THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le *Vendredi 13 décembre 2019* **par :**

Jules Martel - Tordjman

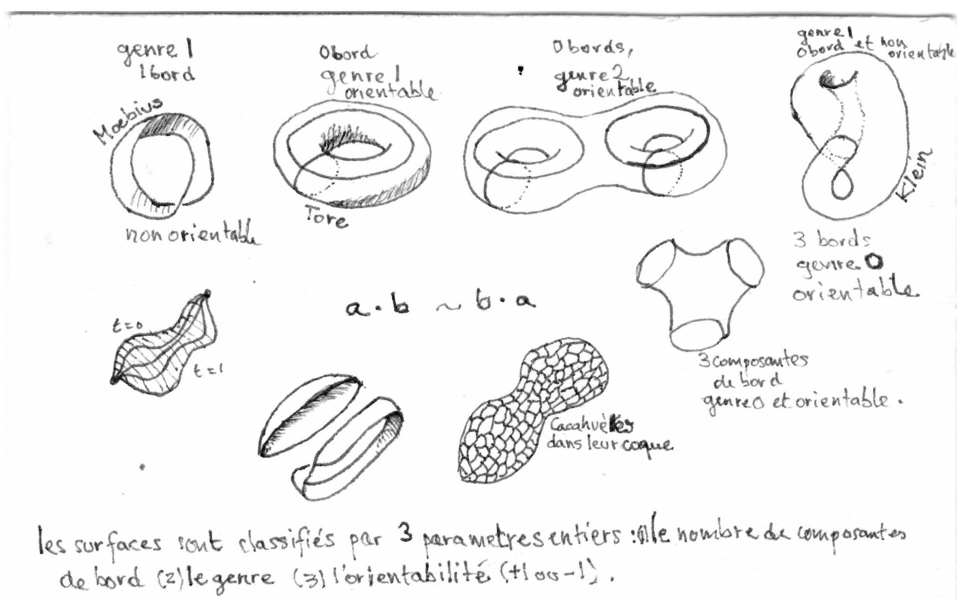
Interprétations homologiques d'invariants quantiques

JURY

Anna Beliakova	Professeur des Universités	Universität Zürich
Christian Blanchet	Professeur Émérite	Université de Paris
Francesco Costantino	Professeur des Universités	Université Paul Sabatier
Thomas Fiedler	Professeur des Universités	Université Paul Sabatier
John Guaschi	Professeur des Universités	Université de Caen
Vadim Schechtman	Professeur des Universités	Université Paul Sabatier
Emmanuel Wagner	Professeur des Universités	Université de Paris

École doctorale et spécialité :
EDMITT
Unité de Recherche :
Institut de Mathématiques de Toulouse (UMR 5219)
Directeur de Thèse :
Francesco Costantino
Rapporteurs :
John Guaschi et *Emmanuel Wagner*

"Tout est relatif..."
Le filochard du Café Chérie, à propos de son galion.



Esquisses de topologie, encre
Jean-Michel Fischer, Toulouse 2017.

Remerciements

Mes premiers mots vont à François Costantino, mon directeur de thèse. C'est un immense honneur d'avoir été l'élève d'un tel mathématicien et d'avoir pu bénéficier de son expertise de la topologie de basse dimension et d'un spectre de connaissance dont l'ampleur m'impressionne. François, la vitesse avec laquelle tu perçois les concepts, en envisages les détails, mènes des calculs, et structures ta pensée logique resteront - et bien au delà de mon travail de thèse - un idéal de vivacité d'esprit vers lequel je chercherai toujours à tendre. Il me suffit parfois de te donner un mot pour que tu aies déjà saisi le raisonnement qu'il engendrera, les techniques qu'il obligera ou les points que nous devrons investiguer. En un ou deux découpages topologiques de basse dimension dont tu partages les secrets, tu sais te convaincre quasi-instantanément de la potentielle véracité (ou non) d'un résultat, et tu m'as appris à affûter ce réflexe. Je te remercie pour cette inspiration que tu suscites mais aussi pour ta droiture à toute épreuve, pour m'avoir suivi, poussé, soutenu, conseillé, accompagné et finalement pour la rigueur que tu as mise dans la relecture de mes écrits.

Je remercie particulièrement John Guaschi et Emmanuel Wagner pour avoir rapporté cette thèse ; je suis très honoré, et désormais fier, qu'ils aient accepté ce lourd travail. Ils sont deux chercheurs dont j'admire la vision des maths, la façon d'en parler ou de les dessiner, autant que j'apprécie leur sympathie, si bien que c'est leurs regards sur mes travaux qui les valorisent et les enrichissent aujourd'hui - à mes yeux. C'est également un honneur de compter Anna Beliakova dans mon jury, qui est venue de Zurich pour assister à ma soutenance, ainsi que Vadim Schechtman dont les travaux sont précurseurs à tout ce manuscrit. Je souhaite adresser un remerciement particulier à Christian Blanchet. Au delà de ta présence aujourd'hui, tu as accepté d'encadrer mon mémoire de master, tu m'as mis le pied à l'étrier de la topologie, tu m'as encouragé à faire une thèse, tu m'y as aidé et j'ai pu sentir ton soutien tout au long de celle ci et au gré de nos nombreuses rencontres, ainsi tes travaux et ton savoir sont omniprésents dans ce manuscrit. Ta présence dans mon jury aujourd'hui est pour moi le couronnement de tout ce cheminement. Enfin, je suis touché que Thomas Fiedler fasse partie de mon jury. Thomas, je te remercie pour cela et par dessus tout pour faire vivre l'école d'hiver (?) la Llagone - Matemale ; j'ai aujourd'hui l'impression de faire partie de sa belle histoire. Les exposés de plus en plus interminables que tu me confies ont forgé mes connaissances et surtout ma capacité à exposer cette science devant un auditoire aussi récalcitrant que pointu, au sein duquel j'ai toujours aperçu ton regard bienveillant. J'y ai fait des rencontres inoubliables et apprécie toujours y retrouver les conversations riches et animées - à propos de maths ou de tout autre chose - qui s'y tiennent avec toi, avec ses inconditionnels Vladimir Verchinine,

Stepan Orevkov (notamment à l'occasion des longues routes/visites sinueuses dans sa voiture) ou avec Raymond ! J'espère rendre hommage à cette école et à sa pédagogie via l'exposé que je donne aujourd'hui.

Je remercie chaleureusement les organisateurs de Winter braids : Paolo Bellingeri, Vincent Florens, Jean-Baptiste Meilhan, et à nouveau Emmanuel Wagner. Cette école est une aubaine pour tous les jeunes chercheurs, et - au delà de cette seule semaine annuelle - l'attention que vous leur portez est unique. J'en ai amplement bénéficié, et vous en suis très reconnaissant.

Pendant mes études à Paris j'ai rencontré des professeurs qui sont probablement un peu responsables de ce que j'achève aujourd'hui. Je pense en particulier à Maxime Wolff et Julien Marché. Maxime, tu as encadré un de mes mémoires qui m'a autant marqué que ma rencontre avec toi, et Julien tu m'as posé des questions lors de mes soutenances pertinentes au point que j'en cherche encore les réponses. Avant tout, vous avez été des enseignants franchement inoubliables, votre savoir ou comment vous le divulguez sont inspirants, et c'est cette expérience faite que l'on se lance dans les maths les yeux fermés.

A Toulouse j'ai rencontré l'équipe de topologie, son séminaire, et j'apprécie grandement partager des discussions avec ses acteurs, je pense à Jean-François Barraud, Jean Raimbaud, Reda Chhaibi, Cyril Lecuire, Stéphane Lamy, Slavyana Geninska, Corentin Boissy.

Je dois des remerciements très particuliers à des professeurs qui ont suivi mon chemin y compris à d'autres endroits que la topologie, qui ont gardé un œil sur mon parcours ou sur mon émancipation dans mon métier d'enseignant-chercheur, je pense à Joan Milles, Pascale Roesch et Emmanuel Paul. Enfin je remercie Laurent Bakri et Guillaume Loizelet avec qui ce fut un plaisir d'enseigner. J'ai une pensée pour tous les élèves qui ont suivi mes cours, et qui m'ont remboursé mes efforts par beaucoup de plaisir.

La recherche à Toulouse ne pourrait être pérenne sans Tamara, Martine, Agnès, Jocelyne, Delphine, Marie-Line, Fabien.

Mais l'IMT c'est aussi ses doctorants et post-doctorants. Il y a d'abord Marc, avec qui je n'ai pas eu de discussions d'algèbre dépassant le programme de licence, mais des échanges, des promenades et surtout des gros baskets certainement plus enrichissants, ainsi je suis content de connaître Alice aujourd'hui. Ensuite, il y a Anne et Kévin qui sont les premiers sur qui je suis tombé, qui m'ont offert une écoute sans borne, et avec qui je suis fier d'être ami désormais. Je me souviens de Guillem, Fabrizio, Jordi, Arthur, André et Suzanna mes aînés. J'ai aujourd'hui un attachement particulier pour les inséparables topologues, Flo et JM, avec qui je ne partage pas seulement mon domaine de recherche ; bonne route à vous ! Je me souviens avoir partagé mon bureau avec Zak, puis Jorge, Elena et finalement Sarah à qui je souhaite une belle thèse ; mais aussi avoir partagé mon mandat au conseil du labo avec Antoine et Baptiste, et je crois qu'on a fait du bon boulot. La liste est longue, mais c'est avec autant d'émotion que je pense à Julie, Damien, Anton, Massimo, Paul-David, Daniel, Sergio, Sarah, Audun, Dat, Laura, Maxime, Dimitri, Dominique, Tu, William, Fabien, Guillaume, Maylis, Valentin, Hugo... et tous les autres. Un immense Merci à vous tous. Merci à Momo pour son écoute, à Shems et aux thesards du Mirail.

Une pensée singulière pour mes frères de thèse, Giulio et Sonny que j'aime rencontrer et avec qui j'apprécie discuter.

Lors de mon passage à l'INSA, Olivier Mazet, Stéphane Genieys et Laure Pédèches ont

porté une attention particulière à mon accueil et à mon suivi, merci. Merci aux ATER : Ali et Ibrahim, à mon cher cobureau Valentin et à tout le GMM pour sa bonne ambiance.

Au sein de mon domaine de recherche j'ai rencontré des chercheurs inspirants et avec qui de simples bribes de conversations participent à mon épanouissement dans ce domaine, je citerais Gregor Masbaum, Catherine Gille, Benjamin Audoux, Dror Bar-Natan, Roland Van der Veen, Hoël Queffelec, Louis-Hadrien Robert, Ramanujan Santharoubane. Mais aussi des jeunes chercheurs qui ont rendu toutes ces conférences très agréables...

Il y a avant tout Léo. Mais je ne serais pas là sans Loulou et Marco, pour qui mon amitié dépasse absolument ce cadre, et sans qui je n'aurais pas passé des moments si intéressants et pourtant si drôles, qu'ils soient mathématiques ou bien tout autre. Je pense également à Maÿlis, Sashka, Renaud, Delphine, Ramanu, Fathi, Cristina, Martin, Daniel, Julien, Ludovico, Edoardo, Anne-Laure, Filip, Livio, Olivier, Anthony...

A l'occasion de mon master parisien j'ai fait des rencontres importantes.

Il y a avant tout Léo. Tu es ma première rencontre de cette époque faste, et j'ai l'impression d'avoir grandi et grandi mathématiquement à tes cotés, je suis fier de ça ; et je garde évidemment des souvenirs impérissables de toutes nos virées bretonnes, parisiennes, sur Loire ou topologiques, avec Léa. Je suis certain de toujours trouver un réconfort et une tendresse chez mon Michou ou mon Henri. Je pense à mes origines mathématiques également avec Théo djodjo, Marine, Louis, Sacha, Cyril, Gaetan, Seauvaget, Nicolina, Boulanger, Vincent, David, Léo, Mathieu, Brian, Hugo... Je me souviens d'Ivens, Anatole et Morgane de l'époque où je faisais également de la Physique.

Avant ça, il me semble avoir fait une prépa avec Clément, Clara et Thomas. Heureusement, depuis nous partageons des souvenirs nettement plus solidement ancrés et géniaux, avec Marmar, Méli ou Raph. Je remercie Laversanne, Édouard, Flavio, Iskander, Alix, Tossi, et la génération du dessus pour avoir passé ces années là à mes cotés. Une pensée émue pour ma prof de Français de l'époque qui pensait que mes études n'aboutiraient jamais.

La suite est plus personnelle... Puisque l'occasion m'est offerte d'être absous de tout ridicule, j'en profite pour dire ma gratitude envers toutes les personnes qui n'ont peut-être pas partagé mes études mais qui ont envahi ma vie et y ont fait rayonné un bonheur tel, que je leur dois aujourd'hui cet aboutissement. Que celui qui craigne la mièvrerie passe son chemin jusqu'à l'introduction.

A mon arrivée à Toulouse j'ai apprécié rencontrer Jane, Got, Dominique, JF, Alain et tous les quizzeurs. Puis j'ai pris goût à cette ville en rencontrant mon Bazou, Jules, Nicole, Thomas, Élo, Abdel, David et tout le corps architectural de Toulouse.

Je veux dire un merci sans contour à Suzie, Yohanna, puis Aurore pour avoir littéralement illuminé mon quotidien. J'ai pu compter sur votre soutien et votre joie de vivre sans pareils, mais néanmoins sublimés par les sourires de Gabriel, Hédi, Dara, Mario, Raph, Julia, Le-kouz, Eli, Laurie et tous les Lions qui ont fait rugir de joie ma maison. Merci pour toute cette atmosphère qu'on a créée. La famille est bien plus large, je dois évidemment parler de la bienveillance de Krikri, et de Fab, Marie, Julie, Nico, Anne, leur petits et bien d'autres.

Plus largement, la vie m'a servi la fratrie que mes parents ont oublié de m'offrir : Fato, Cécile et Corentin, votre présence quotidienne, fidèle, et forcément éternelle constitue le socle de ma vie sur lequel je n'hésite pas à poser mes fesses au premier signe de fatigue. Yassine, Samuel, Théophile, Jean, vous êtes l'arrière garde qui m'octroie une force qu'il m'arrive de croire sans limite. J'espère être à la hauteur de l'amour que vous m'offrez tous. Laurie et Yassine, vous êtes forcément conscients de ce que cette thèse vous doit : vous - la rue Balard - et toutes (!!!) nos escapades ont sauvé ses pires moments, l'ont tenue à flot, et j'ai pu maintenir mes plus beaux caps. Je me souviens alors de vos résolutions au PMU d'Amboise. Flor, tu m'as connu à l'époque ou je balbutiais des maths et cette année nous soutenons nos thèses presque de concert, toi en médecine. Je suis heureux de te compter encore à mes cotés et avec Tim. Mimi et Veroushka vous êtes mes plus chics demoiselles d'honneur. J'aimerais ouvrir mes bras en grand pour vous Alix et Grégoire, la famille s'agrandit. Sofiane, je suis fier d'avoir célébré mon "théorème facile" avec toi, et d'avoir écrit un morceau de nos travaux ensembles... mais aussi de soutenir avant toi. La course à la première Fields / Martel est lancée, et j'ai un checkpoint d'avance, bonne route. Merci à Guillaume W. d'avoir les bons mots, et les mauvais et puis la flemme m'arrête là. C'est un grand bonheur de connaître Paul, Big et toute La Douzette. Sans pouvoir être exhaustif, je remercie Marion, Daphné, Laurine, Yoann, Costa et Clermont-Ferrand, Fababri, Vasco, Marion, mon Flo, Augustin, Claire, Mathieu, Estelle, Quentin, Lorris, Noé, Judith, Arturo, la CCJT, Sammy, Le Drax, Clément M.

Sophie, tu es certainement ma découverte la plus belle, la plus profonde, la plus ambitieuse ou fructueuse parmi celles qui jonchent mes années de thèse mais aussi celle qui me rend le plus fier et le plus heureux. Si ce manuscrit rend publique ma science, le reste nous appartient.

Je ne me sentirais pas aussi riche sans la folie de la Tordjmania, son amour débordant, et à qui je dois au moins la moitié de mon éducation ou de mon identité.

Je remercie Yves, Isabelle et Brigitte, pour m'avoir couvé depuis tout p'tit avec une attention touchante. Tata Sarah et tante Marianne pour Blauzac, la Chalp, et encore pleins d'autres choses, je pense aussi à Olivier.

Merci à Titi et aux sétois. Je me souviens avec émotion de nos conversations avec Éric et papa autour des sciences dures et de la guerre froide, le jeudi midi à la lib'. Merci à Kamel pour son gospel.

Je remercie Dieu... et Émir notamment pour avoir pris le temps de me regarder grandir, où me viennent des souvenirs émus...

J'ai une pensée infiniment peinée pour Alain et Vincent, à qui j'aurais aussi aimé montré ce travail. Ces pensées accompagnent Nat' et Sandra.

J'ai une pensée aimante pour mon grand-père Jean, mamie Ziza, et papi Maklouf à l'heure d'écrire ces mots.

Mamie, Annie, je t'assure que ces maths ne ferons se crasher aucun avion, c'est donc une science dont tu peux croire à l'exactitude, je suis ainsi très fier de les achever devant toi et espère qu'elles te convaincront. Merci de me nourrir encore de ta fougue, de tes sourires et de ton amour.

Papa, ton dernier cadeau fut cet ordi afin que je puisse écrire ma thèse décemment. Le dernier texte que je t'ai lu, près de ton lit, c'est l'introduction (qui suit) de ce travail... Tu t'es

endormi, j'en ai déduit qu'il fallait la retoucher et j'ai fait de mon mieux. Parmi tes derniers mots, je me rappelle un autoritaire “je veux lui payer une belle reliure”, tu verrais ici que celle de l'université n'est pas si mal. J'ai le cœur déchiré car je faisais aussi tout ça pour toi. Maman, c'est à toi d'être témoin de ce que je faisais pendant ces longues années durant lesquelles vous m'avez apporté un soutien sans aucune faille. Je vois dans tes yeux vos regards à tous les deux, et je veux les rendre fiers.

Ainsi - et pour conclure ces litanies - depuis le premier “D” de l'introduction et jusqu'au □ final de ce manuscrit, chaque mot est écrit en l'honneur de mes parents. Vous m'avez montré une richesse, une finesse, comme une vivacité d'esprit, capables de dompter le sérieux comme le pied de nez, ou de jongler avec l'amour, l'humour et la raison. Ce sont des valeurs qui me suffisent pour vivre, elles habillent et nourrissent la logique si bien que je m'en souviendrais à chaque endroit de mes travaux quels qu'ils soient.

Contents

Introduction

Dans ce travail, nous donnons des interprétations homologiques à certains invariants quantiques. Nous définissons et nous interprétons des représentations quantiques des groupes de tresses en termes d'homologies de certains revêtements d'espaces de configurations de points, nous étudions la fidélité de ces représentations et de certaines représentations obtenues des "TQFTs non semi-simples" qui leur sont intimement liées.

Topologie quantique

Les *groupes quantiques* sont des algèbres de Hopf qui proviennent de déformations d'algèbres enveloppantes d'algèbres de Lie. Cette notion apparaît à l'origine dans la littérature physique mais fut formalisée mathématiquement indépendamment par V. Drinfel'd et M. Jimbo autour de 1985. Nous mentionnons [Kas] et [C-P] comme ouvrages de référence sur le sujet, et nous en introduirons quelques aspects en Section 1.3. La catégorie des modules sur un groupe quantique donné est monoïdale et souvent munie d'une R-matrice (une famille de solutions de l'équation de Yang – Baxter), ce qui permet d'obtenir des représentations du groupe des tresses (Section 1.3). Ce dernier possède plusieurs définitions de nature topologique et ces *représentations quantiques de tresses* constituent ainsi le point initial de la *topologie quantique*.

Les théorèmes de Markov et de Lickorish – Wallace et Kirby (Theorem 2.3.7) encouragent l'émancipation de cette théorie. Si le premier permet de passer des tresses aux nœuds en clôturant les tresses, le second permet de passer des nœuds aux variétés différentielles de dimension 3 par la chirurgie de Dehn. Chaque étape a un coût qui impose une restriction aux invariants : les relations de Markov dans le premier cas, et celles de Kirby dans le second. Néanmoins, des représentations du groupe des tresses suffisamment riches peuvent assumer chacune de ces étapes et atteindre le niveau des invariants de variétés de dimension 3. Cette stratégie développée avec des représentations quantiques en données initiales fut l'œuvre de N. Reshetikhin et V. Turaev. Dans [R-T], ils construisent le *foncteur de Reshetikhin – Turaev* $\mathcal{RT}$ qui généralise les représentations quantiques de tresses aux enchevêtrements. Dans [RT2], le foncteur de Reshetikhin – Turaev est généralisé aux variétés de dimension 3. Le foncteur $\mathcal{RT}$ aboutit finalement à l'obtention d'invariants polynomiaux de nœuds, ainsi qu'à des invariants de variétés de dimension 3, tous appelés *invariants quantiques* par extension.

Cette théorie connaîtra encore une évolution significative grâce à la construction de *théories quantiques des champs topologiques (TQFTs)* obtenues à partir de ces invariants quantiques via la construction universelle de Blanchet – Habegger – Masbaum – Vogel [BHMV]. Ces TQFTs permettent de localiser les propriétés des invariants quantiques de variétés, en les rendant fonctoriels vis à vis du recollement de variétés à bord, voir Section 2.3.2 pour une définition plus précise.

Le foncteur $\mathcal{RT}$ permet entre autre de retrouver le fameux polynôme de Jones, invariant de nœuds découvert par V. Jones en 1984 et qui possède une propriété nouvelle en comparaison avec ses prédécesseurs : il permet de différencier un nœud de son image par un miroir. Si le polynôme d'Alexander découvert soixante ans plus tôt possède nombre de définitions topologiques, le contenu topologique de l'invariant de Jones se révèle moins clair. Le même constat s'applique à tous les invariants quantiques : leur construction repose sur les propriétés purement algébriques des catégories de modules sur les groupes quantiques si bien que leur définitions topologiques sont rares. Ainsi, la teneur topologique des invariants quantiques est le sujet principal de plusieurs conjectures importantes du domaine. Donner des interprétations homologiques aux invariants quantiques - ce que nous faisons dans ce travail - s'inscrit dans ce cadre.

Représentations homologiques de tresses

La construction de représentations homologiques du groupe des tresses repose sur le fait que ce dernier agit par difféotopie sur le disque à pointes. Cette action se généralise coordonnée par coordonnée aux espaces de configurations de plusieurs points dans le disque à pointes et devient une représentation linéaire dès lors qu'elle est relevée à l'homologie.

C'est R. Lawrence qui développe cette idée autour de 1990 dans sa thèse, à l'époque déjà dans l'idée de tenter de lever le mystère du contenu topologique du polynôme de Jones. Elle construit une famille graduée de représentations du groupe des tresses sur des groupes d'homologie à coefficients locaux dans un anneau de polynômes de Laurent sur l'espace de configuration de points dans le disque à pointes ([Law]).

C'est à la même époque que la topologie quantique se développe et quelques liens avec la théorie homologique de Lawrence semblent émerger. V. Drinfel'd et T. Kohno relient indépendamment ([Drin] [K0]) les représentations quantiques de tresses à des représentations monodromiques provenant de l'équation de Knizhnik – Zamolodchikov (KZ) mettant en jeu une action topologique des tresses sur des espaces de configuration de points. Nous ferons référence à ce théorème comme celui de Drinfel'd – Kohno. Dans [F-W], G. Felder et C. Wieczerkowski construisent une action du groupe quantique $U_q\mathfrak{sl}(2)$ sur un module engendré par des objets topologiques du disque à pointes - les r-lacets - ainsi qu'une action naturelle du groupe des tresses sur ces modules, qui commute à l'action quantique. L'interprétation homologique de ce module de r-lacets ne reste que conjecturale ([F-W, Conjectures 6.1, 6.2]) tout comme ses liens avec la théorie homologique de Lawrence. Enfin, dans [S-V], V. Schetchtman et A. Varchenko obtiennent des représentations de groupes quantiques sur des groupes d'homologie à coefficients locaux sur des espaces de configuration de points.

Il faudra attendre une décennie avant que les représentations de Lawrence ne gagnent toute leur notoriété grâce aux travaux de D. Krammer et S. Bigelow qui montreront leur fidélité au deuxième niveau de la graduation [Kra] [Big0], celui que nous appellerons *représentation BKL*. Ce sont les premières représentations fidèles de dimension finie des groupes de tresses. Pour faire la preuve de leur fidélité, S. Bigelow introduit des objets homologiques, les fourchettes, qui lui servent de base pour la représentation ainsi qu'un couplage homologique entre ces objets. Il se sert de ces outils pour donner une définition homologique du polynôme de Jones dans [Big3] en reprenant les idées de R. Lawrence. Pour le faire il n'utilise pas le formalisme quantique mais simplement les relations d'écheveau satisfaites par le polynôme de Jones. En utilisant la même stratégie il donnera également une définition homologique au polynôme HOMFLY dans [Big4]. Dans [P-P], L. Paoluzzi et L. Paris montrent que les représentations BKL ne recouvrent qu'une sous-représentation de la représentation homologique (complète) à coefficients dans l'anneau des polynômes de Laurent.

Dans [J-K], C. Jackson et T. Kerler établissent explicitement un isomorphisme entre la représentation BKL et celle sur un sous module du produit tensoriel de modules de Verma sur $U_q\mathfrak{sl}(2)$, à savoir l'action restreinte aux vecteurs *de plus haut poids* et sous-poids 2. Inspiré par ces travaux, dans [Kar], l'auteur s'intéresse au cas où les paramètres quantiques sont des racines de l'unité et étudie la décomposition projective d'un produit tensoriel de modules isomorphes aux représentations de Lawrence, il trouve une action fidèle du centre du groupe de tresses. Dans [K2], T. Kohno montre que les représentations de Lawrence sont isomorphes à celles de monodromie KZ restreintes aux vecteurs de plus hauts poids (théorème de Kohno, 2012), elles mêmes identiques aux représentations sur les vecteurs de plus haut poids des produit de modules de Verma de $U_q\mathfrak{sl}(2)$ obtenues via la R-matrice (théorème de Drinfel'd – Konho, prouvé vingt ans plus tôt). Ceci établit un lien direct et profond entre les représentations de Lawrence et la R-matrice de $U_q\mathfrak{sl}(2)$ qui est résumé dans [Ito, Theorem 4.5]. Cependant, l'isomorphisme de Kohno est valable pour des paramètres génériques (il n'est pas un morphisme sur l'anneau des polynômes de Laurent, mais sur $\mathbb{C}$ lorsque le paramètre quantique est évalué à une valeur "générique") et ne recouvre pas tout le produit tensoriel de modules de Verma, mais seulement l'action restreinte aux vecteurs de plus haut poids dont les bases ne sont pas simples à expliciter ni à utiliser. Dans [Ito1],[Ito2][An] les auteurs utilisent l'isomorphisme de Kohno pour donner des interprétations homologiques à deux grandes familles de polynômes quantiques de nœuds construites à partir des modules de $U_q\mathfrak{sl}(2)$: *les polynômes de Jones colorés* et ceux *d'Alexander colorés* (aussi appelés *invariants ADO*, introduits par Akutsu – Deguchi – Ohtsuki).

Le théorème de Kohno

Nous résumons les liens antérieurs à ce travail, existant entre représentations quantiques de tresses et représentations de Lawrence, en fixant des notations. Soit V le module de Verma de $U_q\mathfrak{sl}(2)$ (on ne fera pas apparaître le paramètre dont il dépend dans cette introduction, afin de ne pas alourdir les notations), pour $n \in \mathbb{N}$, le module $V^{\otimes n}$ est muni d'une action quantique du groupe des tresses $\mathcal{B}_n$. Soit $r \in \mathbb{N}$, $W_{n,r}$ le sous espace de $V^{\otimes n}$ engendré par les vecteurs de sous poids r et $Y_{n,r}$ celui engendré par les vecteurs de plus haut poids de $W_{n,r}$. Les espaces $W_{n,r}$ et $Y_{n,r}$ sont des sous représentations du groupe des tresses, et $V^{\otimes n} = \bigoplus_{r\in\mathbb{N}} W_{n,r}$. Aussi, $Y_{n,r}$ est une sous représentations de $W_{n,r}$ qui est irréductible sur le corps des fractions ([J-K]). Toutes ces définitions sont rigoureusement données dans la Section 1.4.17.

Soit X_r l'espace de configuration de r-points dans le disque épointé de n pointes. Notons $H_r(X_r)$ les groupes d'homologies absolues sur un certain revêtement de X_r (ou système local), dont nous donnons toutes les définitions précises dans la Section 3.2.1. $H_r(X_r)$ est un module sur l'anneau des polynômes de Laurent $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ sur lequel agit le groupe des tresses via la représentation de Lawrence, pour $r \in \mathbb{N}$. Une conséquence des théorèmes de Drinfel'd – Kohno et de Kohno est l'existence d'un isomorphisme :

$$H_r(X_r) \to Y_{n,r} \tag{1}$$

de représentations du groupe $\mathcal{B}_n$. Cet isomorphisme est valable pour tout $r \in \mathbb{N}$ et pour un ensemble générique de paramètres complexes (i.e. seulement en tant que $\mathbb{C}$-espace vectoriels, après avoir évalué le paramètre quantique à une valeur générique). L'isomorphisme est explicité dans [Ito, Theorem 4.5] (dans le cas unicolore, i.e. avec plusieurs copies du même module de Verma) via la base des multi-fourchettes à gauche et une certaine base de l'espace des vecteurs de plus haut poids à droite.

Dans ce travail de thèse, nous étendons les représentations de Lawrence à l'homologie relative ce qui nous permet d'étendre le théorème de Kohno dans plusieurs directions. En premier lieu, nous travaillons sur l'anneau des polynômes de Laurent et surtout nous obtenons un isomorphisme de $\mathcal{B}_n$-modules entre ces groupes d'homologie relative et les espaces de poids $W_{n,r}$ (au lieu de $Y_{n,r}$). Tandis que l'action de $U_q\mathfrak{sl}(2)$ sur le produit tensoriel de modules de Verma n'apparaît pas dans le théorème de Kohno et le Morphisme (1), nous définissons homologiquement l'action de $U_q\mathfrak{sl}(2)$, ce qui rend l'interprétation homologique du produit tensoriel de modules de Verma complète du point de vue des actions de $U_q\mathfrak{sl}(2)$ et de $\mathcal{B}_n$.

Le contenu de ce manuscrit

Dans le premier chapitre, nous exposons le cadre mathématiques de ce travail de thèse. En Section 1.1 nous définissons les objets topologiques étudiés : les groupes modulaires, et nous nous attardons sur le cas particulier des groupes de tresses. Dans la Section 1.2 nous présentons des familles de représentations du groupe des tresses, de nature topologique, tandis qu'en Section 1.3 nous définissons des représentations quantiques de tresses ce qui nous conduit à introduire la notion de groupe quantique ainsi que les objets qui s'y rapportent. Dans la dernière section, Section 1.4, nous portons notre attention sur l'anneau utilisé pour construire un groupe quantique et son influence sur la théorie des représentations. Cela nous invite à introduire les *versions entières* de ces algèbres quantiques qui permettent de travailler avec les polynômes de Laurent. La Remarque 1.4.24 montre que les représentations quantiques du groupe des tresses associées à $U_q\mathfrak{sl}(2)$ sont graduées par les entiers qui correspondent aux "poids" des vecteurs, tandis qu'en Section 1.2.3 nous introduisons une famille de représentations homologiques du groupe des tresses, graduée par les entiers naturels - celles introduites par R. Lawrence dans [Law] - que nous appellerons *représentations de Lawrence*.

Dans le deuxième chapitre, nous nous intéressons aux premiers niveaux de la graduation des représentations quantiques d'une part et homologiques d'autre part. En Section 2.1 nous prouvons que le premier niveau des représentations quantiques est isomorphe aux représentations de Gassner qui correspondent au premier niveau des représentations homologiques de Lawrence dans une version "colorée" (i.e. à plusieurs variables, Theorem 2.1.1, Section 2.1). Cela explicite le théorème de Kohno au premier niveau et dans une version colorée :

$$H_1(X_1) \to Y_{n,1}.$$

En Section 2.2, nous construisons une version colorée des représentations BKL agissant sur $H_2(X_2)$. L'anneau de ses coefficients est celui des polynômes de Laurent en $n+1$ variables (au lieu de deux, ce qui justifie la dénomination de "colorée"). Nous donnons des bases de cette représentation en utilisant du calcul de Fox, et nous calculons les matrices associées aux générateurs du groupe des tresses à l'aide d'une généralisation (colorée) du couplage nouilles-fourchettes introduit par Bigelow pour montrer la fidélité de BKL (Proposition 2.2.5).

Dans [CGP2], Costantino – Geer – Patureau construisent des invariants quantiques de 3-variétés à partir d'une catégorie de modules quantiques non semi-simple sur la version déroulée de $U_q\mathfrak{sl}(2)$, ce qui dénote une nette évolution en comparaison avec ceux construits plus tôt par Reshetikhin – Turaev. Dans [BCGP2], Blanchet – Costantino – Geer – Patureau parviennent à adapter la construction universelle ([BHMV]) à ce formalisme non semi-simple, et obtiennent des TQFTs dites *non semi-simples*. Ces TQFTs sont naturellement graduées et la graduation est préservée par l'action du groupe modulaire. Dans la Section 2.3 nous étudions le premier niveau de la TQFT non semi-simple de la sphère à quatre pointes. Nous relions la représentation obtenue de son groupe modulaire via la TQFT à une représentation de nature homologique, ce qui aboutit à la fidélité de la représentation quantique (Theorem 3, Section 2.3.3). Nous remarquons que les premiers niveaux de la TQFT des sphères à pointes sont reliés aux représentations quantiques de tresses aux racines de l'unité en Section 2.3.4.

Le dernier chapitre étend les représentations de Lawrence via l'homologie relative, il clarifie et généralise leurs liens avec les représentations quantiques de tresses obtenues sur le produit de modules de Verma de $U_q\mathfrak{sl}(2)$ par la R-matrice. En nous inspirant de [F-W], nous étudions des groupes d'homologie localement finie, relative et a coefficients dans un système local abélien sur des espaces de configurations de r - points dans le disque épointé, que nous notons $H_r(X_r, X_r^-)$. Nous munissons ces complexes d'une action du groupe quantique $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ (une version entière de $U_q\mathfrak{sl}(2)$ définie Section 1.4.3) via des actions homologiques de ses générateurs et cela aboutit au résultat suivant.

Theorème 0.0.1 (Theorem 4, Section 3.2.3.3)**.** *Le module sur les polynômes de Laurent* $\mathcal{H} = \bigoplus_{r\in\mathbb{N}} H_r(X_r, X_r^-)$ *est une représentation de* $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$*.*

Dans le Lemme 3.2.8, nous montrons que $H_r(X_r, X_r^-)$ est un module libre sur l'anneau des polynômes de Laurent, et qu'une base (dites "entière") est donnée par la famille des *multi-arcs* définie en Section 3.2.2.2. Cela nous conduit à reconnaître cette représentation de $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ comme étant un produit tensoriel de modules de Verma, ce que nous résumons dans l'énoncé suivant.

Theorème 0.0.2 (Theorem 5, Section 3.2.4.3)**.** *Pour tout* $n \in \mathbb{N}$*, il existe un morphisme de* $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$*-modules :*

$$V^{\otimes n} \to \mathcal{H} = \bigoplus_{r\in\mathbb{N}} H_r(X_r, X_r^-)$$

tel que la base entière classique de $V^{\otimes n}$ est envoyée sur la base des multi-arcs *(définie au préalable Section 3.2.2.2) et montrée comme étant une base entière de toute l'homologie relative. L'entier n correspond au nombre de pointes du disque D_n utilisé pour définir l'espace des configurations X_r.*

Enfin, nous retrouvons une action naturelle du groupe des tresses (par homéomorphismes) sur ces modules homologiques, et nous montrons qu'il s'agit de la représentation obtenue par la R-matrice de la catégorie de modules de $U_q\mathfrak{sl}(2)$.

Theorème 0.0.3 (Theorem 6, Section 3.2.5.2)**.** *Pour tout $n \in \mathbb{N}$ et tout $r \in \mathbb{N}$, le morphisme :*

$$W_{n,r} \to H_r(X_r, X_r^-)$$

induit par le théorème précédent est un isomorphisme de représentations de $\mathcal{B}_n$, si bien que le morphisme :

$$V^{\otimes n} \to \mathcal{H} = \bigoplus_{r \in \mathbb{N}} H_r(X_r, X_r^-)$$

du théorème précédent est un ismorphisme de $U_q\mathfrak{sl}(2)$-modules et *de $\mathcal{B}_n$-modules.*

Nous exhibons des bases entières de l'homologie (i.e. des bases en tant que module sur un anneau entier de polynômes de Laurent). L'action de $U_q\mathfrak{sl}(2)$, ainsi que celle du groupe des tresses, respectent cette structure, tout comme l'isomorphisme vers le produit tensoriel de modules de Verma.

Nous montrons que la suite longue de l'homologie relative devient dans ce modèle, pour $r \in \mathbb{N}$, une suite courte :

$$1 \to H_r(X_r) \to H_r(X_r, X_r^-) \to H_{r-1}(X_r^-) \to 1,$$

de telle sorte que $H_r(X_r, X_r^-)$ étend les représentations de Lawrence (définies sur les modules absolus de gauche $H_r(X_r)$). Ce travail permet donc d'étendre le Théorème de Kohno au delà des vecteurs de plus haut poids, et de retrouver homologiquement tout le produit de modules de Verma de $U_q\mathfrak{sl}(2)$. Les représentations de Lawrence en sont une sous-représentation, ainsi le théorème de Kohno est un corollaire de ce travail. Les hypothèses de généricité sont clarifiées et deviennent algébriques car tous les isomorphismes conservent la structure entière des coefficients, et les liens entre les bases entières (celle des *multi-arcs*) et les différentes bases intervenants dans la littérature sont explicités. Ceci est résumé dans le Corollaire 3.2.76.

Les représentations homologiques obtenues sont une généralisation des représentations de Lawrence, donc elles sont génériquement fidèles. Elles permettent de retrouver homologiquement plusieurs propriétés de la catégorie de modules sur $U_q\mathfrak{sl}(2)$.

Nous illustrons la structure de poids du produit tensoriel de modules de Verma dans le diagramme suivant, au niveau r de la graduation :

$$
\begin{array}{ccc}
\cdots & & \cdots \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
W_{n,r} & \longleftrightarrow & H_r(X_r, X_r^-) \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
W_{n,r+1} & \longleftrightarrow & H_{r+1}(X_{r+1}, X_{r+1}^-) \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
\cdots & & \cdots
\end{array}
.
$$

Les flèches horizontales correspondent aux isomorphismes de représentation de tresses du Théorème 0.0.3, tandis que les flèches verticales correspondent aux actions des générateurs E et F de $U_q\mathfrak{sl}(2)$, quantiques à gauche et homologiques à droite (définition homologique fruit du présent travail, inspirée par [F-W]), qui régissent la structure de poids des modules de Verma. La somme directe de tous les espaces alignés verticalement à gauche donne le produit de modules de Verma $V^{\otimes n}$, tandis que celle des espaces alignés verticalement à droite correspond au module homologique noté $\mathcal{H}$. L'interprétation homologique des générateurs de $U_q\mathfrak{sl}(2)$ en découle, ainsi que celles des relations qu'ils vérifient et de la R-matrice construite à partir de ces générateurs.

Ce modèle homologique (pour les représentations quantiques de tresses) est ensuite appliqué aux nœuds vus comme des clôtures de tresses. Avoir une interprétation de tout le produit de modules de Verma permet d'obtenir une formule des traces (homologiques) pour les polynômes de Jones colorés, qui s'apparente à une somme pondérée de nombres de Lefschetz abélianisés, c'est le contenu du Théorème 7, Section 4.1.4.

Introduction

In this work, we give homological interpretations to a number of quantum invariants. We define and interpret quantum representations of braid groups, in terms of the homologies of some covering spaces of configuration spaces of points. We then study the faithfulness of these representations and the one of some closely related representations arising from *"non semi-simple TQFTs"*.

Quantum topology

Quantum groups are Hopf algebras arising from deformations of enveloping algebras of Lie algebras. While first appearing in works of physics, this notion was formalized independently by V. Drinfel'd and M. Jimbo in 1985. We mention [Kas] and [C-P] as books of reference on the subject, and introduce some aspects of it in Section 1.3.

The category of modules on a given quantum group is monoidal and often equipped with an R-matrix (a family of solutions of the Yang - Baxter equation), which allows representations of the braid groups to be obtained from it (Section 1.3). The braid groups admit several topological definitions; these *quantum representations of braids* thus constitute the starting point of *quantum topology*.

The Markov and Lickorish – Wallace and Kirby theorems (Theorem 2.3.7) encourage the expansion of this theory. The first one enables the transition from braids to knots through braid closures; the second one enables the transition from knots to dimension 3 differentiable manifolds through the Dehn surgery. Each step comes at a cost imposing a restriction on invariants: the Markov relations in the first case, those of Kirby in the second. Nevertheless, sufficiently rich representations of the braid groups can drive each of these steps and achieve the level of dimension 3 manifold invariants. This strategy based on quantum representations as initial data was developed by N. Reshetikhin and V. Turaev. In [R-T], they built the *Reshetikhin – Turaev functor* ($\mathcal{RT}$ functor) which generalizes the quantum representations of braids to tangles. In [RT2], the $\mathcal{RT}$ functor is generalized to 3 dimensional manifolds. Finally, the $\mathcal{RT}$ functor results in polynomial knot invariants, as well as dimension 3 manifold invariants, that are all called *quantum invariants* by extension.

This theory will reach another significative evolution thanks to the construction of *topological quantum field theories (TQFTs)* obtained from these quantum invariants via the universal construction of Blanchet – Habegger – Masbaum – Vogel [BHMV]. These TQFTs provide a localization of quantum invariants properties, by making them functorial regarding gluings of manifolds with boundary, see Section 2.3.2 for a more precise definition.

The $\mathcal{RT}$ functor allows among other things to recover the famous Jones polynomial, an invariant of knots discovered by V. Jones in 1984 and which has a new property compared to its predecessors: it makes it possible to differentiate a knot from its image by a mirror. If the Alexander polynomial discovered sixty years earlier has many topological definitions, the topological content of Jones' invariant is less clear. The same observation applies to all quantum invariants: their construction is based on the purely algebraic properties of the module categories on quantum groups, so that their topological definitions are rare. Thus, the topological content of quantum invariants is the main subject of several important conjectures in the domain. Giving homological interpretations to quantum invariants - what we do in this work - fits into this framework.

Homological representations of the braid groups

Building homological representations of braid groups relies on the fact that it acts by mapping class on the punctured disk. This action generalizes to configuration spaces of several points in the punctured disk, and becomes linear while lifted to homology.

It's R. Lawrence who has developed this idea around 1990 in her thesis, by the time it was already for the purpose of finding topological information in the Jones polynomial. She builds a family of graded representations of the braid groups over homology groups with local coefficients in a ring of Laurent polynomials over the configuration space of points inside the punctured disk ([Law]).

At the same time, the field of quantum topology arised and some links with the homological theory of Lawrence seem to appear. V. Drinfel'd and T. Kohno relate independently ([Drin] [K0]) quantum representations of braid groups with monodromy representations coming from Knizhnik – Zamolodchikov (KZ) equation, involving a topological action of braids on configuration spaces of points. We will refer to this theorem as Drinfel'd – Kohno's. In [F-W], G. Felder and C. Wieczerkowski build an action of the quantum group $U_q\mathfrak{sl}(2)$ on some module generated by topological objects of the punctured disk - r-loops - together with a natural action of the braid groups which commutes with the quantum one. The homological interpretations of this module remain conjectures ([F-W, Conjecture 6.1, 6.2]) as well as its links with Lawrence's theory. Finally, in [S-V], V. Schetchtman and A. Varchenko obtain representations of quantum groups on some local system homology on configuration spaces of points.

