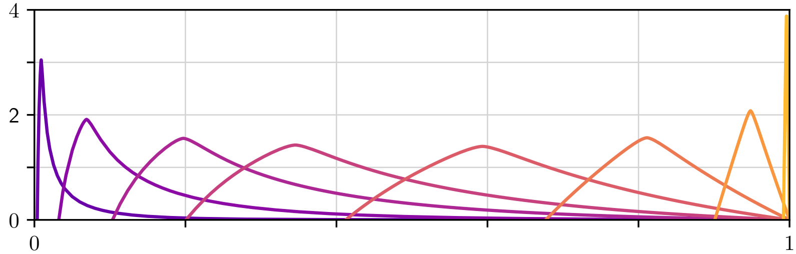

Figure 7.3: $-\operatorname{div}(\nabla u + |\nabla u|\nabla u) = 15u^{14}$, $n = 5$

Example 7.6. *If* $\frac{1}{p}A(x, s^p) = \frac{1}{2}s^2 + \frac{1}{3}s^3$, $n = 5$ *and* $F(x,s) = s^{1.5} + s^4$ *then, similar to the first example, the functional has a nontrivial minimizer* u_1 *and a mountain pass solution* u_2 *on any* $C^{1,1}$ *domain. Arguing as in Example 7.3 both are radially symmetric in case of a ball.*

The numerical simulation in Fig. 7.4 shows that $|u'(1)| > |u_2'(1)|$ *corresponds to mountain pass solutions on annuli.*

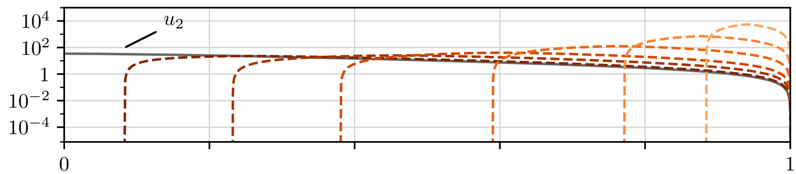

Figure 7.4: $-\operatorname{div}(\nabla u + |\nabla u|\nabla u) = 1.5\sqrt{u} + 4u^3$, $n = 5$ and $|u'(1)| > |u_2'(1)|$.

$|u'(1)| < |u_1'(1)|$ shown in Fig. 7.5 correspond to minimizers on annuli.

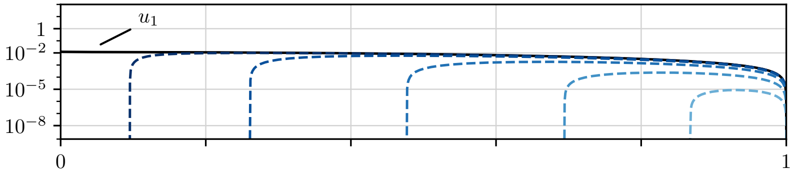

Figure 7.5: $-\operatorname{div}(\nabla u + |\nabla u|\nabla u) = 1.5\sqrt{u} + 4u^3$, $n = 5$ and $|u'(1)| < |u_1'(1)|$.

Since all radially symmetric solutions are in $W_0^{1,3}(\Omega)$ and f satisfies Assumption 1 on page 42 it follows from Theorem 3.34 that all weak solutions are bounded, thus the unbounded solutions shown in Fig. 7.6 with $|u_1'(1)| < |u'(1)| < |u_2'(1)|$ cannot be weak solutions on the ball (or the annulus).

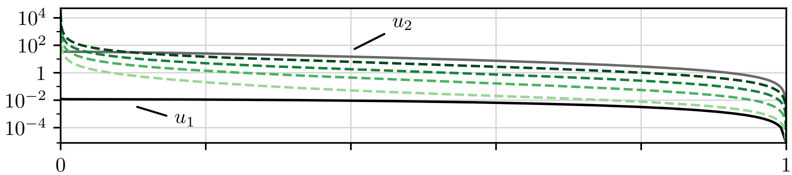

Figure 7.6: $-\operatorname{div}(\nabla u + |\nabla u|\nabla u) = 1.5\sqrt{u} + 4u^3$, $n = 5$ and $|u_1'(1)| < |u'(1)| < |u'(2)|$.

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