It is only a decade after that Lawrence's theory obtained all its notoriety thanks to D. Krammer and S. Bigelow's works, showing their faithfulness at the second level of the grading [Kra], [Big0], the one we refer to as *BKL representation*. This is the first known faithful and finite dimensional representation of the braid groups. To perform the proof of their faithfulness, S. Bigelow introduces homological objects, *forks*, that he uses as a basis for the representation together with a homological pairing between these objects. He then uses these tools to give a homological definition of the Jones polynomial in [Big3] taking back R. Lawrence's ideas. To do the latter, he does not use quantum formalism but skein relations satisfied by the Jones polynomial. He follows the same strategy to give a homological definition of the HOMFLY polynomial in [Big4]. In [P-P], L. Paoluzzi and L. Paris show that the BKL representation only recovers a sub-representation of the entire homological representation with coefficient in the Laurent polynomial ring.

In [J-K], C. Jackson and T. Kerler establish explicitly an isomorphism between the BKL representation and the one on a sub-module of tensor products of $U_q\mathfrak{sl}(2)$ Verma modules, namely the restricted action to *highest weight vectors* and sub-weights 2. Following this, in [Kar], the author is interested in the case where quantum parameters are roots of unity, he studies the projective decomposition of a tensor product of modules isomorphic to Lawrence representations, and he finds a faithful action of the center of the braid groups. In [K2], T. Kohno shows Lawrence's representations are isomorphic to those from KZ monodromy restricted to highest weight vectors (Kohno's theorem, 2012), themselves already shown to be isomorphic to the braid representations on highest weight vectors of tensor products of $U_q\mathfrak{sl}(2)$ Verma modules obtained by the R-matrix (Drinfel'd – Kohno's theorem, proved twenty years earlier). This establishes a direct and deep relation between Lawrence's representations and $U_q\mathfrak{sl}(2)$ R-matrix that is summed up in [Ito, Theorem 4.5]. Yet Kohno's isomorphism works for a generic set of parameters (it is not a morphism on Laurent polynomials ring, but on $\mathbb{C}$ when quantum parameters is evaluated at a "generic" value) and does not recover the whole product of Verma modules, but the restricted action to highest weight vectors for which basis are not easy to compute nor to use. In [Ito1], [Ito2], [An], the authors use Kohno's isomorphism to give homological interpretations for two important families of quantum knot polynomials built from $U_q\mathfrak{sl}(2)$ modules: *colored Jones polynomials* and *colored Alexander polynomials* (also known as *ADO invariants*, first introduced by Akutsu – Deguchi – Ohtsuki).

Kohno's theorem

We recall links existing - before this work - between quantum representations of braids and Lawrence's representations, fixing notations. Let V be the Verma module of $U_q\mathfrak{sl}(2)$ (we won't put the parameter it depends on in notations, so as to simplify them), for $n \in \mathbb{N}$, the module $V^{\otimes n}$ is endowed with a quantum action of the braid group $\mathcal{B}_n$. Let $r \in \mathbb{N}$, $W_{n,r}$ be the sub space of $V^{\otimes n}$ generated by vectors of sub weight r and $Y_{n,r}$ be the one generated by highest weight vectors of $W_{n,r}$. Spaces $W_{n,r}$ and $Y_{n,r}$ are sub representations of braid groups, and $V^{\otimes n} = \bigoplus_{r\in\mathbb{N}} W_{n,r}$. The representation $Y_{n,r}$ is known to be an irreducible braid sub-representation of $W_{n,r}$ when working on the fraction field ([J-K]). All these definitions are rigorously given in Section 1.4.17.

Let X_r be the configuration space of r-points taken in the punctured disk with n punctures D_n. Let $H_r(X_r)$ be the absolute homology groups on some covering space of X_r (or local system), for which we give all precise definitions in Section 3.2.1. $H_r(X_r)$ is a module over the Laurent polynomials ring $\mathbb{Z}\left[q^{\pm 1}, t^{\pm 1}\right]$ over which the braid groups act via Lawrence's representations, for $r \in \mathbb{N}$. The existence of the following morphism is a consequence of Drinfel'd – Kohno and Kohno's theorems:

$$H_r(X_r) \to Y_{n,r}. \tag{2}$$

It is a morphism of representations of $\mathcal{B}_n$. This isomorphism holds for all $r \in \mathbb{N}$ and for a generic set of complex parameters (i.e. only as $\mathbb{C}$ vector spaces after evaluation of the quantum parameter to a generic value). The isomorphism is explicitly formulated in [Ito, Theorem 4.5] (in the unicolored case, i.e. using several copies of the same Verma module instead of different ones) via the multi-fork basis on the left and some given basis of the highest weight space on the right.

In this work, we extend Lawrence's representations to relative homology which allows us to extend Kohno's theorem in different directions. First, we work with the Laurent polynomials ring but moreover we obtain an isomorphism of $\mathcal{B}_n$-modules between these relative homology modules and $W_{n,r}$ (instead of $Y_{n,r}$). Although the $U_q\mathfrak{sl}(2)$ action on product of Verma modules does not appear in Kohno's theorem and Morphism (2), we define homologically the action of $U_q\mathfrak{sl}(2)$, which makes the homological interpretation of Verma modules' tensor products complete regarding the $U_q\mathfrak{sl}(2)$-action and the $\mathcal{B}_n$-action.

The content of this manuscript

In the first chapter, we present the mathematical framework of this thesis. In Section 1.1 we define topological objects of interest: mapping class groups, and we focus on the special case of braid groups. In Section 1.2 we introduce families of representations of braid groups, of topological nature, while in Section 1.3 we define quantum representations of braids which leads us to the introduction of the notion of quantum group and related objects. In the last section, Section 1.4, we pay attention to the ring used to build a quantum group and its influence on the representation theory. Remark 1.4.24 shows quantum representations of braids associated to $U_q\mathfrak{sl}(2)$ are graded by integers corresponding to “weights” of vectors, while in Section 1.2.3 we introduce a family of homological representations of braid groups, graded by integers - those introduced by R. Lawrence in [Law] - that we call *Lawrence's representations.*

In the second chapter, we are interested in the first level of the grading of quantum representations on one hand and homological ones on the other. In Section 2.1 we prove that the first level of quantum representations is isomorphic to the *Gassner representation* that corresponds to the first level of Lawrence's homological representations in a “colored” version (i.e. multi-variables, Theorem 2.1.1, Section 2.1). This emphasizes the first level of Kohno's theorem in a colored version:

$$H_1(X_1) \to Y_{n,1}.$$

In Section 2.2, we build a colored version of BKL representations acting upon $H_2(X_2)$. Its ring of coefficients is Laurent polynomials in $n+1$ variables (instead of two, which justifies the “colored” denomination). We give basis for these representations using Fox calculus, and we compute matrices associated to braid generators using a (colored) generalization of the fork-noodle pairing introduced by Bigelow to show BKL faithfulness (Proposition 2.2.5).

In [CGP2], Costantino – Geer – Patureau build quantum invariants of 3 manifolds from a non semi-simple quantum category of modules on the *unrolled* version of $U_q\mathfrak{sl}(2)$, and it presents a strong improvement in comparison with those built earlier by Reshetikhin – Turaev. In [BCGP2], Blanchet – Costantino – Geer – Patureau succeed in adapting universal construction ([BHMV]) to this non semi-simple formalism, they obtain *non semi-simple TQFTs.* These TQFTs are naturally graded and the grading is preserved by the mapping class group action. In Section 2.3 we study the first level of the non semi-simple TQFT of the sphere with four punctures. We relate the obtained representation of its mapping class group to a representation of homological flavor, it leads to the faithfulness of the quantum representation (Theorem 3, Section 2.3.3). We then remark that first levels of punctured spheres' TQFTs are related to quantum representations of braid groups at roots of unity in Section 2.3.4.

The last chapter extends Lawrence's representations via relative homology, it clarifies and generalizes their links with quantum representations of braid groups obtained on tensor products of $U_q\mathfrak{sl}(2)$ Verma's by use of the R-matrix. Inspired by [F-W], we study groups of homology, locally finite, relative and having coefficients in an abelian local system over configuration spaces of r - points in the punctured disk, which we denote by $H_r(X_r, X_r^-)$. We endow these complexes with an action of the quantum group $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ (an integral version of $U_q\mathfrak{sl}(2)$ defined in Section 1.4.3) via homological actions of its generators, that leads to the following result.

Theorem 0.0.4 (Theorem 4, Section 3.2.3.3)**.** *The module $\mathcal{H} = \bigoplus_{r\in\mathbb{N}} H_r(X_r, X_r^-)$ over Laurent polynomials is a representation of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$.*

In Lemma 3.2.8, we show that $H_r(X_r, X_r^-)$ is a free module on Laurent polynomials ring, and that a basis (said "integral") is given by the family of *multi-arcs* defined in Section 3.2.2.2. This helps us recognizing this $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ representation as a tensor product of Verma modules, what we sum-up in the following statement.

Theorem 0.0.5 (Theorem 5, Section 3.2.4.3)**.** *For all $n \in \mathbb{N}$, there exists a morphism of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$-modules :*

$$V^{\otimes n} \to \mathcal{H} = \bigoplus_{r\in\mathbb{N}} H_r(X_r, X_r^-)$$

such that the classical integral basis of $V^{\otimes n}$ is sent to the multi-arcs *basis shown to be an integral basis of the whole relative homology. The integer n corresponds to the number of punctures of the disk D_n used to define the configuration space X_r.*

Finally, we find a natural action of braid groups (by homeomorphisms) over these homological modules, and we show that it is the R-matrix representation obtained using $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ Verma modules.

Theorem 0.0.6 (Theorem 6, Section 3.2.5.2)**.** *For all $n \in \mathbb{N}$ and all $r \in \mathbb{N}$, the morphism :*

$$W_{n,r} \to H_r(X_r, X_r^-)$$

induced by the previous theorem is an isomorphism of $\mathcal{B}_n$ - representations, so much that the morphism:

$$V^{\otimes n} \to \mathcal{H} = \bigoplus_{r\in\mathbb{N}} H_r(X_r, X_r^-)$$

from previous theorem is a morphism of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$-modules and *of $\mathcal{B}_n$-modules.*

We give integral basis of homology (i.e. basis as module on an integral ring of Laurent polynomials). The $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$-action and the $\mathcal{B}_n$-action preserve this structure, so does the isomorphism to the tensor product of Verma modules.

We show that the long exact sequence of relative homology becomes, in this model, a short one:

$$1 \to H_r(X_r) \to H_r(X_r, X_r^-) \to H_{r-1}(X_r^-) \to 1,$$

so that $H_r(X_r, X_r^-)$ extend Lawrence's representations on the left (defined on absolute modules $H_r(X_r)$). This work allows then an extension of Kohno's theorem beyond highest weight vectors, and to recover homologically the entire tensor product of $U_q\mathfrak{sl}(2)$ Verma modules. Lawrence's representations are sub-representations of it so that Kohno's theorem is a corollary of this work. Generic hypothesis are clarified and become algebraic thanks to the fact that all isomorphisms preserve the integral structure of coefficients, and the links between integral basis (multi-arcs) and basis one finds in the literature are exposed. All of this is summed-up in Corollary 3.2.76.

The obtained homological representations are a generalization of Lawrence's representations so they are generically faithful. They allow a homological recovering of several properties of the category of $U_q\mathfrak{sl}(2)$-modules.

We illustrate the weight structure of tensor product of Verma modules in the following diagram, at level r of the grading:

$$\begin{array}{ccc}
\cdots & & \cdots \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
W_{n,r} & \longleftrightarrow & H_r(X_r, X_r^-) \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
W_{n,r+1} & \longleftrightarrow & H_{r+1}(X_{r+1}, X_{r+1}^-) \\
E \circlearrowleft \; \circlearrowright F & & E \circlearrowleft \; \circlearrowright F \\
\cdots & & \cdots
\end{array}.$$

Horizontal arrows correspond to isomorphisms of braid representations from Theorem 0.0.6, while vertical arrows correspond to $U_q\mathfrak{sl}(2)$ generators action E, F: the quantum ones on the left side and the homological ones (homological definitions inspired by [F-W] are given in this work) on the right side, that rules the weight structure on Verma modules. The direct sum of all spaces aligned vertically on the left gives the Verma module $V^{\otimes n}$, while the one of all spaces aligned on the right corresponds to the homological module $\mathcal{H}$. The homological interpretation of $U_q\mathfrak{sl}(2)$ generators follows, together with the ones of relations they satisfy and the R-matrix built using these generators.

This homological model (for quantum representations of braids) is then applied to knots seen as braids closures. Having an interpretation of the entire tensor product of Vermas allows to get a (homological) trace formula for colored Jones polynomials, that looks like a weighted sum of abelianized Lefschetz numbers, and this is the content of Theorem 7, Section 4.1.4.

Chapter 1

Background: Topology and Algebra

1.1 Topology: Mapping class groups and braid groups

1.1.1 Mapping class group

In this section we give the different definitions of the mapping class group and related tools and we define the braid groups. We follow [F-M].

Let S be an oriented surface, we denote by $Homeo^+(S, \partial S)$ the group of orientation-preserving homeomophisms that fix the boundary pointwise, endowed by compact-open topology, and let $Homeo_0(S, \partial S)$ be the connected component of the identity.

Definition 1.1.1 (Mapping Class Group). *The* Mapping Class Group *of S is the group of isotopy classes of orientation preserving homeomorphisms of S. Namely:*

$$\begin{aligned} Mod(S) &= \pi_0(Homeo^+(S, \partial S)) \\ &= Homeo^+(S, \partial S)/\, Homeo_0(S, \partial S) \end{aligned} .$$

In this definition it is equivalent to consider isotopy instead of homotopy or diffeomorphisms instead of homeomorphisms, it would result to the same group.

Notations.

- The only non-compact surfaces we will deal with will be punctured surfaces. Thus, the surface will always be topologically the connected sum of $g \geq 0$ tori with $b \geq 0$ boundary components (or b removed disks) and $n \geq 0$ points removed from the interior. We will denote such a surface $S^b_{g,n}$ (without b if its boundary is empty).
- When the surface has no boundary component we will use the notation $M(g, n)$ to designate its mapping class group, where g is its genus and n the number of punctures.

Remark 1.1.2.

- Isotopies are fixing the boundary at all time.
- Instead of considering punctures, we can consider homeomorphisms preserving the set of points, with isotopies fixing each points and homotopies sending unmarked points to unmarked points.

- Two differences between punctures and boundary components: the first one is that a homeomorphism can permute the punctures but has to fix the boundary components pointwise. The other one is that an isotopy must fix the boundary component, when it can rotate the neighborhood of a puncture. We will translate this relation in term of exact sequences later.

- Once the punctures are indexed, there is an natural homomorphism, named perm from now on, and defined as follows:

$$\text{perm} : Mod(S^b_{g,n}) \to \mathfrak{S}_n$$

  sending a mapping class to the induced permutation of the marked points (using the "marked points model"). We will often consider that the punctures are indexed, and will name them $p_1, p_2, \ldots, p_n$.

Now we give the first examples. Beginning with one boundary component.

Example 1.1.3 (Disk)**.** Two results concerning the two dimensional disk D^2:

- The *Alexander trick* states that any homeomorphism of the disk fixing the boundary is isotopic to the identity, so that $Mod(D^2)$ is trivial.

- Considering a homeomorphism fixing the center of the disk, the same proof shows that D^2 with one puncture also has a trivial mapping class group.

The first example with only punctures is the following.

Example 1.1.4 (2 and 3-punctured sphere)**.** The mapping class group of the sphere with 3 points removed, namely $M(0,3)$, is exactly $\mathfrak{S}_3$, the group of permutation of three elements. In the case of two punctures one has $M(0,2) = \mathbb{Z}/2\mathbb{Z}$.

The first non trivial example for closed surface is the torus, it is a main example as it inspires the general classification of mapping classes.

Example 1.1.5 (Torus)**.** Let T^2 be the torus (genus 1 closed surface). There is a homomorphism:

$$Mod(T^2) \to SL(2,\mathbb{Z})$$

given by the action of mapping classes on $H_1(T^2,\mathbb{Z})$. By theorem it is in fact an isomorphism, see [F-M, Theorem 2.5].

The above example gives another crucial way to study mapping class groups, using its action on homology groups. We call this point of view *homological representation of the mapping class group* and recovering these kind of representations of mapping class groups will be central in this work. In the case of braids it produces a whole family of famous representation (Burau, Gassner, Lawrence, Bigelow-Krammer...) that we will try to relate to quantum representations of mapping class groups.

The 4-punctured sphere has a proper and nice way to be studied that gives an easy way to visualize its mapping class group. This will be used later on to study the quantum representation of it, so that we give the example here.

Example 1.1.6 (Sphere with 4 punctures)**.** This example is treated in [F-M, Section 2.2.5], but we mainly follow [AMU] here which provides matrices for generators that will be useful to study the representations.

The idea is to study $M(0,4)$ from the torus. Think of the torus as the square with opposite faces identified, say that the left lower vertex is in $\begin{pmatrix}0\\0\end{pmatrix}$. Let ι be the π-rotation that fixes the square and that has four fixed points ($\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}1/2\\0\end{pmatrix}$, $\begin{pmatrix}0\\1/2\end{pmatrix}$, $\begin{pmatrix}1/2\\1/2\end{pmatrix}$), we call it the *hyperelliptic involution* . The quotient of the torus by the action of ι is an orbifold that is topogically a sphere with 4 ramified points. We will use this sphere with this four marked points in order to use the well known mapping class group of the torus.

Let $A \in SL(2,\mathbb{Z})$ be a matrix, $v \in \left(\frac{1}{2}\mathbb{Z}\right)^2$ a vector, and $\phi_{A,v}$ be the transformation of $\mathbb{R}^2$:

$$x \mapsto Ax + v.$$

This defines a diffeomorphism of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ that commutes with $\iota = -\mathrm{Id}$. We keep the notation $\phi_{A,v}$ to designate the diffeomorphism of the quotient $T^2/\{\pm\mathrm{Id}\}$, the 4-punctured sphere. By Theorem 3.1 of [AMU], this association is surjective. More precisely, all the braid generators of $M(0,4)$, from a standard presentation given below, are reached by diffeomorphisms of the form $\phi_{A,v}$. The study of the kernel is given by the exact sequence of [AMU, Corollary 3.3]:

$$1 \to N \to M(0,4) \to PSL(2,\mathbb{Z}) \to 1$$

where a mapping class coming from some $\phi_{A,v}$ is sent to the matrix $\pm A \in PSL(2,\mathbb{Z})$, and $N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Moreover, from [F-M, Proposition 2.7], the sequence splits so that $M(0,4)$ is the semi-direct product $PSL(2,\mathbb{Z}) \ltimes N$.

The latter is done in a more algebraic manner in [Bir] using explicitly the general presentation of the mapping class groups of punctured spheres. From Theorem 1.1.8 stated below, we get the following presentation for $M(0,4)$. Let $\sigma_1, \sigma_2, \sigma_3$ be the three generators together with the following relations:

$$\sigma_1\sigma_3 = \sigma_3\sigma_1 \tag{1.1}$$

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \tag{1.2}$$

$$\sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2 \tag{1.3}$$

$$(\sigma_1\sigma_2\sigma_3)^4 = 1 \tag{1.4}$$

$$\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 = 1 \tag{1.5}$$

Let G be the subgroup of $M(0,4)$ generated by σ_1 and σ_2, and let N be the subgroup generated by $a = \sigma_1\sigma_3^{-1}$ and $b = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$.

Lemma 1.1.7 ([Bir, Lemma 5.4.1])**.** *The group $M(0,4)$ is the semi direct product of the normal subgroup N and the subgroup G.*

Then one gets that G is isomorphic to $PSL(2,\mathbb{Z})$ under the following association:

$$\sigma_1 \leftrightarrow A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\ ,\ \sigma_2 \leftrightarrow B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It is easy to check that the elements a and b commutes and are of order 2 by simple applications of Relations 1.1. This gives that N is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

For the general case of the sphere with n punctures, there is the following theorem, giving a general presentation of the mapping class group.

Theorem 1.1.8. *If $n \geq 2$, then $M(0,n)$ admits a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ together with the following defining relations:*

$$\begin{array}{rcll} \sigma_i\sigma_j &=& \sigma_j\sigma_i, & |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i &=& \sigma_{i+1}\sigma_i\sigma_{i+1} & \\ (\sigma_1\sigma_2\cdots\sigma_{n-1})^n &=& 1 & \\ \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 &=& 1 & \end{array}$$

In the above theorem, σ_i corresponds to the half Dehn twist along some arc relating p_i and p_{i+1}, for $i = 1, \ldots, n-1$. We will sometimes refer to these generators as *braid generators*, as they satisfy the *braid relations* (defined in next section). We define the half Dehn twists involved in the above theorem.

Definition 1.1.9 (Half Dehn twist)**.** *Let α be an arc in a surface M having endpoints in a subset $Q \subset M$. By half Dehn twist along α we mean the homeomorphism:*

$$\tau_\alpha : (M,Q) \to (M,Q)$$

which is obtained as the result of the isotopy of the identity map $\mathrm{Id} : M \to M$ rotating α in M about its midpoint by the angle π in the direction provided by the orientation of M. The half-twist τ_α is the identity outside a small neighborhood of α in M.

1.1.2 Braid Group

In this section we review three equivalent definitions of the braid group. The first definition is the Artin definition, using a presentation with generators and relations, and it is the one commonly used to draw braid diagrams. The second one shows that the braid group is a fundamental group, and the last one that it is a mapping class group. Let $n \in \mathbb{N}$, we define in each context $\mathcal{B}_n$ the braid group of braids with n strands.

1.1.2.1 Artin braid group

We define the braid group using its Artin system of generators and relations.

Definition 1.1.10 (Braid Group.)**.** *The* braid group $\mathcal{B}_n$ *is the group generated by* $n-1$ *generators* $\sigma_1, \ldots, \sigma_{n-1}$ *satisfying the so called* "braid relations"*:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \textit{ if } |i-j| \leq 2 \textit{ and,}$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \textit{ for } i = 1, \ldots, n-2.$$

Remark 1.1.11. $\mathcal{B}_1$ is trivial and B_2 is the infnite cyclic group with one generator. By stating $x = \sigma_1\sigma_2\sigma_1$ and $y = \sigma_1\sigma_2$, we get the following modified presentation of $\mathcal{B}_3$:

$$\left\langle x, y \mid x^2 = y^3 \right\rangle$$

so that one recognizes the fundamental group of the trefoil knot.

Definition 1.1.12. *Let* $\mathfrak{S}_n$ *be the permutations group of* n *elements, the morphism* perm *is defined as follows:*

$$\text{perm} : \begin{cases} \mathcal{B}_n & \rightarrow & \mathfrak{S}_n \\ \sigma_i & \mapsto & s_i = (i, i+1) \end{cases}$$

where s_i*, for some* $i = 1, \ldots, n$ *refers to the transposition that permutes* i *and* $i+1$*.*

It is easy to see that the morphism perm is well defined and surjective as the s_i's generate $\mathfrak{S}_n$.

1.1.2.2 Braid diagrams

Definition 1.1.13 (Geometric braids.)**.** *A* geometric braid *on* n *strands is a set* $b \subset \mathbb{R}^2 \times I$ *formed by* n *disjoint topological intervals, called strands of* b*, such that the projection* $\mathbb{R}^2 \times I \rightarrow I$ *maps each string homeomorphically onto* I*. And so that:*

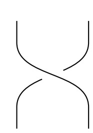
$$b \cap (\mathbb{R}^2 \times 0) = \{(1,0,0), (2,0,0), \ldots, (n,0,0)\} \textit{ and}$$
$$b \cap (\mathbb{R}^2 \times 1) = \{(1,0,1), (2,0,1), \ldots, (n,0,1)\}.$$

Two geometric braids b and b' are said *isotopic* if one can be deformed continuously into the other staying in the class of braids at each time. The isotopy classes are called *braids* on n strands.

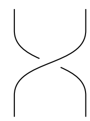
Let b_1 and b_2 be two braids. Their product $b_1 b_2$ is defined to be the geometric braid $(x, y, t) \in \mathbb{R}^2 \times I$ with $(x, y, 2t) \in b_1$ if $0 \leq t \leq \frac{1}{2}$ and $(x, y, 2t-1) \in b_2$ if $\frac{1}{2} \leq t \leq 1$. This product behaves well with isotopy of braids, so that it defines a product on the braids on n strands. This product has a neutral element, namely the *trivial braid* :

$$\{(1,0), \ldots, (n,0)\} \times I \in \mathbb{R}^2 \times I$$

For $1 \leq i \leq n-1$, we define $\bar{\sigma}_i$ to be the braid that have the following diagram:



This braid has $\bar{\sigma}_i^{-1}$ as an inverse, which correspond to the following diagram:



Then, the morphism:

$$\begin{cases} \mathcal{B}_n & \to & \{\text{braids on } n \text{ strands}\} \\ \sigma_i & \mapsto & \bar{\sigma}_i \end{cases}$$

is an isomorphism. From now on, we use the notation σ_i to designate both the braid and the Artin generator, and $\mathcal{B}_n$ to talk about the braid group in general.

Remark 1.1.14. The morphism perm : $\mathcal{B}_n \to \mathfrak{S}_n$ has the following meaning: a braid b is sent by perm to the permutation that sends each $i \in \{1, \dots, n\}$ to the only $j \in \{1, \dots, n\}$ such that the strand attached to $(i, 0, 0)$ at the bottom has $(j, 0, 1)$ as top endpoint.

1.1.2.3 Pure braid group.

The *Pure braid group on n strands* is made of the braids with all strands having the same endpoints ($(i, 0, 0)$ is attached to $(i, 0, 1)$ by a string, for all $i \in \{1, \dots, n\}$). It is a subgroup of $\mathcal{B}_n$ that we denote $\mathcal{PB}_n$.

Definition 1.1.15 (Pure braid group)**.** *Let $n \in \mathbb{N}^*$, the* pure braid group *is defined as follows:*

$$\mathcal{PB}_n = Ker(\text{perm} : \mathcal{B}_n \to \mathfrak{S}_n).$$

Proposition 1.1.16 ([Bir, (1.11)])**.** *$\mathcal{PB}_n$ is generated by elements $A_{i,j}$ for $1 \leq i < j \leq n$ expressed using standard generators of $\mathcal{B}_n$ as follows:*

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}.$$

Remark 1.1.17. One can notice that these pure generators are conjugated in $\mathcal{B}_n$ (but not in $\mathcal{PB}_n$!) using the elements:

$$\alpha_{i,j} = \sigma_{j-1}\sigma_{j-2} \cdots \sigma_i$$

so that we have for $1 \leq i < j < k \leq n$:

$$\alpha_{j,k}A_{i,j}\alpha_{j,k}^{-1} = A_{i,k} \ , \ \alpha_{i,k}A_{i,j}\alpha_{i,k}^{-1} = A_{j,k}$$

for which a drawing is convincing.

There are two important morphisms related with each other: first the injection ι of $\mathcal{B}_n$ into $\mathcal{B}_{n+1}$ obtained by adding one straight string having $(n+1, 0, 0)$ and $(n+1, 0, 1)$ as endpoints and not crossing any other strand. The second morphism concerns only the pure braid group, it is the forgetting morphism $f_n : \mathcal{PB}_n \to \mathcal{PB}_{n-1}$ sending a pure braid b on n strands to the braid $f_n(b)$ on $n-1$ strands obtained by removing the $(n+1)$'th strand of b. It is obvious that:

$$f_n \circ \iota = \text{Id}_{\mathcal{PB}_{n-1}}.$$

Let U_n be the kernel of f_n, as f_n has a section, we have that $\mathcal{PB}_n = \mathcal{PB}_{n-1} \ltimes U_n$.

Proposition 1.1.18. *The group U_n is free on the $n-1$ generators $\{A_{i,n}\}_{i=1,2,\ldots,n-1}$.*

This has the following consequence.

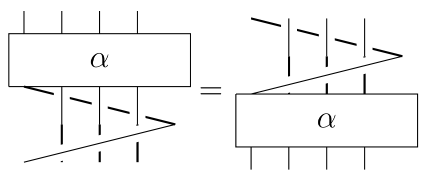
Proposition 1.1.19 ([Bir, (1.12)])**.** *The list of the following relations using the generators $A_{i,j}$ for $1 \leq i < j \leq n$ completes a presentation of $\mathcal{PB}_n$:*

$$A_{r,s}^{-1} A_{i,j} A_{r,s} = \begin{cases} A_{i,j} & \text{if} \quad s < i \text{ or } i < r < s < j \\ A_{r,j} A_{i,j} A_{r,j}^{-1} & \text{if} \quad s = i \\ A_{r,j} A_{s,j} A_{i,j} A_{s,j}^{-1} A_{r,j}^{-1} & \text{if} \quad i = r < s < j \\ A_{r,j} A_{s,j} A_{r,j}^{-1} A_{s,j}^{-1} A_{i,j} A_{s,j} A_{r,j} A_{s,j}^{-1} A_{r,j}^{-1} & \text{if} \quad s < i \text{ or } r < i < s < j \end{cases}$$

Proposition 1.1.20. *For $n \geq 3$, the center of the braid group $Z(\mathcal{B}_n) = Z(\mathcal{PB}_n)$ is the infinite cyclic group generated by the element $\theta_n = \Delta_n^2$ with:*

$$\Delta_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1$$

A nice way to verify that the element θ_n is central consists in the following picture.



1.1.2.4 Fundamental group of configuration spaces

First we recall the definition of configuration spaces. Let M be a topological space, we set $F_n(M)$ to be the *configuration space of ordered n-tuples of points in M.*

$$F_n(M) = \{(z_1, \ldots, z_n) \in M^n \text{ s.t. } z_i \neq z_j \text{ for } i \neq j\}$$

Its fundamental group is called the *pure braid group of M on n strands.*

Proposition 1.1.21. *For $M = \mathbb{R}^2$ we recover the pure braid group $\mathcal{PB}_n$ defined above. Namely:*

$$\pi_1(F_n(M)) \cong \mathcal{PB}_n.$$

To make the identification, we associate to a pure geometric braid $b \in \mathbb{R}^2 \times I$ the path: $I \to F_n(\mathbb{R}^2)$ defined by: $t \mapsto (u_1(t), \ldots, u_n(t))$ with the condition that the i'th string of the braid b meets $\mathbb{R}^2 \times \{t\}$ in $(u(t), t)$ for all $i = 1, \ldots, n$. This path begins and ends at the point:

$$q_n = ((1,0), \ldots, (n,0))$$

which we use as basepoint for $F_n(\mathbb{R}^2)$.

Conversely, if $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ is a path of $F_n(\mathbb{R}^2)$, we can define from it the pure braid:

$$\bigcup_{i=1}^{n} \bigcup_{t \in I} (\alpha_i(t), t).$$

These two constructions are inverse of each other so that $\mathcal{PB}_n = \pi_1(F_n(\mathbb{R}^2), q_n)$.

To recover the entire braid group $\mathcal{B}_n$ we need the action of $\mathfrak{S}_n$ over $F_n(M)$ by permutation of coordinates. From this action we define the quotient $C_n(M)$ to be the *configuration space of unordered n-tuples of points in M* :

$$C_n(M) = F_n(M)/\mathfrak{S}_n$$

Proposition 1.1.22. *For $M = \mathbb{R}^2$, $\mathcal{B}_n = \pi_1(C_n(\mathbb{R}^2), \tilde{q_n})$ where $\tilde{q_n}$ is the class of q_n under the $\mathfrak{S}_n$-action.*

1.1.2.5 Mapping class group of the punctured disk.

Let D_n be the closed disk with n marked points, called *"the punctured disk"* from now on.

Proposition 1.1.23. *Let $n \in \mathbb{N}^*$, one has:*

$$B_n \cong Mod(D_n) = M(0, n) = Mod(\Gamma_{0,n}),$$

using the mapping class group notations defined above.

Let $Q_n = \{(1, 0), \ldots, (n, 0)\} \subset \mathbb{R}^2$ the set of marked points, and D_n a topological disk containing Q_n and oriented counterclockwise. For $i = 1, \ldots, n-1$ we set the following spanning arcs:

$$\alpha_i = [i, i+1] \times \{0\} \in D_n.$$

The half Dehn twists $\tau_{\alpha_1}, \ldots \tau_{\alpha_{n-1}}$ (see Definition 1.1.9) are well known to satisfy the braid relations, and the following well defined morphism:

$$\eta : \begin{cases} \mathcal{B}_n & \to & Mod(D_n) \\ \sigma_i & \mapsto & \tau_{\alpha_i} \end{cases}$$

is an isomorphism.

1.1.3 Exact sequences among mapping class groups

1.1.3.1 Birman exact sequence

If S is a surface, let (S, x) be the surface S with x a marked point. There is a natural map:

$$Forget : Mod(S, x) \to Mod(S)$$

called the forgetful map.

Let α be a loop in S based in x. Let α be an isotopy of point, by Proposition 1.11 of [F-M] it is extendable to an isotopy of the whole surface, we can define ϕ_α to be the homeomorphism obtained at the end of the isotopy. This homeomorphism defines a class in $Mod(S,x)$ that we define to be $Push(\alpha)$. It can be shown to be well defined and to provide the morphism:

$$Push : \pi_1(S,x) \to Mod(S,x).$$

Proposition 1.1.24 ([F-M, Theorem 4.6])**.** *If S has negative Euler characteristic, then the* Birman exact sequence *holds:*

$$1 \to \pi_1(S,x) \xrightarrow{Push} Mod(S,x) \xrightarrow{Forget} Mod(S) \to 1.$$

If S has some marked points, the Birman exact sequence restrict to the pure mapping class group. If $S_{g,n}$ refers to the surface of genus g with n marked points, and if $PMod(S_{g,n})$ designates its pure mapping class group, the Birman exact sequence becomes:

$$1 \to \pi_1(S_{g,n}) \xrightarrow{Push} PMod(S_{g,n+1}) \xrightarrow{Forget} PMod(S_{g,n}) \to 1.$$

1.1.3.2 Capping the boundary

Let S be a surface with boundary, and S' be the surface obtained from S by closing one boundary component with a punctured disk. Set p_0 to be the puncture of the capping disk. Let β be the loop in S' corresponding to the boundary component of S. The group $Mod(S,\{p_1,\dots,p_k\})$ is the subgroup of $Mod(S)$ consisting of elements that fix the marked points $p_1,\dots,p_k$, where $k \geq 0$, while $Mod(S',\{p_0,p_1,\dots,p_k\})$ is the subgroup of $Mod(S')$ consisting of elements that fix the marked points $p_0,p_1,\dots,p_k$. Then let $Cap : Mod(S,\{p_1,\dots,p_k\}) \to Mod(S',\{p_0,p_1,\dots,p_k\})$ be the induced homomorphism defined as follows: let f be a homeomorphism of S fixing ∂S and representing a class of $Mod(S,\{p_1,\dots,p_k\})$, and $\hat{f}$ be the homeomorphism of S' which coincides with f in S and is the identity outside. Then Cap sends the class of f to the one of $\hat{f}$ which turns out to be in $Mod(S',\{p_0,p_1,\dots,p_k\})$.

Proposition 1.1.25 ([F-M, 4.2.5])**.** *The morphism Cap satisfies the following exact sequence:*

$$1 \to \langle \tau_\beta \rangle \to Mod(S,\{p_1,\dots,p_k\}) \xrightarrow{Cap} Mod(S',\{p_0,p_1,\dots,p_k\}) \to 1$$

where τ_β refers to the Dehn twist along β and the first injection is the inclusion.

1.2 Topological representations of braid groups

Two different kind of representations are mainly studied in this work, and we try to find relations between them. The first family, called *homological representations* are using the fact that the braid group is the mapping class group of the punctured disk and involves actions on homology groups of topological spaces built from it. The second family of *quantum* *representations* uses the generators and the braid relations, and are built from a category of modules over some quantum group providing an R-matrix that satisfies an equation corresponding to the braid relation, namely the *Yang-Baxter equation.*

Definition 1.2.1 (Representation of the braid group)**.** *Let $n \in \mathbb{N}^*$, and $\mathbb{C}[\mathcal{B}_n]$ be the group algebra of $\mathcal{B}_n$ with coefficient in $\mathbb{C}$. A* representation of $\mathcal{B}_n$ *is an algebra morphism:*

$$\mathbb{C}[\mathcal{B}_n] \to \mathrm{End}_{\mathbb{C}}(V)$$

where V is a complex vector space.

Definition 1.2.2 (Induced representation from the pure braid group)**.** *Let r be a representation of $\mathcal{PB}_n$:*

$$r : \mathbb{C}[\mathcal{PB}_n] \to \mathrm{End}_{\mathbb{C}}(V)$$

There exists a natural induced representation $Ind(r)$ of $\mathcal{B}_n$ over the space:

$$Ind(V) = \mathbb{C}[\mathcal{B}_n] \otimes_{\mathbb{C}[\mathcal{PB}_n]} V$$

where the action of $\mathcal{PB}_n$ is given by product on the left of the tensor product and by r on the right.

Example 1.2.3. The representation perm of $\mathcal{B}_n$ over $\mathbb{C}[\mathfrak{S}_n]$ is induced from the trivial representation of $\mathcal{PB}_n$.

1.2.1 Automorphism of the free groups

1.2.1.1 Braid group as a free automorphism sub-group

Let n be an integer, and $F_n = \langle x_1, \ldots, x_n \rangle$ be the free group on n generators. The braid group B_n acts on F_n by automorphisms. To see this, let's define $\tilde{\sigma}_i$ for $i = 1, \ldots, n-1$ as the following automorphism of F_n:

$$\tilde{\sigma}_i(x_k) = \begin{cases} x_{k+1} & \text{if } k = i \\ x_k x_{k-1} x_k^{-1} & \text{if } k = i+1 \\ x_k & \text{otherwise} \end{cases}.$$

These $\tilde{\sigma}_i$'s verify the braid relations so that one obtains the representation:

$$\begin{array}{ccc} \mathcal{B}_n & \to & Aut(F_n) \\ \sigma_i & \mapsto & \tilde{\sigma}_i \end{array}.$$

This action is faithful so that $\mathcal{B}_n$ is a subgroup of $Aut(F_n)$. The group F_n is identified with the fundamental group of the n-times punctured disk, with basepoint d taken in the boundary of the disk. The generator x_i of F_n, for $i = 1, \ldots, n$, is then identified with the loop that goes from d, passes once clockwise around the puncture p_i and going back to d, not encircling any other puncture. Let f be a self homeomorphism of D_n, as it fixes d, it yields an automorphism of $F_n = \pi_1(D_n, d)$ that only depends on the isotopy class of f, so that the action on F_n is well defined on $Mod(D_n)$. One verifies that this action is the one sending the half Dehn twist σ_i to the automorphism $\tilde{\sigma}_i$ for $i = 1, \ldots, n$.

1.2.1.2 Magnus representations.

Definition 1.2.4 (Fox free differential calculus)**.** *For each $j = 1, \ldots, n$ there is a mapping:*

$$\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \to \mathbb{Z}F_n$$

given by:

$$\frac{\partial}{\partial x_j}\left(x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_r}^{\epsilon_r}\right) = \sum_{i=1}^{r} \epsilon_i \delta_{\mu_i,j} x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_i}^{(\epsilon_i - 1)/2},$$

and

$$\frac{\partial}{\partial x_j}\left(\sum a_g g\right) = \sum a_g \frac{\partial}{\partial x_j}(g)\,, \;\; g \in F_n \;\; a_g \in \mathbb{Z},$$

where $\epsilon_i = \pm 1$, δ is the Kronecker symbol, and $\mathbb{Z}F_n$ is the the group ring of F_n.

Let Φ be a homomorphism acting on F_n and A_Φ be any group of automorphisms of F_n which satisfy:

$$\Phi(x) = \Phi(a(x))$$

for each $x \in F_n$ and $a \in A_\Phi$.

Definition 1.2.5 (Magnus representation, [Bir, Theorem 3.9])**.** *Let $a \in A_\Phi$ and $[a]^\Phi$ be the following $n \times n$ matrix:*

$$[a]^\Phi = \left[\Phi\left(\frac{\partial(a(x_i))}{\partial x_j}\right)\right]_{i,j}.$$

Then the morphism:

$$\begin{array}{rcl} A_\Phi & \to & \mathcal{M}\left(n, \mathbb{Z}F_n\right) \\ a & \mapsto & [a]^\Phi \end{array}$$

is a well defined group homomorphism, called a Magnus representation.

Let Z_n be the free abelian group of rank n with free basis $t_1, \ldots, t_n$ and $\mathfrak{a}$ be the following morphism:

$$\mathfrak{a} \begin{array}{rcl} F_n & \to & Z_n \\ x_i & \mapsto & t_i \end{array}.$$

Definition 1.2.6 (Gassner representation of the pure braid group)**.** *Let $1 \leq r < s \leq n$ and $A_{r,s} \in \mathcal{PB}_n$ the corresponding generator of the pure braid group on n strands. Let $[A_{r,s}]$ be the following matrix:*

$$[A_{r,s}]^{\mathfrak{a}} = \left[\mathfrak{a}\left(\frac{\partial(\widetilde{A_{r,s}}(x_i))}{\partial x_j}\right)\right]_{i,j}.$$

Then the morphism:

$$\begin{array}{rcl} \mathcal{PB}_n & \to & \mathcal{M}\left(n, \mathbb{Z}(Z_n)\right) \\ A_{r,s} & \mapsto & [A_{r,s}]^{\mathfrak{a}} \end{array}$$

is a Magnus representation, called the Gassner representation of the pure braid group.

We will present this representation in a more concrete way, giving explicit matrices, in next section.

Lemma 1.2.7 ([Bir, Lemma 3.11.1])**.** *The Gassner representation is reducible to an* $(n-1)\times(n-1)$ *representation.*

Sketch of proof. Let $g_i = x_1 \cdots x_i \in F_n$, this provides a change of generator basis for F_n. The matrices:

$$\left[\mathfrak{a}\left(\frac{\partial(\widetilde{A_{r,s}}(g_i))}{\partial g_j}\right)\right]_{i,j}$$

correspond to Gassner matrices given in another basis associated to the g_i's. After computation one remarks that the last rows and columns for all these matrices is $(0,\ldots,1)$ so that it can be deleted. □

Remark 1.2.8. Let $t = t_1 = \cdots = t_n$, then the Gassner representation becomes the Burau representation. See [Bir, Section 3.3], or details in next section.

Remark 1.2.9 (Enright representations)**.** One can define inductively higher Fox free derivatives, see [Bir, Equation 3.17]. From this notion, there exists higher representations of automorphims group of F_n, namely the Enright representations, see [Bir, Theorem 3.8]. This family is graded, the grading is the degree of Fox derivation.

1.2.2 The Gassner representation

In this section, we give a concrete definition of Gassner representation, first defined in Definition 1.2.6, following the work of [B-N]. In [B-N], the Gassner representation is built as a "multi-color" Burau one. Let t be a formal variable and $U_{n,i}(t)$ be the standard Burau matrix associated to σ_i, the i'th standard generator of $\mathcal{B}_n$. It consists of an $n\times n$ identity matrix where one replaces the 2×2 block obtained with the i'th and $i+1$'th rows and columns by the standard block:

$$\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}.$$

Definition 1.2.10 ([B-N])**.** *Let* $b = \prod_{\alpha=1}^{k} \sigma_{i_\alpha}^{s_\alpha}$ *be a braid written as a product of standard generators. Let* Γ *be the following product of matrices:*

$$\Gamma(b) = \prod_{\alpha=1}^{k} U_{n,i_\alpha}(t_{j_\alpha})^{s_\alpha}$$

where j_α *is the index of the "passing over" strand at the* $\#\alpha$ *crossing, and* $t_1,\ldots,t_n$ *are set to be formal variables.*

Proposition 1.2.11 ([B-N])**.** *The map:*

$$\Gamma : \begin{array}{ccc} \mathcal{B}_n & \to & \mathcal{M}_n(\mathbb{Z}[t^{\pm 1}]) \\ b & \mapsto & \Gamma(b) \end{array}$$

is well defined.

The map Γ is well defined but not multiplicative, i.e. not an algebra morphism. Namely $\Gamma(ab) \neq \Gamma(a)\Gamma(b)$ when a and b are braids in general.

Proposition 1.2.12. *The morphism Γ becomes multiplicative when restricted to the pure braids, so that it yields a representation of $\mathcal{PB}_n$.*

We build an induced representation of $\mathcal{B}_n$ over $\mathbb{C}[\mathfrak{S}_n] \otimes \mathbb{C}^n$. Let $(g_1, g_2, \ldots, g_n)$ be the canonical basis of the involved copy of $\mathbb{C}^n$, then we define the induced Gassner representation as follows: morphism

Definition 1.2.13 (Gassner representation of $\mathcal{B}_n$)**.** *The* induced Gassner *(see Definition 1.2.2) representation of B_n, denoted* $\mathrm{Gassner}_n$ *is defined using the following endomorphisms associated to standard generators and extended to all the braids multiplicatively.*

$$Gassner_n(\sigma_i) : \begin{cases} \mathbb{C}[\mathfrak{S}_n] \otimes \mathbb{C}^n & \rightarrow \quad \mathbb{C}[\mathfrak{S}_n] \otimes \mathbb{C}^n \\ \tau \otimes v & \mapsto \quad (i, i+1) \circ \tau \otimes U_{n,i}(t_{\tau^{-1}(i+1)})(v) \end{cases}$$

where σ_i is, again, the i^{th} standard generator of B_n. It's a representation over a space of dimension $n! \times n$.

This representation contains the Gassner representation of pure braids. It also contains the Burau representation as it was already the case for Γ, we state this in the following remark.

Remark 1.2.14.

- If a is a pure braid, $Gassner_n(a)$ is block diagonal and $\Gamma(a)$ is the first upper left block of $Gassner_n(a)$, corresponding to its restriction to $\mathbb{C}[()] \otimes \mathbb{C}^n$, $()$ stands for the identity permutation.
- If all the variables are set to be equal to one variable, namely $t_1 = \cdots = t_n = t$, then Γ is the Burau representation.

The Burau representation is known to be faithful for $n = 2, 3$, unfaithful for $n \geq 5$, and it remains an open question for $n = 4$. The natural question coming from the study of Burau is if the Gassner representation is faithful, as it is richer than Burau. It is in fact still an open question.

This question is entirely contained in the question whether Γ is a faithful representation of PB_n or not. The explication is the following remark:

Remark 1.2.15. The image of $\mathbb{C}[()] \otimes \mathbb{C}^n$ under the action of a braid a is contained in the space $\mathbb{C}[\mathrm{perm}(a)] \otimes \mathbb{C}^n$. This ensures that in order to get the identity matrix from $Gassner_n$, the braid a must be pure.

The latter is a direct consequence of Definition 1.2.2. The faithfulness of the Gassner representations is reduced to the following open question.

Open Question. *Is Γ faithful as a representation of PB_n?*

We end this presentation with a word about faithfulness of Gassner representations. We recall the Birman exact sequence 1.1.24 in the case of the punctured disk that involves the pure braid group $\mathcal{PB}_n$:

$$1 \to F_{n-1} \to \mathcal{PB}_n \to \mathcal{PB}_{n-1} \to 1,$$

and is called the *Fadell – Neuwirth* exact sequence. Indeed, let D_n be the disk with n punctures, this exact sequence is the Birman exact sequence after remarking that the pure braid group is the pure mapping class group of D_n, and that the π_1 of D_{n-1} is a free group in $n-1$ generators denoted F_{n-1}. Moreover this pure Birman exact sequence splits so that $\mathcal{PB}_n$ is the semi direct product of $\mathcal{PB}_{n-1}$ with F_{n-1}. Let Γ_n be the Gassner representation of the pure braid group $\mathcal{PB}_n$, then one can check that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{PB}_n & \xrightarrow{\hat{Forget}} & \mathcal{PB}_{n-1} \\ \downarrow{\Gamma_n} & & \downarrow{\Gamma_{n-1}} \\ \Gamma_n(\mathcal{PB}_n) & \longrightarrow & \Gamma_{n-1}(\mathcal{PB}_{n-1}) \end{array}$$

where the lower horizontal arrow consists in setting t_n to be 1 and deleting last row and column of the matrix. This old fact allows a treatment of the faithfulness question by recursion on n. In some sense the Gassner representation commutes with the $Forget$ map so that the recursion property is reduced to the faithfulness of the induced representation of Γ_n over F_{n-1} ([Knu, Section 2.2] for a presentation of these facts). It was used in a series of articles to refine the kernel of Gassner representations. The theorem giving the finest kernel the author knows is the following:

Theorem 1.2.16 ([Knu, Theorem 3.4])**.** *The kernel of the action of Γ_n over F_{n-1} lies in $[C^3F_{n-1}, C^2F_{n-1}]$ where $C^\bullet F_{n-1}$ stands for the terms of the lower central series of F_{n-1}.*

1.2.3 The Bigelow-Krammer-Lawrence representation

1.2.3.1 Construction

The general concept of Lawrence's representations, see [Ito1], is to make the braid group act on a homology group of a certain covering of the configuration space of several points.

Definition 1.2.17 (Configuration space of the punctured disk)**.** *Let n, m be integers. The configuration space $C_{n,m}$ of m unordered points in D_n is defined as follows:*

$$C_{n,m} = \{(z_1, \ldots, z_m) \in D_n^m \text{ s.t. } z_i \neq z_j \text{ for } i \neq j\}/\mathfrak{S}_m$$

where $\mathfrak{S}_m$ acts by permutation on the order of coordinates.

Next proposition can be found in [K2] (relation 2.1) for an algebraic description, or in Proposition 1.3 of [P-P] for a concrete computation using a cell complex.

Proposition 1.2.18 ([P-P, Proposition 1.3]). *The first homology group of this space, namely $H_1(C_{n,m}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^n \oplus \mathbb{Z}$ where the first n generators correspond to meridians of the hyperplans $\{z_1 = p_i\}$ while the last generator corresponds to a meridian of the discriminantal arrangment $\bigcup_{1 \le i < j \le n} \{z_i = z_j\}$.*

Definition 1.2.19 ($\mathbb{Z}^2$-cover). *Let α be the homomorphism:*

$$\alpha : \pi_1(C_{n,m}) \xrightarrow{Hurewicz} H_1(C_{n,m}) = \mathbb{Z}^n \oplus \mathbb{Z} \xrightarrow{C} \mathbb{Z} \oplus \mathbb{Z} = \langle q \rangle \oplus \langle t \rangle$$

where the second map is defined by $C(x_1, \ldots, x_n, d) = (x_1 + \ldots + x_n, d)$. Let $\pi : \widetilde{C_{n,m}} \to C_{n,m}$ be the covering space corresponding to the kernel of α. By identification of q and t as deck transformations, $H_m(\widetilde{Cn, m}, \mathbb{Z})$ is a free $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module.

Since $Ker\alpha$ is invariant under the B_n action, B_n acts on $H_m(\widetilde{Cn, m}, \mathbb{Z})$ as $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module automorphisms. This provides the representations we were looking for. We give a more concrete definition of this set up in what follows in the case $m = 2$. In Chapter 3 we will deal extensively with the case $m \in \mathbb{N}$, so that this first step helps becoming familiar with the framework.

Let $m = 2$. This is the case known as *Bigelow-Krammer-Lawrence* representation, and sometimes called *BKL representation* in what follows. It provides the first known faithful representation of the braid groups, the latter is proved in [Big0] and [Kra]. The details of the construction come from [K-T] and [Big0].

Let $C = C_{n,2}$ be the configuration space of unordered pairs of points, $d_1, d_2 \in \partial D_n$ and $c = \{d_1, d_2\}$ be the base point of C. In the sequel, an unordered pair of distinct points $x, y \in D_n$ is denoted $\{x, y\}$. A path $\xi : I \to C$ is a pair of paths $\xi = \{\xi_1, \xi_2\}$ where $\xi_1, \xi_2 : I \to C$. As we are looking to unordered pairs of points, there are two possibilities for a path ξ to be a loop:

$$\xi_1(0) = \xi_1(1) \text{ and } \xi_2(0) = \xi_2(1)$$

so that both the ξ_i's are loops, or:

$$\xi_1(0) = \xi_2(1) \text{ and } \xi_2(0) = \xi_1(1),$$

here ξ_1 and ξ_2 permutes their endpoints (so that they are not loops) but the product $\xi_1\xi_2$ is a loop. We define two numerical invariants of loops in C, namely w and u.

The first one, w, is defined for the two cases of a loop $\xi = \{\xi_1, \xi_2\}$ as follows :

- If ξ_1 and ξ_2 both are loops, then we define $w(\xi) = w(\xi_1) + w(\xi_2)$ where $w(\xi_i)$ is the total winding number (meaning around all the punctures) of ξ_i.
- For the other case we define $w(\xi) = w(\xi_1\xi_2)$ the total winding number of the loop $\xi_1\xi_2$.

To define the invariant u, we remark that the map:

$$\begin{cases} I & \to & S^1 \\ s & \mapsto & \frac{\xi_1(s) - \xi_2(s)}{|\xi_1(s) - \xi_2(s)|} \end{cases}$$

sends $s = 0, 1$ to the same points or to opposite ones. Hence, the square of this function provides a loop of S^1, $u(\xi)$ is the index of it. Note that $u(\xi)$ is even if the ξ_i's are loops, odd otherwise. These classic invariants are additive with respect to product of loops and preserved under homotopy.

Here is an analytic definition of those invariants:

$$w(\xi) = \frac{1}{2\pi i} \sum_{j=1}^{n} \left(\int_{\xi_1} \frac{dz}{z - p_j} + \int_{\xi_2} \frac{dz}{z - p_j} \right)$$

$$u(\xi) = \frac{1}{\pi i} \int_{\xi_2 - \xi_1} \frac{dz}{z}$$

The map:

$$\phi : \xi \to q^{w(\xi)} t^{u(\xi)}$$

is a surjective group homomorphism from $\pi_1(C)$ to $\mathbb{Z}^2 = \mathbb{Z}\langle q, t \rangle$.

Let $\widetilde{C} \to C$ be the covering corresponding to the kernel of ϕ, with q and t acting as commuting deck transformations on $\widetilde{C}$, and we choose once a lift $\tilde{c}$ of the base point c. This turns $\mathcal{H} = H_2(\widetilde{C}, \mathbb{Z})$ into a module over $R = \mathbb{Z}\left[q^{\pm 1}, t^{\pm 1}\right]$. In fact the module $\mathcal{H}$ is the locally finite homology or Borel-Moore homology, see definitions in Section 5.1.

Let f be a self-homeomorphism of D_n, it induces a homeomorphism $\hat{f} : C \to C$ by:

$$\hat{f}(\{x, y\}) = \{f(x), f(y)\}$$

Note that $\hat{f}(c) = c$ as d_1 and d_2 are picked in the boundary of D_n, and we can define the induced automorphism $f_\#$ of $\pi_1(C, c)$.

Lemma 1.2.20. $\phi \circ f_\# = \phi$

Proof. One need to verify that both invariants w and u are preserved by $f_\#$.

For w, it comes from the fact that the equality $w \circ f_\# = w$ holds for small loops encircling the punctures, and then for arbitrary loops since it only depends on the homology class in the first homology group of D_n which is generated by these small loops.

For the second one u, $u \circ f_\# = u$ holds because this invariant does not "see" the punctures, i.e. u factors through the embedding $D_n \to D^2$. Forgetting the punctures, all homeomorphisms are isotopic to the identity, so that $u \circ f_\# = u$. □

This lemma implies that $\hat{f}$ lifts uniquely to a map $\tilde{f} : \widetilde{C} \to \widetilde{C}$ fixing any lift of c, and that $\tilde{f}$ commutes with covering transformations. Therefore it induces an R-linear automorphism f_* of $\mathcal{H}$, the Bigelow-Krammer-Lawrence representation follows:

$$BKL : \begin{array}{ccc} B_n = Mod(D_n) & \to & Aut_R(\mathcal{H}) \\ f & \mapsto & f_* \end{array}$$

Theorem 1.2.21 ([Big0] [Kra])**.** *The Bigelow-Krammer-Lawrence representation is faithful for all $n > 1$.*

It's the first known example of a faithful linear representation of the braid groups, while the Lawrence's representations are anounced to be faithful in [Z2] for greater values of m. The case $m = 1$ is the Burau representation known not to be faithful for $n \geq 5$.

To prove the faithfulness, Bigelow used tools named *forks* and *noodles* introduced by Krammer, and a pairing between them. This pairing is also useful to compute matrices of the representations, so that we introduce this framework here.

1.2.3.2 Forks and Noodles

The following definition will be generalized in Chapter 3, Section 3.2.2.

Definition 1.2.22 (Fork, $m = 2$). *A* fork *is an embedded tree $F \in D_n$ with four vertices d_1, p_i, p_j, and z such that $F \cap \partial D_n = \{d_1\}$, F intersects the punctures only in p_i, p_j, and all three edges have z as a vertex.*

- *The edge containing d_1 is called the handle of F and denoted $H(F)$.*
- *The union of other two edges is called the* tine *of F and denoted $T(F)$.*
- *The tine is oriented in such a way that it has the handle lying on its right.*

For any fork F we construct an associated surface $\widetilde{\Sigma}$ in $\widetilde{C}$ as follows. First let F' be the parallel fork of F with a parallel tine with same endpoints and parallel handle based on d_2. We define the following surface of C:

$$\Sigma(F) = \{\{x, y\} \text{ s.t. } x \in T(F) \backslash \{p_1, \ldots, p_n\}\ , y \in T(F') \backslash \{p_1, \ldots, p_n\}\}$$

In order to get a surface of $\widetilde{C}$ we need to chose a lift of $\Sigma(F)$. We use the handle to do so. Let $\tilde{\beta}$ be the lift beginning at $\tilde{c}$ of $\{\beta_1, \beta_2\}$ where β_1, β_2 are respectively the handle of F and F' starting on d_1 and d_2. Let $\widetilde{\Sigma}(F)$ be the lift of $\Sigma(F)$ which contains $\tilde{\beta}(1)$. This will be call the *handle process* in the general set-up of Section 3.2.2.2.

Definition 1.2.23 (Noodle). *A* Noodle *is an arc embedded in D_n going from d_1 to d_2.*

We construct a surface associated to N as follows:

$$\Sigma(N) = \{\{x, y\} \in C \text{ s.t. } x, y \in N\},$$

and then we choose $\widetilde{\Sigma}(N)$ to be the lift of $\Sigma(N)$ which contains $\tilde{c}$.

1.2.3.3 Pairing between forks and noodles.

Let F be a fork, N be a noodle, $\widetilde{\Sigma}(F)$ and $\widetilde{\Sigma}(N)$ be the surfaces built from them respectively.

First let's suppose (w.l.o.g.) that F and N intersect transversely in some points $z_1, \ldots, z_l$, and F' and N intersect transversely in $z'_1, \ldots, z'_l$ such that z_i and z'_i are joint by a short arc of N not containing any other intersection point. Surfaces $\widetilde{\Sigma}(F)$ and $\widetilde{\Sigma}(N)$ do not intersect necessarily because of the choice of the lift. But there exists a unique monomial $m_{i,j} = q^{a_{i,j}} t^{b_{i,j}}$ such that $m_{i,j}\widetilde{\Sigma}(N)$ intersects $\widetilde{\Sigma}(F)$ at a point lying over $\{z_i, z'_j\}$. Let $\epsilon_{i,j}$ be the sign of the intersection.

Definition 1.2.24. *We define the pairing as follows:*

$$\langle N, F\rangle = \sum_{i=1}^{l}\sum_{j=1}^{l}\epsilon_{i,j}m_{i,j}.$$

To compute explicitly $m_{i,j}$ we define a path of $\tilde{C}$ using composition of the following arcs:

- α_1 from d_1 to the handle of F, α_2 from d_2 to the handle of F',
- β_1 from z to z_i along $T(F)$, β_2 from z' to z'_j along $T(F')$,
- γ_1 from z_i to one of the d_i's in such a way that it doesn't cross z'_j,
- γ_2 from z'_j to one of the d_i's in such a way that it doesn't cross z_i.

Then we define the loop $\delta_{i,j}$:

$$\delta_{i,j} = \{\alpha_1, \alpha_2\}\{\beta_1, \beta_2\}\{\gamma_1, \gamma_2\}$$

Let $\widetilde{\delta}_{i,j}$ be the lift of $\delta_{i,j}$ beginning at $\tilde{c}$. Then we have (see [Big0]):

$$m_{i,j} = \phi(\delta_{i,j}),$$

and [Big0, Claim 3.3]:

$$\epsilon_{i,j} = -m_{i,i}m_{j,j}m_{i,j}(q = 1, t = 1).$$

Bigelow's proof of the faithfulness involves the following two lemmas.

Lemma 1.2.25 (Basic Lemma)**.** *If $[\sigma]$ lies in the kernel of the Bigelow Krammer-Lawrence representation, then*

$$\langle N, F\rangle = \langle N, \sigma(F)\rangle$$

for any fork F and noodle N.

Lemma 1.2.26 (Key Lemma)**.** *Let N be a noodle and let F be a fork. Then $\langle N, F\rangle = 0$ if and only if N and $T(F)$ do not intersect (up to isotopy).*

1.2.3.4 Matrices for the BKL-representation

In order to compute the homology of $\widetilde{C}$, and as we will need to compute it for another covering of C, we need to deal with $\pi_1(C)$. We give a presentation of it, according to [Big0].

For $j = 1, \ldots, n$, let ζ_j be the loop based at d_1 and passing once counterclockwise around p_j. Let x_j be the loop $\{\zeta_j, d_2\}$ of C. Let τ_1 be an arc from d_1 to d_2 and τ_2 from d_2 to d_1 such that $\tau_1\tau_2$ is a simple closed curve oriented counterclockwise and enclosing no puncture points, and let y be the loop $\{\tau_1, \tau_2\}$ of C. We define the set $\mathcal{G}$:

$$\mathcal{G} = \{x_1, \ldots, x_n, y\}.$$

Now we define some relations, for $j \in \{1, \ldots, n\}$:

$$r_{j,j} = [x_j, yx_jy],$$

and for $1 \leq j < k \leq n$:

$$r_{j,k} = \left[x_j, yx_ky^{-1}\right]$$

where the bracket refers to the commutator, and we define the set $\mathcal{R} = \{r_{j,k} \text{ for } 1 \leq j \leq k \leq n\}$.

Proposition 1.2.27 ([Big0]). *Let K be the Cayley Complex of $\langle \mathcal{G} | \mathcal{R} \rangle$. Then C is homotopically equivalent to K. It follows that a presentation of $\pi_1(C)$ is given by: $\langle \mathcal{G} | \mathcal{R} \rangle$.*

From this presentation, Bigelow has computed $H_2(\widetilde{C})$. It leads to matrices of the Bigelow-Krammer-Lawrence representation.

Theorem 1.2.28 ([Big0, Theorem 4.1]). *$H_2(\widetilde{C})$ is a free $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module of dimension $\binom{n}{2}$. There is a basis:*

$$\{v_{j,k} : 1 \leq j \leq k \leq n\}$$

on which the standard generator σ_i of B_n acts as follows:

$$\sigma_i(v_{j,k}) = \begin{cases} v_{j,k}, & i \notin \{j-1, j, k, k-1, k\} \\ qv_{i,k} + (q^2 - q)v_{i,j} + (1-q)v_{j,k}, & i = j-1 \\ v_{j+1,k}, & i = j \neq k-1 \\ qv_{j,i} + (1-q)v_{j,k} + (q^2-q)tv_{i,k}, & i = k-1 \neq j \\ v_{j,k+1}, & i = k \\ -tq^2 v_{j,k}, & i = j = k-1 \end{cases}.$$

We will follow this procedure to build colored BKL-representation in Section 2.2, and to obtain the matrices as well. We will construct a generalization of these representations in Chapter 3, for which we will extend largely these objects to bigger $m \in \mathbb{N}^*$, embedding them in bigger modules.

1.3 Quantum Algebra and Braid representations

1.3.1 Braid group representation from quasi-triangular Hopf algebras

In this section, we give an idea of how to get braid group representations from a special algebraic structure, namely the *quasi-triangular* bialgebras. First we recall notations and properties needed to define a bialgebra. This section is based on the book [Kas].

1.3.1.1 Quasi triangular Hopf algebras

Definition 1.3.1 (Bialgebra)**.** *A* bialgebra *is a quintuple* $(H,\mu,\eta,\Delta,\epsilon)$ *where* (H,μ,η) *is an algebra,* (H,Δ,ϵ) *is a coalgebra verifying the equivalent conditions:*

- μ *and* η *are coalgebra-morphisms.*
- Δ *and* ϵ *are algebra-morphisms.*

Definition 1.3.2 (Hopf algebra)**.** *An endomorphism* S *of* H *is called an* antipode *for* H *if:*

$$S\star \mathrm{Id}_H = \mathrm{Id}_H \star S = \eta\circ\epsilon.$$

Where $\star$ *designates the convolution product, namely the following composition of morphisms:*

$$f\star g : H \xrightarrow{\Delta} H\otimes H \xrightarrow{f\otimes g} H\otimes H \xrightarrow{\mu} H$$

where f *and* g *are endomorphisms of* H*.*

A Hopf algebra *is a bialgebra together with an antipode. A morphism of Hopf algebra is a morphism between the underlying bialgebras commuting with the antipode.*

Main examples of Hopf algebras consist in *quantum groups*. They come from a quantization of the product of enveloping algebras of Lie algebras. The most famous is the quantization of the envelloping algebra $U(\mathfrak{sl}(2))$. We give a first definition of it.

Definition 1.3.3 ($U_q\mathfrak{sl}(2)$)**.** *Let* q *be a complex parameter. We define* $U_q = U_q(\mathfrak{sl}(2))$ *as the* $\mathbb{C}$*-algebra generated by the four generators* E, F, K, K^{-1} *together with relations:*

$$KK^{-1} = K^{-1}K = 1$$
$$KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F$$
$$[E,F] = \frac{K-K^{-1}}{q-q^{-1}}.$$

This algebra is Noetherian with no zero divisor. The set $\{E^iF^jK^l\}_{i,j\in\mathbb{N};l\in\mathbb{Z}}$ is a basis of U_q. We endow U_q with a coalgebra structure defining Δ and ϵ as follows:

$$\begin{aligned}\Delta(E) = 1\otimes E + E\otimes K,&\quad \Delta(F) = K^{-1}\otimes F + F\otimes 1\\ \Delta(K) = K\otimes K,&\quad \Delta(K^{-1}) = K^{-1}\otimes K^{-1}\\ \epsilon(E)=\epsilon(F)=0,&\quad \epsilon(K)=\epsilon(K^{-1})=1\end{aligned}$$

and we define an antipode as follows:

$$S(E) = EK^{-1}, S(F) = -KF, S(K) = K^{-1}, S(K^{-1}) = K.$$

This Hopf algebra structure is neither commutative nor cocommutative, and quantum groups constitute a family of nice examples of this kind.

We now state the fundamental result about the category of algebra representation of U_q, namely the category of U_q-module, in the generic case.

Theorem 1.3.4 ([Kas, Theorem VII.2.2])**.** *If q is not a root of unity, any finite-dimensional U_q-module is semisimple.*

In this section we define $U_q\mathfrak{sl}(2)$ as a $\mathbb{C}$-algebra while in next Section 1.4 we will take care of the ground ring to refine the algebra properties.

Quantum representations of braid groups are representations coming from inherent objects of the category of modules over a quantum group. We now describe these objects, starting with the notion of R-matrix in a general sense.

Definition 1.3.5. *Let V be a vector space. A linear automorphism c of $V \otimes V$ is said to be an R-matrix if it is a solution of the* Yang Baxter *equation*

$$(c \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes c)(c \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes c)(c \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes c)$$

that holds in the automorphism group of $V \otimes V \otimes V$.

For any vector space V, the flip $\tau_{V,V} \in Aut(V \otimes V)$ defined by $\tau(v_1 \otimes v_2) = v_2 \otimes v_1$ yields the most trivial R-matrix.

Finaly we reach the notion of *quasi-triangular bialgebras* that refers to bialgebras providing R-matrices. The definition follows.

Definition 1.3.6. *A bialgebra H is* quasi triangular *if there exists an invertible element R of the algebra $H \otimes H$, called a* universal R-matrix*, such that for all $x \in H$ we have:*

$$\Delta^{op}(x) = R\Delta(x)R^{-1}$$

where $\Delta^{op} = \tau_{H,H} \circ \Delta$, and such that the two relations below hold:

$$(\Delta \otimes \mathrm{Id}_H)(R) = R_{13}R_{23}$$

$$(\mathrm{Id}_H \otimes \Delta)(R) = R_{13}R_{12}$$

where we used the following notations, if $R = \sum_i s_i \otimes t_i$:

$$R_{23} = \sum_i 1 \otimes s_i \otimes t_i, R_{13} = \sum_i s_i \otimes 1 \otimes t_i, R_{12} = \sum_i s_i \otimes t_i \otimes 1.$$

Such bialgebras produce solutions of the Yang-Baxter equation, using their modules as follows.

Let V and W be two H-modules. We define what we call a *braiding* from the element R, to be the following H-module isomorphism $c^R_{V,W}$ between $V \otimes W$ and $W \otimes V$:

$$c^R_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w))$$

where $v \in V$ and $w \in W$. The properties satisfied by R in $H \otimes H$ imply several properties of the braiding, one of them being what follows for U, V, W three H-modules:

$$(c^R_{V,W} \otimes \mathrm{Id}_U)(\mathrm{Id}_V \otimes c^R_{U,W})(c^R_{U,V} \otimes \mathrm{Id}_W) = (\mathrm{Id}_W \otimes c^R_{U,V})(c^R_{U,W} \otimes \mathrm{Id}_V)(\mathrm{Id}_U \otimes c^R_{V,W}).$$

So that for $U = V = W$ we have a solution of the Yang-Baxter equation.

The *Drinfeld - Jimbo* construction produces a quantum enveloping algebra which is braided from any semi-simple Lie algebra. We just give an example coming from $U_q(\mathfrak{sl}(2))$ in the case if q is a root of unity of order d odd. More precisely we define $\bar{U}_q$ to be the quotient of U_q by the ideal generated by the central elements: $E^d, F^d, K^d - 1$. We get from Drinfled-Jimbo construction that U_q is a quasi-triangular Hopf algebra, with a universal R-matrix having the following expression:

$$\bar{R} = \frac{1}{d} \sum_{0 \leq i,j,k \leq d-1} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E^k K^i \otimes F^k K^j$$

where we used the quantum factorial defined by $[k]! = [k]\,[k-1]\cdots[1]$ and $[k] = \frac{q^k - q^{-k}}{q-q^{-1}}$.

1.3.1.2 Braid group and tangles representations

From quasi-triangular Hopf algebras, there is a natural construction of braid group representations.

Let V be a vector space, c a linear automorphism of $V \otimes V$, with $n > 1$ an integer. Then for $1 \leq i \leq n-1$ we define a linear automorphism c_i of $V^{\otimes n}$ by:

$$c_i = \begin{cases} c \otimes \mathrm{Id}_{V^{\otimes (n-2)}} & if \quad i = 1 \\ \mathrm{Id}_{V^{\otimes (i-1)}} \otimes c \otimes \mathrm{Id}_{V^{\otimes (n-i-1)}} & if \quad 1 < i < n-1 \\ \mathrm{Id}_{V^{\otimes (n-2)}} \otimes c & if \quad i = n \end{cases}$$

Proposition 1.3.7. *Let $c \in Aut(V \otimes V)$ be a solution of the Yang-Baxter equation. Then, for any $n > 0$ there exists a unique group morphism $\rho_n^c : \mathcal{B}_n \to Aut(V^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for $i = 1, \dots, n-1$.*

Definition 1.3.8 (Tangles)**.** *A* tangle *with k inputs and l outputs is a finite system of disjoint smoothly embedded oriented arcs and circles in $\mathbb{R}^2 \times [0,1]$ such that the endpoints of the arcs are the points $(1,0,0), \dots, (k,0,0)$ and $(1,0,1), \dots, (l,0,1)$. The circles lie in $\mathbb{R}^2 \times (0,1)$.*

A tangle is framed *if it is equipped with a non-singular normal vector field equal in the endpoints of the arcs to the vector $(0,-1,0)$.*

Two (framed) (k,l)-tangles L_1 and L_2 are said to be isotopic *if L_1 may be smoothly deformed into L_2 staying in the class of (framed) (k,l)-tangles during the deformation.*

Remark 1.3.9. Braids are special types of tangles. Namely: a braid of $\mathcal{B}_n$ is a (n,n)-tangle with no circle.

Definition 1.3.10 (Category of Tangles)**.** *The category of tangles $\mathcal{T}$ is the category whose objects are non negative integers, and a morphism from k to l is a (k,l)-tangle. Let $f : k \to l$ and $g : l \to m$, the morphism fg is represented by the tangle obtained by attaching f on the top of g.*

Definition 1.3.11 (Category of colored tangles)**.** *Let C be a category. A C-colored tangle is a tangle where every component is equipped with an object of C. More precisely, the category of C-colored tangle $\mathcal{T}_C$ is the category whose objects are finite sequences $((V_1, \epsilon_1), \ldots, (V_m, \epsilon_m))$, where $V_1, \ldots, V_m$ are objects of C and $\epsilon_1, \ldots, \epsilon_m \in \{+, -\}$. A morphism $\eta \to \eta'$ is an isotopy type of a C-colored framed tangle such that η (resp. η') is the sequence of colors and directions of those tangles which hit the bottom (resp. top) boundary endpoints. (The sign $+$ stands for the downward direction, while the $-$ the upward.)*

This category is monoidal. The tensor product of two sequences η and η' is given by the concatenation of sequences. The tensor product of two morphisms f and g is obtained by placing the colored framed tangle representing f to the left of the one representing g.

Theorem 1.3.12 (Reshetikhin-Turaev functor, [R-T])**.** *Let $\mathcal{C}$ be the category of $U_q\mathfrak{sl}(2)$-modules. There exists a monoidal functor $\mathcal{RT}$ between $\mathcal{T}_{\mathcal{C}}$ and $\mathcal{C}$.*

The above theorem is loosely stated as one must refine the category of $U_q\mathfrak{sl}(2)$-modules before applying the construction for it to work. We will see cases of $U_q\mathfrak{sl}(2)$-modules categories for which the theorem holds. Historically there are two $U_q\mathfrak{sl}(2)$-modules category (for infinite versions of $U_q\mathfrak{sl}(2)$) for which this theorem holds: the semi-simple theory first introduced in [R-T], and the non semi-simple one introduced for instance in [CGP2]. For a categorical approach to the non semi-simple construction of a Reshetikhin-Turaev functor, see [DR], while in this work we are interested in the concrete $U_q\mathfrak{sl}(2)$ case. This functor then also provides braid representations as braids are a sub-category of tangles.

Proposition 1.3.13. *The braid representations coming from the $U_q\mathfrak{sl}(2)$ R-matrix (Proposition 1.3.7) are restrictions of the functor $\mathcal{RT}$ to braids.*

We will use the general term of *quantum representations* or $U_q\mathfrak{sl}(2)$*-representations* of the braid group in what follows to designate the representations built from the $\mathcal{RT}$-functor, or equivalently by use of the R-matrix.

Remark 1.3.14 (Restriction to $Ker\, E$ and to weight spaces.)**.** The above proposition states that the braids act over tensor products of $U_q\mathfrak{sl}(2)$-modules as $U_q\mathfrak{sl}(2)$-module morphisms by use of the braiding. This has the two following consequences that we will use extensively in what follows:

(Weight spaces) Let $\lambda \in \mathbb{C}$. The restriction of a quantum representation to the $U_q\mathfrak{sl}(2)$-submodule defined by $Ker(K - \lambda \mathrm{Id})$ (corresponding to eigenvectors for the action of K and the eigenvalue λ) is a representation of the braid group. The elements of this submodule will be designated by *weight vectors of weight* λ.

(Highest weights) The restriction of a quantum representation to the $U_q\mathfrak{sl}(2)$-submodule defined by $Ker\, E$ is a representation of the braid group. The elements of this submodule will be designated by *highest weight vectors*.

1.3.2 Category of $\overline{U}_q^H\mathfrak{sl}(2)$-modules: the ADO polynomial set-up

We present now a slightly modified version of the quantum enveloping algebra of $\mathfrak{sl}(2)$, that is presented in large details in [CGP1] for instance. From now on, let q be a root of unity of pair degree: i.e. such that $q^{2r}=1$ for some integer $r \geq 2$.

1.3.2.1 The algebra $U_q^H\mathfrak{sl}(2)$

Let $U_q^H\mathfrak{sl}(2)$ be the $\mathbb{C}$-algebra U_q of Definition 1.3.3 improved with one more generator H, so given by generators E, F, K, K^{-1}, H and the relations from U_q together with relations:

$$HK = KH, \qquad [H,E] = 2E, \qquad [H,F] = -2F.$$

The Hopf algebra structure of $\overline{U}_q^H\mathfrak{sl}(2)$ comes from the one of U_q extended by:

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \epsilon(H) = 0, \qquad S(H) = -H.$$

Definition 1.3.15 ($\overline{U}_q^H\mathfrak{sl}(2)$)**.** *$\overline{U}_q^H\mathfrak{sl}(2)$ is the Hopf algebra $U_q^H\mathfrak{sl}(2)$ modulo the relations $E^r = F^r = 0$.*

Let V be a finite dimensional $\overline{U}_q^H\mathfrak{sl}(2)$-module. An eigenvalue $\lambda \in \mathbb{C}$ of the action $H : V \to V$ is called a *weight* of V and the associated eigenspace is called a *weight space*. We call V a *weight module* if V splits as a direct sum of weight spaces and if K acts as the exponential of H on V, namely $Kv = q^{\lambda}v$ if v is a vector of weight λ.

Definition 1.3.16 ($\overline{U}_q^H\mathfrak{sl}(2)$ braiding, [CGP1, Subsection 2.2])**.** *Let $\mathscr{C}$ be the category of finite dimensional weight $\overline{U}_q^H\mathfrak{sl}(2)$-modules, and let V and W be two elements of this category. Let R be the R-matrix defined by Ohtsuki in [Oht] with the expression that can be found in [CGP1, Equation (5)]. It is not an element of $\overline{U}_q^H\mathfrak{sl}(2) \otimes \overline{U}_q^H\mathfrak{sl}(2)$ so it is not a universal R-matrix, but it yields an operator on $V \otimes W$ as follows:*

$$\mathrm{R} = q^{H\otimes H} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n,$$

where the action $q^{H\otimes H}$ is described for v and v' two weight vectors of weights λ and λ' respectively as follows:

$$q^{H\otimes H}(v \otimes v') = q^{\lambda\lambda'} v \otimes v'.$$

This way, R is a well defined linear map, and gives rise to a braiding:

$$c_{V,W} : \begin{array}{ccc} V \otimes W & \to & W \otimes V \\ v \otimes w & \mapsto & \tau(\mathrm{R}(v \otimes x)) \end{array}$$

where τ is defined by $\tau(v \otimes w) = w \otimes v$.

The above definition uses the following quantum numbers.

Definition 1.3.17. *For $n \in \mathbb{N}$:*

$$\{x\} = q^x - q^{-x} \text{ and } \{n\}! = \{n\}\{n-1\}\cdots\{1\}$$

1.3.2.2 Simple $\overline{U}_q^H\mathfrak{sl}(2)$-modules

We focus on a special class of finite dimensional weight modules, the one we will use for the quantum representations construction below. For each $\lambda \in \mathbb{C}$ there exists a unique $\overline{U}_q^H\mathfrak{sl}(2)$-module V_λ which is r-dimensional and of highest weight $\lambda + r - 1$. The module V_λ has a basis $\{e_0^\lambda, \dots, e_{r-1}^\lambda\}$ whose action is given by

$$H.e_i^\lambda = (\lambda + r - 1 - 2i)e_i^\lambda, \quad E.e_i^\lambda = \frac{\{i\}\{i-\lambda\}}{\{1\}^2} e_{i-1}^\lambda, \quad F.e_i^\lambda = e_{i+1}^\lambda.$$

The module V_λ is called *typical* if $\lambda \in (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$, *atypical* otherwise. If V_λ is typical then it is simple, and is generated (as a module) by any of the basis vectors e_i^λ. For an eigenvector for the action of H (the e_i^λ's for instance) we call *weight* its eigenvalue. One remarks from the expression of the action that the weights decrease 2 by 2 from e_0^λ of weight $\lambda + r - 1$, to e_{r-1}^λ of weight $\lambda - r + 1$ and so on, so that λ is the *"middle weight"* of V_λ.

Let V_λ be the module of "middle" weight λ and of dimension r, for a $\lambda \in \mathbb{C}$ and $\{e_0^\lambda, e_1^\lambda, \dots, e_{r-1}^\lambda\}$ its standard basis.

Definition 1.3.18. *Let λ and μ be elements of $\mathbb{C}$. We define the morphism $\mathcal{R}$ from $V_\lambda \otimes V_\mu$ to $V_\mu \otimes V_\lambda$ as follows:*

$$\mathcal{R}(\lambda, \mu) = c_{V_\lambda, V_\mu}$$

.

This operator $\mathcal{R}$ used as an R-matrix, provides braid representations. In the following sections, we focus on some special representations built from it.

1.3.2.3 Sub-space of sub-maximal weights

Let's fix n to be an integer which will be in the following the number of strands of the braids. Let $\lambda_1, \dots, \lambda_n$ be elements of $\mathbb{C}$ and let $W_1^{\lambda_1,\dots,\lambda_n} = \mathrm{Span}\,(f_1, f_2, \dots, f_n)$ be the subspace of $V^{\lambda_1,\dots,\lambda_n} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$ spanned by $\{f_1, f_2, \cdots, f_n\}$, where the f_i's are defined as follow:

$$f_1 = e_1^{\lambda_1} \otimes e_0^{\lambda_2} \otimes e_0^{\lambda_3} \otimes \dots \otimes e_0^{\lambda_r}$$

$$f_2 = e_0^{\lambda_1} \otimes e_1^{\lambda_2} \otimes \dots \otimes e_0^{\lambda_r}$$

and so on, with:

$$f_i = e_0^{\lambda_1} \otimes e_0^{\lambda_2} \otimes \dots \otimes e_1^{\lambda_i} \dots \otimes e_0^{\lambda_r}.$$

These vectors are built as the tensor products of $n-1$ maximal weight vectors plus one of weight ("sub-maximal") $\lambda + r - 3$, namely $e_1^{\lambda_i}$, inserted on the i-th position of the tensor product.

The vectors f_i's all have the same weight (eigenvalue regarding the action of H): $\sum_{i=1}^n (\lambda_i + r - 1) - 2$. Then we call W_1 the subspace of *"sub-maximal weight vectors"*.

Remark 1.3.19. From Remark 1.3.14, the space W_1 is a sub-representation of the braid group. This can be seen also directly from the expression of the R-matrix.

The following remark describes the action of the R-matrix on the tensor product of two weight vectors in all the cases we need.

Remark 1.3.20. Since $E(e_0)=0$, if $i+j\leq 1$, then:

$$\mathrm{R}(e_i\otimes e_j)=q^{H\otimes H/2}(\mathrm{Id}\otimes\mathrm{Id}+E\otimes F)e_i\otimes e_j.$$

The sub-maximal weight vectors fulfill the conditions of this formula.

1.3.2.4 The ADO set-up for braid representations and knot invariants

The category $\mathscr{C}$ is a category of $U_q\mathfrak{sl}(2)$ modules for which the $\mathcal{RT}$ functor is shown to work, see [BCGP2]. Namely let β be a braid, and $\lambda_1,\dots,\lambda_n$ be complex parameters.

Remark 1.3.21 (ADO representations of the braid group)**.** The restriction of the Reshetikhin-Turaev functor to braid associates a morphism of $\overline{U}_q^H\mathfrak{sl}(2)$-modules:

$$\mathcal{RT}(\beta)\in\mathrm{Hom}_{\overline{U}_q^H\mathfrak{sl}(2)}\left(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n},V_{\mathrm{perm}(\beta)(\lambda_1)}\otimes\cdots\otimes V_{\mathrm{perm}(\beta)(\lambda_n)}\right)$$

as typical and atypical modules are objects of $\mathscr{C}$. In particular, it provides a representation of $\mathcal{PB}_n$ on $\mathrm{End}_{\overline{U}_q^H\mathfrak{sl}(2)}(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n})$.

Definition 1.3.22 (ADO polynomials, [ADO])**.** *Let $\mathscr{C}'$ be the sub-category of $\mathscr{C}$ made of typical modules. A modified version of the Reshetikhin-Turaev functor restricted to $\mathscr{C}'$-colored knots provides a knot invariants first introduced in [ADO], see [Ito1] for another reference. This family of invariants is known as* colored Alexander polynomials *or* ADO polynomials.

Remark 1.3.23. Since the braiding is a morphism of $\overline{U}_q^H\mathfrak{sl}(2)$-modules, it preserves weights, see Remark 1.3.14. The latter guarantees that W_1 is a sub representation of $\mathcal{PB}_n$.

Then, from now on we will study the action of braids over the sub-maximal weight space W_1. Let's call the braiding PRep_n (where n is still the number of strands), and briefly expose this representation over $\mathcal{PB}_2$ which will give an idea of the standard block of the matrices associated to the standard generators in the general case.

Example 1.3.24 (Braiding for $\mathcal{PB}_2$ in the sub-maximal weight basis)**.**

$$\begin{array}{rcl}\mathrm{PRep}_2(f_1)&=&q^{(\lambda_1+r-1)(\lambda_2+r-3)/2}\{1-\lambda_1\}f_1+q^{(\lambda_1+r-3)(\lambda_2+r-1)/2}f_2\\&=&q^{(\lambda_1+r-1)(\lambda_2+r-1)/2}(q^{-(\lambda_2+r-1)}\{1-\lambda_1\}f_1+q^{-(\lambda_1+r-1)}f_2)\end{array}$$

$$\mathrm{PRep}_2(f_2)=q^{(\lambda_1+r-1)(\lambda_2+r-3)/2}f_1=q^{(\lambda_1+r-1)(\lambda_2+r-1)/2}q^{-(\lambda_1+r-1)}f_1.$$

Where we have used the quantum brace defined as: $\{x\}=q^x-q^{-x}$.
Now we can write this action with the basis $\{f_1,f_2\}$, and we get a matrix depending on λ_1 and λ_2 which will be the standard block used to construct our representation in the case of n strands.

$$\begin{array}{rcl}\mathrm{PRep}_2(\{f_1,f_2\})&=&q^{(\lambda_1+r-1)(\lambda_2+r-1)/2}\begin{pmatrix}q^{-(\lambda_1+r-1)}\{1-\lambda_1\}&q^{-(\lambda_1+r-1)}\\q^{-(\lambda_2+r-1)}&0\end{pmatrix}\\&=&q^{(\lambda_1+r-1)(\lambda_2+r-1)/2}\,\mathrm{Block}(\lambda_1,\lambda_2)\end{array}$$

The matrix named Block will be useful in the sequel.

As $\mathcal{PB}_n$ is less convenient than $\mathcal{B}_n$ because of its generators which are quite more complicated, we will build the induced representation of the entire $\mathcal{B}_n$ in the next section.

1.3.2.5 Braid group representation

Definition 1.3.25. *Let* quant_n *be the induced representation of* $\mathcal{B}_n$ *over* $Ind(W_1) = \mathbb{C}\left[\mathcal{B}_n\right] \otimes_{\mathbb{C}[\mathcal{PB}_n]} W_1$*, see Definition 1.2.2.*

Remark 1.3.26. Let $\tau \in \mathfrak{S}_n$, and W^τ be the sub-maximal weight subspace of:

$$V^\tau = V_{\lambda_{\tau(1)}} \otimes \cdots \otimes V_{\lambda_{\tau(n)}}.$$

Then one can check that:

$$Ind(W_1) = \bigoplus_{\tau \in \mathfrak{S}_n} W^\tau = \mathbb{C}\left[\mathfrak{S}_n\right] \otimes W_1.$$

The second equality is given by the natural isomorphism:

$$\tau \otimes f_i \mapsto f_i^\tau$$

where:

$$f_i^\tau = e_0^{\lambda_{\tau(1)}} \otimes \cdots \otimes e_1^{\lambda_{\tau(i)}} \otimes \cdots \otimes e_0^{\lambda_{\tau(n)}}.$$

In this context, $\mathrm{quant}_n(\sigma_i)$, is the following morphism:

$$\begin{array}{ccc} W & \to & W \\ \tau \otimes v & \mapsto & ((i,i+1) \circ \tau) \otimes [\mathrm{Id}^{\otimes i-1} \otimes \mathcal{R}(\lambda_{\tau^{-1}(i)}, \lambda_{\tau^{-1}(i+1)}) \otimes \mathrm{Id}^{\otimes n-i-2}]v \end{array}.$$

Let's look at the action of a generator σ_i for a certain i over a vector of type $\tau \otimes f_j$ if f_j is different from f_i and f_{i+1}. Let's consider here, to help the reader, the action of σ_1 over a f_j with $j > 2$ so that the beginning of the expression of f_j is $f_j = e_0 \otimes e_0 \otimes \cdots$. As we have:

$$[\mathcal{R}(\lambda_{\tau^{-1}(1)}, \lambda_{\tau^{-1}(2)}) \otimes \mathrm{Id}^{\otimes n-2}] e_o \otimes e_0 \otimes \cdots = q^{(\lambda_{\tau^{-1}(1)}+r-1)(\lambda_{\tau^{-1}(2)}+r-1)/2} e_o \otimes e_0 \otimes \cdots$$

The action of σ_1 on the vector $\tau \otimes f_j$ is, by linearity:

$$\mathrm{quant}_n(\sigma_1)(\tau \otimes f_j) = q^{(\lambda_{\tau^{-1}(1)}+r-1)(\lambda_{\tau^{-1}(2)}+r-1)/2}((1,2)\tau \otimes f_j).$$

In order to normalize those actions, so to get entire identity blocks in the matrices, we are going to modify a bit quant_n and to define the representation we will focus on in the sequel.

Definition 1.3.27. *Let* σ_i *be the standard generator of* $\mathcal{B}_n$*, we define the representation* Quant_n *of* $\mathcal{B}_n$ *by defining* $\mathrm{Quant}_n(\sigma_i)$ *as follows:*

$$\mathrm{Quant}_n(\sigma_i) : \begin{array}{ccc} W & \to & W \\ \tau \otimes v & \mapsto & q^{-(\lambda_{\tau^{-1}(1)}+r-1)(\lambda_{\tau^{-1}(2)}+r-1)/2}\, \mathrm{quant}_n(\sigma_i)(\tau \otimes v) \end{array}.$$

It is easy to verify that this modification of quant still defines a representation. Indeed, the two new coefficients appearing, after the modification, in front of the action of $\sigma_i\sigma_{i+1}\sigma_i$ and $\sigma_{i+1}\sigma_i\sigma_{i+1}$ on a vector $\tau \otimes v$ are respectively:

$$q^{(\lambda_{\tau^{-1}(i)}+r-1)(\lambda_{\tau^{-1}(i+1)}+r-1)/2} \times q^{(\lambda_{((i,i+1)\tau^{-1})(i+1)}+r-1)(\lambda_{((i,i+1)\tau^{-1})(i+2)}+r-1)/2}$$
$$\times q^{(\lambda_{((i+1,i+2)(i,i+1)\tau^{-1})(i)}+r-1)(\lambda_{((i+1,i+2)(i,i+1)\tau^{-1})(i+1)}+r-1)/2}$$

and

$$q^{(\lambda_{\tau^{-1}(i+1)}+r-1)(\lambda_{\tau^{-1}(i+2)}+r-1)/2} \times q^{(\lambda_{((i+1,i+2)\tau^{-1})(i)}+r-1)(\lambda_{((i+1,i+2)\tau^{-1})(i+1)}+r-1)/2}$$
$$\times q^{(\lambda_{((i,i+1)(i+1,i+2)\tau^{-1})(i+1)}+r-1)(\lambda_{((i,i+1)(i+1,i+2)\tau^{-1})(i+2)}+r-1)/2}$$

Which both are equal to:

$$q^{(\lambda_{\tau^{-1}(i)}+r-1)(\lambda_{\tau^{-1}(i+1)}+r-1)/2} q^{(\lambda_{\tau^{-1}(i+1)}+r-1)(\lambda_{\tau^{-1}(i+2)}+r-1)/2} q^{(\lambda_{\tau^{-1}(i)}+r-1)(\lambda_{\tau^{-1}(i+2)}+r-1)/2}.$$

This equality guarantees that the modified representation still satisfies the braiding, and that we still have a representation of $\mathcal{B}_n$.

Remark 1.3.28 (Quadratic normalization and framing)**.** The normalization coefficient applied in Definition 1.3.26 is a quadratic term in the variables λ_i's. It removes all the quadratic terms in the expression of the matrix associated to a generator. In fact this term is necessary to get representation of the framed braid group, see Definition 1.3.8. The quadratic terms can be removed whenever one wants a representation of the unframed braid group $\mathcal{B}_n$. This normalization will be always considered in this work, as we won't be concerned by the framing.

We give one numerical example to illustrate and to give an idea of how matrices of the induced representation are organized regarding permutations.

Numerical example. As an example, here is the action of the first generator of $\mathcal{B}_3$. It is built using SageMath and its given order over $\mathcal{B}_3$. The order over $\mathfrak{S}_3$ is the following: $(), (2,3), (1,2), (1,2,3), (1,3,2), (1,3)$ (with $()$ designating the identity permutation), and the vectors are ordered as follow: $f_1^{()}, f_2^{()}, f_3^{()}, f_1^{(2,3)}, \dots$ and so on.

Using this basis, $\mathrm{Quant}_3(\sigma_1)$ is the matrix:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -s_2^2+1 & -s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_3^2+1 & -s_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-s_1^2+1 & -s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -s_1^2+1 & -s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -s_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_3^2+1 & -s_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_2^2+1 & -s_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

We used the following change of formal variables: $s_i = q^{-(\lambda_i - 1)}$ that will be used in Section 2.1, and the important fact that $q^r = -1$.

In Section 2.1 we prove the following theorem, see Theorem 2.1.1.

Theorem 1 (Theorem 2.1.1)**.** *This sub-maximal weight representation of the braid group is the Gassner representation.*

1.3.3 Lawrence representations are contained in the non-semi simple quantum ones

In [Ito1], it is shown that the BKL representation is a sub-representation of the quantum representation obtained with the category of $\overline{U}_q^H\mathfrak{sl}(2)$-modules.

Let q be a $2r$ root of unity, and λ a complex number such that $\{\lambda - i\} \neq 0$ for all $i \in \mathbb{Z}$. Let V'_λ be the corresponding typical module of middle weight $\lambda - r + 1$ (V'_λ is of maximal weight λ). We let ϕ be the representation of $\mathcal{B}_n$ using the R-matrix coming with the category $\mathscr{C}$ of $\overline{U}_q^H\mathfrak{sl}(2)$-modules.

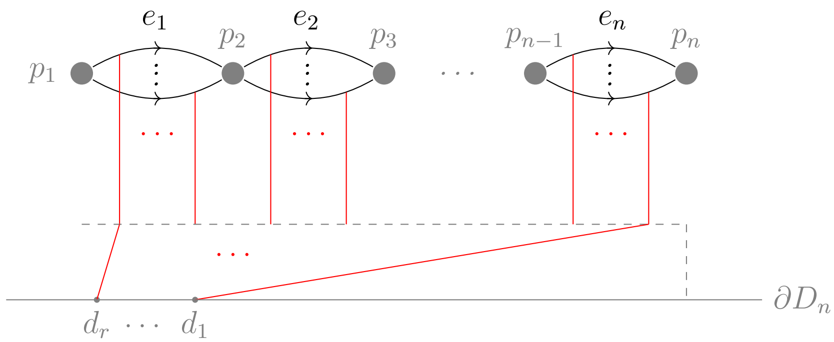
$$\phi : \left\{ \begin{array}{ccl} \mathcal{B}_n & \to & GL({V'}_\lambda^{\otimes n}) \\ \sigma_i & \mapsto & \phi(\sigma_i) = \mathrm{Id}^{\otimes(i-1)} \otimes \mathcal{R}(\lambda,\lambda) \otimes \mathrm{Id}^{\otimes(n-i-1)}. \end{array} \right.$$

Let $X^r_{n,m}$ be the sub-representation spanned by $\{e^\lambda_{i_1} \otimes \ldots \otimes e^\lambda_{i_n}$ s.t. $i_1 + \ldots + i_n = m\}$. From the previous sections, we remark that the space of sub-maximal weights is $X_{n,1}$, and that all the vectors of $X_{n,m}$ have the same weight, more precisely $n\lambda - 2m$, i.e. they lie in $Ker\,(H - (n\lambda - 2m)\mathrm{Id})$. Set $Y^r_{n,m}$ to be the space $Ker\,E \cap X_{n,m}$. Since $\mathcal{B}_n$ acts by $\overline{U}_q^H\mathfrak{sl}(2)$-morphism, $Y_{n,m}$ is a $\mathcal{B}_n$-subrepresentation.

The BKL-like representation used in [Ito1] are subrepresentations of $H_m(\widetilde{C_{n,m}}, \mathbb{Z})$. The BKL representations we have studied in detail are actually $H_2(C_{n,2}, \mathbb{Z})$, and as we have seen that forks define classes of this $\mathbb{Z}[\mathfrak{q}^{\pm 1}, \mathfrak{t}^{\pm 1}]$-module, we can define *multiforks* corresponding to classes of $H_m(\widetilde{C_{n,m}}, \mathbb{Z})$ in general. Here we used formal variables $\mathfrak{q}$ and $\mathfrak{t}$ in order not to make the confusion with the q and t from the quantum side, and all homology modules are locally finite chains homologies (Definition 5.1.1).

Definition 1.3.29 (Multi-fork, [Ito1])**.** *An m-multifork is an m-tuple of forks (see Definition 1.2.22) such that the set of tines and the set of handles are embedded (without crossing).*

Let $\mathcal{E}_{n,m} = \{(e_1, \ldots, e_{n-1}) \in \mathbb{N}^{n-1}$ s.t. $e_1 + \ldots + e_{n-1} = m\}$. For each $\mathbf{e}$ in $\mathcal{E}_{n,m}$ we associate a multifork $\mathcal{F}_{\mathbf{e}} = \{F_1, \ldots, F_m\}$ called a standard multifork, and such that F_i corresponds to e_i parallel forks going from the punctures p_i and p_{i+1} directly without encircling any other *puncture, see Figure 1.6.*



(1.6)

This definition will be generalized in Chapter 3.

We can associate to multiforks, in the spirit of what we did for forks, homology classes in $H_m^{lf}(\widetilde{C_{n,m}}, \mathbb{Z})$ (see Section 3.2.2.2 for an exhaustive construction, and Remark 5.1.3 for an idea of why forks are cycles). If we set $\mathcal{H}_{n,m}$ to be the subspace generated by all multiforks, one can show that it generically admits the standard multiforks as a basis (this fact will be largely detailed in Section 3.2.2.4). As $\mathcal{H}_{n,m}$ is invariant under the $\mathcal{B}_n$ action, we define it to be the Lawrence's representation $L_{n,m}$. In [Ito1], the author defines the truncated Lawrence representation as follows for the case of t set to be a root of unity.

Suppose that $-\mathfrak{t}$ is set to be the root of unity $\zeta_N^2 = e^{2\pi\sqrt{-1}/N}$ for an integer N. Set $\mathcal{E}_{n,m}^{\geq N} = \{(e_1, \ldots, e_{n-1}) \in \mathcal{E}_{n,m}$ s.t. $e_i \geq N$ for some $i\}$ and $\mathcal{H}_{n,m}^N$ to be the subspace of $\mathcal{H}_{n,m}$ spanned by $\{\mathcal{F}_{\mathbf{e}}$ s.t. $\mathbf{e} \in \mathcal{E}_{n,m}^{\geq N}\}$. Finally we set $\overline{\mathcal{H}_{n,m}^N} = \mathcal{H}_{n,m}/\mathcal{H}_{n,m}^N$. It is a fact that the action $L_{n,m}$ behaves well with the quotient, so that the truncated Lawrence representation are defined to be:

$$l_{n,m}^N : \mathcal{B}_n \to GL(\overline{\mathcal{H}_{n,m}^N}).$$

After the specialization of $\mathfrak{t}$, what we get is a $\mathbb{Z}[\mathfrak{q}^{\pm 1}]$-module action.

The following theorem states that truncated Lawrence's representation are of quantum nature.

Theorem 1.3.30 ([Ito1, Theorem 4.2])**.** *For an n braid $\beta \in \mathcal{B}_n$ the following matrices are equal:*

$$\phi_{n,m}(\beta)_{|Y} = l_{n,m}^N(\beta)_{|\mathfrak{q}=\zeta^{-2\lambda}\ ,\ \mathfrak{t}=-\zeta^2}$$

The above theorem is a consequence of [K2], relating $\mathcal{B}_n$ representation over product of quantum Verma modules (defined in the next section) and Lawrence's homological representation. The case $m = 2$ of the latter theorem is detailed in [J-K], but in a slightly different fork basis.

1.4 Quantum algebra: ground ring, specialization and integral versions

In this section we reconstruct the algebra $U_q\mathfrak{sl}(2)$ keeping track of the algebra structure. Depending on what kind of ring we need to work with, the structure of the quantum algebra can deeply change together with its different categories of modules. We will see how the ring is important in order to specialize the theory to complex numbers for example, what kind of genericity one has to restrict to by considering a large ring. This should emphasize the interest to deal with integral versions (in a sense that will be defined) of the algebra, and lead to different definitions of the latter.

1.4.1 Rational theory and specialization issues

The most generic definition of $U_q\mathfrak{sl}(2)$ is as a vector space over a rational field.

Definition 1.4.1. *The algebra $U_q\mathfrak{sl}(2)$ is the algebra over $\mathbb{Q}(q)$ generated by elements E, F and $K^{\pm 1}$, satisfying the following relations:*

$$KK^{-1} = K^{-1}K = 1$$

$$KEK^{-1} = q^2E, \ KFK^{-1} = q^{-2}F$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The definition generalizes to any field of characteristic 0 instead of $\mathbb{Q}$, for example $\mathbb{C}$, without a deeply change of the theory, so that we will use $\mathbb{Q}$ or $\mathbb{C}$.

There is an adapted *"Poincaré-Birkhoff-Witt"* basis.

Proposition 1.4.2 ([Kas, Proposition IV.1.4])**.** *As a $\mathbb{Q}(q)$ algebra, the following family:*

$$\left\{F^a K^b E^{a+n} \ , \, a \in \mathbb{N} \ a + n \in \mathbb{N} \ b \in \mathbb{Z}\right\}$$

is a basis of $U_q\mathfrak{sl}(2)$.

The finite dimensional theory of module is semi-simple.

Theorem 1.4.3 ([Kas])**.** *Let $\mathcal{C}$ be the category made of $U_q\mathfrak{sl}(2)$ modules of finite dimension. Then $\mathcal{C}$ is semi-simple. Its simple modules are called $S_i^{\pm}$ for a choice of $i \in \mathbb{N}$. It is a* highest weight module *of highest weight $\pm q^i$. The latter means that the action of K is diagonalizable and that there exists an eigenvector for the eigenvalue $\pm q^i$ over which the action of E is 0.*

Remark 1.4.4 (Specialization issue)**.** The process of *specialization* of the parameter q is algebraically the following. Let $\xi \in \mathbb{C}$ be a complex number. By specialization of q to the parameter ξ one considers the morphism:

$$eval : \begin{array}{ccc} \mathbb{Q}(q) & \to & \mathbb{C} \\ q & \mapsto & \xi \end{array}$$

and the following complex vector space:

$$U_\xi = \mathbb{C} \otimes_{eval} U_q\mathfrak{sl}(2).$$

We make the remark here that the morphism *eval* is well defined only if ξ is a transcendental number. This is the first example of issue one encounter while working with $\mathbb{Q}(q)$ as ground ring. More precisely, For the purpose of passing from the $\mathcal{RT}$ functor to invariants of 3 dimensional manifolds, one has to deal with q being a root of unity, for which the ground ring $\mathbb{Q}(q)$ is not appropriate.

The above remark justifies the definition of integral versions of $U_q\mathfrak{sl}(2)$, the aim of next subsection.

1.4.2 Integral versions of $U_q\mathfrak{sl}(2)$

Definition 1.4.5 (Integral version, [C-P, § 9.2])**.** *Let $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$ be the ring of Laurent polynomials in the single variable q. An* integral version *of $U_q\mathfrak{sl}(2)$ is an $\mathcal{R}$-subalgebra $U_\mathcal{R}$ of $U_q\mathfrak{sl}(2)$ such that the natural map:*

$$U_\mathcal{R} \otimes_\mathcal{R} \mathbb{Q}(q) \to U_q\mathfrak{sl}(2)$$

is an isomorphism of $\mathbb{Q}(q)$ algebras.

Let $\xi \in \mathbb{C}^*$ then the specialization of $U_\mathcal{R}$ to ξ means the following vector space:

$$U_\xi = \mathbb{C} \otimes_{eval} U_\mathcal{R}.$$

One can replace $\mathbb{C}$ by $\mathbb{Q}(\xi)$ or even $\mathbb{Z}[\xi^{\pm 1}]$ if necessary.

There exists different integral versions of $U_q\mathfrak{sl}(2)$ in the literature that provide highly different representation theories, and specializations. from now on and until the end of this section, $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$.

Definition 1.4.6 (Kac - De Concini - Procesi version, [DCP])**.** *Let $U_q^{KCP}\mathfrak{sl}(2)$ be the $\mathcal{R}$-subalgebra of $U_q\mathfrak{sl}(2)$ generated by E, F and $K^{\pm 1}$. It is an integral version of $U_q\mathfrak{sl}(2)$ called the Kac - De Concini - Procesi version of $U_q\mathfrak{sl}(2)$.*

Theorem 1.4.7. *The algebra $U_q^{KCP}\mathfrak{sl}(2)$ has the following set as basis over $\mathcal{R}$:*

$$\left\{ F^a K^b \left(\frac{K - K^{-1}}{q - q^{-1}} \right) E^d \right\}_{a,d \in \mathbb{N}, b \in \mathbb{Z}}$$

Theorem 1.4.8 ([Bas, Theorem 2.9])**.** *Let $\xi = e^{2i\pi/p}$ and, U_ξ be the specialized $U_q^{KCP}\mathfrak{sl}(2)$. The center of U_ξ is the following:*

$$Z(U_\xi) = \mathbb{Q}(\xi)\langle E^p, F^p, K^p, \Omega \rangle$$

where:

$$\Omega = FE + \frac{qK - q^{-1}K^{-1}}{(q - q^{-1})^2}$$

is the Casimir element *of $U_q\mathfrak{sl}(2)$.*

Definition 1.4.9 (Lusztig Version, [Lus2], [C-P, § 9.3])**.** *Let $n \in \mathbb{N}^*$, the* divided powers *of E and F are the following elements of $U_q\mathfrak{sl}(2)$:*

$$E^{(n)} = \frac{E}{[n]!} \text{ and } F^{(n)} = \frac{F}{[n]!}$$

where $[n]! = [n]\cdots[1]$, and $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. Let $U_q^L\mathfrak{sl}(2)$ be the $\mathcal{R}$-subalgebra of $U_q\mathfrak{sl}(2)$ generated by $E^{(n)}$, $F^{(n)}$ for $n \in \mathbb{N}^*$ and $K^{\pm 1}$. Then, $U_q^L\mathfrak{sl}(2)$ is an integral version for $U_q\mathfrak{sl}(2)$ called the Lusztig version of $U_q\mathfrak{sl}(2)$*.*

Proposition 1.4.10 ([C-P, Proposition 9.3.3])**.** *The algebra $U_q^L\mathfrak{sl}(2)$ admits the following set as an $\mathcal{R}$-basis:*

$$U_q^L\mathfrak{sl}(2) = \mathcal{R}\left\langle F^{(a)} K^b \begin{bmatrix} K, c \\ n \end{bmatrix} E^{(d)} \right\rangle_{a,d,c \in \mathbb{N}, b \in \mathbb{Z}}$$

where:

$$\begin{bmatrix} K, c \\ n \end{bmatrix} = \prod_{s=1}^{n} \frac{K q^{c+1-s} - K^{-1} q^{s-1-c}}{q^s - q^{-s}}.$$

Proposition 1.4.11 ([C-P, § 9.3])**.** *Let $\xi = e^{2i\pi/p}$ and, U_ξ be the specialized $U_q^L\mathfrak{sl}(2)$. Then:*

- $E^p = F^p = 0$
- *K^p is central, while $K^{2p} = 1$.*

For a classification of finite dimensional modules of $U_q^L\mathfrak{sl}(2)$ see [BFGT].

Proposition 1.4.12 ([Len])**.** *The unrolled quantum group $\overline{U}_q^H\mathfrak{sl}(2)$ is embedded inside the Lusztig version $U_q^L\mathfrak{sl}(2)$.*

1.4.3 Half-Lusztig version

In this section, we define an integral version for $U_q\mathfrak{sl}(2)$ that will be central in Chapter 3. This integral version is similiar to the one introduced by Lusztig and presented in Definition 1.4.9. The difference is that we consider only the divided powers of F as generators, not those of E. This version is introduced in [Hab] and [J-K] (with subtle differences in the definitions of divided powers for F). We follow the one of [J-K], so that we first define their divided powers, presenting a minor difference from the original ones of Lusztig. Let:

$$F^{(n)} = \frac{(q - q^{-1})^n}{[n]_q!} F^n$$

be the element of $U_q\mathfrak{sl}(2)$. The ring $\mathcal{R}$ is still the ring of integral Laurent polynomials in the variable q.

Definition 1.4.13 (Half Lusztig algebra, [Hab], [J-K])**.** *Let $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ be the $\mathcal{R}$-subalgebra of $U_q\mathfrak{sl}(2)$ generated by E, $K^{\pm 1}$ and $F^{(n)}$ for $n \in \mathbb{N}^*$. We call it a* half-Lusztig version *of $U_q\mathfrak{sl}(2)$, the word half to illustrate that we consider only half of divided powers as generators.*

Remark 1.4.14 (Relations in $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$, [J-K, (16) (17)])**.** The relations among generators involving divided powers are the following:

$$KF^{(n)}K^{-1} = q^{-2n}F^{(n)}$$

$$\left[E, F^{(n+1)}\right] = F^{(n)}\left(q^{-n}K - q^{n}K^{-1}\right) \text{ and } F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q F^{(n+m)}$$

where $\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \frac{[n+m]_q!}{[n]_q![m]_q!}$. The other relations are the one from Definition 1.3.3.

The coproduct is given by:

$$\Delta(K) = K \otimes K \ , \ \Delta(E) = E \otimes K + 1 \otimes E,$$

$$\Delta(F^{(n)}) = \sum_{j=0}^{n} q^{-j(n-j)}K^{j-n}F^{(j)} \otimes F^{(n-j)}.$$

Proposition 1.4.15. *The algebra $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ admits the following set as an $\mathcal{R}$-basis:*

$$\left\{K^l E^m F^{(n)}, l \in \mathbb{Z}, m, n \in \mathbb{N}\right\}.$$

1.4.4 Verma modules and braiding

Now we define a special family of universal objects in the category of $U_q\mathfrak{sl}(2)$-modules, we express their presentation in the special case of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ and we give a braiding for this family of modules. Namely, the *Verma modules* are infinite dimensional modules which have a universal (among quantum groups) definition. We translate this definition in the case of the integral version $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$.

Definition 1.4.16 (Universal integral Verma modules, [C-P, § 10.1.A])**.** *Let U be an integral version of $U_q\mathfrak{sl}(2)$ and s be a variable. The Verma module V^s is the infinite U-module defined as follows:*

$$V^s = \left(U \otimes \mathbb{Z}\left[s^{\pm 1}\right]\right) \Big/ \mathcal{I}$$

where $\mathcal{I}$ is the left ideal generated by E and $K - s1$.

In [J-K], the authors give an explicit presentation of the integral Verma-module of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$.

Definition 1.4.17 (Verma modules for $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$, [J-K, (18)]). *Let V^s be the Verma module of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$. It is the infinite $\mathbb{Z}\left[q^{\pm 1}, s^{\pm 1}\right]$-module, generated by vectors $\{v_0, v_1 \ldots\}$, and endowed with an action of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$, generators acting as follows:*

$$K \cdot v_j = sq^{-2j} v_{j-1} \text{ and } E \cdot v_j = v_{j-1}$$

$$F^{(n)} v_j = \left(\left[\begin{array}{c} n+j \\ j \end{array} \right]_q \prod_{k=0}^{n-1} sq^{-k-j} - s^{-1} q^{j+k} \right) v_{j+n}.$$

Remark 1.4.18. Some remarks about notations.

- By specializing $s = q^{\alpha}$, one recognizes the Verma module of highest weight α often presented like this in the literature.
- The way of defining the universal Verma module in Definition 1.4.16 is to put the highest weight inside the ring action, not seeing it as a parameter.

Definition 1.4.19 (R-matrix, [J-K, (21)]). *Let $s = q^{\alpha}$, $t = q^{\alpha'}$. The operator $q^{H \otimes H/2}$ is the following:*

$$q^{H \otimes H/2} : \begin{array}{ccc} V^s \otimes V^t & \to & V^s \otimes V^t \\ v_i \otimes v_j & \mapsto & q^{(\alpha - 2i)(\alpha' - 2j)} v_i \otimes v_j \end{array}.$$

We define the following R-matrix:

$$R : q^{H \otimes H/2} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}$$

which will be well defined as an operator on Verma modules, see the following proposition.

Proposition 1.4.20 ([J-K, Theorem 7]). *Let V^s and V^t be Verma modules of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$. Let* R *be the following operators:*

$$\mathrm{R} : q^{-\alpha\alpha'/2} T \circ R$$

where T is the twist defined by $T(v \otimes w) = w \otimes v$. Then R *provides a braiding for $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ integral Verma modules.*

Remark 1.4.21. Again, we have normalized the action by the factor $q^{-\alpha\alpha'/2}$ that corresponds to a framing information, as we are considering unframed braids, see Remark 1.3.27.

Corollary 1.4.22 ([J-K, Theorem 7]). *The morphism:*

$$Q : \begin{array}{ccc} \mathbb{C}\left[\mathcal{B}_n\right] & \to & \operatorname{End}_{\mathcal{R}, U_q^{\frac{L}{2}}\mathfrak{sl}(2)} \left(V^{s \otimes n}\right) \\ \sigma_i & \mapsto & 1^{\otimes i-1} \otimes \mathrm{R} \otimes 1^{\otimes n-i-2} \end{array}$$

is an $\mathcal{R}$-algebra morphism. It provides a representation of $\mathcal{B}_n$ such that its action commutes with the one of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$.

Remark 1.4.23. One can consider a braid action over $V^{s_1} \otimes \cdots \otimes V^{s_n}$ so that the morphism Q is well defined but becomes multiplicative (i.e. algebra morphism) only when restricted to the pure braid group $\mathcal{PB}_n$. Then one can consider the induced representation of $\mathcal{B}_n$, see Definition 1.2.2, or restrict to a representation of the colored braid groupoid.

Remark 1.4.24. Two remarks from Remark 1.3.14:

- For $r \in \mathbb{N}$, the subspace $W_{n,r} = Ker(K - s^n q^{-2r})$ of $(V^s)^{\otimes n}$ provides a sub-representation of $\mathcal{B}_n$.
- The subspace $Y_{n,r} = W_{n,r} \cap Ker\, E \subset W_{n,r}$ provides a sub-representation of $\mathcal{B}_n$.

Theorem 1.4.25 (Irreducibility of highest weight modules, [J-K, Theorem 21])**.** *The $\mathcal{B}_n$-representations $Y_{n,r}$ are irreducible over the fraction field $\mathbb{Q}(q, s)$.*

Chapter 2

Homological and quantum representations: first strata

The representations of $\mathcal{B}_n$ over typical modules of $\overline{U}_q^H\mathfrak{sl}(2)$, as well as the one over Verma modules of $U_q\mathfrak{sl}(2)$, are graded by weight of vectors see Sections 1.3.2 and 1.4.4. Both gradings have the same first strata that we investigate concretely in this chapter. In Section 2.1 we show that the Gassner representation is the sub-representation formed by the sub-maximal weight vectors, while in Section 2.2 we build a colored version of the BKL representation that corresponds to vectors of sub-sub-maximal weights.

In the non semi-simple TQFT's built in [BCGP2], representations of the mapping class groups live among the category of graded vector spaces. Again, the representation is graded so that it is natural to look at the first strata trying to find homological standard construction. In Section 2.3 we recognize representation of homological nature in the first stratum of the quantum non semi-simple representation of the mapping class group $M(0,4)$. The latter leads to the faithfulness of the representation.

2.1 The non semi-simple TQFT's contain the Gassner representations.

In this section, we will show that the Gassner representation is contained in the non semi-simple TQFT's representation. More precisely, the representation Gassner described in Section 1.2.2 is algebraically the same as the representation Quant built in Section 1.3, for any number n of strands for $\mathcal{B}_n$.

We recall the context of both representations, namely:

- from Section 1.2.2 that Gassner_n is a representation of $\mathcal{B}_n$:

$$\text{Gassner}_n : \mathcal{B}_n \to \text{End}\left(\mathbb{C}[\mathfrak{S}_n] \otimes Vect(g_1, \ldots, g_n)\right)$$

  involving formal variables $t_1, \ldots, t_n$.

- from Section 1.3 that Quant_n is a representation of $\mathcal{B}_n$:

$$\mathrm{Quant}_n : \mathcal{B}_n \to \mathrm{End}\left(\mathbb{C}[\mathfrak{S}_n] \otimes Vect(f_1, \ldots, f_n) = W\right)$$

  involving formal variables $\lambda_1, \ldots, \lambda_n$.

In order to relate the representations Quant and Gassner, we need first to connect variables. To do so, we use new "colors" $s_1, \ldots, s_n$ which are related to the λ_i's and the t_i's as follows:

$$s_i = \sqrt{t_i} = q^{-(\lambda_i - 1)} \ , \ \forall i.$$

Then let Φ be the following morphism relating both representations :

$$\begin{array}{ccc} W & \to & \mathbb{C}[\mathfrak{S}_n] \otimes Vect(g_1, \ldots, g_n) \\ \tau \otimes f_i & \mapsto & (-1)^i \frac{1 - s^2_{\tau^{-1}(i)}}{\prod_{j=i}^n s_{\tau^{-1}(j)}} \tau \otimes g_i \end{array}$$

where we used the basis $\tau \otimes f_i$ for W and $\tau \otimes g_i$ for $\mathbb{C}[\mathfrak{S}_n] \otimes \mathbb{C}^n$, τ being a permutation in $\mathfrak{S}_n$.

Theorem 2.1.1. *Gassner representations are of quantum type, namely the morphism* Φ *conjugates* Quant *to* Gassner*, in the sense that for all* $n \in \mathbb{N}$*, for all* $\alpha \in \mathcal{B}_n$ *and for all* $w \in W$ *we have the equality:*

$$\mathrm{Gassner}_n(\alpha) \circ \Phi(w) = \Phi \circ \mathrm{Quant}_n(\alpha)(w)$$

Proof. Let σ_k be a standard generator of $\mathcal{B}_n$, $\tau \in \mathfrak{S}_n$ and f_i so that $\tau \otimes f_i$ to be an element of the basis of W mentioned above.
Remark that if i is different from k and $k+1$ then, as Quant and Gassner both act by identity over $\tau \otimes f_i$, the equality is trivial.

- *Case 1:* $i = k$. Let's compute the two sides of the commutation equality. We begin with $\Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_k)$:

$$\mathrm{Quant}_n(\sigma_k)(\tau \otimes f_k) = (1 - s^2_{\tau^{-1}(k)})((k, k+1)\tau \otimes f_k) - s_{\tau^{-1}(k+1)}((k, k+1)\tau \otimes f_{k+1})$$

  Then, the composition by Φ gives:

$$\begin{array}{rcl} \Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_k) & = & \Phi\left((1 - s^2_{\tau^{-1}(k)})((k, k+1)\tau \otimes f_k) - s_{\tau^{-1}(k+1)}((k, k+1)\tau \otimes f_{k+1})\right) \\ & = & A \cdot (k, k+1)\tau \otimes g_k + B \cdot (k, k+1)\tau \otimes g_{k+1} \end{array}$$

  where:

$$A = (-1)^k (1 - s^2_{\tau^{-1}(k)}) \frac{1 - s^2_{\tau^{-1}(k+1)}}{\prod_{j=k}^n s_{\tau^{-1}(j)}}$$

$$\begin{aligned} B &= -(-1)^{k+1} s_{\tau^{-1}(k+1)} \frac{1-s^2_{\tau^{-1}(k)}}{s_{\tau^{-1}(k)} \prod_{j=k+2}^n s_{\tau^{-1}(j)}} \\ &= (-1)^k s^2_{\tau^{-1}(k+1)} \frac{1-s^2_{\tau^{-1}(k)}}{\prod_{j=k}^n s_{\tau^{-1}(j)}} \end{aligned}$$

Now we compute $\mathrm{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_k)$:

$$\begin{aligned} \mathrm{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_k) &= (-1)^k \frac{1-s^2_{\tau^{-1}(k)}}{\prod_{j=k}^n s_{\tau^{-1}(j)}} \mathrm{Gassner}_n(\sigma_k)(\tau \otimes g_k) \\ &= (-1)^k \frac{1-s^2_{\tau^{-1}(k)}}{\prod_{j=k}^n s_{\tau^{-1}(j)}} \Big(1 - s^2_{\tau^{-1}(k+1)}(k,k+1)\tau \otimes g_k \\ &\quad + s^2_{\tau^{-1}(k+1)}(k,k+1)\tau \otimes g_{k+1}\Big) \\ &= \Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_k). \end{aligned}$$

The last equality comes from the expression of $\Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_k)$ obtained above, and provides the equality we want.

- *Case 2: $i = k+1$.* We begin with the computation of $\Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_{k+1})$:

$$\begin{aligned} \Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_{k+1}) &= \Phi\left(-s_{\tau^{-1}(k)}(k,k+1)\tau \otimes f_k\right) \\ &= (-1)^{k+1} s_{\tau^{-1}(k)} \frac{1-s^2_{\tau^{-1}(k+1)}}{\prod_{j=k}^n s_{\tau^{-1}(j)}} (k,k+1)\tau \otimes g_k \\ &= (-1)^{k+1} \frac{1-s^2_{\tau^{-1}(k+1)}}{\prod_{j=k+1}^n s_{\tau^{-1}(j)}} (k,k+1)\tau \otimes g_k \end{aligned}.$$

Now we compute $\mathrm{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_{k+1})$:

$$\begin{aligned} \mathrm{Gassner}_n(\sigma_k) \circ \Phi(\tau \otimes f_{k+1}) &= \mathrm{Gassner}_n(\sigma_k)\left((-1)^{k+1} \frac{1-s^2_{\tau^{-1}(k+1)}}{\prod_{j=k+1}^n s_{\tau^{-1}(j)}} \tau \otimes f_k\right) \\ &= (-1)^{k+1} \frac{1-s^2_{\tau^{-1}(k+1)}}{\prod_{j=k+1}^n s_{\tau^{-1}(j)}} \tau \otimes g_k \\ &= \Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_{k+1}) \end{aligned}.$$

The last equality coming from the expression of $\Phi \circ \mathrm{Quant}_n(\sigma_k)(\tau \otimes f_{k+1})$ obtained above, and provides the equality we want.

We have proved that for any generator σ_k of $\mathcal{B}_n$, its representation by Quant_n and by $\mathrm{Gassner}_n$ are conjugated by Φ, as the equality of the proposition holds for all the basis vectors of W. As Quant_n and $\mathrm{Gassner}_n$ are representations, the theorem is proved for all braids. □

2.2 Colored BKL representations

In this section, we construct BKL-like homological representations of braid groups, called *colored BKL representations*. We follow the construction of [K-T] and [Big0] that inspires a generalization of it. We follow ideas of [Big0] to compute the matrices of these representations. This construction corresponds to the level $r = 2$ of the one presented in Chapter 3, Section 3.2. Although the obtained representations are the same, the following construction is different: it involves Fox calculus for the computation of the local system, and uses a pairing to compute matrices.

2.2.1 Construction and Faithfulness

We recall from Section 1.2.3 that C designates the configuration space of unordered pairs of points of D_n, we note $\{x, y\}$ an element of C ($\{x, y\} = \{y, x\}$), and $c = \{d_1, d_2\}$ a base point of C with the d_i's lying in the boundary of D_n. The difference with Section 1.2.3, is that we keep all n variables corresponding to the meridians $\{\{z_1, z_2\}$ s.t. $z_1 = p_i\}$ (generators of $H_1(C)$) in the abelianized local system as follows.

Definition 2.2.1. *We consider the Hurewicz morphism:*

$$Hurewicz : \pi_1(C) \to H_1(C) = \mathbb{Z}^n \oplus \mathbb{Z} = \langle q_1 \rangle \otimes \cdots \otimes \langle q_n \rangle \oplus \langle t \rangle,$$

and we denote by $\widetilde{C}$ the covering corresponding to the kernel of this map, namely the maximal abelian cover. Now $H_2(\widetilde{C})$ is a $\mathbb{Z}\left[q_1^{\pm 1}, \ldots, q_n^{\pm 1}, t^{\pm 1}\right]$ over which $\mathcal{PB}_n$ acts as $\mathbb{Z}\left[q_1^{\pm 1}, \ldots, q_n^{\pm 1}, t^{\pm 1}\right]$-module automorphisms. This action is the so called colored BKL representation.

We define invariants w_i of homotopy classes of loops in C for all $i \in \{1, \ldots, n\}$ and for the two cases of a loop $\xi = \{\xi_1, \xi_2\}$ of C:

- If ξ_1 and ξ_2 both are loops, then we define $w_i(\xi) = w_i(\xi_1) + w_i(\xi_2)$ where $w_i(\xi_i)$ is the winding number around the puncture p_i of a loop of D_n.
- For the case where ξ_1 and ξ_2 permute base points we define $w_i(\xi) = w_i(\xi_1\xi_2)$ to be the winding number around the puncture p_i of the loop $\xi_1\xi_2$.

Let also u be the same invariant as in Section 1.2.3, namely the index (speaking of a loop of S^1) of the square of the following application:

$$\begin{array}{rcl} I & \to & S^1 \\ s & \mapsto & \frac{\xi_1(s)-\xi_2(s)}{|\xi_1(s)-\xi_2(s)|} \end{array}$$

These invariants can equivalently be defined as follows:

$$w_i(\xi) = \frac{1}{2\pi i}\left(\int_{\xi_1} \frac{dz}{z - p_i} + \int_{\xi_2} \frac{dz}{z - p_i}\right)$$

and:

$$u(\xi) = \frac{1}{\pi i}\int_{\xi_2 - \xi_1} \frac{dz}{z}$$

The map:

$$\phi : \xi \to q_1^{w_1(\xi)} \cdots q_n^{w_n(\xi)} t^{u(\xi)}$$

is a surjective group homomorphism from $\pi_1(C)$ to the free abelian group with $(n+1)$ generators $q_1, \ldots, q_n, t$.

Then $\widetilde{C} \to C$ is the covering map corresponding to the kernel of ϕ, and $\mathcal{H} = H_2^{lf}(\widetilde{C}, \mathbb{Z})$ a module over $\mathcal{R} = \mathbb{Z}\left[q_1^{\pm 1}, \ldots, q_n^{\pm 1}, t^{\pm 1}\right]$, once we choose a lift $\tilde{c}$ of the base point c.

We recall that if f is a self-homeomorphism of D_n, it induces a homeomorphism $\hat{f} : C \to C$ by:

$$\hat{f}(\{x, y\}) = \{f(x), f(y)\} \tag{2.1}$$

Note that $\hat{f}(c) = c$ as d_1 and d_2 are picked in the boundary of D_n. We define the induced automorphism $f_\#$ of $\pi_1(C, c)$. Again, the following holds.

Lemma 2.2.2. *Let f be a self-diffeomorphism not permuting punctures. Then $\phi \circ f_\# = \phi$.*

Proof. The proof is the same as Lemma 1.2.20. Here we need that f does not permute the punctures, which is induced by the condition that the diffeomorphism is the identity in homology. Otherwise, a small circle encircling a puncture moved by f will count $+1$ for a different winding number before and after the application of f.

□

This lemma implies that, in the case where f does not permute punctures, $\hat{f}$ lifts uniquely to a map $\tilde{f} : \widetilde{C} \to \widetilde{C}$ fixing any lift of c, and that $\tilde{f}$ commutes with covering deck-transformations. Therefore it induces an $\mathcal{R}$-linear automorphism f_* of $\mathcal{H}$.

Definition 2.2.3 (Colored BKL representation)**.** *The* colored Bigelow-Krammer-Lawrence *representation is:*

$$PB_n \to \underset{R}{Aut}(\mathcal{H}) \ , \ [f] \mapsto f_*$$

where PB_n refers to the pure mapping class group of the punctured disk, which corresponds exactly to homeomorphisms that fix the punctures pointwise.

What follows immediately, is that by specializing every variables q_i to the same variable q we obtain the BKL representation of PB_n as a subgroup of B_n, so that the following holds.

Proposition 2.2.4. *The colored BKL representation of PB_n is faithful*

2.2.2 Pairing between forks and noodles

Using notations of Section 1.2.3, let F be a fork and N a noodle, and let $\widetilde{\Sigma}(F)$ and $\widetilde{\Sigma}(N)$ the associated surfaces of $\widetilde{C}$.

Suppose that $T(F)$ and N intersect transversely in some points $z_1, \dots, z_l$, and $T(F')$ and N intersect transversely in $z'_1, \dots, z'_l$ such that z_i and z'_i are joint by a short piece of N not containing any other intersection point. Surfaces $\widetilde{\Sigma}(F)$ and $\widetilde{\Sigma}(N)$ do not intersect necessarily because of the choice of the lift, but there exists a unique monomial $m_{i,j} = \prod_{k\in\{1,\dots,l\}} q_k^{w_k(\xi_{i,j})} t^{u_{i,j}}$ such that $m_{i,j}\widetilde{\Sigma}(N)$ intersects $\widetilde{\Sigma}(F)$ at a point lying over $\{z_i, z'_j\}$. Let $\epsilon_{i,j}$ be the sign of the intersection. We define the pairing as follows:

$$\langle N, F \rangle = \sum_{i=1}^{l} \sum_{j=1}^{l} \epsilon_{i,j} m_{i,j}. \tag{2.2}$$

There is again a practical way to compute this pairing. To compute $m_{i,j}$ we stick with the path $\delta_{i,j}$ defined in Section 1.2.3, and we let $\widetilde{\delta}_{i,j}$ be the lift of $\delta_{i,j}$ beginning at $\widetilde{c}$. This path goes first from $\widetilde{c}$ to $\widetilde{\Sigma}(F)$ then to the lift of $\{z_i, z_j'\}$ lying over $\widetilde{\Sigma}(F) \cap m_{i,j}\widetilde{\Sigma}(N)$, so that it ends in $m_{i,j}\tilde{c}$. It is a path from $\tilde{c}$ to $m_{i,j}\tilde{c}$. Then we have:

$$m_{i,j} = \phi(\delta_{i,j}),$$

and

$$\epsilon_{i,j} = -(-1)^{u_{i,i}+u_{j,j}+u_{i,j}}$$

as the intersection sign is computable in C (does not depend on which covering one lifts the surfaces to), it is the same as for BKL representations, see [Big0, Equation (1)].

2.2.3 Matrices for colored BKL representations

Inspired by Part 4 of [Big0] we give explicit matrices for colored BKL representations.

Proposition 2.2.5. *$\mathcal{H}$ is a free $\mathcal{R}$-module. It has a basis:*

$$\{v_{j,k} : 1 \leq j \leq k \leq n\}$$

The group $\mathcal{B}_n$ acts on $\mathcal{H} \otimes \mathcal{R}[\mathfrak{S}_n]$ by the induced action from $\mathcal{PB}_n$. We give the action of the standard generators σ_i on $\mathcal{H} \otimes 1$, let $\tau = (i, i+1)$.

$$\sigma_i(v_{j,k} \otimes 1) = \begin{cases} v_{j,k} \otimes \tau, & i \notin \{j-1, j, k, k-1, k\} \\ q_{i+1} v_{i,k} \otimes \tau + (q_j^2 - q_j) v_{i,j} \otimes \tau + (1 - q_j) v_{j,k} \otimes \tau, & i = j-1 \\ v_{j+1,k} \otimes \tau, & i = j \neq k-1 \\ q_{i+1} v_{j,i} \otimes \tau + (1 - q_{i+1}) v_{j,k} \otimes \tau + (q_{i+1}^2 - q_{i+1}) t v_{i,k} \otimes \tau, & i = k-1 \neq j \\ v_{j,k+1} \otimes \tau, & i = k \\ -t q_{i+1}^2 v_{j,k} \otimes \tau, & i = j = k-1 \end{cases}$$

Proof. We begin with recalling Proposition 1.2.27, and the Cayley complex K of the presentation $\langle \mathcal{G} | \mathcal{R} \rangle$, which is homotopy equivalent to C.

For $j = 1, \ldots, n$, we let x_j be the loop $\{\zeta_j, d_2\}$ of C and y be the loop $\{\tau_1 \tau_2\}$ of C. The set $\mathcal{G}$ was defined as follows:

$$\mathcal{G} = \{x_1, \ldots, x_n, y\}.$$

The set of relations was $\mathcal{R} = \{r_{j,k}$ for $1 \leq j \leq k \leq n\}$, with for $j \in \{1, \ldots, n\}$:

$$r_{j,j} = [x_j, y x_j y].$$

and for $1 \leq j < k \leq n$:

$$r_{j,k} = \left[x_j, y x_k y^{-1}\right].$$

Now we can compute $\mathcal{H}$ using the Fox derivatives (see Definition 1.2.4). We let C_1 and C_2 be the free $\mathcal{R}$-modules with basis $\{e_g : g \in \mathcal{G}\}$ and $\{f_r : r \in \mathcal{R}\}$ respectively. For any word in $\mathcal{G}$, we define $[w] \in C_1$ according to these rules:

$$\begin{array}{rcl} [1] & = & 0 \\ [gw] & = & e_g + \phi(g)\,[w] \\ [g^{-1}w] & = & \phi(g)^{-1}([w] - e_g) \end{array}$$

for $g \in \mathcal{G}$. Then $H_2(\widetilde{C})$ is the kernel of the map $\partial : C_2 \to C_1$ defined by $\partial f_r = [r]$.

The computation gives:

$$\partial f_r = \begin{cases} (q_j t + 1)((1-t)[x_j] + (q_j - 1)[y]) & \text{if } r = r_{j,j} \\ (1 - q_k)[x_j] + (1 - q_k)(q_j - 1)[y] + t(q_j - 1)[x_k] & \text{if } r = r_{j,k} \end{cases}$$

If we restrict the morphism to the space $Vect(f_{j,j}, f_{j,k}, f_{k,k})$, we get the matrix:

$$\begin{pmatrix} (1-t)(q_j t + 1) & (1 - q_k) & 0 \\ (q_j - 1)(q_j t + 1) & (1 - q_k)(q_j - 1) & (q_k - 1)(q_k t + 1) \\ 0 & t(q_j - 1) & (1-t)(q_k t + 1) \end{pmatrix}$$

which corresponds to the only non-vanishing blocks of the application ∂. Each block has a rank one kernel generated by the vector:

$$v_{j,k} = -(1 - q_k)(q_k t + 1) f_{j,j} + (1-t)(q_k t + 1)(q_j t + 1) f_{j,k} - t(q_j - 1)(q_j t + 1) f_{k,k}$$

so that we get a basis of $H_2(\widetilde{C})$, namely $\{v_{j,k} : 1 \leq j < k \leq n\}$.

Now a nice way to compute the matrices for the action of σ_i, is to find forks $F_{j,k}$ which correspond to the vector $v_{j,k}$, and to use the pairing with some noodles to get the expression of vectors in the fork basis. In what follows we still abusively use F to designate both the fork and the associated homology class of $\widehat{\Sigma}(F)$.

Let's fix d_1 and d_2 lying in the lower half plane of the boundary of D_n.

Definition 2.2.6 (Standard fork)**.** *For each $1 \leq j < k \leq n$, let $F_{j,k}$ be the fork that lies entirely in the lower half of D_n such that the endpoints of $T(F_{j,k})$ are the punctures p_j and p_k, we usually call it* a standard fork*.*

Remark 2.2.7. There exists $\lambda \in \mathcal{R}$ such that for all $j, k \in \{1, \dots, n\}$:

$$F_{j,k} = \lambda v_{j,k}$$

(in terms of homology classes). The proof of this fact is exactly the same as the one for the unicolored version, see [Big0] proof of Theorem 4.1. The latter is done remarking that it is sufficient to consider the homology module restricted to the disk containing $F_{j,k}$, its endpoints, and no other puncture.

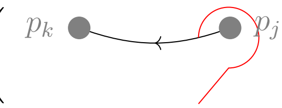
By Remark 2.2.7, we compute the braid action over standard forks. There are cases where $\sigma_i(F_{j,k})$ is directly a standard fork, namely:

- $i \notin \{j-1, j, k-1, k\}$
- $i = j \neq k - 1$

- $i = k$

In the case $i = j = k-1$, the fork $\sigma_i(F_{j,k})$ has the same tine edge as $F_{j,k}$ with opposite orientation:

$$\sigma_i(F_{j,k}) = \left(\quad \right)$$



where in red is represented the handle, and in black the tine. The handle joins the boundary in d_1 outside the parenthesis. It follows that it represents the same surface in C as $F_{j,k}$ with opposite orientation. Then the classes in $\widetilde{C}$ differ by a covering transformation. We get that $\sigma_i(v_{j,k}) = -tq_j^2 v_{j,k}$. The precise computation is made in Chapter 3, in Example 3.2.20, using the *handle rule* introduced in Remark 3.2.19 that deals with a change of handle. The remaining cases are $i = j-1$ and $i = k-1 \neq j$. The following claim restricts the linear combination, and is proved exactly the same way as Claim 4.2 of [Big0]:

Claim 2.2.8 ([Big0, Claim 4.2])**.** *$\sigma_i(v_{j,k})$ is a linear combination of $v_{j',k'}$ with $j', k' \in \{i, i+1, j, k\}$*

In the case $i = j-1$ for instance, this claim implies that there exists $A, B, C \in \mathcal{R}$ s.t. :

$$\sigma_i(F_{j,k}) = AF_{i,j} + BF_{j,k} + CF_{i,k}$$

To get A, B, C we pair with noodles. As it only depends on homological class of the surface associated to fork, by pairing some appropriate noodles with the studied forks in one hand and with the standard fork involved in its decomposition on the other, we are able to compute the coefficients of the linear combination. In Example 2.2.9, we perform this computation in one of the two remaining cases.

Example 2.2.9. Let F be the fork corresponding to the image of $F_{2,4}$ after applying the homeomorphism corresponding to the generator σ_1 of B_n. Considering the Claim 2.2.8 we can restrict ourselves to B_4 and the study of D_4 with only four punctures. This example is enough to deduce the general expression of the action of σ_i on the vector $v_{j,k}$ in the case $i = j-1$, which is one of the two remaining cases not entirely treated in the proof of Proposition 2.2.5.

First we use Claim 2.2.8 to deduce that the class in $H_2(\widetilde{C})$ associated to F has a linear decomposition in terms of standard forks $F_{1,2}, F_{1,4}$ and $F_{2,4}$. We use the following notations:

$$F = AF_{1,2} + BF_{1,4} + CF_{2,4}$$

where $A, B, C \in \mathcal{R}$ are the coefficients we are looking for. We compute A, B, C using the pairing 2.2.

Remark 2.2.10. In order to compute invariants of loops $\delta_{i,j}$'s (see subsection 2.2.2 and Definition 1.2.24), it is useful to draw both paths (ξ_1, ξ_2) composing it to see immediately the value of the invariants w_i but for the last invariant u the parametrization is crucial, so we need to think about the movie of the loop. We draw some in Figure 2.2.

Let N_i be the noodle starting at d_1 and passing once clockwise around the puncture p_i before coming back to d_2. We get the easy following computation of the pairing with standard forks:

Remark 2.2.11.

$$\langle N_i, F_{j,k}\rangle = \begin{cases} -q_i & \text{if } i = j \\ q_i^{-1}t^{-1} & \text{if } i = k \\ q_i^{-1}t^{-1} - t^{-1} + 1 + q_i & \text{if } j < i < k \\ 0 & \text{otherwise} \end{cases}$$

Similarly we compute:

$$\langle N_1, F\rangle = -q_1 q_2^2$$
$$\langle N_4, F\rangle = q_4^{-1}t^{-1}.$$

We detail the computation of the pairing of F with N_3 (one can realize that it involves exactly the same paths that for $\langle N_i, F_{j,k}\rangle$ above with $j < i < k$). The situation is depicted in Figure 2.1.

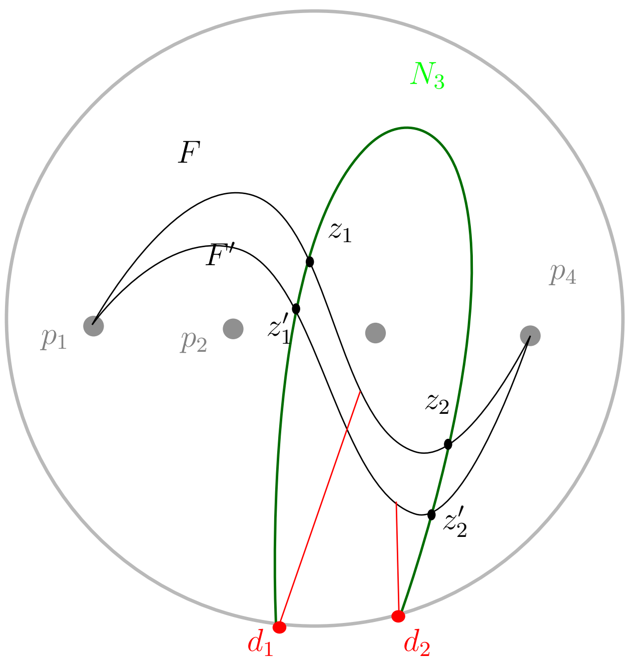


Figure 2.1 – Intersection of fork F with noodle N_3.

F and N_3 have two intersection points, the pairing involves four terms:

- for $\delta_{1,1}$ we get $m_{1,1} = q_3^{-1}t^{-1}$ so that $u_{1,1} = -1$ and that $\epsilon_{1,1} = 1$,
- for $\delta_{2,2}$ we get $m_{2,2} = q_3$ so that $u_{2,2} = 0$ and that $\epsilon_{1,1} = -1$,

- for $\delta_{1,2}$ we get $m_{1,2} = 1$ so that $u_{1,2} = 0$ and that $\epsilon_{1,2} = -(-1)^{u_{1,1}+u_{2,2}+u_{1,2}} = 1$,
- for $\delta_{1,2}$ we get $m_{2,1} = t^{-1}$ so that $u_{2,1} = -1$ and that $\epsilon_{2,1} = -(-1)^{u_{1,1}+u_{2,2}+u_{2,1}} = -1$.

Beside $\delta_{1,2}$ that is trivial, we draw $\delta_{1,1}$, $\delta_{2,2}$ and $\delta_{2,1}$ in Figure 2.2 from which above computations are immediate. Finally:

$$\langle N_3, F\rangle = q_3^{-1}t^{-1} - t^{-1} + 1 + q_3.$$

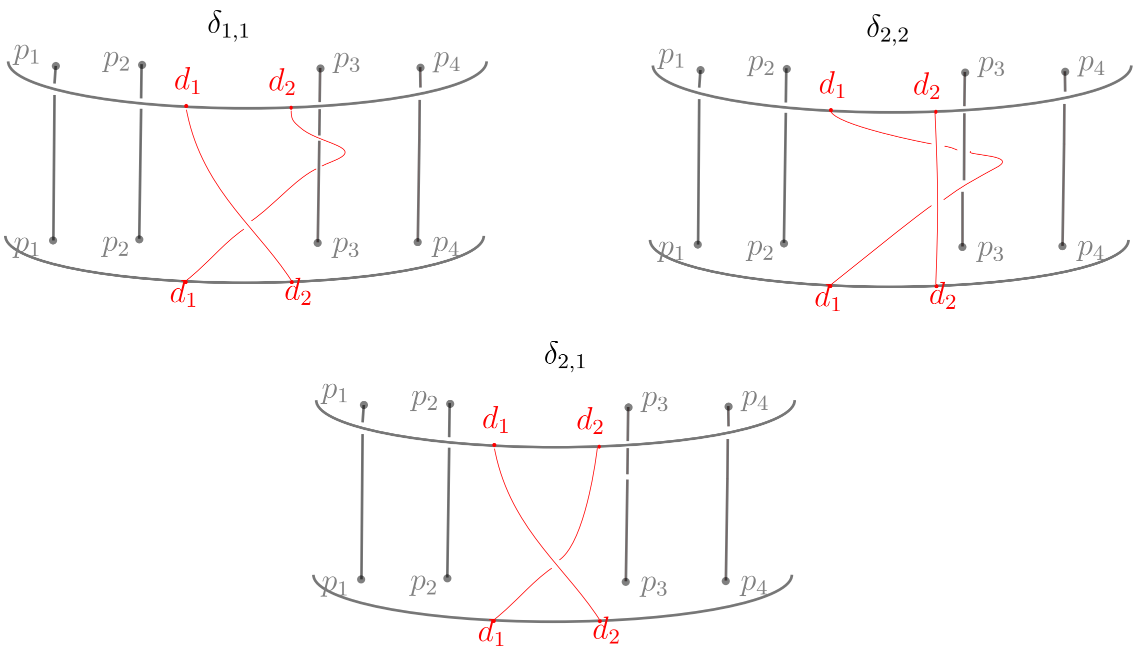


Figure 2.2 – $\delta_{1,1}, \delta_{22}$ and $\delta_{2,2}$

Replacing the computations above in the expression:

$$\langle N_i, F\rangle = A\langle N_i, F_{1,2}\rangle + B\langle N_i, F_{1,4}\rangle + C\langle N_i, F_{2,4}\rangle$$

with $i = 1$ we get the condition:

$$A + B = q_2^2$$

and with $i = 3$:

$$B + C = 1.$$

We need one more condition. We obtain it by pairing with the noodle $N_{2,3}$ defined as the noodle starting at d_1 and running around the punctures p_2 and p_3 before coming back to d_2 (see Figure 2.3, noodle oriented from left to right).

We get the pairings:

- $\langle N_{2,3}, F_{1,2}\rangle = (q_2q_3)^{-1}t^{-1}$,

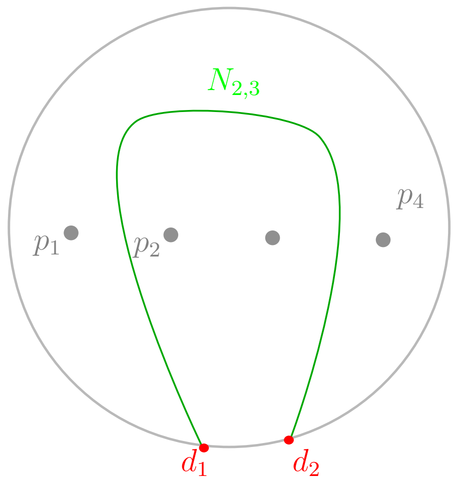


Figure 2.3 – Noodle $N_{2,3}$.

- $\langle N_{2,3}, F_{1,4}\rangle = (q_2q_3)^{-1}t^{-1} - t^{-1} + 1 - q_2q_3$,
- $\langle N_{2,3}, F_{2,4}\rangle = -q_2q_3$,
- $\langle N_{2,3}, F\rangle = q_2q_3^{-1}t^{-1} - q_2t^{-1} + q_2 - q_2q_3 = q_2(1-q_3)(q_3^{-1}t^{-1}+1)$.

By identification, we finally obtain:

$$A = q_2^2 - q_2 \ , \ B = q_2 \ , \ C = 1 - q_2.$$

□

Proposition 2.2.5 allows computation of matrices. The action described in the proposition is not multiplicative as the permutation induced by a braid shuffles the punctures and the corresponding variables in the action. In Chapter 3, Sections 3.2.2.3 and 3.2.2.4, we give homological tools that simplifies the computation of matrices and recovers the above proposition.

We end this section by a computational approach to these matrices. Let $BKL_i(q,t)$ be the matrix representing the action of σ_i in the (unicolored) Bigelow-Krammer-Lawrence representation written in the basis $\{v_{j,k}\}$ using the lexicographic order. See [Big0] Section 4 or Theorem 1.2.28. It has entries in $\mathbb{Z}[q^{\pm1}, t^{\pm1}]$.

Then it's a basic matrix computation that verifies the following remark.

Remark 2.2.12. Let $q_1, \ldots, q_n$ be variables. Then:

$$BKL_i(q_{i+1},t)BKL_j(q_{j+1},t) = BKL_j(q_{j+1},t)BKL_i(q_{i+1},t) \text{ for } |i-j| \geq 2$$

$$BKL_{i+1}(q_{i+1},t)BKL_i(q_{i+2},t)BKL_{i+1}(q_{i+2},t) = BKL_i(q_{i+2},t)BKL_{i+1}(q_{i+2},t)BKL_i(q_{i+1},t).$$

One can check this by a straightforward matrix computation.

Now we can define the colored BKL matrix associated to a braid.

Definition 2.2.13. *Let α be a braid having the following word decomposition in the standard generators: $\alpha = \prod_{m=1}^{k} \sigma_{i_m}^{s_m}$ where s_m are signs. Let j_m be the index of the " under " strand at the m'th crossing in α, braids read from right to left. Let the matrix $cBKL(\alpha)$ associated to the braid α be:*

$$cBKL(\alpha) := \prod_{m=1}^{k} BKL_{i_m}(q_{j_m+1}, t)^{s_m}. \tag{2.3}$$

Remark 2.2.12 shows that $cBKL$ is a well defined map between the braid group and the matrix group, but it is not multiplicative. For pure braids, $cBKL$ becomes a homomorphism and what we get is a representation of PB_n:

$$cBKL : \begin{array}{lll} PB_n & \to & GL_{\binom{n}{2}}\left(\mathbb{Z}\left[q_1^{\pm 1}, \ldots, q_n^{\pm 1}, t\right]\right) \\ \alpha & \mapsto & cBKL(\alpha). \end{array}$$

Remark 2.2.12 is a computational proof that this is a representation, i.e. that it satisfies braid relations. From Proposition 2.2.5 we remark that it is the colored BKL representation, corresponding to the initial homological definition (Proposition 2.2.5). This is remarking that the only variable involved in the action of σ_i in Proposition 2.2.5 corresponds to the underpassing strand. Specializing all q_i's to a single variable q recovers the unicolored BKL-representations.

Remark 2.2.14. In Section 1.2.2, we have presented a construction of the Gassner representation as a generalization of the Burau representation. Namely we used the standard Burau block of matrix but one has to use the variable t_i if the strand i is passing above, i.e. the coloring follows strands. Here the conclusion is the same: the colored BKL representation uses the BKL standard block but with formal variables following the index of the strands (it is clear in Formula 2.3).

2.2.4 Colored BKL in the general framework.

The colored BKL representations of the pure braid group are a colored generalization of the BKL representation involving $n+1$ formal variables instead of two. They are faithful ([Kra1], [Big0]) and known to be quantum representations of the braid group over sub-sub-maximal weight sub-modules from [K1, Theorem 3.1]. In Chapter 3 we will generalize these representations, recovering them as sub representations of larger homological representations. We will also recover the property that they are quantum representations. Namely, the colored BKL representation are sub representations of the homological action of the braid group over $\mathcal{H}_r^{\mathrm{rel}\,-}$ defined in Section 3.2.5, for $r = 2$. By Theorem 5, they are finite dimensional sub representations of the product of $U_q\mathfrak{sl}(2)$ Verma modules. We also give conditions for (generalized) forks to be a basis of the entire homological module in Corollary 3.2.29.

Although they are recovered by the homological representations of Chapter 3, in this section we had a slightly different approach. We built the local system using a Cayley complex homotopically equivalent to the corresponding covering, and we used Fox calculus to find basis vectors of the homology modules. It is well known that Gassner (and Burau) representations constitute the first stratum of the Enright representations (Remark 1.2.9), the first stratum of the Lawrence representation, and the first stratum of a family of quantum representation (Section 2.1.1). As the colored BKL representations are the second stratum of the quantum family and of Lawrence's representations, one would ask if the second stratum of the Enright representations contains the colored BKL representation.

Open Question. *Do Enright representations recover colored BKL representations?*

We will generalize this question at the end of Chapter 3, involving all strata of the three families: namely Enright, homological (Lawrence) and quantum representations. We mention this question here in the precise case of colored BKL as the Fox calculus approach may be more convenient to deal with Enright representations that are defined from Fox higher derivatives.

2.3 Non semi-simple representations of $M(0,4)$

In this section, we build the non semi-simple TQFT's representations of $M(0,4)$ (the mapping class group of the four times punctured sphere using a precise basis). Then we state how they contain the hyperelliptic representations of $M(0,4)$ presented in Example 1.1.6. This leads to the faithfulness of the representation.

2.3.1 Recalls on representation of $PSL(2,\mathbb{Z})$

We consider the three following presentations of groups:

$$G_1 = \left\langle a, b \mid aba = bab \ , \ (aba)^4 = 1 \right\rangle$$

$$G_2 = \left\langle s, t \mid s^2 = t^3 \ , \ t^4 = 1 \right\rangle$$

$$H = \left\langle s, t \mid s^2 = t^3 = 1 \right\rangle$$

and let f be the following morphism:

$$f : \begin{cases} G_1 & \rightarrow & SL(2,\mathbb{Z}) \\ a & \mapsto & A \\ b & \mapsto & B \end{cases}$$

with:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \ B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then f is a homomorphism.

Fact 2.3.1. • *The groups G_1 and G_2 are isomorphic, up to the following inverse subsitutions:*

$$s = ab \ , \ t = aba \text{ and } a = s^{-1}t \ , \ b = t^{-1}s^2$$

we call it G from now on. Moreover the group G is isomorphic to the quotient of the braid group $\mathcal{B}_3$ by the central subgroup generated by $(\sigma_1\sigma_2\sigma_1)^4$.

- *The group H is isomorphic to the quotient of G by the group generated by $s^3 = t^2$, so that it is isomorphic to the quotient of $\mathcal{B}_3$ by its central subgroup $Z(\mathcal{B}_3)$.*

We introduce here the following matrices, in order to relate the different presentations to the matrix representation f:

$$S = f(s) = AB = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \ , T = f(t) = ABA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

We get $T^2 = -I_2$, so that f provides a morphism $\bar{f} : H \to PSL(2, \mathbb{Z})$.

Proposition 2.3.2. *The morphisms $f : G \to SL(2, \mathbb{Z})$ and $\bar{f} = H \to PSL(2, \mathbb{Z})$ are isomorphisms.*

2.3.2 Recalls on non semi-simple TQFT's

In [BCGP2], the authors construct a TQFT from the non semi-simple category $\mathscr{C}$ of $\overline{U}_q^H\mathfrak{sl}(2)$ weight modules. We present here a non exhaustive summary of the construction, recalling first what is a TQFT and what information is included in these theories. This part is here to give ideas before fixing notations for the precise case of interest (which will be done in next section).

Definition 2.3.3 (Category of cobordisms)**.** *An oriented $(n+1)$-manifold M with boundary decomposed as $\partial M = -\Sigma_1 \bigsqcup \Sigma_2$, where Σ_1, Σ_2 are oriented n-manifolds, and $-\Sigma_1$ means Σ_1 with reversed orientation, is called a cobordism from Σ_1 to Σ_2. Given a cobordism M_1, from Σ_1 to Σ, and a cobordism M_2, from Σ to Σ_2, one can glue these together along Σ to obtain a cobordism from Σ_1 to Σ_2. Let the category $\mathcal{C}ob_{n+1}$ be the one whose objects are the oriented n-manifolds, whose morphisms are equivalence classes of cobordisms, and where gluing plays the role of composition. Two cobordisms from Σ_1 to Σ_2 are called equivalent if they are isomorphic rel. boundary (i.e. the isomorphism is required to be the identity on Σ_1 and Σ_2). Taking equivalence classes ensures that composition is associative, and the product manifold $[0, 1] \times \Sigma$ plays the role of the identity morphism of Σ. Observe that this category has an involution (given by orientation reversal) and a monoidality structure (given by disjoint union).*

Remark 2.3.4. This is the basic definition of the category of cobordisms. We usually restrict it to compact surfaces and allow richer cobordism. *"Extra decorations"* of cobordisms can be of the following types: cobordism containing a banded link, decorated points in the surface, or cohomology class associated with objects. Then an appropriate generalization of the notion of isomorphism of cobordism is required.

Definition 2.3.5 ((2 + 1)-TQFT)**.** *Let* $\mathcal{C}ob_{2+1}$ *be the category of* 3*-dimensional cobordisms, and* k *be a commutative ring. A* (2 + 1)-TQFT *is a functor:*

$$V : \mathcal{C}ob_{2+1} \to k - modules$$

satisfying:

(Monoidality) $V(\Sigma_1 \bigsqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$.

(Duality) $V(-\Sigma_1) = V(\Sigma_1)^*$ *where* $*$ *stands for the dual module.*

(Unit) $V(\emptyset) = k$.

Again this is the initial definition of TQFT, while extra decorated cobordisms need an adapted definition of a TQFT functor.

Remark 2.3.6 (Mapping class group representations)**.** Let Σ be a surface, Φ a diffeomorphism of it and V a $(2+1)$-TQFT. The mapping cylinder of Φ is the following manifold:

$$I_\Phi = (\Sigma \times [0,1])$$

with $(x,1) \sim x$ and $(x,0) \sim \Phi(x)$. The manifold I_Φ is a cobordism between Σ and itself, so that $V(I_\Phi)$ is an endomorphism of $V(\Sigma)$. The functoriality of V together with the notion of isomorphism of cobordisms imply that:

$$\begin{array}{rcl} Mod(\Sigma) & \to & \mathrm{End}(V(\Sigma)) \\ \Phi & \mapsto & V(\Phi) := V(I_\Phi) \end{array}$$

is a representation of $Mod(\Sigma)$ over $V(\Sigma)$. This remark shows that a TQFT functor provides a representation of the mapping class group for every surfaces.

From certain categories of quantum groups modules, the Reshetikhin – Turaev functor $\mathcal{RT}$ (see Definition 1.3.12) provides invariants of links. The following is well known.

Theorem 2.3.7 (Lickorish – Wallace and Kirby Theorem)**.** *Any closed, orientable, connected* 3*-manifold may be obtained by performing* Dehn surgery *on a framed link in the* 3*-sphere with* ± 1 *surgery coefficients. Two framed links give the same manifold if and only if they are related by a series of* Kirby moves*.*

We don't give the definitions of Dehn surgery nor Kirby moves, see [BHMV] for instance. We want to emphasize that the $\mathcal{RT}$ functor gives a quantum invariant of framed link that has been generalized in [RT2] to 3-manifold invariants, applying the above theorem. We call such invariants of manifolds *quantum invariants* by extension. In [RT2] the initial category of quantum groups modules is semi-simple.

In [BHMV], the authors present a *universal construction* of TQFT. Namely they suggest a technique to get a TQFT from a family of quantum invariants of three manifolds, using a natural pairing. The technique works with the Reshetikhin-Turaev invariants from [RT2] and gave rise to the semi-simple Reshetikhin-Turaev TQFT's. In [CGP2] the authors succeed in constructing a quantum invariant from a non semi-simple category of quantum groups modules. An adaptation of the universal construction is performed in [BCGP2] and provides a *non semi-simple TQFT*.

Theorem 2.3.8 ([BCGP2, Theorem 1.1]). *There exists a monoidal functor $\mathbb{V} : \mathcal{C}ob \to \mathcal{G}r\mathcal{V}ect$ from the category of decorated surfaces and decorated cobordisms to the category of finite dimensional $\mathbb{Z}$-graded vector spaces. This functor is built from the non semi-simple category $\mathscr{C}$ of $\overline{U}_q^H\mathfrak{sl}(2)$ weight modules. The category $\mathscr{C}$ is used to decorate the cobordisms, see the definition of decorations in [BCGP2, Subsection 3.3].*

Moreover the mapping class group representations (Remark 2.3.6) preserve the grading.

The non semi-simplicity of $\mathscr{C}$ implies richer topological information than in the case of the classical Reshetikhin-Turaev TQFTs (semi simple). For instance, the following theorem about mapping class group representations is a strong improvement compared to the original Reshetikhin-Turaev TQFT's.

Theorem 2.3.9 ([BCGP2, Theorem 1.3]). *The action of a Dehn twist along a non-separating curve of a surface Σ has infinite order on $\mathbb{V}(\Sigma)$.*

The latter suggests that the new family of representations of mapping class groups provided by the non semi-simple TQFTs is richer and one would be interested in the question of their faithfulness. We investigate a precise case in the following section. We end these recalls by a far from exhaustive presentation of the universal construction providing non semi-simple TQFTs from $\mathcal{RT}$-functor.

Remark 2.3.10 (Sketch of non semi simple TQFT construction). We present loosely the universal construction introduced in [BHMV] and performed in a more sophisticated way in [BCGP2] giving rise to non semi-simple TQFTs.

Let Σ be a surface, $V_1(\Sigma)$ be the complex vector space generated by all cobordisms between $\emptyset$ and Σ and $\widehat{V_1}$ the one generated by all cobordisms between Σ and $\emptyset$. There is a pairing:

$$\begin{array}{rcl} V_1(\Sigma) \times \widehat{V_1}(\Sigma) & \to & \mathbb{C} \\ (C, C') & \mapsto & N_r^0(C \sharp_\Sigma C') \end{array}$$

where $C\sharp_\Sigma C'$ is the closed 3-dim manifold obtained by gluing C and C' along their common boundary, and N_r^0 is the quantum invariant of closed manifold constructed in [CGP2]. By making the quotient of V_1 by the kernel of this pairing, one obtains the TQFT module associated to Σ. To deal with details of the pairing and of the quotient in the case of decorated cobordisms, one should follow [BCGP2]. In what follows we will refer to [BCGP2] notations to fit with the $\mathscr{C}$-decorated formalism.

2.3.3 TQFT-representations of $M(0,4)$.

We follow [BCGP2] to give a basis of the vector space associated via the non semi-simple TQFT functor to the sphere with 4 punctures. The definitions of typical module of $\overline{U}_q^H\mathfrak{sl}(2)$ can be found in Section 1.3.2 while tools as Clebsch-Gordan quantum coefficients and 6j-symbols are taken from [C-M, CGP2]. From now on, we let S_4 be the sphere containing four marked points p_1, p_2, p_3, p_4. In order to compute the TQFT, we shall decorate the punctures using $\overline{U}_q^H\mathfrak{sl}(2)$

simple modules parametrized by complex numbers. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be complex parameters in $(\mathbb{C}\backslash\mathbb{Z}) \bigcup r\mathbb{Z}$, considering q to be a root of unity such that $q^{2r} = 1$. We recall from Section 1.3.2 that $\mathscr{C}$ designates the category of $\overline{U}_q^H \mathfrak{sl}(2)$ weight modules. Let $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be S_4 decorated by modules $V_{\lambda_i} \in \mathscr{C}$ associated to each p_i, with $i = 1, 2, 3, 4$ (to fit with the decorated formalism [BCGP2, Subsection 3.3]). See Section 1.3.2 for the definition of the V_{λ_i}'s. If $V = ((V_{\lambda_1}, +), (V_{\lambda_2}, +), (V_{\lambda_3}, +), (V_{\lambda_4}, +))$, then $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ refers to the sphere with four punctures decorated by V, namely S^2_V using notations from Section 6.1 of [BCGP2]. We only give ideas of the construction.

Proposition 2.3.11 ([BCGP2, Proposition 6.1]). *Let $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the 0-graded vector space associated to $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ via the non semi-simple TQFT functor $\mathbb{V}$ (from Theorem 2.3.8). Algebraically, the space $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is isomorphic to $\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, F(V))$ where $F(V) = V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \otimes V_{\lambda_4}$ is a module of $\mathscr{C}$, and $\mathbb{I}$ the identity object of $\mathscr{C}$, namely the one dimensional $\overline{U}_q^H \mathfrak{sl}(2)$-module.*

Idea of the proof. The idea of this proposition is that the space $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ provided by the universal construction is generated by cobordisms between the empty set and $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ (Remark 2.3.10). The latter correspond to $\mathscr{C}$-decorated ribbon tangles embedded inside the 3-dimensional ball having $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as boundary (tangles ending at punctures). Then the $\mathcal{RT}$-functor fully interprets these ribbon tangles as elements of $\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, F(V))$. □

To transform a vector v of $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ under the action of an element of the mapping class group one must just glue the corresponding mapping cylinder to the $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ at the extremity of the vector v (interpreted as a cobordism) so to get a new cobordism between the empty set and $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ defined to be the image of v under the mapping class action. The latter corresponds to the gluing of a ribbon sphere braid to the ribbon tangle corresponding to v. The ribbon aspect of the theory forces one to work with arrows instead of punctures $p_1, \ldots, p_4$, as extremities of ribbons are arrows. Then one must consider mapping classes fixing the arrows at the punctures, which correspond to mapping classes of the sphere with boundary components instead of punctures, but one can verify the following remark allowing us to deal with the whole mapping classes of the punctured sphere.

Remark 2.3.12. A simple full Dehn twist around one puncture colored by a $\overline{U}_q^H \mathfrak{sl}(2)$ simple module gives a full twist to the arrow. In terms of morphism of the category $\mathscr{C}$, this corresponds (through the $\mathcal{RT}$-functor) to a morphism from a simple module to itself. By Schur's Lemma, such morphism is diagonal.

Hence, a solution to avoid a restriction of the mapping class group is to consider the projective representations over the TQFT space $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ associated to $S_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, keeping simple punctures and forgetting the arrows at punctures. We will stick to this from now on and until the end of this section. In this framework of projective representations, it is not necessary to consider ribbons anymore, we simply consider $\mathscr{C}$-colored tangles. We introduce $\mathscr{C}$-decorated trivalent graphs that will determine a basis of the TQFT later on.

Definition 2.3.13 ($\mathcal{H}$ and $\mathcal{I}$ graphs)**.** *Let $\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)$ be the decorated graph on the left of Figure 2.4 embedded into the 3-dimensional ball having the punctures p_i's as endpoints in S_4 - the boundary of the ball. In this graph, β is another complex parameter. A decoration $\lambda \in \mathbb{C}$ refers to the module V_λ. In Figure 2.4 the graph $\mathcal{I}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \gamma)$ is also represented on the right, which corresponds to another vector of $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ used below. In what follows, we use a graph $\mathcal{G}$ to refer to its image $\mathcal{V}(\mathcal{G}) \in \mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, F(V))$ if no confusion arises in equations.*

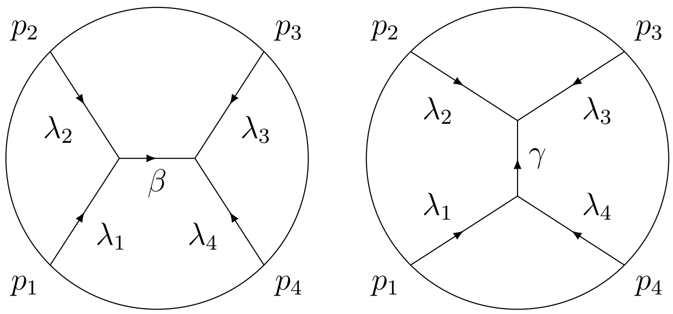


Figure 2.4 – Graphs $\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)$ and $\mathcal{I}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \gamma)$

Remark 2.3.14. As we only consider simple-module coloring of punctures, we can use the Clebsch-Gordan decomposition of tensor products of simple modules (see [C-M, Section 1.3]) to establish a correspondence between $\mathscr{C}$-tangles and admissible trivalent graphs embedded in the ball, colored with elements of $\mathscr{C}$ and ending at punctures. The word *admissible* refers here to the fact that the trivalent graphs must satisfy a relation at each node provided by the Clebsch-Gordan formula. Indeed, let V_a and V_b be two typical modules of middle weights a and $b \in (\mathbb{C} \setminus \mathbb{Z}) \bigcup r\mathbb{Z}$. For $a, b \in \mathbb{C}$ generic, it holds: $V_a \otimes V_b = \bigoplus_{a+b-c \in H_r} V_c$ with $H_r = \{r-1, r-3, \ldots, -r+1\}$, and and that any $\overline{U}_q^H \mathfrak{sl}(2)$ module map $V_c \to V_a \otimes V_b$ is a scalar multiple of the inclusion map of V_c into $V_a \otimes V_b$ given in Theorem 1.7 of [C-M].

From Proposition 2.3.11 and from the construction of the TQFT functor $\mathbb{V}$ from [BCGP2] presented in Theorem 2.3.8, one can check the following fact:

Fact 2.3.15 ($\mathcal{H}$ graphs basis)**.** *Let $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the* 0*-graded TQFT space associated to $S_4(\lambda_1, \ldots, \lambda_4)$ by functor $\mathbb{V}$ (Theorem 2.3.8). Then $\mathcal{V}$ is isomorphic to the vector space generated by all $\mathscr{C}$-decorated trivalent graphs inside the ball having ends at punctures, modulo the whole set of Relations (N a–j) of [CGP2, Section 2.2]. Moreover a basis of $\mathcal{V}$ is given by the set of all graphs $\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)$ satisfying the node condition (or* admissible condition*), namely that the sum of parameters arriving to each vertex must be in H_r.*

Idea of the proof. The first step would be to interpret $\mathscr{C}$-decorated tangles embedded in the ball (decorated cobordisms from the empty set to the sphere) as $\mathscr{C}$-decorated trivalent graphs. This fact is an inherent tool in the quantum-module category $\mathscr{C}$ and is a classical property of the non semi-simple $\mathcal{RT}$-functor. It works the same for quantum invariants of manifold from [CGP2]. Namely, from the decomposition of tensor products of simple objects given by the Clebsch-Gordan formula (Remark 2.3.14), one obtains the formula of Proposition 2.3.17 between graphs stated below. This gives a hint to pass from tangles to trivalent graphs. Once this step is done, one has to show that the family of $\mathcal{H}$-graphs yields a basis of this space of graphs.

One can check that Relations (N a–j) from [CGP2, Section 2.2] ensure that this family of graphs generates $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ (i.e. that any admissible trivalent graph can be expressed as a linear combination of the $\mathcal{H}$ graphs using these relations).

There is a little more work to get that the family is linearly independent. For instance, in the proof of [BCGP2, Proposition 6.1], the pairing:

$$\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, F(V)) \times \mathrm{Hom}_{\mathscr{C}}(F(V), \mathbb{I}) \to \mathbb{C}$$

is shown to be non-degenerate (as $F(V)$ is a projective $U_q\mathfrak{sl}(2)$-module). The pairing corresponds - in terms of cobordisms - to the one (schematically) presented in Remark 2.3.10. One can compute it using $\mathcal{H}$ graphs and their dual graphs corresponding to elements of $\mathrm{Hom}_{\mathscr{C}}(F(V), \mathbb{I})$ and deduce the linear independence of these families.

This type of proof is performed in [BCGP2, Section 6.3] to give a basis for the TQFT of empty surfaces. □

Remark 2.3.16 ($\mathcal{I}$ graphs basis)**.** The graphs $\mathcal{I}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \gamma_\pm)$ correspond to another basis of $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, with the admissible values for γ (see Remark 2.3.20 below for instance).

In our case, the node conditions ("admissible conditions") are the following ones:

$$\lambda_1 + \lambda_2 - \beta \in H_r \tag{2.4}$$

$$\lambda_3 + \lambda_4 + \beta \in H_r, \tag{2.5}$$

so a basis of $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is given by the set $\{\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)\}$ with all possible parameters β such that Conditions 2.4 and 2.5 are satisfied.

Notations (0-graded, level 2 TQFT)**.** We suppose from now on that $r = 2$, then $H_r = \{-1, +1\}$. Suppose also that $\lambda_4 = -(\lambda_1 + \lambda_2 + \lambda_3)$, then we are left with three free parameters, namely $\lambda_1, \lambda_2, \lambda_3$. This set-up corresponds to the 0-graded TQFT in the case $r = 2$ (often referred to as the "level 2" non semi-simple TQFT), we denote the corresponding 0-graded space $\mathcal{V}(\lambda_1, \ldots, \lambda_4)$.

There are two possible graphs given by the two possible values for β. Let:

$$\beta_+ = \lambda_1 + \lambda_2 + 1$$

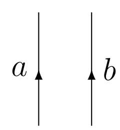
$$\beta_- = \lambda_1 + \lambda_2 - 1.$$

We use the notations $\mathcal{H}_+(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\mathcal{H}_-(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ to refer to the graphs $\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)$ with $\beta = \beta_+ = \lambda_1 + \lambda_2 + 1$ and $\beta = \beta_- = \lambda_1 + \lambda_2 - 1$ respectively;

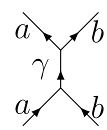
and $\mathcal{I}_+(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\mathcal{I}_-(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for the graphs $\mathcal{I}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \gamma)$ with $\gamma = \gamma_+ = \lambda_1 + \lambda_4 + 1$ and $\gamma = \gamma_- = \lambda_1 + \lambda_4 - 1$ respectively. We could have removed λ_4 from the arguments, as λ_4 is fixed, depending on the other arguments. This fact remains true even when we permute punctures, so that it permutes colors of the graphs, but the last argument will always be the opposite of the sum of the others, as $\lambda_i = -\sum_{j \neq i} \lambda_j$ for $i = 1, 2, 3, 4$.

From Relations (N a–j) [CGP2, Section 2.2] and mentioned to define $\mathcal{V}$, we will need three of them to build the representation that we recall in the three following propositions.

Proposition 2.3.17 ([CGP2, Equation (N i)])**.** *The following equality holds in* $\mathcal{V}(\lambda_1, \ldots, \lambda_4)$*:*



$$= \sum_{\gamma \in a+b+H_r} \mathsf{d}(\gamma)$$



where graphs are considered to be the same outside the part of the picture drawn in the small ball considered here.

Proposition 2.3.18 ([CGP2, Equation (N j)])**.** *There is the change of basis formula (between* $\mathcal{H}$ *and* $\mathcal{I}$*) that is obtained using what we call* $6j$*-symbols as follows:*

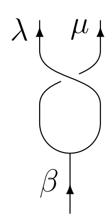
$$\mathcal{H}_\pm(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \sum_{\epsilon = \pm 1} \mathsf{d}(\lambda_1 + \lambda_4 + \epsilon) \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_1 + \lambda_2 \pm 1 \\ \lambda_3 & -\lambda_4 & \lambda_1 + \lambda_4 + \epsilon \end{vmatrix} \mathcal{I}_\epsilon(\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

where:

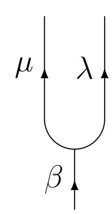
$$\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} = (-1)^{r-1+B_{165}} \frac{\{B_{345}\}!\,\{B_{123}\}!}{\{B_{246}\}!\,\{B_{165}\}!} \begin{bmatrix} j_3 + r - 1 \\ A_{123} + 1 - r \end{bmatrix} \begin{bmatrix} j_3 + r - 1 \\ B_{354} \end{bmatrix}^{-1} \times$$
$$\times \sum_{z=m}^{M} (-1)^z \begin{bmatrix} A_{165} + 1 \\ j_5 + z + r \end{bmatrix} \begin{bmatrix} B_{156} + z \\ B_{156} \end{bmatrix} \begin{bmatrix} B_{264} + B_{345} - z \\ B_{264} \end{bmatrix} \begin{bmatrix} B_{453} + z \\ B_{462} \end{bmatrix}$$

where $A_{xyz} = \frac{j_x + j_y + j_z + 3(r-1)}{2}$, $B_{xyz} = \frac{j_x + j_y - j_z + r - 1}{2}$, $m = \max(0, \frac{\mathrm{j_3 + j_6 - j_2 - j_5}}{2})$ *and* $M = \min(\mathrm{B_{435}, B_{165}})$.

Proposition 2.3.19 ([CGP2, Equation (N g)])**.** *The following equality holds in* $\mathcal{V}(\lambda_1, \ldots, \lambda_4)$*:*

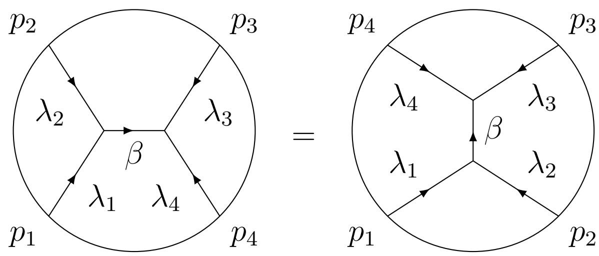


$$= q^{\frac{-\lambda^2 - \mu^2 + \beta^2 + (r-1)^2}{4}} \times$$



where we suppose that the graphs are the same everywhere outside the small ball drawn here.

Remark 2.3.20. We have the following equality, given by an obvious symmetry:



Using the notations introduced above, we get: $\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta) = \mathcal{I}(\lambda_1, \lambda_4, \lambda_3, \lambda_4, \beta)$.

The last three properties allow us to compute the action in the $\mathcal{H}$ basis.

Proposition 2.3.21. *The action of standard generators $\sigma_1, \sigma_2, \sigma_3$ of $M(0,4)$ (Example 1.1.6) over $\mathcal{V}(\lambda_1, \dots, \lambda_4)$ in the $\mathcal{H}$-basis are given by the following formulas:*

$$\sigma_1(\mathcal{H}_\pm(\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = q^{\frac{-\lambda_1^2-\lambda_2^2+\beta_\pm^2+(r-1)}{4}} \mathcal{H}_\pm(\lambda_2, \lambda_1, \lambda_3, \lambda_4),$$

$$\sigma_3(\mathcal{H}_\pm(\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = q^{\frac{-\lambda_3^2-\lambda_4^2+\beta_\pm^2+(r-1)}{4}} \mathcal{H}_\pm(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

and:

$$\sigma_2(\mathcal{H}_\pm(\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = f^\pm_{2,+}(\lambda_1, \lambda_3, \lambda_2, \lambda_4)\mathcal{H}_+(\lambda_1, \lambda_3, \lambda_2, \lambda_4) + f^\pm_{2,-}(\lambda_1, \lambda_3, \lambda_2, \lambda_4)\mathcal{H}_-(\lambda_1, \lambda_3, \lambda_2, \lambda_4)$$

where:

$$f^\pm_{2,+}(\lambda_1, \lambda_3, \lambda_2, \lambda_4) = \left(\sum_{\substack{\gamma=\lambda_1+\\ \lambda_4\pm 1}} \mathsf{d}(b_+)\mathsf{d}(\gamma) q^{\frac{-\lambda_2^2-\lambda_3^2+\gamma^2+(r-1)}{4}} \begin{vmatrix} \lambda_1 & \lambda_4 & \gamma \\ \lambda_2 & -\lambda_3 & b_+ \end{vmatrix} \begin{vmatrix} \lambda_1 & \lambda_2 & \beta_\pm \\ \lambda_3 & -\lambda_4 & \gamma \end{vmatrix} \right)$$

$$f^\pm_{2,-}(\lambda_1, \lambda_3, \lambda_2, \lambda_4) = \left(\sum_{\substack{\gamma=\lambda_1+\\ \lambda_4\pm 1}} \mathsf{d}(b_-)\mathsf{d}(\gamma) q^{\frac{-\lambda_2^2-\lambda_3^2+\gamma^2+(r-1)}{4}} \begin{vmatrix} \lambda_1 & \lambda_4 & \gamma \\ \lambda_2 & -\lambda_3 & b_- \end{vmatrix} \begin{vmatrix} \lambda_1 & \lambda_2 & \beta_\pm \\ \lambda_3 & -\lambda_4 & \gamma \end{vmatrix} \right)$$

Proof. The idea to get the images of half-Dehn twists by the TQFT is to apply the twist to the graphs $\mathcal{H}_\pm(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ corresponding to basis vectors.

Remark 2.3.22. Suppose that the twist involves a permutation τ of the punctures. Let $\tau(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be $(\lambda_{\tau(1)}, \dots, \lambda_{\tau(4)})$ for $\tau \in \mathfrak{S}_4$. We want to express the obtained graph in terms of the vectors $\mathcal{H}_\pm(\tau(\lambda_1, \lambda_3, \lambda_2, \lambda_4))$ which yield a basis of the TQFT space $\mathcal{V}(\tau(\lambda_1, \lambda_2, \lambda_3, \lambda_4))$ associated to the punctured sphere with permuted punctures.

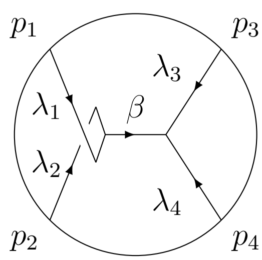
This is done using the rules presented above. We use notations of Proposition 1.1.8 for the generators of $M(0,4)$. As σ_1 refers to the half-Dehn twist along $[p_1, p_2]$, we get that:

$$\sigma_1(\mathcal{H}_{\pm}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = q^{\frac{-\lambda_1^2 - \lambda_2^2 + \beta_{\pm}^2 + (r-1)}{4}} \mathcal{H}_{\pm}(\lambda_2, \lambda_1, \lambda_3, \lambda_4)$$

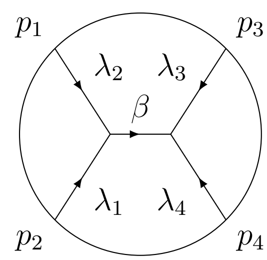
with $\beta_{\pm}$ defined above.

In terms of graphs, the latter is illustrated below.

$\sigma_1(\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta)) =$



$= q^{\frac{-\lambda_1^2 - \lambda_2^2 + \beta^2 + (r-1)^2}{4}}$



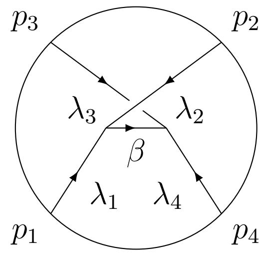
which is straightforward from Proposition 2.3.19.

The same works for σ_4 so that one obtains:

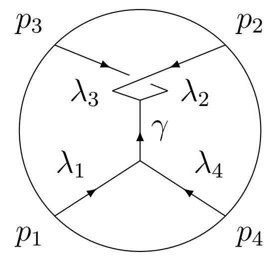
$$\sigma_4(\mathcal{H}_{\pm}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = q^{\frac{-\lambda_3^2 - \lambda_4^2 + \beta_{\pm}^2 + (r-1)}{4}} \mathcal{H}_{\pm}(\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

To compute σ_2 which corresponds to the half Dehn-twist along $[p_2, p_3]$, there is a little more work. The shortest way to express $\sigma_2(\mathcal{H}_{\pm}(\lambda_1, \lambda_2, \lambda_3, \lambda_4))$ in terms of graphs $\mathcal{H}_{\pm}(\lambda_1, \lambda_3, \lambda_2, \lambda_4)$ is to pass through the $\mathcal{I}$ graphs as follows:

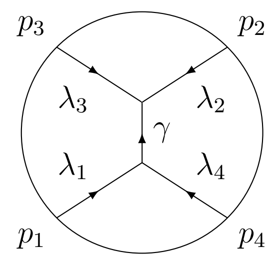
$$\sigma_2(\mathcal{H}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\beta)) =$$



$$= \sum_{\gamma=\lambda_1+\lambda_4\pm 1} \mathsf{d}(\gamma) \begin{vmatrix} \lambda_1 & \lambda_2 & \beta \\ \lambda_3 & -\lambda_4 & \gamma \end{vmatrix} \times$$

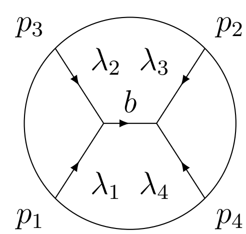


$$= \sum_{\gamma=\lambda_1+\lambda_4\pm 1} q^{\frac{-\lambda_2^2-\lambda_3^2+\gamma^2+(r-1)}{4}} \mathsf{d}(\gamma) \begin{vmatrix} \lambda_1 & \lambda_2 & \beta \\ \lambda_3 & -\lambda_4 & \gamma \end{vmatrix} \times$$



The second equality comes from Proposition 2.3.17 while the last one from Proposition 2.3.19. The last graph must be expressed then back in terms of $\mathcal{H}_\pm(\lambda_1,\lambda_3,\lambda_2,\lambda_4)$ using Proposition 2.3.18. Finally, we get the following expression for $\sigma_2(\mathcal{H}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\beta))$:

$$\sum_{\gamma=\lambda_1+\lambda_4\pm 1} q^{\frac{-\lambda_2^2-\lambda_3^2+\gamma^2+(r-1)}{4}} \mathsf{d}(\gamma) \begin{vmatrix} \lambda_1 & \lambda_2 & \beta \\ \lambda_3 & -\lambda_4 & \gamma \end{vmatrix} \left(\sum_{\substack{b=\lambda_1 \\ +\lambda_3\pm 1}} \mathsf{d}(b) \begin{vmatrix} \lambda_1 & \lambda_4 & \gamma \\ \lambda_2 & -\lambda_3 & b \end{vmatrix} \right.$$



$$\left. \vphantom{\sum} \right)$$

We reorganize terms in order to get a more readable formula for the image of both vectors $\mathcal{H}_+$ and $\mathcal{H}_-$ expressed in the basis we were looking for:

$$\begin{aligned}
&\sigma_2(\mathcal{H}_\pm(\lambda_1,\lambda_2,\lambda_3,\lambda_4)) = \\
&= \left(\sum_{\substack{\gamma=\lambda_1+ \\ \lambda_4\pm 1}} \mathsf{d}(b_+)\mathsf{d}(\gamma) q^{\frac{-\lambda_2^2-\lambda_3^2+\gamma^2+(r-1)}{4}} \begin{vmatrix} \lambda_1 & \lambda_4 & \gamma \\ \lambda_2 & -\lambda_3 & b_+ \end{vmatrix} \begin{vmatrix} \lambda_1 & \lambda_2 & \beta_\pm \\ \lambda_3 & -\lambda_4 & \gamma \end{vmatrix} \right) \mathcal{H}_+(\lambda_1,\lambda_3,\lambda_2,\lambda_4) \\
&+ \left(\sum_{\substack{\gamma=\lambda_1+ \\ \lambda_4\pm 1}} \mathsf{d}(b_-)\mathsf{d}(\gamma) q^{\frac{-\lambda_2^2-\lambda_3^2+\gamma^2+(r-1)}{4}} \begin{vmatrix} \lambda_1 & \lambda_4 & \gamma \\ \lambda_2 & -\lambda_3 & b_- \end{vmatrix} \begin{vmatrix} \lambda_1 & \lambda_2 & \beta_\pm \\ \lambda_3 & -\lambda_4 & \gamma \end{vmatrix} \right) \mathcal{H}_-(\lambda_1,\lambda_3,\lambda_2,\lambda_4) \\
&= f_{2,+}^{\pm}(\lambda_1,\lambda_3,\lambda_2,\lambda_4)\mathcal{H}_+(\lambda_1,\lambda_3,\lambda_2,\lambda_4) + f_{2,-}^{\pm}(\lambda_1,\lambda_3,\lambda_2,\lambda_4)\mathcal{H}_-(\lambda_1,\lambda_3,\lambda_2,\lambda_4).
\end{aligned}$$

□

Using these descriptions of the action of σ_i, for $i = 1, 2, 3$, over $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ we associate to it an operator in $PGL\left(\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \mathcal{V}\left(\tau(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\right)\right)$ with $\tau = \mathrm{perm}(\sigma_i) = (i, i+1) \in \mathfrak{S}_4$ permuting variables the way σ_i permutes punctures. As we are not dealing with endomorphisms, we don't have a representation of the mapping class group, but a projective one over $\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \otimes \mathbb{C}[\mathfrak{S}_4]$. The latter is the induced representation of the pure mapping class group (consisting in the Torelli group made of mapping classes not permuting punctures), and uses a basis $\{\mathcal{H}_\pm(\tau(\lambda_1, \lambda_2, \lambda_3, \lambda_4)), \tau \in \mathfrak{S}_4\}$ consisting in graphs $\mathcal{H}_\pm$ with colors permuted by permutations of $\mathfrak{S}_4$. This definition of the induced representation is in the spirit of Definition 1.2.2 for braids.

Definition 2.3.23. *We define the following representation:*

$$\Phi : \begin{cases} M(0,4) & \to & PGL(\mathcal{V}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \otimes \mathbb{C}[\mathfrak{S}_4]) \\ \sigma_i & \mapsto & \Phi(\sigma_i) \end{cases}$$

where:

$$\begin{aligned}\Phi(\sigma_i) : \mathcal{H}_\pm(\tau(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto & f^\pm_{i,+}(\tau(\lambda_1, \lambda_3, \lambda_2, \lambda_4))\mathcal{H}_+\left((i, i+1) \circ \tau\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right)\right) \\ & + f^\pm_{i,-}(\tau(\lambda_1, \lambda_3, \lambda_2, \lambda_4))\mathcal{H}_-\left((i, i+1) \circ \tau\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4\right)\right).\end{aligned}$$

Remark 2.3.24 (Normalization)**.** As we are considering a projective action, we are going to normalize the representation canceling some factors. We will simplify quadratic terms interpreted as framing information by the $\mathcal{RT}$-functor (Remark 1.3.27) and that we don't take into consideration in this work as we did for braids. All the quadratic terms in λ_i, for $i = 1, 2, 3, 4$, appear as factors of the operators. For instance in the expression of $\sigma_1(\mathcal{H}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \beta))$, there is only β^2 depending on the basis vector, so that the associated operator has $q^{\frac{-\lambda_1^2 - \lambda_2^2 + (r-1) + (\lambda_1+\lambda_2)^2 + 1}{2}}$ as factor. For σ_2 we see that in the coefficients $q^{\frac{-\lambda_2^2 - \lambda_3^2 + \gamma^2 + (r-1)}{2}}$, there is only γ^2 that varies with the basis vector. After developing both possible expressions for γ, we remark that $q^{\frac{-\lambda_1^2 - \lambda_4^2 + (r-1) + (\lambda_1+\lambda_4)^2 + 1}{2}}$ factors the expression. For σ_3, we factorize by $q^{\frac{-\lambda_3^2 - \lambda_4^2 + (r-1) + (\lambda_3+\lambda_4)^2 + 1}{2}}$, so that we modify slightly the representation getting rid of these factor coefficients in the expression of matrices of the corresponding operators.

After the computation of the $6j$-symbols, we get matrices at level $r = 2$, replacing λ_4 by $-(\lambda_1 + \lambda_2 + \lambda_3)$. We make the change of variables: $A_i = q^{2\lambda_i}$ for $i = 1, 2, 3$ and we end up with the following expressions:

$$M_1(A_1, A_2, A_3) = \underset{\mathcal{B}_{(1)}, \mathcal{B}_{(1,2)}}{Mat} \Phi(\sigma_1) = \begin{pmatrix} \sqrt{A_1 A_2} & 0 \\ 0 & \frac{1}{\sqrt{A_1 A_2}} \end{pmatrix} \tag{2.6}$$

$$M_2(A_1, A_2, A_3) = \underset{\mathcal{B}_{(1)}, \mathcal{B}_{(2,3)}}{Mat} \Phi(\sigma_2) = (A_2^2 A_3^2 - 1) \begin{pmatrix} \frac{-(1+A_3^2)}{A_1 A_2 A_3^2} & \frac{-(1+A_1^2)}{A_1 A_3} \\ \frac{A_1^2 A_2^2 A_3^2 + 1}{A_1 A_2^2 A_3} & \frac{(A_2^2+1) A_1}{A_2} \end{pmatrix} \tag{2.7}$$

where $\mathcal{B}_\tau$ designates the basis $\{\mathcal{H}_\pm(\tau(\lambda_1, \lambda_2, \lambda_3, \lambda_4))\}$ for $\tau \in \mathfrak{S}_4$, and $Mat_{B,B'}$ is the block corresponding to the image of B in B'.

From now on, we restrict to the unicolored case, with $A_1 = A_2 = A_3 = A$, then $Mat_{\mathcal{B}_{(1)},\mathcal{B}_{(2,3)}}\,\Phi(\sigma_1) = Mat_{\mathcal{B}_{(1)},\mathcal{B}_{(3,4)}}\,\Phi(\sigma_3)$.

We recall the exact sequence giving the homological representation of $M(0,4)$ presented in Example 1.1.6 together with the arrows associated to the representations involved here:

$$\begin{array}{ccccccc} 1 \longrightarrow \mathbb{Z}/2\times\mathbb{Z}/2 \to & M(0,4) = G \ltimes N & \xrightarrow{\iota} PSL(2,\mathbb{Z}) \longrightarrow 1 \\ & \downarrow \Phi & \\ & PGL(\mathcal{V}) & \end{array} \tag{2.8}$$

where G is the subgroup generated by σ_1, σ_2 and N the one generated by $\alpha = \sigma_1\sigma_3^{-1}$ and $\beta = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$. The space $\mathcal{V}$ designates the TQFT vector space associated to S_4 colored with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, last color still being the opposite of the sum of the others, tensored with permutations.

The following theorem concerns the restriction of this action to the group $G = \langle\sigma_1, \sigma_2\rangle$ which is isomorphic to $PSL(2,\mathbb{Z})$.

Theorem 2. *The representation $\Phi_{|G}$ of G provides a faithful representation of $PSL(2,\mathbb{Z})$.*

Proof. We recall the notations and framework. The morphism Φ is the quantum representation of $M(0,4)$ built on the sphere with 3 marked points colored by λ, the last one by -3λ, with λ a generic color of the category $\mathscr{C}$, namely $\lambda \in (\mathbb{C}\backslash\mathbb{Z}) \bigcup r\mathbb{Z}$ and $A = q^{-2\lambda}$.

Let b be a mapping class in G, we have that $\mathrm{perm}(b)$ is contained in the permutations that stabilized the last point, namely p_4 colored by -3λ. Then $\Phi(b)(\mathcal{V}(\lambda,\lambda,\lambda,-3\lambda)\otimes\mathbb{C}\left[()\right]) \subset (\mathcal{V}(\lambda,\lambda,\lambda,-3\lambda)\otimes\mathbb{C}\left[()\right])$, if $()$ designates the identity permutation. This shows that for $\Phi_{|G}$ we can restrict ourselves to an action in $PGL\left((\mathcal{V}(\lambda,\lambda,\lambda,-3\lambda)\right) = PGL\left(\mathcal{V}\right)$ so that we still get a representation of G. We end the proof considering this representation.

We are going to work with the s, t generators of $PSL(2,\mathbb{Z})$, which are the images of $\sigma_1\sigma_2$ and $\sigma_1\sigma_2\sigma_1$ respectively under the morphism ι. Let QS and QT be their images under the quantum representation:

$$\begin{aligned} QS(A) &= \underset{\mathcal{V}=Vect(\mathcal{H}_\pm(\lambda,\lambda,\lambda,-3\lambda))}{Mat}(\Phi(\sigma_1\sigma_2)) = \frac{1}{A^2-1}\begin{pmatrix} -1 & -A^2 \\ \frac{(A^2-1)^2}{A^2}+1 & A^2 \end{pmatrix} \\ QT(A) &= \underset{\mathcal{V}=Vect(\mathcal{H}_\pm(\lambda,\lambda,\lambda,-3\lambda))}{Mat}(\Phi(\sigma_1\sigma_2\sigma_1)) = \frac{1}{A^2-1}\begin{pmatrix} -A & -A \\ \frac{(A^2-1)^2}{A}+A & A \end{pmatrix}. \end{aligned}$$

These matrices are obtained from the appropriate products of matrices $M_i(A,A,A)$ with $i = 1,2$ and after some renormalization making the determinant of QS and QT being equal to 1, keep using the fact that we are considering projective matrices well defined up to multiplication by a scalar. We remark that they are well defined for $A \neq 0, \pm 1$.

One can verify that $QS(A)^3 = QT(A)^2 = -\mathrm{Id}$ so that they are sent to the unit element of $PSL(2,\mathbb{Z})$, this guarantees that we have a representation of $PSL(2,\mathbb{Z})$. Let P be the following matrix:

$$P = \begin{pmatrix} 0 & 1 \\ \frac{A^2-1}{A} & -1 \end{pmatrix}$$

It is invertible for a generic choice of A ($A \neq \pm 1, 0$).

Let's consider the representation Ψ of G obtained by conjugation of Φ by P, we have the following:

$$CS(A) := P^{-1}QS(A)P = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = S$$
$$CT(A) := P^{-1}QT(A)P = \begin{pmatrix} 0 & \frac{1}{A} \\ -A & 1 \end{pmatrix}$$

For $A = 1$ we get $CT(1) = T$ and we get that Ψ is the standard representation of $PSL(2,\mathbb{Z})$, which is faithful by definition.

For $A = 1$ we are not in the generic case, and Φ is not well defined (nor conjugated to Ψ).

Using the fact that the entries of matrices are meromorphic functions in the parameter A, and the density of possibilities for the choice of A we get that these representations are generically faithful as follows.

Let $g \in M(0,4)$ and $G(A)$ its image under the above representation Ψ. As the entries are holomorphic in A, let L_g be the domain where $G(A) \neq \mathrm{Id}$. As $G(A) = \mathrm{Id}$ corresponds to zeros of holomorphic functions, it corresponds to isolated values of A so that L_g is dense in $\mathbb{C}$. This is due to the fact that Ψ is faithful for $A = 1$ which guarantees that the functions considered are not zero everywhere and that its zeros are isolated. Hence, Ψ is faithful for:

$$A \in \bigcap_{g \in M(0,4)} L_g,$$

which is a countable intersection of dense spaces, hence dense by Baire's theorem. This proves that the representation Ψ is generically faithful, and since it is generically conjugated to Φ, Φ is also a generically faithful representation of $PSL(2,\mathbb{Z})$. □

Theorem 3. *The projective representation Φ of $M(0,4)$ is faithful.*

Proof. Let $h \in M(0,4)$. Suppose h is in the kernel of Φ. As $M(0,4) = G \ltimes N$, there exists a unique decomposition $h = g \cdot a$ with $g \in G$ and $a \in N$. For h to be in the kernel of Φ, $\mathrm{perm}(h)$ must be the identity permutation. This comes from the fact that only pure mapping classes are sent to block diagonal matrices. It implies $\mathrm{perm}(g) = \mathrm{perm}(a)^{-1}$. We've noticed at the beginning of the proof of Proposition 2 that $\mathrm{perm}(g)$ fixes p_4, so that a must also fix p_4. The element a is one of the following: $\alpha, \beta, \alpha\beta$ of 1 (α and β are the generators of N recalled above). One remarks that α sends p_4 in p_3, β sends p_4 in p_2 and by $\alpha\beta$, p_4 is sent in p_1. This shows that the only possibility for a is 1 (i.e. the only element fixing p_4). Then $h = g \in G$ and can't be in the kernel of Φ from Proposition 2. □

Corollary 2.3.25. *Let $\Phi \in M(0,4)$ be a pseudo-Anosov mapping class. The stretching factor of Φ is detected by the TQFT representation $\mathbb{V}$.*

Proof. From Theorem 3, the TQFT representation detects the representation of $M(0,4)$ in $PSL(2,\mathbb{Z})$ and from Lemma 3.6 of [AMU] the stretching factor is detected (as an eigenvalue) in the $PSL(2,\mathbb{Z})$ representation of $M(0,4)$. $\square$

2.3.4 TQFT representations of bigger mapping class groups

First we relate quantum (pure) braid representations with non semi-simple TQFT representations of the (pure) mapping class groups of punctured spheres. We restrict the study to *pure* groups for an easier reading, but everything can be generalized to whole groups by considering the induced representations (see Definition 1.2.2).

Let q be a $2r^{th}$ root of unity, $\lambda_1, \ldots, \lambda_n \in (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$ be complex colors, $V_{\lambda_1}, \ldots V_{\lambda_n}$ be the associated typical modules of $\mathscr{C}$, and set $V = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. Let $\beta \in \mathcal{PB}_n$ be a braid and $\mathcal{RT}(\beta) \in \mathrm{End}(V)$ its ADO-type representation introduced in Remark 1.3.20 defined using the functor $\mathcal{RT}$. Let $\mathcal{V}$ be the (0-graded) TQFT vector space associated by $\mathbb{V}$ to the sphere with $n+1$ punctures, $p_1, \ldots, p_{n+1}$ and such that: p_1 is decorated by V_{λ_1} and so on until p_n is decorated by V_{λ_n} and p_{n+1} is decorated by V^* the dual space of V. We recall the capping morphism from Proposition 1.1.25 in the case of the disk with n marked points:

$$Cap : \mathcal{PB}_n \to PM(0, n+1)$$

where $\mathcal{PB}_n$ is the pure braid group on n strands, and $PM(0,n)$ is the pure mapping class group of the sphere with $(n+1)$ punctures. Namely, the latter corresponds to the Torelli group of the punctured sphere and is made of mapping classes that leave the punctures fixed pointwise. The morphism $T = \mathbb{V} \circ Cap$ restricted to $\mathcal{V}$ (the 0-graded sub-space) provides a representation of $\mathcal{PB}_n$ ($\mathbb{V}$ is the TQFT functor).

We recall from [BCGP2, Proposition 6.1] that the (0-graded) TQFT space $\mathcal{V}$ is isomorphic to $\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, V \otimes V^*)$.

Let ϕ be the following injective morphism:

$$\phi : \begin{array}{ccc} \mathrm{End}(V) & \to & \mathrm{End}\left(\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, V \otimes V^*)\right) \\ M & \to & M \otimes \mathrm{Id} \end{array}$$

Lemma 2.3.26. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{PB}_n & \xrightarrow{\mathcal{RT}} & \mathrm{End}(V) \\ & \searrow^{T} & \downarrow \phi \\ & & \mathrm{End}\left(\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, V \otimes V^*)\right) \end{array}$$

Proof. For a braid $\beta \in \mathcal{PB}_n$ we must show that:

$$T(\beta) = \mathcal{RT}(\beta) \otimes \mathrm{Id}_{V^*}.$$

By composition by Cap, $\mathcal{PB}_n$ acts over the $(n+1)$ punctured sphere by mapping classes fixing p_{n+1}. The mapping cylinder associated to an element of $\mathcal{PB}_n$ is the identity cobordism in a small disk containing p_{n+1} as the only puncture. By gluing this cylinder to a cobordism generating the TQFT interpreted as an element of $\mathrm{Hom}_{\mathscr{C}}(\mathbb{I}, V \otimes V^*)$, it is easy to see that the morphism is the identity over the $U_q\mathfrak{sl}(2)$-module decorating p_{n+1}, namely V^*, and on the the n other punctures it is by construction obtained by applying the $\mathcal{RT}$-functor. It proves the lemma. $\square$

Proposition 2.3.27. *If there exists a $2r^{th}$ root of unity q and colors $\lambda_1, \ldots, \lambda_n \in (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$ such that the representation $\mathcal{RT}$ of $\mathcal{PB}_n$ is faithful then the representation of $PM(0, n+1)$ (suitably decorated as above) provided by $\mathbb{V}$ is faithful.*

Proof. The proof is a direct consequence of Lemma 2.3.26, of the surjectivity of Cap and of the injectivity of ϕ. Namely, as for any $\beta \in \mathcal{PB}_n$:

$$\mathbb{V} \circ Cap(\beta) = \mathcal{RT}(\beta) \otimes \mathrm{Id}_{V^*}$$

(previous lemma) and that any element in $M(0, 4)$ can be written $Cap(\beta)$ for some β, if $\mathcal{RT}$ is faithful so is $\mathbb{V}$. $\square$

The latter shows that the question of the faithfulness of punctured sphere mapping class group TQFT representations is included in the following open question.

Open Question. *Are the ADO representations of braid groups introduced in Remark 1.3.20 faithful?*

From Section 1.3.3, Kohno's Theorem [K2] and Theorem 5 of Chapter 3 of this work, it is well known that quantum representations are recovered by Lawrence's representations. In [Big0] and [Kra], it is proved that BKL representations of Section 1.2.3 are generically faithful, and from [Z2] that the family of Lawrence representations are faithful in general (except for the Burau level). The word *generically* stands for a generic set of parameters $(q, \lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n+1}$. For instance the faithfulness proof of [Big0] relies on the key lemma recalled in Lemma 1.2.26. This lemma uses extensively the Laurent polynomial structure of coefficients, and a study of some coefficient of the noodle-fork pairing defined to be the maximal coefficient with respect to some lexical order on monomials. This argument crashes down whenever one wants to specialize the proof for q being a root of unity. In this sense, the quantum representations are "generically faithful" but the question whether they are faithful at roots of unity (ADO set-up) is still open, so is the question of the faithfulness of TQFT representations of the punctured spheres.

The case of the torus was studied in [BCGP2] and led to an analog of Theorem 3.

Theorem 2.3.28 ([BCGP2, Theorem 6.28])**.** *The non semi-simple TQFT projective representation of the mapping class group of the torus, provided by the functor $\mathbb{V}$, is faithful modulo its center.*

These "small" cases (in terms of genus) are first steps for an answer to the following general question.

Question ([BCGP2, Question 1.7.(1)])**.** *Let Σ be a surface. Is the non semi simple TQFT representation of $Mod(\Sigma)$ over $\mathbb{V}(\Sigma)$ faithful?*

Chapter 3

Homological model for quantum representations

In this chapter we give a homological framework that recovers some aspects of modules categories of quantum groups such as the quantum action and the braiding.

In Section 3.1 we define the context with a more general definition of configuration spaces of points (compared to Definition 1.2.17), together with their associated colored braid groupoid that allows construction of local system on it. This general framework is here to provide ideas for generalizations of the results of Section 3.2.

In Section 3.2 we work with the space X_r of configurations of r-points inside the n-punctured disk, which is a special case of the general framework mentioned above. We define the maximal abelian cover of it and we study the homology modules $\mathcal{H}_\bullet^{\mathrm{rel}\,-}$ with coefficients in the local system corresponding to the maximal abelian cover. We prove that these modules over some Laurent polynomial ring are endowed by an action of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ (defined in Section 1.4.3) in Theorem 4. We recognize more precisely that it is a tensor product of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ Verma-modules in Theorem 5. The involved homology modules are naturally endowed by a mapping class action of the braid group. In Theorem 6 we show that this action is the one given by the $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$-braiding over tensor products of Verma modules, see Subsection 1.4.4. In Subsection 3.2.6 we give ideas to improve this model such as: recovering more properties of the $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$-module category, recovering other quantum algebras, recovering other braid representations or obtain modules over a non-abelian ring.

In Section 4 we pass from quantum braid representation to the level of quantum knot invariant. Namely, the colored Jones polynomials are polynomial knot invariants that can be computed from quantum braid representations. We apply our homological model for quantum braid representations to obtain homological interpretation of colored Jones invariants. In Theorem 7 we prove that the colored Jones polynomials can be expressed as a weighted sum of abelianized Lefschetz numbers.

3.1 General framework: configuration space and local systems.

In this section, following [F-W], we define a family of configuration spaces, their associated notion of colored braid groupoid, together with directions to construct local systems on these spaces. The aim of this part is to define a general framework of configuration spaces, to see that their fundamental group is related to braid groups, so that representations of braid groups provide local systems on configuration spaces.

3.1.1 Configuration space of points and classification of base points.

Definition 3.1.1 (Configuration space)**.** *Let X be a connected 2 - dimensional manifold, and $\mathbf{n} = (n_1, \ldots, n_k)$ be a set of k positive integers such that $n_1 + \cdots + n_k = N$. The* unordered configuration space *is:*

$$\mathcal{C}_{\mathbf{n}}(X) = \left(X^N \setminus \bigcup_{i<j} \{z_i = z_j\} \right) / \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$$

where the permutations act on the order of coordinates so that we can think of an element as a sequence $(Z_1, \ldots, Z_k)$ of pairwise disjoint subsets of X with cardinalities $|Z_i| = n_i$.

For the purpose of this work, we deal mainly with $X = \mathbb{C}$ or $X = D$ where D is the unit disk.

3.1.2 Colored Braid Groupoid.

All the background regarding links between fundamental groupoid and topology can be found in [Br], where one can find the correspondence between topological coverings and the fundamental groupoid. First, we recall the categorical definition of a groupoid.

Definition 3.1.2 (Groupoid)**.** *A* groupoid *G is a category inside which every morphism is invertible.*

Example 3.1.3. This notion of groupoid is used in topology to generalize the one of fundamental group.

(i) The fundamental groupoid $\Pi_1(M)$ of a topological space M is the groupoid whose set of objects is M and whose morphisms from x to y are the homotopy-classes $[\gamma]$ of continuous maps $\gamma : [0, 1] \to M$ with endpoints map to x and y (which the homotopies are required to fix). Composition is by concatenation (and reparametrization) of representative maps.

(ii) Let O be a subset of a topological space M, there exists a sub-groupoid of the fundamental groupoid:

$$G = \bigcup_{\alpha \in O} G_\alpha \subset \Pi_1(M),$$

it consists in the groupoid of paths having endpoints in O.

(iii) When $O = \{x\}$ is a single point, then the corresponding sub-groupoid is the fundamental group based in x.

Definition 3.1.4 (Colored braid groupoid)**.** *Let $N \in \mathbb{N}^*$. The colored braid groupoid on N strands is the groupoid whose set of objects is $\mathfrak{S}_N$ and morphisms between τ_1 and $\tau_2 \in \mathfrak{S}_N$ are braids β satisfying:*

$$\tau_1 \operatorname{perm}(\beta) = \tau_2$$

where perm *is the morphism that sends a braid to its induced permutation.*

Remark 3.1.5. The braid group is the fundamental group of a configuration space, for x an arbitrary base point:

$$\mathcal{B}_N = \pi_1(\mathcal{C}_N(\mathbb{C}), x)$$

see Subsection 1.1.2.4.

Remark 3.1.6. Let $\sigma_i, i = 1\ldots, N$ be the standard generators of $\mathcal{B}_N$. Then the system:

$$\sigma_i^\alpha : \alpha \to \operatorname{perm}(\sigma_i)\alpha$$

of morphisms for $\alpha \in \mathfrak{S}_N$, provides generating morphisms of G.

Remark 3.1.7 (**n**-colored braid groupoid)**.** Let $\mathbf{n} \in \{(n_1, \ldots, n_k) \text{ s.t. } n_1 + \cdots + n_k = N\}$ and $\mathcal{C}_{\mathbf{n}}(X)$ be the configuration space defined in 3.1.1 with $X = \mathbb{C}$. Let x be a chosen base point of $\mathcal{C}_{\mathbf{n}}(X)$, and O_x be its orbit under the action of $\mathfrak{S}_N$ so that O_x can be thought as the set:

$$O_x = \mathfrak{S}_N \,/ \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k} \,.$$

We define $\mathcal{B}_{\mathbf{n}}(X, x)$ to be the groupoid whose set of objects is O_x and morphisms consist in homotopy classes of paths having endpoints in O_x. It is a sub-groupoid of the fundamental groupoid of $\mathcal{C}_{\mathbf{n}}(\mathbb{C})$, see Example 3.1.3 (ii). As O_x is a quotient of $\mathfrak{S}_N$, $\mathcal{B}_{\mathbf{n}}(X, x)$ is called the **n**-colored braid groupoid in the spirit of Definition 3.1.4.

3.1.3 Representation of the colored braid groupoid.

Definition 3.1.8 (Representation of a groupoid)**.** *A* representation *of a groupoid G is a functor from G to the category $Vect$ of vector spaces.*

What is called an R-matrix representation of the braid group (see Section 1.3) can be generalized to the braid groupoid considering a family of labeled R-matrices instead of a single one.

Definition 3.1.9. *Let $U_\lambda, \lambda = 1, \ldots, N$ be a family of vector spaces, together with a family $\mathrm{R}_{\lambda,\mu}$ of invertible elements of $\operatorname{Hom}(U_\lambda, U_\mu)$, for each pair of indexes λ, μ. Let B be the colored* *braid groupoid on N strands defined in Definition 3.1.4. An R-matrix representation of B, is a representation on the following spaces:*

$$V_\alpha = U_{\alpha(1)} \otimes \cdots \otimes U_{\alpha(N)}$$

for $\alpha \in \mathfrak{S}_N$, together with the following operators associated to generating morphisms of B (Remark 3.1.6:

$$\rho(\sigma_i^\alpha) = P\,\mathrm{R}^{i,i+1}_{\alpha(i),\alpha(i+1)} \text{ with } P(u \otimes v) = v \otimes u.$$

The operator $P\,\mathrm{R}^{i,i+1}_{\lambda,\mu}$ stands for $P\,\mathrm{R}_{\lambda,\mu}$ acting on the i^{th} and j^{th} factors of the tensor product.

Proposition 3.1.10. *Using previous definition's set up, we get an R-matrix representation of B if and only if the R-matrices satisfy the following Yang-Baxter equation:*

$$\mathrm{R}^{2,3}_{\mu,\nu}\,\mathrm{R}^{1,3}_{\lambda,\nu}\,\mathrm{R}^{1,2}_{\lambda,\mu} = \mathrm{R}^{1,2}_{\lambda,\mu}\,\mathrm{R}^{1,3}_{\lambda,\nu}\,\mathrm{R}^{2,3}_{\mu,\nu}.$$

Remark 3.1.11. This definition of R-matrix representation of the colored braid groupoid can be adapted to the **n**-colored braid groupoid defined in Remark 3.1.7.

We mention two particular families of such representations of colored braid groupoid, that will be discussed in this chapter.

Example 3.1.12 (1-dimensional representations)**.** Using notations of Definition 3.1.9, we fix $U_\lambda = \mathbb{C}, \lambda = 1, \ldots, N$ and a family of complex numbers $q_{\lambda,\mu}$. Then :

$$\rho(\sigma_i^\alpha) = q_{\overline{\alpha}(i),\overline{\alpha}(i+1)}$$

defines an R-matrix representation of $\mathcal{B}_{n_1,\ldots,n_k}(\mathbb{C})$.

Example 3.1.13 (Higher dimensional quantum representations)**.** The quantum group $U_q\mathfrak{sl}(2)$ provides examples of such families via its (appropriate) category of modules. For instance, the category $\mathscr{C}$ of $\overline{U}_q^H\mathfrak{sl}(2)$-modules (Section 1.3.2) provides a family of R-matrices that satisfy the Yang-Baxter equation (Definition 1.3.16). Let $V_{\lambda_1}, \ldots, V_{\lambda_N}$ be typical modules (λ_i's are complex parameters), the latter allows one to get a representation of the colored braid groupoid over:

$$\bigcup_{\alpha \in \mathfrak{S}_N} V_{\lambda_{\alpha(1)}} \otimes \cdots \otimes V_{\lambda_{\alpha(N)}}.$$

Remark 3.1.14. The representations of Example 3.1.13 are equivalent to the representations of $\mathcal{B}_n$ induced by the ones of $\mathcal{PB}_n$ over:

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}.$$

The result of Section 3.2 are obtained from a one-dimensional representation corresponding to the set-up of Example 3.1.12. Example 3.1.13 is introduced to give perspective of improvement such as constructing a local system with an associated ring of coefficients that is non-abelian.

3.1.4 Local systems.

Let X be a topological space, and x a base point. From a representation of $\pi_1(X,x)$ one can build a locally trivial fiber bundle with a flat connection (this will be done in this section). On the other hand, from a locally trivial bundle with flat connection, one obtains a representation of $\pi_1(X,x)$ by monodromy. Hence the notions of representation of $\pi_1(X,x)$ and of locally trivial fiber bundle with flat connection are equivalent, and often reunited under the name of *local system*.

This has a slight generalization, and one can build a local system from a representation of some subgroupoid of the fundamental groupoid. Ingredients of such construction are mentioned in Example 6.10 of [Mac], but in the general case of Lie groupoids. Here we give an idea of such a construction in the special case of colored braid groupoid representation.

Definition 3.1.15. *Let O be a subset of a topological space M, and $G = \bigcup_{\alpha\in O} G_\alpha \subset \Pi_1(M)$ the groupoid corresponding to paths having endpoints in O. Let ρ be a representation of G on the family $(V_\alpha)_{\alpha\in O}$ of vector spaces. For $\alpha \in O$ let $\widetilde{M_\alpha}$ be the universal cover corresponding to the base point α, so that G acts on the right of $\coprod_{\alpha\in O} \widetilde{M_\alpha}$ by composition of paths. Namely, if $\eta_{\alpha\beta} \in G_{\alpha\beta}$ for $\alpha, \beta \in O$, then $\eta_{\alpha\beta}$ provides a map:*

$$\widetilde{M_\alpha} \to \widetilde{M_\alpha}$$

by pre-composition by $\eta_{\alpha\beta}$. We define the local system associated to ρ to be the following vector bundle:

$$L_\rho = \coprod_{\alpha\in O} \widetilde{M_\alpha} \times V_\alpha /\sim$$

with identifications $(\widetilde{m}_\alpha, \rho_{\alpha\beta}(\eta_{\alpha\beta})v_\beta) \sim (\widetilde{m}_\alpha \cdot \eta_{\alpha\beta}, v_\beta)$, for $\eta_{\alpha\beta} \in G_{\alpha\beta}$.

Remark 3.1.16. Two remarks putting this definition in the general theory of local system.

- Such a local system is the same as a flat vector bundle over M together with a family of vector spaces V_α and isomorphisms of the fibers over α with V_α such that parallel transport operators are given by ρ. Local horizontal sections are continuous sections which locally can be written as $m \to (\tilde{m}, v)$, with constant v.

- When O is a single point then G is the fundamental group based on it. In this case, the construction is the standard local system one, where one obtains a flat fiber bundle from a representation of the fundamental group.

Applying this construction to colored braid groupoid representations from Example 3.1.12 or 3.1.13 provides local system over configuration spaces. From example 3.1.13, and this definition of local system, we see that quantum R-matrices can serve to construct a local system on configuration spaces.

This construction of local system is informal while in next section we construct a local system from a based fundamental group representation.

3.2 A homological model for $U_q\mathfrak{sl}(2)$

In this section we work on a special case of configuration space. We apply Borel-Moore homology to them, with coefficients in the local system corresponding to the maximal abelian cover. We present the framework in Subsection 3.2.1. We study the algebraic structure of Borel-Moore homology modules in Subsection 3.2.2. We construct homological operators in Subsection 3.2.3 and then we prove that they realize an $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$-module in Theorem 4. In Subsection 3.2.4 we compute the action in a particular basis and we recognize a tensor product of Verma modules in Theorem 5. Finally we construct the mapping class braid action and we recover the $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ braiding in Theorem 3.2.5.

3.2.1 Adapted framework for the $U_q\mathfrak{sl}(2)$ case.

3.2.1.1 The configuration space

Using notations from Section 3.1, we define a special topological space out of the configuration space setting.

Notations. Let $r \in \mathbb{N}$, $n \in \mathbb{N}$, and D be the unit disk. We define the following fibration:

$$p_{r,n} : \begin{cases} \mathcal{C}_{r,1,\ldots,1}(D) & \to & \mathcal{C}_{1,\ldots,1}(D) \\ (\{z_1, \ldots, z_r\}, w_1, \ldots, w_n) & \mapsto & (w_1, \ldots, w_n). \end{cases}$$

where the $(1, \ldots, 1)$ refers to n coordinates. Let $(w_1, \ldots, w_n)$ be a point in $\mathcal{C}_{1,\ldots,1}(D)$. We define the following space of interest:

$$X_r(w_1, \ldots, w_n) \quad = \quad p_{r,n}^{-1}(w_1, \ldots, w_n). \tag{3.1}$$

The points $w_1, \ldots, w_n$ will always be chosen so that they lie in the interior of D. Let $p_1, \ldots, p_n$ be points in the interior of a disk, say the n first integers inside a disk D' of radius $n+1$, and consider $D_n = D' \setminus \{p_1, \ldots, p_n\}$. It is clear that $X_r(w_1, \ldots, w_n)$ and $\mathcal{C}_r(D_n)$ are homotopically equivalent.

Remark 3.2.1. Let $m \in \mathbb{N}^*$, one observes that spaces $X_m(w_1, \ldots, w_n)$ from the above definition and $C_{n,m}$ from Definition 1.2.17 of Chapter 1 are topologically equivalent.

3.2.1.2 The local system

To fit with the set up of Section 3.1, we define first the local system in this context. Let D be the unit disk, and $w_1, \ldots, w_n$ be distinct points in its interior. Let $X_r(w_1, \ldots, w_n)$ be defined as in Relation 3.1. Let $\alpha_1, \ldots, \alpha_n$ be complex numbers (sometimes referred to as *"colors"*), and $q \in \mathbb{C} \setminus 0$. We consider the one dimensional representation, called ρ_r from now on, of Example 3.1.12 with the following specialization:

$$\mathrm{R}_{1,1} = -q^2 \ , \ \mathrm{R}_{1,j} = \mathrm{R}_{j,1} = q^{1-\lambda_j} \text{ for } j = 1, \ldots, n+1.$$

This yields a representation of $\mathcal{B}_{r,1,\ldots,1}(\mathbb{C})$, and a local system L_ρ over $\mathcal{C}_{r,1,\ldots,1}(\mathbb{C})$ and over $\mathcal{C}_{r,1,\ldots,1}(D)$ by restriction.

Definition 3.2.2. *Let $L_r(w_1,\ldots,w_n)$ be the restriction of the local system L_ρ to $X_r(w_1,\ldots,w_n)$.*

For the purpose of this work, it is sufficient to work with a local system defined as a representation of the based fundamental group (instead of a groupoid representation). The construction will then depend on a lift of this base point. We choose conventions for the base point in the following notations, fixing them for the rest of the chapter.

Notations (Base point)**.** Let r, n be positive integers, X_r be the configuration space of r-points inside the n-punctured disk D_n, with punctures named $w_1,\ldots,w_n$ and lying on the real line in the interior of the disk. Set $w_0 = -1$, and $X_r^- \subset X_r$ the configurations with one coordinate in w_0.

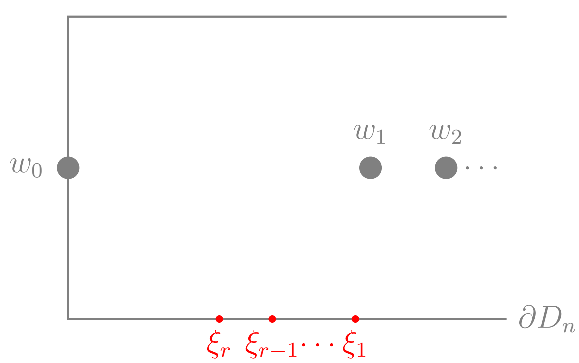
- Let $\boldsymbol{\xi}^r$ be the chosen base point of X_r, verifying:

$$\boldsymbol{\xi}^r = \{\xi_1,\ldots,\xi_r\}$$

such that $\xi_i \in \partial D_n$, $\forall i$ and:

$$\Re(\xi_r) = w_1 - \frac{d}{2}, \Re(\xi_{r-1}) = w_1 - \frac{d}{3}, \ldots, \Re(\xi_1) = w_1 - \frac{d}{r+1}$$

where $d = |w_1 - w_0|$. We illustrate their position in the following figure.



In what follows, distances between the ξ_i's may be deformed in drawings but the order on real parts remains the important fact.

- Let Φ^r be the following homeomorphism:

$$\Phi^r : \begin{cases} X_r \setminus X_r^- & \to & X_{r+1}^- \\ Z & \mapsto & Z \cup w_0 \\ \boldsymbol{\xi}^r & \mapsto & \{\xi_1,\ldots,\xi_r,w_0\} \end{cases}.$$

- Φ^r induces:

$$\Phi_*^r : \pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r) \to \pi_1(X_{r+1}^-, \{\boldsymbol{\xi}^r, w_0\}).$$

To change the base point in the latter space to $\boldsymbol{\xi}^{r+1}$, we move w_0 along ∂D_n and all the points in $\boldsymbol{\xi}^r$ through a path φ^r defined as:

$$\varphi^r : \begin{array}{ccl} I & \to & X_{r+1} \\ t & \mapsto & \varphi^r(t) = \{\varphi_1(t), \ldots, \varphi_r(t), \varphi_{r+1}(t)\} \end{array}$$

where φ_1 goes from w_0 to ξ_{r+1} along ∂D_n in the counterclockwise sense, φ_2 goes from ξ_{r+1} to ξ_r along ∂D_n, and so on, ending with φ_{r+1} going from ξ_2 to ξ_1 along ∂D_n.

- We let then Φ^r be the composition of the above Φ^r_* and the isomorphism induced by the change of base point through precomposition by φ^r.

$$\Phi^r : \ \pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r) \ \to \ \pi_1(X_{r+1}^-, \boldsymbol{\xi}^{r+1}) \ .$$

  This morphism induces a right shift of coordinates of the base point (the new coordinate arriving at the leftmost).

In what follows we will often omit the indexes r in φ^r, $\boldsymbol{\xi}^r$ and Φ^r, to simplify notations when no confusion is possible.

We present in detail the local system in terms of a representation of the group $\pi_1(X_r, \boldsymbol{\xi}^r)$ which simplifies computations of next section. First we give a presentation of $\pi_1(X_r, \boldsymbol{\xi}^r)$ as a braid sub-group, which can be deduced from the one given in the introduction of [Z1], and will be explain with drawings.

Remark 3.2.3. The group $\pi_1(X_r, \boldsymbol{\xi}^r)$ is isomorphic to the subgroup of $\mathcal{B}_{r+n}$ generated by:

$$\langle \sigma_1, \ldots, \sigma_{r-1}, B_{r,1}, \ldots, B_{r,n} \rangle$$

where the σ_i ($i = 1, \ldots, r-1$) are standard generators of $\mathcal{B}_{r+n}$, and $B_{r,k}$ (for $k = 1, \ldots, n$) is the following pure braid:

$$B_{r,k} = \sigma_r \cdots \sigma_{r+k-2} \sigma_{r+k-1}^2 \sigma_{r+k-2}^{-1} \cdots \sigma_r^{-1}.$$

To see the correspondence between loops in X_r and generators of the above braid sub group we draw two examples.

Example 3.2.4. Two types of braid generators for $\pi_1(X_r, \boldsymbol{\xi}^r)$ are given in Remark 3.2.3, which correspond to two types of loops generating $\pi_1(X_r, \boldsymbol{\xi}^r)$. We give examples for both kind.

- The braid σ_1 corresponds to a loop swapping ξ_r and ξ_{r-1} letting other base point coordinates fixed. This can be seen by drawing the movie of the loops in Figure 3.1.
- The braid $B_{r,k}$ for $k \in \{1, \ldots, n\}$ corresponds to ξ_1 running once around w_k before going back keeping other base point coordinates fixed. The correspondence in terms of standard braid generators can be seen by drawing the movie of this loop in Figure 3.2.

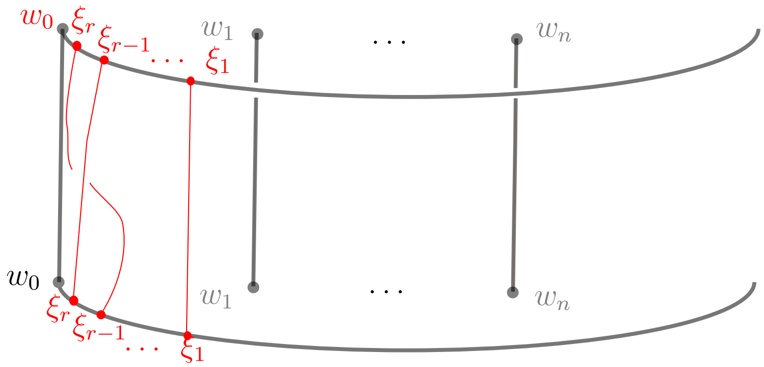


Figure 3.1 – Generator σ_1.

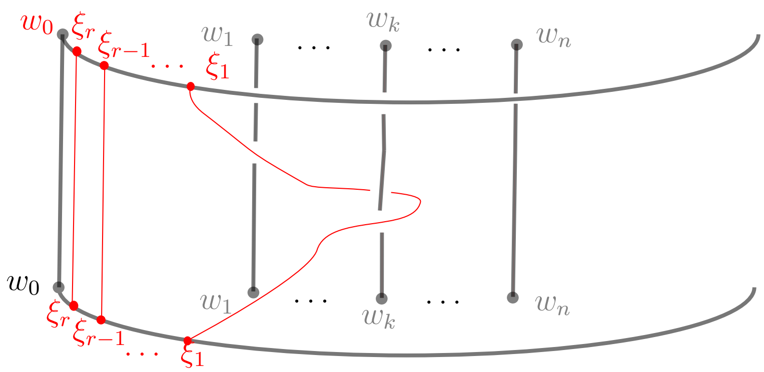


Figure 3.2 – Generator $B_{r,k}$

Using this set up, we define the local system in this context.

Definition 3.2.5 (Local system L_r.)**.** *The local system $L_r(w_1,\ldots,w_n)$ is defined by the following algebra morphism:*

$$\rho_r : \begin{array}{rcl} \mathbb{Z}\left[\pi_1(X_r,\boldsymbol{\xi}^r)\right] & \to & \mathbb{Z}\left[q^{\pm\alpha_i},t^{\pm 1}\right]_{i=1,\ldots,n} \\ \sigma_i & \mapsto & t \\ B_{r,k} & \mapsto & q^{\alpha_k}. \end{array}$$

When no confusion is possible we will omit the dependence in $(w_1,\ldots,w_n)$ in the notations to simplify them.

Remark 3.2.6.
- This local system is equivalent to the one defined in Definition 2.2.1 in the case $r=2$. The correspondence between loops is clear.
- As it is a one dimensional local system it is abelian in the sense that:

$$\rho_r(s_1s_1) = \rho_r(s_1)\rho_r(s_2) = \rho_r(s_2)\rho_r(s_1)$$

for $s_1,s_2 \in \pi_1(X_r,\boldsymbol{\xi}^r)$. Moreover this local system corresponds to the maximal abelian cover of X_r.

- We've seen (Definition 3.2.2) that it is a system local obtained from a representation of the family presented in Example 3.1.12, while building a local system from the family of Example 3.1.13 would lead to a non-abelian local system and a more sophisticated (non abelian) ring of coefficients.

We will use homology modules with coefficients in this local system, so that we fix notations from now on.

Notations. Let $r \in \mathbb{N}$ and $\mathcal{R} = \mathbb{Z}\left[q^{\pm\alpha_i}, t^{\pm 1}\right]_{i=1,\ldots,n}$. We let H^{lf} designates the homology of locally finite chains (see Section 5.1), and we use the following notations for homology with local coefficients:

$$\begin{aligned} \mathcal{H}_r^{\text{abs}} &= \mathrm{H}_r^{lf}\left(X_r; L_r\right) \\ \mathcal{H}_r^{-} &= \mathrm{H}_r^{lf}\left(X_r^-; L_r\right) \\ \mathcal{H}_r^{\text{rel}\,-} &= \mathrm{H}_r^{lf}\left(X_r, X_r^-; L_r\right) \end{aligned}$$

The last one corresponds to the homology associated to the relative complex of locally finite chains. We will use the letter C instead of $\mathcal{H}$ to designate the associated locally finite chain complexes.

Remark 3.2.7. We recall that the representation ρ_r defining the local system L_r is canonically equivalent to the construction of a covering map over X_r. Namely, one can consider the universal cover $\widetilde{X_r}$ of X_r, upon which there is an action of $\pi_1(X_r)$. By making the quotient of $\widetilde{X_r}$ by the action of $Ker\,\rho_r \in \pi_1(X_r)$, one gets a cover $\widehat{X_r}$ of X_r. The group of deck transformations is then isomorphic to $Im(\rho_r) = \mathbb{Z}^{n+1}$. There are three equivalent ways to build the chain complex with local coefficients in L_r:

$$C_\bullet(X_r; L_r) \simeq C_\bullet(\widetilde{X_r}, \mathbb{Z}) \otimes_{\pi_1(X_r)} \mathcal{R} \simeq C_\bullet(\widehat{X_r}).$$

The first one corresponds to complex with coefficients in a locally trivial bundle. In the middle one, the action of $\pi_1(X_r)$ is the one over the universal cover on the left, and given by ρ_r on the right. The last one corresponds to singular chain complex of $\widehat{X_r}$ with the deck transformations action of $\mathcal{R}$.

We will use L_r or ρ_r to designate both the representation of $\pi_1(X_r)$ or the covering $\widehat{X_r}$ together with the deck transformations group action, depending on what we need.

3.2.2 Computation of the homology with local coefficients

3.2.2.1 Algebraic structure of the homological complex.

In this section we apply Borel-Moore homology to the local system presented in previous section. Definitions of locally finite and Borel-Moore homology and the link between them are presented in the Appendix, Chapter 5.

Proposition 3.2.8. *For $r \in \mathbb{N}^*$, the module $\mathcal{H}_r^{rel\,-}$ is a free $\mathcal{R}$-module of dimension $\binom{n+r-1}{r}$. Moreover, it is the only non vanishing module of the complex $\mathrm{H}_\bullet^{lf}\left(X_r, X_r^-; L_r\right)$.*

Proof. All over the proof, the local ring of coefficients will remain L_r so that we omit it in the notations. Let $X_r^{\mathbb{R}}$ be the set $\{x_1, \ldots, x_r\} \in X_r$ such that $x_1, \ldots, x_r$ lie in the segment $[w_0, w_n[$. Set $X_r^{\mathbb{R},-} = X_r^{\mathbb{R}} \cap X_r^-$. We use these simpler spaces to compute the homology, thanks to the following lemma that can be seen as a Bigelow interpretation of the Salvetti retract complex associated to hyperplanes arrangement from [Sal]. This method is adapted from Lemma 3.1 of [Big1].

Lemma 3.2.9 (Bigelow's trick)**.** *The following map:*

$$\mathrm{H}_\bullet^{lf}\left(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r\right) \to \mathrm{H}_\bullet^{lf}\left(X_r, X_r^-; L_r\right) \tag{3.2}$$

induced by inclusion is an isomorphism.

Proof of Lemma 3.2.9. Let $\epsilon > 0$ and A_ϵ be the set of $\{x_1, \ldots, x_r\} \in X_r$ such that $|x_i - x_j| \geq \epsilon$ and $|x_i - w_k| \geq \epsilon$ for all distinct $i, j = 1, \ldots, r$ and $k = 1, \ldots, n$. This family of compact sets yields a basis of compact sets for X_r so that it suffices to show that for all sufficiently small ϵ the map:

$$H_\bullet\left(X_r^{\mathbb{R}}, \left(X_r^{\mathbb{R}} \setminus A_\epsilon\right) \cup X_r^{\mathbb{R},-}\right) \to H_\bullet\left(X_r, \left(X_r \setminus A_\epsilon\right) \cup X_r^-\right)$$

induced by inclusion is an isomorphism. This is sufficient by means of the inductive limit over compact sets definition of Borel-Moore homology introduced in Section 5.1.

Let $D_n' \subset D_n$ be a closed $\epsilon/2$ neighborhood of the interval $[w_0, w_n[$. Let X_r' be the configuration space of r points in D_n', and $X_r'^- = X_r' \cap X_r^-$ be the ones with a coordinate in w_0. We have the following property.

Lemma 3.2.10 (Compressing trick)**.** *The map:*

$$H_\bullet\left(X_r', (X_r' \setminus A_\epsilon) \cup X_r'^-\right) \to H_\bullet\left(X_r, (X_r \setminus A_\epsilon) \cup X_r^-\right)$$

induced by inclusion is an isomorphism.

Proof. To see this, note that the obvious homotopy shrinking X_r to X_r' is a homotopy of the pairs involved. In other words, points in $X_r \setminus A_\epsilon$ corresponding to close points, stay in it because the homotopy is a contraction. We will refer to this process as the *compressing trick* later on. ☐

Let V be the set of $\{x_1, \ldots, x_r\} \in X_r$ with either $\Re(x_i) = \Re(x_j)$ for some $i, j \in \{1, \ldots, r\}$ or $\Re(x_i) = w_k$ for some $i \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, n\}$. Let $U = X_r' \setminus V$. Note that V is a closed subset contained in $X_r' \setminus A_\epsilon$ which is the interior of $(X_r' \setminus A_\epsilon) \cup X_r'^-$. This shows that V satisfies the required hypothesis to perform the excision of the pair (Theorem 5.2.1), so that the following map:

$$H_\bullet\left(U, (U \setminus A_\epsilon) \cup \left(X_r'^- \cap U\right)\right) \to H_\bullet\left(X_r', (X_r' \setminus A_\epsilon) \cup X_r'^-\right)$$

induced by inclusion is an isomorphism by the excision theorem.

Finally there is an obvious *vertical line* deformation retraction that sends U to $X_r^{\mathbb{R}}$ taking $\{x_1, \ldots, x_r\}$ to $\{\Re(x_1), \ldots, \Re(x_r)\}$. This is again a contraction homotopy so that $U \setminus A_\epsilon$ is preserved and $X'_r \cap U$ is sent to $X_r^{\mathbb{R},-}$. This retraction guarantees that the map:

$$H_\bullet\left(X_r^{\mathbb{R}}, \left(X_r^{\mathbb{R}} \setminus A_\epsilon\right) \cup X_r^{\mathbb{R},-}\right) \to H_\bullet\left(U, (U \setminus A_\epsilon) \cup \left(X_r'^{-} \cap U\right)\right)$$

induced by inclusion is an isomorphism, and concludes the proof of Lemma 3.2.9. □

To end the proof it remains to compute the complex $\mathrm{H}_\bullet^{lf}\left(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r\right)$. Let $A_\epsilon^{\mathbb{R}} \in X_r^{\mathbb{R}}$ be the set of configurations $\{x_1, \ldots, x_r\}$ of $X_r^{\mathbb{R}}$ such that $|x_i - x_j| \geq \epsilon$ and $|x_i - w_k| \geq \epsilon$ where $i, j = 1, \ldots, r$ and $k = 1, \ldots, n$. Let $A_\epsilon^{\mathbb{R},w_0}$ be $A_\epsilon^{\mathbb{R}}$ with the additional condition that $|x_i - w_0| \geq \epsilon$ for $i = 1, \ldots, r$. We are going to show that for sufficiently small ϵ, the following complex:

$$H_\bullet\left(X_r^{\mathbb{R}}, \left(X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}\right) \cup X_r^{\mathbb{R},-}; L_r\right)$$

is isomorphic to the Borel-Moore one of a disjoint union of open balls. This will end the computation of $\mathrm{H}_\bullet^{lf}\left(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r\right)$ by definition of Borel-Moore homology. To do so, first we remark that the following spaces are homotopically equivalent:

$$\left(X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}\right) \cup X_r^{\mathbb{R},-} =$$

$$= \left\{ \{x_1, \ldots, x_r\} \in X_r^{\mathbb{R}} \text{ s.t. } \begin{array}{l} |x_i - x_j| < \epsilon \text{ for } i, j = 1, \ldots, r \\ \text{or } |x_i - w_k| < \epsilon \text{ for } k = 1, \ldots, n \\ \text{or } x_i = w_0 \end{array} \right\}$$

$$\simeq \left\{ \{x_1, \ldots, x_r\} \in X_r^{\mathbb{R}} \text{ s.t. } \begin{array}{l} |x_i - x_j| < \epsilon \text{ for } i, j = 1, \ldots, r \\ \text{or } |x_i - w_k| < \epsilon \text{ for } k = 1, \ldots, n \\ \text{or } |x_i - w_0| < \epsilon \end{array} \right\}$$

$$= X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}.$$

This shows that the two following complexes are isomorphic:

$$H_\bullet\left(X_r^{\mathbb{R}}, \left(X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}\right) \cup X_r^{\mathbb{R},-}; L_r\right) \simeq H_\bullet\left(X_r^{\mathbb{R}}, X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}; L_r\right).$$

Then one remarks that $X_r^{\mathbb{R},-}$ is closed in $A_\epsilon^{\mathbb{R},w_0}$ so that we can perform the excision and that the map:

$$H_\bullet\left(X_r^{\mathbb{R}} \setminus X_r^{\mathbb{R},-}, \left(X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}\right) \setminus X_r^{\mathbb{R},-}; L_r\right) \to H_\bullet\left(X_r^{\mathbb{R}}, X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R},w_0}; L_r\right)$$

induced by inclusion is an isomorphism. Let $X_r^{\mathbb{R}}(w_0) \subset X_r^{\mathbb{R}}$ be the space of configurations without any coordinate in w_0. The space $X_r^{\mathbb{R}}(w_0)$ is exactly the space of configurations of r points in $]w_0, w_n[$ such that every coordinate is different from w_k for $k = 0, \ldots, n$. For sufficiently small ϵ, we have shown that the two complexes:

$$H_\bullet\left(X_r^{\mathbb{R}}, \left(X_r^{\mathbb{R}} \setminus A_\epsilon^{\mathbb{R}}\right) \cup X_r^{\mathbb{R},-}; L_r\right) \simeq H_\bullet\left(X_r^{\mathbb{R}}(w_0), X_r^{\mathbb{R}}(w_0) \setminus A_\epsilon^{\mathbb{R},w_0}; L_r\right)$$

are isomorphic. Then, as the family of $A_\epsilon^{\mathbb{R},w_0}$ is a compact sets basis for $X_r^{\mathbb{R}}(w_0)$, we end up with the complexes:

$$\mathrm{H}_\bullet^{lf}\left(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r\right) \simeq \mathrm{H}_\bullet^{lf}\left(X_r^{\mathbb{R}}(w_0); L_r\right)$$

being isomorphic. To conclude the computation we take Bigelow's decomposition of the space of configuration of r points with $n+1$ punctures in open balls, defined as follows and as it is done in [Big1].

Definition 3.2.11. *Let $E_{n,r}^0 = \{(k_0, \ldots, k_{n-1}) \in \mathbb{N}^n$ s.t. $\sum k_i = r\}$ be the set of partitions of r in n integers.*

For $\mathbf{k} \in E_{n,r}^0$, let $U_{\mathbf{k}}$ be the set of all $\{x_1, \ldots, x_r\} \in X_r$ such that $x_1, \ldots, x_r \in]w_0, w_n[$ and:

$$\sharp\left(\{x_1, \ldots, x_r\} \cap]w_i, w_{i+1}[\right) = k_i$$

for $i = 0, \ldots, n-1$. This is an open r-ball of X_r, and one notes that:

$$X_r^{\mathbb{R}}(w_0) = \bigsqcup_{\mathbf{k} \in E_{n,r}^0} U_{\mathbf{k}}.$$

From this disjoint union of open balls, we deduce that $\mathrm{H}_r^{lf}\left(X_r^{\mathbb{R}}(w_0); L_r\right)$ is the direct sum of $\sharp E_{n,r}^0 = \binom{n+r-1}{r}$ copies of $\mathcal{R}$ while all other $\mathrm{H}_k^{lf}\left(X_r^{\mathbb{R}}(w_0); L_r\right)$ for $k \neq r$ vanishes. The complex $\mathrm{H}_\bullet^{lf}\left(X_r^{\mathbb{R}}, X_r^{\mathbb{R},-}; L_r\right)$ has the same decomposition which concludes the proof.

□

Bigelow's trick was initially used to show the following in Lemma 3.1 of [Big1].

Proposition 3.2.12 (Lemma 3.1 [Big1])**.** *The morphism:*

$$\mathrm{H}_\bullet^{lf}\left(X_r^{\mathbb{R}}(w_0); L_r\right) \to \mathrm{H}_\bullet^{lf}\left(X_r(w_0); L_r\right)$$

induced by inclusion is an isomorphism of complexes.

From this and from the proof of Proposition 3.2.8, one gets the following corollary.

Corollary 3.2.13.
- *The morphism: $\mathrm{H}_\bullet^{lf}\left(X_r(w_0); L_r\right) \to \mathrm{H}_\bullet^{lf}\left(X_r, X_r^-; L_r\right)$ induced by inclusion is an isomorphism.*
- *The family $\mathcal{U} = (U_{\mathbf{k}})_{\mathbf{k} \in E_{n,r}^0}$ yields a basis of $\mathcal{H}_r^{rel\,-}$ as an $\mathcal{R}$-module.*

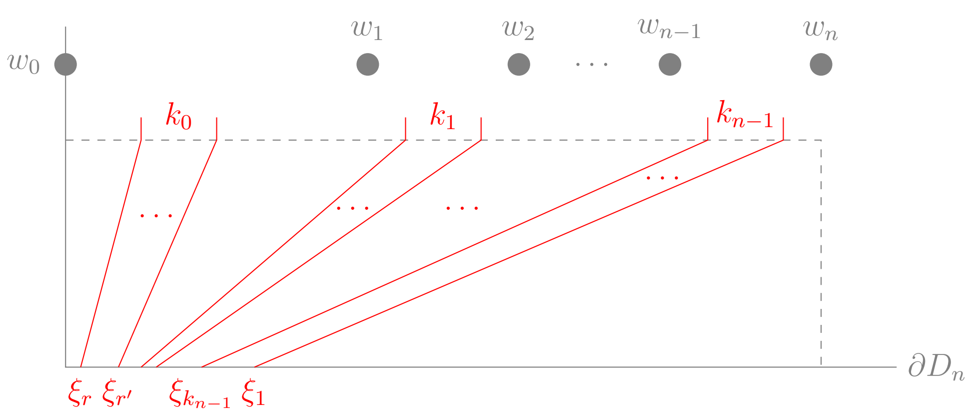
We conclude this part with two remarks about the proof of Proposition 3.2.8.

Remark 3.2.14.
- The proof of Proposition 3.2.8 is constructive in the sense that it provides a process to express homology classes in the $\mathcal{U}$ basis. This will be used in next sections.
- All along the proof of Proposition 3.2.8, the local system does not change, no morphism of the latter is needed. The proof relies only on topological operations such as excisions and homotopy equivalences. In some sense the proof is rigid regarding the local ring of coefficients, and should be adaptable with another one.

3.2.2.2 Homological families.

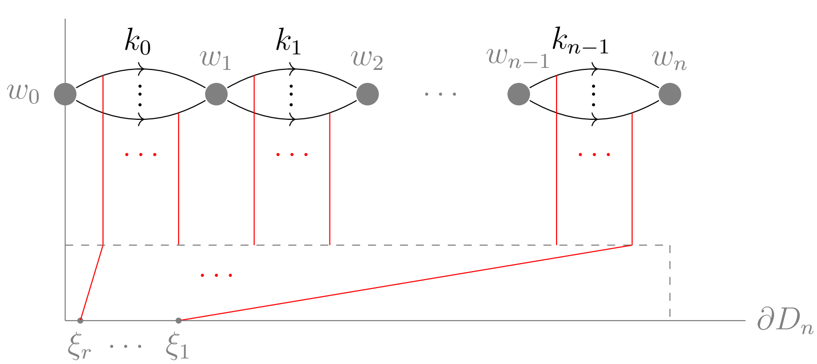
We define four families of topological objects that will later correspond to classes in $\mathcal{H}_r^{\mathrm{rel}\,-}$, all indexed by $E^0_{n,r}$ (Definition 3.2.11). Let $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_r\}$ be the base point of X_r chosen so that all ξ_i's lie in the boundary of D_n and so that $\Re(w_0) < \Re(\xi_r) < \cdots < \Re(\xi_1) < \Re(w_1)$. All the local system construction of homology classes depends on a choice for a lift of $\boldsymbol{\xi}$ that we make here, namely let $\widetilde{\boldsymbol{\xi}}$ be a lift of $\boldsymbol{\xi}$ in the cover corresponding to the local system L_r. For a different choice $\widetilde{\boldsymbol{\xi}'}$ of lift, all the classes are multiplied by the same (invertible) monomial $\rho_r(\widetilde{\boldsymbol{\xi}} \to \widetilde{\boldsymbol{\xi}'})$, namely the local system coefficient of a path joining $\rho_r(\widetilde{\boldsymbol{\xi}})$ and $\rho_r(\widetilde{\boldsymbol{\xi}'})$.

Notations. In what follows we draw topological objects inside the punctured disk, without drawing the boundary of the disk entirely, for an easier reading. The gray color is used to draw the punctured disk. Red arcs are going from a coordinate of the base point $\boldsymbol{\xi}$ of X_r lying in its boundary to a black arc. Dashed black arcs correspond to arcs where several configuration points are embedded, while a plain arc corresponds to just one configuration point (this will be important to define the associated homology classes). Black arcs are oriented, from left to right if nothing is specified and if no confusion arises. Finally, for all the following objects, the red arcs will end up going like the following picture inside the dashed box, so that all families of red arcs are attached to the base point $\{\xi_1, \ldots, \xi_r\}$ of X_r.



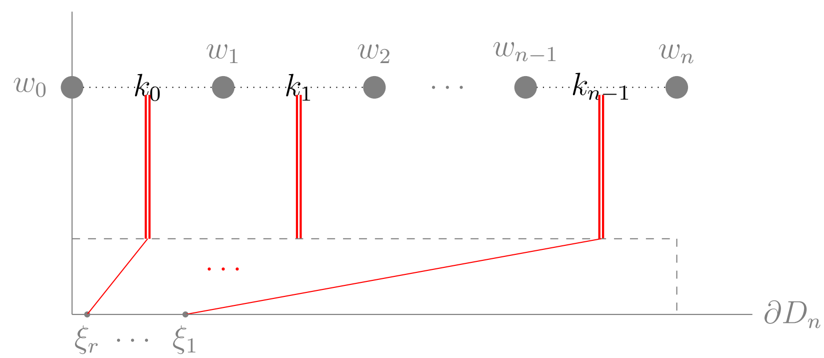
Here, $r' = r - k_0$ is set to simplify the picture. We now list the families of objects of interest.

Multi-Forks. First we recall the definition of *(standard) multifork* that can be found in [Ito]. For $\mathbf{k} \in E^0_{n,r}$ we let $F(k_0, \ldots, k_{n-1})$ be the following picture:



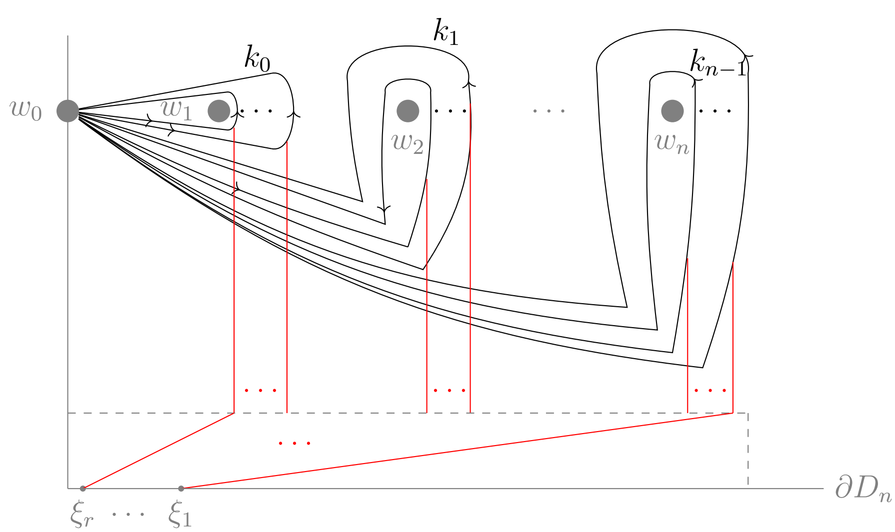
Here we mean k_0 black parallel arcs between w_0 and w_1, k_1 between w_1 and w_2 and so on. These arcs are called *tines*. Every tine is connected to the boundary by a red arc that is called *a handle*. In this standard multifork drawing, all the handles are going straight to the boundary without any crossings. We call $\mathcal{F} = (F(k_0, \ldots, k_{n-1}))_{\mathbf{k} \in E^0_{n,r}}$ the family of all standard multiforks.

Code sequences. We call *code sequence* an element of $\mathcal{U}$. The definition of these objects comes from [Big1]. For $\mathbf{k} = (k_0, \ldots, k_{n-1}) \in E^0_{n,r}$ we use the following drawing to illustrate $U_{\mathbf{k}} = U(k_0, \ldots, k_{n-1})$ introduced in Definition 3.2.8.



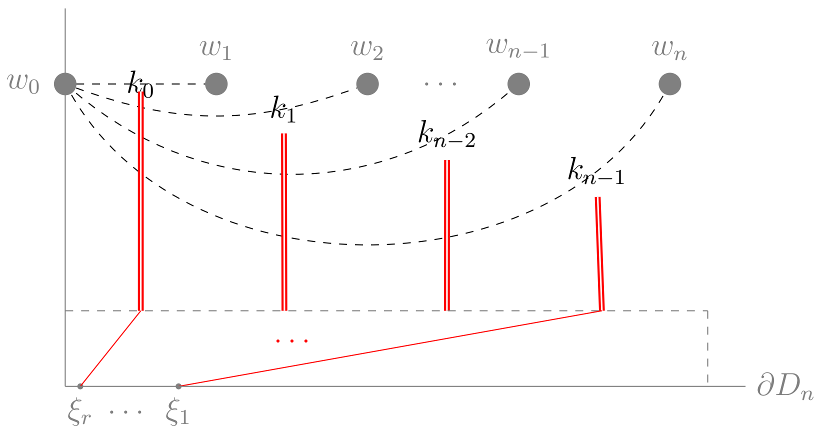
The indexes k_i's are here to illustrate the fact that k_i configuration points are embedded in the corresponding dashed segment. We have attached to an indexed k_i dashed arc what we call a (k_i)-handle to the picture. It is represented by a little red tube which is a simpler representation used for k_i parallel fork-handles. As the elements of $\mathcal{U}$ generate the homology as an $\mathcal{R}$-module, these handles will be used to specify a generator.

r-Loops. This family was introduced in [F-W]. For $\mathbf{k} \in E^0_{n,r}$ we call $L(k_0, \ldots, k_{n-1})$ a *(standard) r-loops*, the object corresponding to the following drawing.



In the picture, k_0 arcs are nested into each other around w_1, k_1 around w_2 and so on. Forks' handles are again considered, and going from left to right while black arcs are going from the inside to the outside. We call $\mathcal{L} = (L(k_0,\ldots,k_{n-1}))_{\mathbf{k}\in E^0_{n,r}}$ the family of all standard r-loops.

Multi-Arcs. This family of objects is new in the literature. For $\mathbf{k} \in E^0_{n,r}$ we define a multi-arc $A'(k_0,\ldots,k_{n-1})$ to be the following picture:



where the k_i's indicate that k_i configuration points are embedded in the corresponding dashed arc. As for code sequences, there is a (k_i)-handle arriving to a dashed arc indexed by k_i which corresponds to k_i parallel fork handles, this will be used to define the associated homology class. We call $\mathcal{A}' = (A'(k_0,\ldots,k_{n-1}))_{\mathbf{k}\in E^0_{n,r}}$ the family of all standard multi-arcs.

These families are separated into two types: the black parts of forks and r-loops have r connected components, while the ones of code sequences and arcs have n of them. This fact leads to two types of construction of homology classes associated to these objects.

orks and r-loops. Let X designates the letter F or L to treat both cases at the same time. Let $\mathbf{k} \in E^0_{n,r}$ and for all $i = 1,\ldots,r$, let:

$$\phi_i : I_i \to D_n$$

be the embedding of the black arc number i of $X(k_0,\ldots,k_{n-1})$, where I_i is a unit interval. There are r black arcs for the r connected components. Let $I = [0,1]$ be the unit interval. There is a natural application:

$$\phi^r : \begin{array}{rcl} I^r = I_1 \times \cdots \times I_r & \to & X_r \\ (t_1,\ldots,t_r) & \mapsto & \{\phi_1(t_1),\ldots,\phi_r(t_r)\} \end{array}$$

which is a singular locally finite r-chain of X_r - and moreover a cycle in Borel-Moore homology (see Remark 5.1.3 for the idea that Borel-Moore homology "does not see punctures at infinity"). To get a cycle in the local system homology, one has to choose a lift of the chain to the corresponding cover. The way to do so is using the red fork-handle of $X(k_0, \ldots, k_{n-1})$ to which is canonically associated a path:

$$\mathbf{h} = \{h_1, \ldots, h_r\} : I \to X_r$$

joining the base point $\boldsymbol{\xi}$ to the r-chain. At the cover level there is a unique lift $\widetilde{\mathbf{h}}$ of $\mathbf{h}$ that starts at $\widetilde{\boldsymbol{\xi}}$. By choosing the lift of $X(k_0, \ldots, k_{n-1})$ passing by $\widetilde{\boldsymbol{\xi}}(1)$, it defines a cycle in $\mathrm{C}_r^{\mathrm{rel}\,-}$, and we still call $X(k_0, \ldots, k_{n-1})$ the associated class in $\mathcal{H}_r^{\mathrm{rel}\,-}$ as we will only use this class out of the original object.

Notations. We will refer to the utilization of handles to choose a lift as the *handle process*.

Codes and Arcs. Let X be the letter U or A to treat both cases at the same time. Let $\mathbf{k} \in E^0_{n,r}$ and for all $i = 1, \ldots, n$, let:

$$\phi_i : I_i \to D_n$$

be the embedding of the dashed black arc number i of $X(k_0, \ldots, k_{n-1})$ indexed by k_{i-1}, where I_i is a unit interval. Let Δ^k be the standard (open) k simplex:

$$\Delta^k = \{0 < t_1 < \cdots < t_k < 1\}$$

for $k \in \mathbb{N}^*$. For all i, we consider the map $\phi^{k_{i-1}}$:

$$\phi^{k_{i-1}} : \begin{array}{ccc} \Delta^{k_{i-1}} & \to & X_{k_{i-1}} \\ (t_1 \ldots, t_{k_{i-1}}) & \mapsto & \{\phi_i(t_1), \ldots, \phi_i(t_{k_{i-1}})\} \end{array}$$

that is a singular locally finite k_{i-1}-chain and moreover a cycle in $X_{k_{i-1}}$. There is a cycle associated to each dashed arc, so that by considering the product of maps $\phi^{k_{i-1}}$ for $i = 1, \ldots, n$ with target in X_r, one generalizes this fact by associating an r-cycle of X_r to each object $X(k_0, \ldots, k_{n-1})$.

We use the same handle process as before with (k_i)-handles to get a cycle in $\mathcal{H}_r^{\mathrm{rel}\,-}$ that we still call $X(k_0, \ldots, k_{n-1})$.

Remark 3.2.15. If ϕ_i and ϕ'_i are two parametrizations of the dashed arc $D^{k_{i-1}}$, then ϕ_i and ϕ'_i are homotopic, so are the associated maps $\phi^{k_{i-1}}$ and $\phi'^{k_{i-1}}$. Then, the homology classes associated to $\phi^{k_{i-1}}$ and $\phi'^{k_{i-1}}$ are equal and this guarantees that objects are well defined.

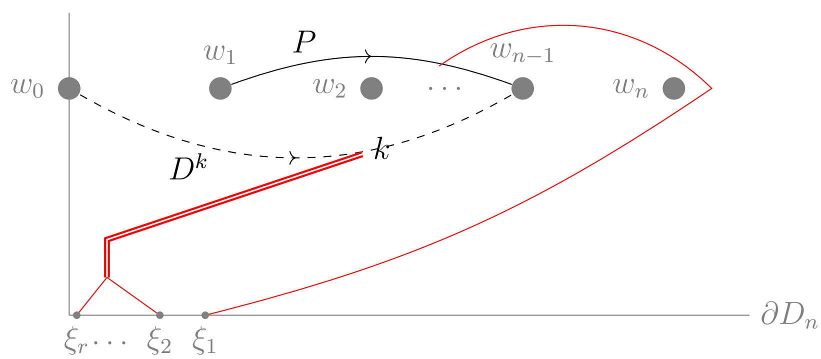
Remark 3.2.16. If ϕ^{k_1} and ϕ^{k_2} corresponds to chains with disjoint supports, there exists an associated chain $\left[\phi^{k_1} \times \phi^{k_2}\right] \in X_{k_1+k_2}$.

To be able to treat these two kinds of objects at the same time we must define classes which are a mix of these two. This is what we do in the following definition.

Definition 3.2.17 (Mixed class)**.** *Let $r \in \mathbb{N}^*$. We call an* r-mixed class $\mathcal{M}(D_1^{k_1}, \ldots, D_d^{k_d}, P_1, \ldots, P_p)$ *the following data:*

- *$d, p, k_1, \dots, k_d$ are positive integers such that $k_i > 1$ for $i = 1, \dots, d$ and $\sum k_i + p = r$.*
- *For $i = 1, \dots, d$, $D_i^{k_i}$ is a dashed black embedded arc inside D_n having endpoints in w_{j_1}, w_{j_2} for $j_1, j_2 \in \{0, \dots, n\}$. The dashed arc is indexed by k_i. It also comes with a red (k_i)-handle joining the dashed arc to $\xi_a, \xi_{a-1}, \dots, \xi_{a-k_i}$ for some $a \in \{k_i, \dots, r\}$.*
- *$P_1, \dots, P_p$ are plain black arcs joining two different w_k's. They all come with a red fork-handle joining the black arc to some ξ_a for some $a \in \{1, \dots, r\}$.*
- *The union of all handles is embedded inside D_n and all ξ_i's are reached, for $i = 1, \dots, r$.*
- *The union of all black arcs (dashed and plain) is embedded inside D_n.*

Example 3.2.18. We draw an example of an $\mathcal{M}(D^k, P)$ mixed class.



There is again a canonical way to assign a class in $\mathcal{H}_r^{\mathrm{rel}\,-}$ to an r-mixed class $\mathcal{M}(D_1^{k_1}, \dots, P_p)$. Following what we did for plain and dashed arcs before, there is a natural application:

$$(\Delta_{k_1} \times \cdots \times \Delta_{k_d} \times I^p) \to X_r$$

associated to dashed and plain arcs as follows: sending the first k_1 coordinates to $D_1^{k_1}$, next k_2's to $D_2^{k_2}$ and so on, and sending the last p coordinates as follows: one in P_1, the next one in P_2 and so on. This application defines a cycle in $C_r^{lf}(X_r, X_r^-, \mathbb{Z})$ and we choose a lift of it using the handle process. We call $\mathcal{M}(D_1^{k_1}, \dots, P_p)$ both the drawing and its associated class in $\mathcal{H}_r^{\mathrm{rel}\,-}$.

3.2.2.3 Local system homology techniques.

In this section we state three properties that will allow us to perform all the homology computations we need in the next sections. The first property deals with a change of handle for a fixed mixed class.

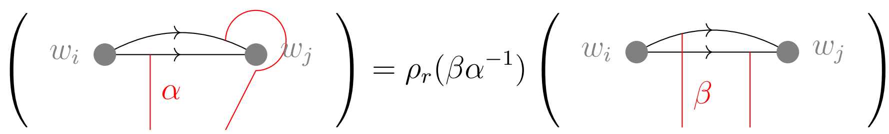
Remark 3.2.19 (Handle rule)**.** Let B be a singular locally finite r-cycle of $C_r(X_r, X_r^-, \mathbb{Z})$. We've seen a process to choose a lift of B to the homology with local coefficients in L_r, using a handle which is a path joining $\boldsymbol{\xi}$ and $x \in B$. Let α and β be two different paths joining $\boldsymbol{\xi}$

and B. Let $\widetilde{B}^{\alpha}$ and $\widetilde{B}^{\beta}$ be the lifts of B chosen using α and β respectively. By the *handle rule* we have that in $\mathcal{H}_r^{\mathrm{rel}\,-}$:

$$\widetilde{B}^{\alpha} = \rho_r(\beta\alpha^{-1})\widetilde{B}^{\beta}$$

where ρ_r is the representation of $\pi_1(X_r, \boldsymbol{\xi}^r)$ used to construct L_r in Definition 3.2.5. It expresses how the local system coordinate of a homological class is translated after a change of handle.

Example 3.2.20. We have the following equality between these 2-forks corresponding to classes in $\mathcal{H}_2^{\mathrm{rel}\,-}$:



with $\rho_r(\beta\alpha^{-1}) = tq^{-2\alpha_j}$. Indeed, we suppose that the drawing is empty everywhere outside the parenthesis besides the red handles α and β that join the base point $\boldsymbol{\xi}$ in the boundary. We suppose also that α and β follow exactly same paths outside the parenthesis. This allows us to draw the colored braid $\beta\alpha^{-1}$ in Figure 3.3.

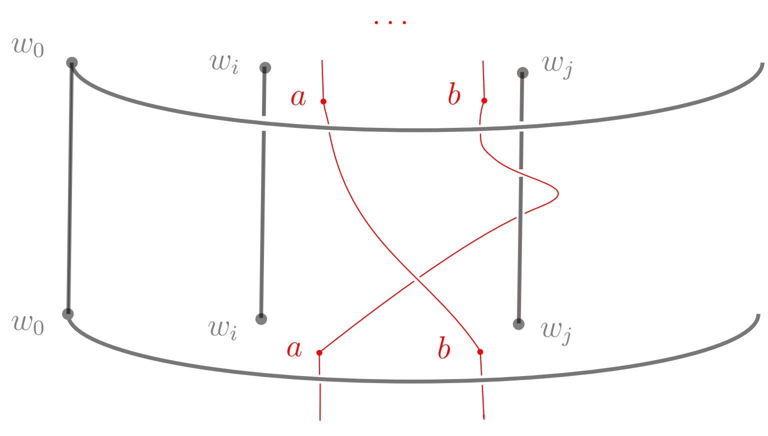


Figure 3.3 – The braid $\beta\alpha^{-1}$

The figure continues outside of this box, but as the path to the base point is the same for α and β the path upper box is the inverse of the lower one. As the local system is abelian, the out box parts of the braid won't contribute to $\rho_r(\beta\alpha^{-1})$. Considering the definition of ρ_r one sees that the local system coordinate of the above path is $tq^{-2\alpha_j}$ so is the one of $\beta\alpha^{-1}$.

We reformulate the compressing trick used in the proof of Proposition 3.2.8 in a more general version.

Proposition 3.2.21 (Compressing trick)**.** *Let $D_p \subset D_n$ (and $D_p^0 \subset D_n$ respectively) be a topological punctured disk with punctures $w_{n_1}, \ldots, w_{n_p}$ and $n_i = 1, \ldots, n$ for $i = 1, \ldots, p$ (resp.*

D_p^0 contains also w_0). Let $X_r(D_p)$ (resp. $X_r(D_p^0)$) be the space of configuration of r points inside D_p (resp D_p^0). Let D'_p (resp D'^0_p) be an ϵ-neighborhood of the segment joining the points $w_{n_1}, \ldots, w_{n_p}$ (resp. having an end in w_0) and contained in the real axis, with ϵ small enough to have $D'_p \subset D_p$. Then the morphisms:

$$H_\bullet\left(X_r(D'_p)\right) \to H_\bullet\left(X_r(D_p)\right)$$

and

$$H_\bullet\left(X_r(D'^0_p), X_r(D'^0_p)^-\right) \to H_\bullet\left(X_r(D^0_p), X_r(D^0_p)^-\right)$$

induced by inclusion are isomorphisms (the module $X_r(D'^0_p)^-$ stands for configurations with one point in w_0). All the homology modules are Borel-Moore ones (or equivalently of locally finite chains) and considered with coefficients in the local system L_r restricted to the space of interest, so that we omit it in the notations.

Proof. The proof is exactly the same as the one of Lemma 3.2.10 but performed inside D_p (resp. D_p^0). □

Proposition 3.2.22 (Combing process.). *Let $M = M(D_1^{k_1}, \ldots, D_d^{k_d}, P_1, \ldots, P_p)$ be a mixed class from Definition 3.2.17, such that the (k_0)-handle reaches $D_1^{k_1}$ in x and the handle reaching P_1 reaches it in x'. Let $P_1 = P_1^- \cup_{x'} P_1^+$ and $D_1^{k_1} = D_1^- \cup_x D_1^+$ be subdivisions of arcs following orientations. Let P (resp. D) be an arc joining x' (resp. x) to some $w \in \{w_0, \ldots, w_n\}$, and such that P and D are disjoint from all the $D_i^{k_1}$'s and the P_i's. We have the following (see Examples 3.2.23 and 3.2.24):*

Plain combing. Let M^- and M^+ be the following classes obtained from M and P:

$$M^- = M(D_1^{k_1}, \ldots, D_d^{k_d}, P_1^- \star P, P_2, \ldots, P_p)$$

$$M^+ = M(D_1^{k_1}, \ldots, D_d^{k_d}, P^{-1} \star P_1^+, P_2, \ldots, P_p)$$

where $\star$ denotes the concatenation of paths, and the handles are preserved from M. There is the following homological relation:

$$M = M^- + M^+.$$

Dashed combing. Let $l \in \{0, \ldots, k_1\}$, and M^l be the following class obtained from M and P:

$$M^l = M\left(\left(D_1^- \star D\right)^l, \left(D^{-1} \star D_1^+\right)^{k_1 - l}, D_2^{k_2}, \ldots, P_p\right)$$

so that the initial arc D_1 is divided into two, one indexed by l the other one by $k_1 - l$. Handles are preserved from M, except for the (k_1) handle tube that is divided into two tubes: one (l)-handle joining $\left(D_1^- \star D\right)^l$ in x and one $(k_1 - l)$-handle joining $\left(\left(D_1^+ \star D\right)^{-1}\right)^{k_1 - l}$ in x. There is the following homological relation:

$$M = \sum_{l=0}^{k_1} M^l.$$

Proof. We separate both cases.

Plain combing. The equality $M = M^- + M^+$ is straightforward in $H_r^{lf}(X_r, X_r^-, \mathbb{Z})$ and the lifts agree with each other as the handles are the same.

Dashed combing. Suppose the mixed class $M = M(D_1^{k_1})$ is made of only one dashed arc. Let ϕ^{k_1}:

$$\phi^{k_1} : \begin{array}{rcl} \Delta^{k_1} & \to & X_{k_1} \\ (t_1, \ldots, t_{k_1}) & \mapsto & \{\phi(t_i), i = 1, \ldots, k_1\} \end{array}$$

be the chain naturally associated with the indexed k_1 dashed arc of the considered mixed class, where ϕ is a parametrization of D^{k_1}. We subdivide the simplex: for $l \in \{0, \ldots, k_1\}$ let $\Delta^{k_1,l}$ be:

$$\Delta^{k_1,l} = \{(t_1, \ldots, t_{k_1}) \in \Delta^k \text{ s.t. } t_l < \phi^{-1}(x) < t_{l+1}\}$$

which image by ϕ^{k_1} corresponds to configurations for which the handle together with D arrive between images of t_l and t_{l+1}. Let $\phi^{k_1,l}$ be the restriction of ϕ^{k_1} to $\Delta^{k_1,l}$. Let:

$$h_t : I \to D_n$$

be an isotopy (rel. endpoints) sending the arc D^{k_1} to the right one of Figure 3.4 (arcs oriented from left to right).

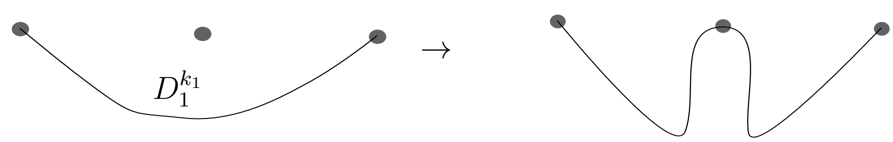


Figure 3.4 – The isotopy h_t.

For all t in I, let $\phi_t^{k_1}$ be the following map:

$$\phi_t^{k_1} : \begin{array}{rcl} \Delta^{k_1} & \to & X_{k_1} \\ (t_1, \ldots, t_{k_1}) & \mapsto & \{h_t \circ \phi(t_i), i = 1, \ldots, k_1\} \end{array}$$

and let $\phi_t^{k_1,l}$ be the following map:

$$\phi_t^{k_1,l} : \begin{array}{rcl} \Delta^{k_1,l} & \to & X_{k_1} \\ (t_1, \ldots, t_{k_1}) & \mapsto & \{h_t \circ \phi(t_i), i = 1, \ldots, k_1\}, \end{array}$$

namely the restriction to $\Delta^{k_1,l}$. Let $\left[\phi_t^{k_1}\right]$ and $\left[\phi_t^{k_1,l}\right]$ be the corresponding simplicial chains. One remarks that $\phi_0^{k_1,l} = \phi^{k_1,l}$ and $\phi_0^{k_1} = \phi^{k_1}$. In terms of chains we have the following equality holding for all $t \in I$:

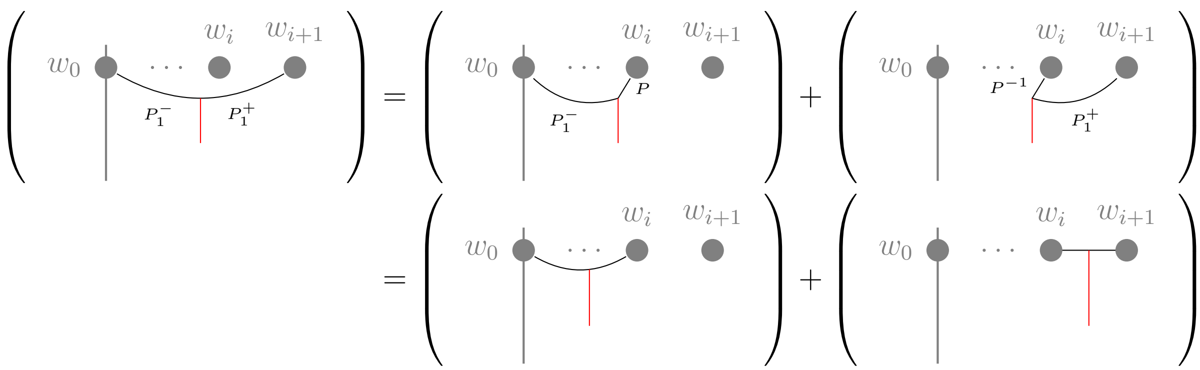
$$\left[\phi_t^{k_1}\right] = \sum_l \left[\phi_t^{k_1,l}\right],$$

this is because $\{\Delta^{k_1,l}, l = 0, \ldots, k_1\}$ is a subdivision of Δ^{k_1}. For $t = 0$ this chain is $[\phi^{k_1}]$ while for $t = 1$, terms of the sum are Borel-Moore cycles homologous to M^l. It shows that $[\phi^{k_1}]$ and $\sum_l M^l$ are homotopic so that the relation $M = \sum_{l=0}^{k_1} M^l$ holds in $H_r^{lf}(X_r, X_r^-, \mathbb{Z})$. Then - as before - the lifting process is unchanged as handles are preserved. It proves the proposition for a mixed class composed by one dashed arc, and it generalizes to all mixed class as only the first component is involved in the combing.

□

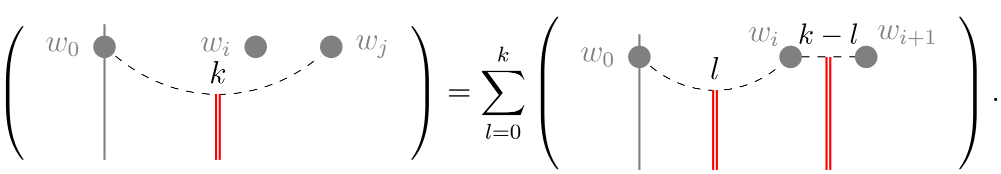
Two examples of combings that will be used many times.

Example 3.2.23 (Breaking a plain arc)**.** By considering a path joining the red handle to w_i one can check the following equality of homology class (all arcs oriented from left to right):



where drawings are the same outside the boxes. To obtain the second line we have applied small isotopies not changing the homology class. One remarks that before the small isotopies are applied, the handle is unchanged.

Example 3.2.24 (Breaking a dashed arc)**.** By considering a path joining the red handle to w_i one can check the following equality of homology class:



where drawings are the same outside the boxes.

3.2.2.4 Relations among families of homology classes.

We still call $\mathcal{F}, \mathcal{U}, \mathcal{L}, \mathcal{A}'$ the families of elements in $\mathcal{H}_r^{\text{rel}\,-}$, and we recall from Corollary 3.2.13 that $\mathcal{U}$ is a basis of $\mathcal{H}_r^{\text{rel}\,-}$ as an $\mathcal{R}$-module. In this section we will perform homology computations in $\mathcal{H}_r^{\text{rel}\,-}$.

To pass from forks to codes, we will apply the compressing trick from Proposition 3.2.21 until a fork meets a dashed arc. We will need the following model.

Notations. Since we work with Borel-Moore homology with local coefficients, we concentrate on the following complex:

$$H_\bullet\left(X_r, (X_r \setminus A_\epsilon) \cup X_r^-; L_r\right)$$

for a small ϵ, with A_ϵ defined as in the proof of Proposition 3.2.8. A dashed arc indexed by $k > 1$ corresponds to an embedding of k points (a k-simplex) inside the arc.

As the order of points does not matter - working in X_r, one can think of the dashed arc as in Figure 3.5.

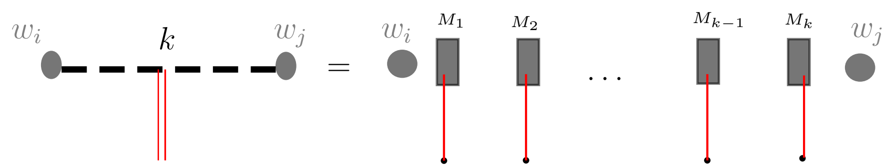


Figure 3.5 – Dashed arc model.

On the left side we see a standard piece of an element of $\mathcal{U}$ and on the right side, one can think of this element as the image of one point by the following embedding:

$$\Delta^k \to]w_i, w_j[$$

where M_i is the image of t_i, the i^{th} coordinate of Δ^k. The M_i's are represented by gray boxes to keep in mind that we work relatively to $X_r \setminus A_\epsilon$. Every point is lifted to the maximal abelian covering (L_r) using the red handle reaching it. A first diffeomorphism of D_n has been applied, allowing one to imagine this picture with w_i facing w_j. This diffeomorphism does not change homology classes.

We recall definitions of q-analogs that we will use extensively from now on.

Definition 3.2.25. *Let i be a positive integer. We define the following elements of $\mathbb{Z}\left[t^{\pm 1}\right] \subset \mathcal{R}$.*

$$(i)_t = (1 + t + \cdots + t^{i-1}) = \frac{1-t^i}{1-t},$$

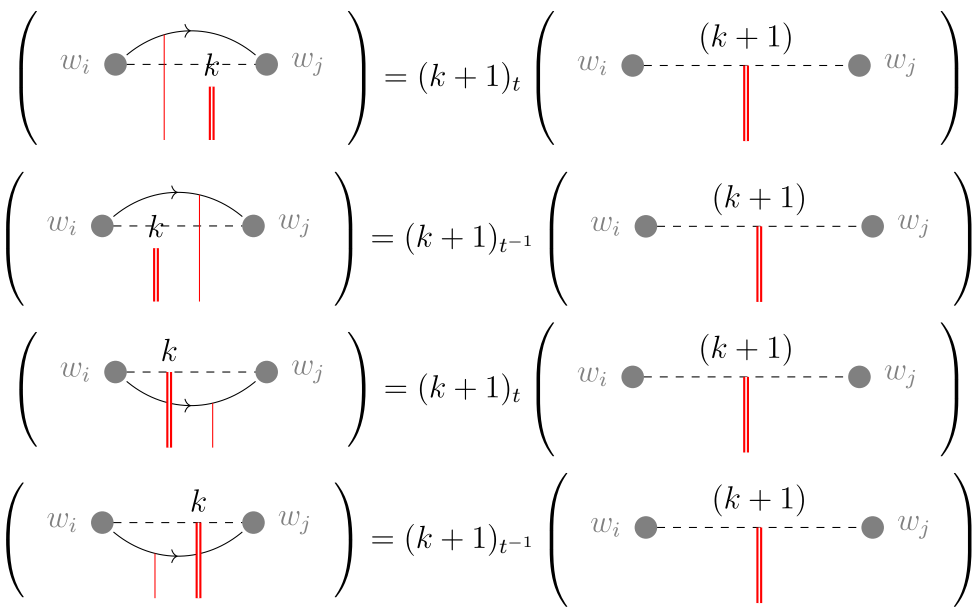
$$(k)_t! = \prod_{i=1}^{k} (i)_t,$$

$$(a,t)_n = (1-a)(1-at)\cdots(1-at^{n-1}),$$

$$\binom{k}{l}_t = \frac{(k)_t!}{(k-l)_t!(l)_t!} = \frac{(t,t)_k}{(t,t)_l(t,t)_{k-l}}.$$

We give crucial homological relations that will relate forks and codes (and later loops and arcs) and allow one to compute actions in next section.

Lemma 3.2.26. *Let $k > 1$ be an integer. The following equalities hold in $\mathcal{H}_\bullet^{rel\,-}$:*



where we suppose that the classes are the same everywhere outside the parenthesis, red handles joining same base points following same paths.

Proof. We prove the first equality - last three correspond to symmetric situations so they are proved similarly. The idea of the proof is an application of the compressing trick of Proposition 3.2.21, which consists in applying a homotopy compressing the disk until points cannot approach each other vertically anymore without meeting. Namely, let D be the disk depicted in the parenthesis. While compressing D to an open $\frac{\epsilon}{2}$-neighborhood D' of $[w_i, w_j]$, the plain arc from the top will approach the dashed arc. As we work in Borel-Moore homology, so relatively to $X_r \setminus A_\epsilon$ for a small ϵ, at some points, the point lying on the plain arc will cut the dashed arc to put its ϵ-neighborhood in. As there are k points lying on the dashed arc, there are $k+1$ possibilities of cuts (near $]w_0, M_1[,]M_1, M_2[, \ldots,]M_{k-1}, M_k[$ or $]M_k, w_j[$). The situation may be summed up as the equality of Figure 3.6. In the figure, we distinguish the point M from the plain arc coming between M_{i-1} and M_i in the sum.

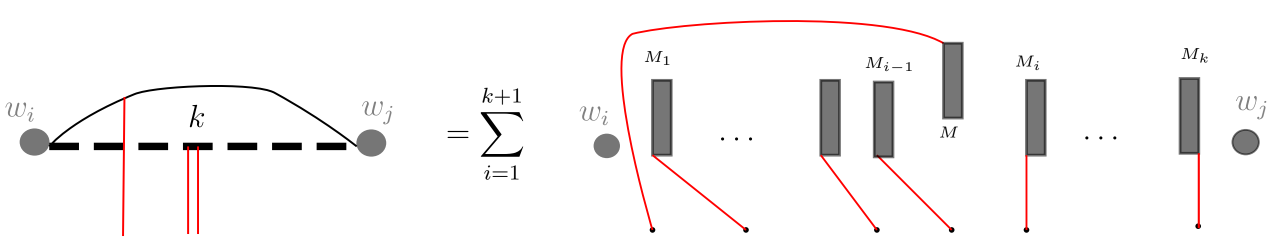


Figure 3.6 – Homological relation.

To be more precise, let ϕ^k be the chain:

$$\Delta^k \to X_k$$

associated to the indexed k dashed arc. And ψ:

$$I \to D_n$$

the one associated to the plain one. Then:

$$\Psi = \{\phi, \psi^k\} : \Delta^k \times I \to X_{k+1}$$

is the chain associated to the left object of the equality we are proving. For $i = 1, \ldots, k+1$, let Δ_i be:

$$\Delta_i = \{(t_1, \ldots, t_k, t) \in \Delta^k \times I \text{ s.t. } t_{i-1} < t < t_i\}$$

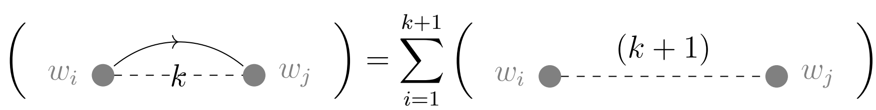
and Ψ_i be the restriction of Ψ to Δ_i. In terms of chains we have the equality:

$$[\Psi] = \sum_i [\Psi_i],$$

as the set $\{\Delta_i, i = 1, \ldots, k+1\}$ is a subdivision of $\Delta^k \times I$. Every Δ_i is naturally homeomorphic to the standard simplex Δ^{k+1}. By homotoping the plain arc to the dashed one, one obtains a homotopy from Ψ_i to ϕ^{k+1}, for all $i \in \{1, \ldots, k+1\}$. Then:

$$[\Psi] = \sum_{i=1}^{k+1} \left[\phi^{k+1}\right].$$

This shows that the relation:



holds in $H(X_r, X_r^-, \mathbb{Z})$. This can be seen as Figure 3.6 without handles. (A mixed class without handles corresponds to an unlifted homology class.)

Now it's just a matter of reorganizing the handles in the elements of the sum in Figure 3.6 to get a dashed arc model. Using the handle rule, one can check that for $i \in \{1, \ldots, k+1\}$ we have the equality of Figure 3.7 in the local system homology.

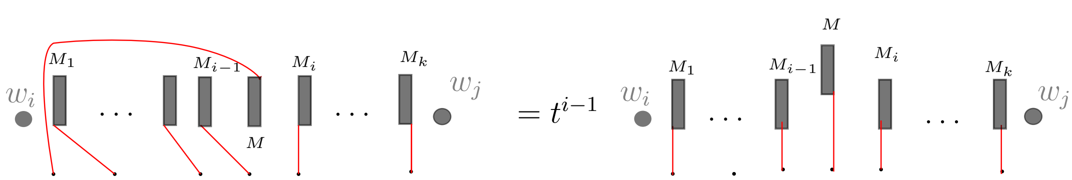


Figure 3.7 – Local system relation.

To see this, we draw the colored braid associated to this change of handle, in Figure 3.8, so that one verifies its local coordinate to be t^{i-1} (as $(i-1)$ red strands are passing successively in front of the i^{th} one).

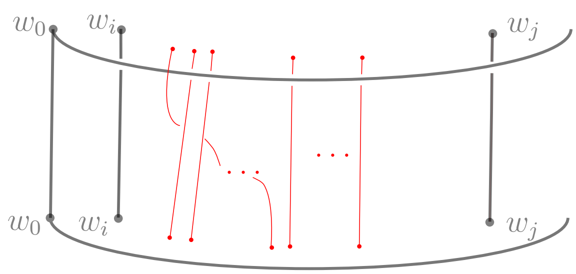
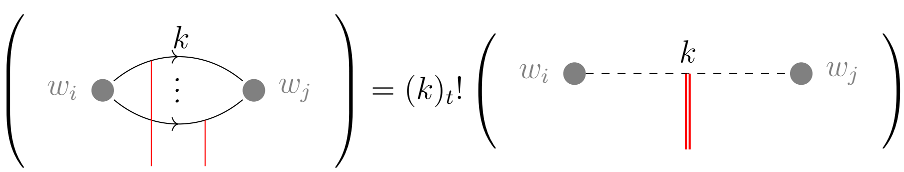


Figure 3.8 – Handle rule.

Again, in the picture, one has to imagine that the red handles are going back to the base point before and after this box following same paths so that it does not contribute to the local system coefficient. This concludes the proof of the first relation provided by the lemma. ☐

From this we deduce several corollaries. A first straightforward consequence of Lemma 3.2.26 is the relation between the families $\mathcal{F}$ and $\mathcal{U}$.

Corollary 3.2.27. *Let $k > 1$ be an integers, the following equality holds in $\mathcal{H}_{\bullet}^{rel\,-}$:*



.

Proof. The proof is made by recursion on k. The recursion property is given by Lemma 3.2.26. ☐

From this result, the proof of the following is immediate.

Corollary 3.2.28. *Let $\mathbf{k} \in E^0_{n,r}$, there is the following relation between the standards fork and code sequence associated to* $\mathbf{k}$.

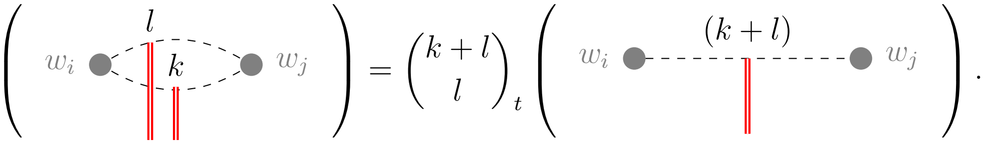
$$F(k_0, \ldots, k_{n-1}) = \left(\prod_{i=0}^{n-1} (k_i)_t! \right) U(k_0, \ldots, k_{n-1}).$$

This recovers the consequences of Kohno's theorem that can be found in [Ito], stating that the family of multiforks is generically a basis of $H_r(X_r(w_0); L_r)$. We state this precisely in our context in the following corollary.

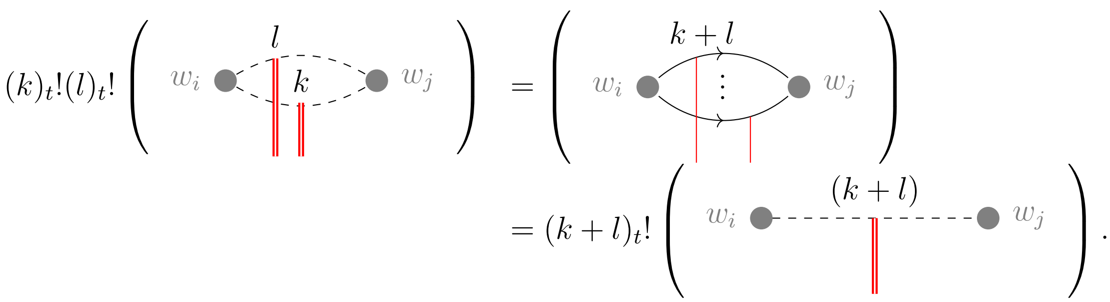
Corollary 3.2.29. *The family $\mathcal{F}$ is a basis of $H_r(X_r(w_0), L_r) = \mathcal{H}_r^{rel\,-}$ whenever one works over a ring R where all the $(i)_t!$ are invertible for i an integer lower or equal to r.*

Lemma 3.2.26 allows also one to compute the fusion between two dashed arcs.

Corollary 3.2.30. *For integers $k, l > 1$, there is the following relation between mixed classes:*



Proof. The two following equalities are direct consequences of previous Corollary 3.2.26.



One concludes using the integral equality:

$$(k+l)_t! = (k)_t!(l)_t! \binom{k+l}{l}_t$$

and simplification by $(k)_t!(l)_t!$. ◻

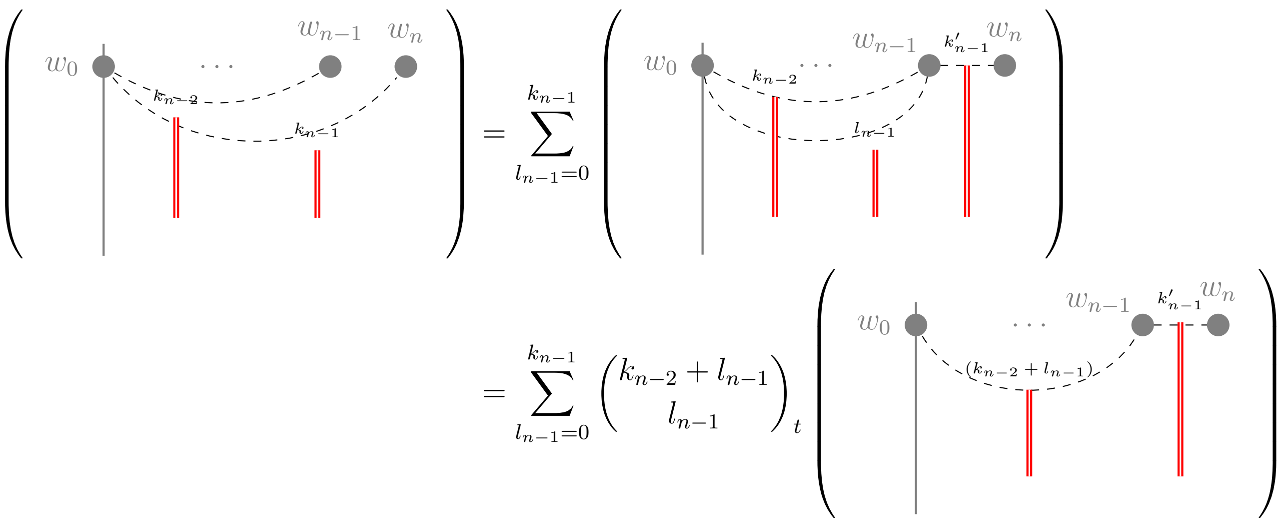
Now we prove a proposition relating multi-arcs with code sequences.

Proposition 3.2.31. *Let $\mathbf{k} \in E^0_{n,r}$. There is the following relation between the standard multi-arc and the standard code sequence associated to $\mathbf{k}$.*

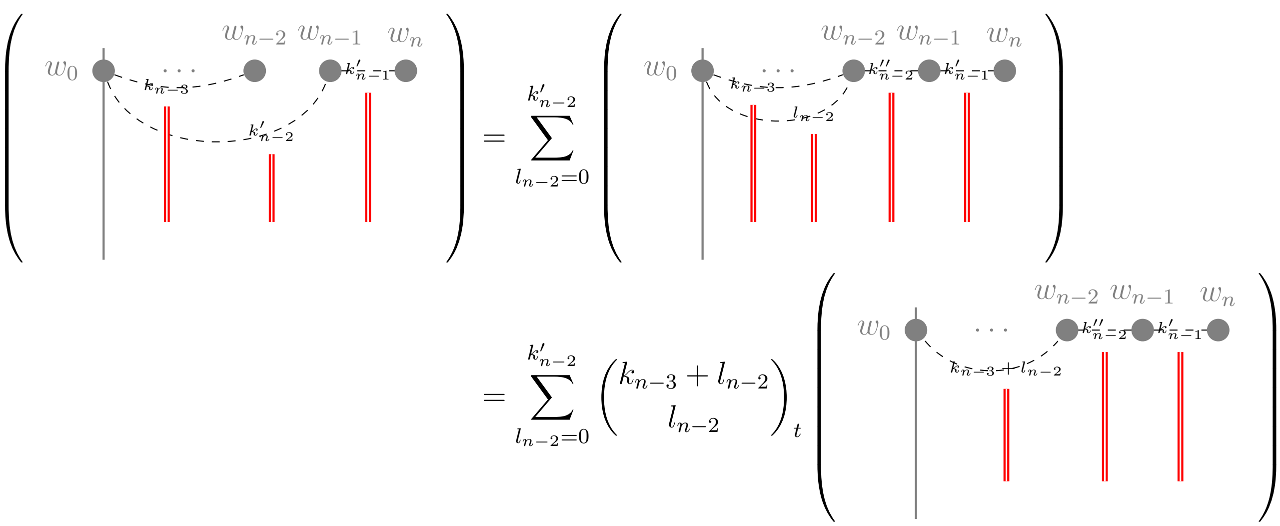
$$A'(k_0, \dots, k_{n-1}) = \sum_{l_{n-1}=0}^{k_{n-1}} \sum_{l_{n-2}=0}^{k_{n-2}+l_{n-1}} \cdots \sum_{l_1=0}^{k_1+l_2} \left(\prod_{i=0}^{n-2} \binom{k_i + l_{i+1}}{l_{i+1}}_t U(k'_0, k''_1, \dots, k''_{n-2}, k'_{n-1}) \right)$$

where $k'_0 = k_0 + l_1$, $k'_{n-1} = k_{n-1} - l_{n-1}$ and $k''_i = k_i + l_{i+1} - l_i$ for $i = 1, \dots, n-2$.

Proof. Let $\mathbf{k} \in E^0_{n,r}$ and A' its associated multi-arcs. We treat one by one the dashed arcs of A' starting by the one ending at w_n then the one ending at w_{n-1} and so on. The first step is the following:



with $k'_{n-1} = k_{n-1} - l_{n-1}$. The first equality is a breaking of dashed arc, see Example 3.2.24. The second equality is a direct application of Corollary 3.2.30. The end of the proof is an iteration of this process. Next step is the following, with $k'_{n-2} = k_{n-2} + l_{n-1}$:



where $k''_{n-2} = k'_{n-2} - l_{n-2}$. A complete iteration of this process gives the formula of the proposition. □

By looking at the diagonal terms of the matrix expressing mutli-arcs in the code sequence basis, one gets the following corollary.

Corollary 3.2.32 (Basis of multi-arcs)**.** *The family $\mathcal{A}'$ of multi-arcs is a basis of $\mathcal{H}_r^{rel\,-}$ as an $\mathcal{R}$-module.*

Proof. Let $E^0_{n,r}$ being given the lexical order. This yields an order on families $\mathcal{A}'$ and $\mathcal{U}$. One can see from Proposition 3.2.31 that with this order, the matrix expressing multi-arcs in the code sequence basis is upper-triangular. The determinant of this matrix is given by the product of diagonal terms. The diagonal terms are the binomial in the sum of the formula of Proposition 3.2.31 corresponding to $l_i = 0$ for all $i \in \{1, \dots, n-1\}$. In these cases, the binomials are equal to 1 so that the determinant of the matrix is 1. As $\mathcal{U}$ is a basis and the change of basis determinant is invertible, the proof is complete. □

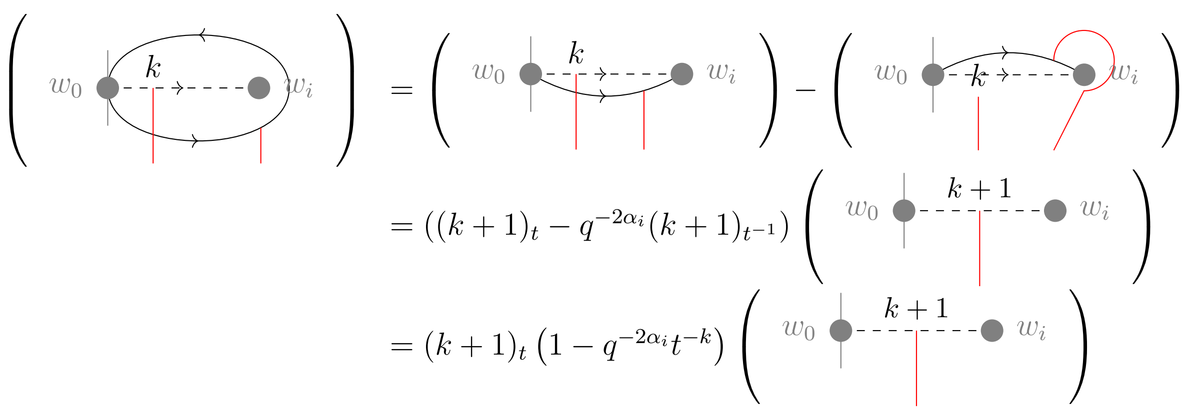
The last proposition of this section relates the r-loops family to the multi-arcs one, which concludes the total picture of relations between the four standards families indexed by $E^0_{n,r}$ present in the literature.

Proposition 3.2.33. *Let* $\mathbf{k} \in E^0_{n,r}$. *There is the following relation between the standards multi-arcs and r-loops associated to* $\mathbf{k}$.

$$L(k_0, \dots, k_{n-1}) = \left(\prod_{i=0}^{n-1} (k_i)_t! \prod_{k=0}^{k_i} \left(1 - q^{-2\alpha_i} t^{-k}\right) \right) A'(k_0, \dots, k_{n-1}).$$

Proof.

Remark 3.2.34. First we observe the following equalities between homology classes:



where everything stands inside a small neighborhood of the picture, without perturbating the rest of the class contained outside of it. The first equality comes from a breaking of plain arc, see Example 3.2.23. The second one is a consequence first of the application of a handle rule to get vertical handles, and then relations of Lemma 3.2.26.

To prove the proposition, one treats separately the loops winding around w_1, from those winding around w_2 etc. Every case is a straightforward recursion, by the above remark, and leads to the formula of the proposition. □

This answers Conjecture 6.1 of [F-W]. In fact it is a more precise statement saying exactly under which conditions the family of r-loops is a basis of the homology.

Corollary 3.2.35 ([F-W, Conjecture 6.1])**.** *If R is a ring in which all the* $(1 - q^{-2\alpha_i} t^{-k})$ *are invertible for all* $i = 1, \dots, n$ *and so are all the* $(k)_t!$ *(for* $k \leq r$*), then* $\mathcal{H}^{rel\,-}_r$ *is a free R-module with the family* $\mathcal{L}$ *of r-loops as basis.*

Actually, the lifts of the r-loops chosen in [F-W] are not exactly the same as ours, namely the handles we've chosen do not correspond to their choice of lift. But, as we have seen, a change of lift corresponds to the multiplication by an invertible monomial of $\mathcal{R}$, so that the conditions to be a basis are the same.

3.2.3 Homological representation of $U_q\mathfrak{sl}(2)$

The goal of this section is to define homological operators E, F and K, acting over $\bigoplus_{r\in\mathbb{N}^*}\mathcal{H}_r^{\text{rel }-}$ and to verify that they realize an algebra representation of $U_q\mathfrak{sl}(2)$. In this section we will need other quantum numbers.

Definition 3.2.36. *Let i be a positive integer. We define the following elements of $\mathbb{Z}\left[q^{\pm 1}\right]$.*

$$[i]_q = \frac{q^i - q^{-i}}{q - q^{-1}},$$

$$[k]_q! = \prod_{i=1}^{k} [i]_q ,$$

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[k-l]_q!\,[l]_q!}.$$

Remark 3.2.37. Let $t = q^{-2}$, so that the following relations hold in $\mathbb{Z}\left[q^{\pm 1}\right]$:

$$(i)_t = q^{1-i}\,[i]_q ,$$

$$(k+1)_t! = q^{\frac{-k(k-1)}{2}}\,[k+1]_q!,$$

$$\binom{k+l}{l}_t = q^{-kl}\begin{bmatrix} k+l \\ l \end{bmatrix}_q .$$

3.2.3.1 Action of $F^{(1)}$, and its divided powers.

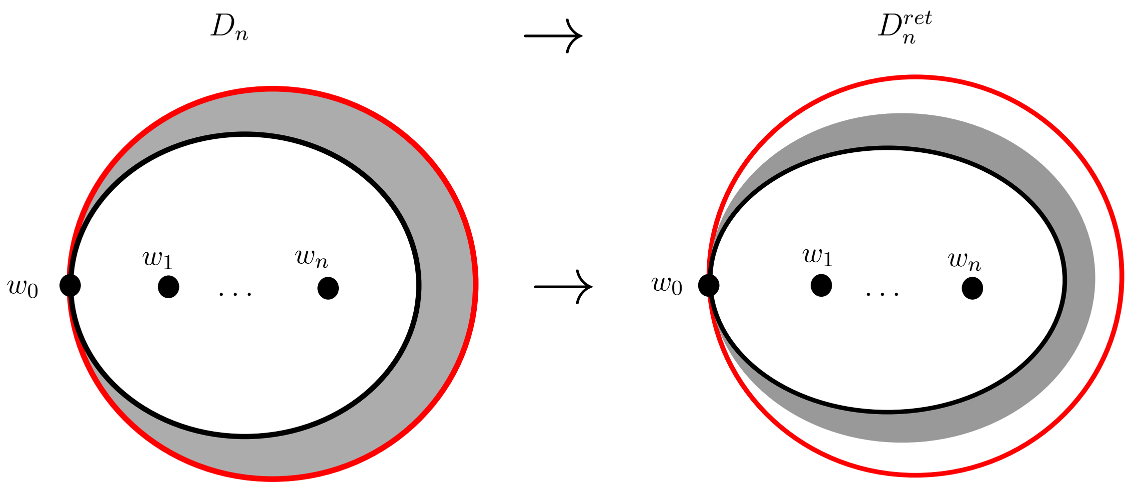
We start with $F^{(1)}$, we are looking for an operator from $\mathcal{H}_r^{\text{rel }-}$ to $\mathcal{H}_{r+1}^{\text{rel }-}$. We need to increase by one the degree of a chain while passing from X_r to X_{r+1} for the topological space.

Remark 3.2.38 (Collar retraction)**.** Let ret be the following continuous map from the left disk (D_n) with red boundary to the right one (D_n^{ret}) consisting in compressing the gray part, while keeping the interior white part fixed.

There exists a homotopy sending D_n to D_n^{ret}. One can perform the retraction at the level of configuration space component by component so that the continuous map:

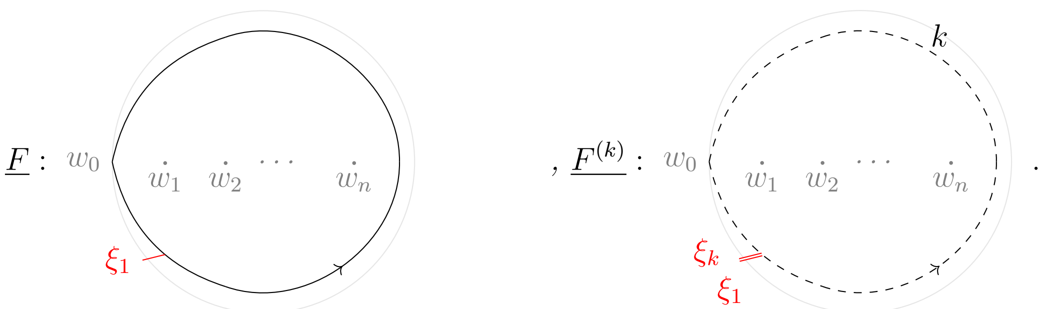
$$ret : \left\{ \begin{array}{rcl} X_r & \to & X_r^{ret} \\ \{x_1, \ldots, x_r\} & \mapsto & \{ret(x_1), \ldots, ret(x_r)\} \end{array} \right.$$

makes X_r and X_r^{ret} homotopically equivalent. This last fact together with the fact that ret preserves X_r^- ensures that ret induces the homology endomorphism $ret_* = \text{Id} \in \text{End}\left(H_\bullet(X_r, X_r^-, \mathbb{Z})\right)$.



We recall that the base point of X_{r+1} satisfies $\Re(w_0) < \Re(\xi_{r+1}) < \Re(\xi_r) < \ldots < \Re(\xi_1) < \Re(w_1)$, so that the last $(r+1)$ base point's coordinate is on the left side of others.

Definition 3.2.39 (Tines, Handles)**.** *Let $k > 2$ be an integer, $\underline{F}$ and $\underline{F^{(k)}}$ be the following cycles in $\mathcal{H}$:*



Let $T(\underline{F}) : I \to D_n$ and $T(\underline{F^{(k)}}) : I^k \to D_n$ embeddings of I and I^k in the black arcs (plain and dashed respectively) of the pictures, namely small deformations (pushing inside D_n) of the arc running once along the boundary. Let $H(\underline{F}) : I \to D_n$ and $H(\underline{F^{(k)}}) : I^k \to D_n$ their red handle and (k)-handle respectively, as in the picture.

We will define the operators by suitably "adding" the classes of the above objects, as follows.

Let $\widetilde{c}$ be a chain in $\mathrm{C}_r^{\mathrm{rel}\,-} = \mathrm{C}_r^{lf}(X_r, X_r^-; L_r)$, and $\widetilde{\boldsymbol{\gamma}} = \{\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_r\}$ be a path from $\widetilde{c}$ to $\widetilde{\boldsymbol{\xi}^r}$, the base point's lift. Let $c \in \mathrm{C}_r^{lf}(X_r, X_r^-; \mathbb{Z})$ be the image of $\widetilde{c}$ under the covering map ρ_r, and $\boldsymbol{\gamma}$ the image of $\widetilde{\boldsymbol{\gamma}}$ under the covering map. The chain $\widetilde{c}$ is fully determined by the data c and $\boldsymbol{\gamma}$.

We define the following morphisms at the level of locally finite chain complex with local coefficients:

$$F' : \left\{ \begin{array}{ccc} \mathrm{C}_r^{\mathrm{rel}\,-} & \to & \mathrm{C}_{r+1}^{\mathrm{rel}\,-} \\ \widetilde{c} = (c, \boldsymbol{\gamma}) & \mapsto & F'(\widetilde{c}) = \left[\left(ret(c), T(\underline{F}) \right), \left(\boldsymbol{\gamma} \cup H(\underline{F}) \right) \right] \end{array} \right.$$

$$F'^{(k)} : \begin{cases} \mathrm{C}_r^{\mathrm{rel}\,-} & \to & \mathrm{C}_{r+k}^{\mathrm{rel}\,-} \\ \widetilde{c} = (c, \boldsymbol{\gamma}) & \mapsto & F'(\widetilde{c}) = \left[\left(ret(c), T(\underline{F^{(k)}}) \right), \left(\boldsymbol{\gamma} \cup H(\underline{F^{(k)}}) \right) \right] \end{cases}$$

Remark 3.2.40 (Base point)**.** We mention the fact that the path $\boldsymbol{\gamma}$ starts at $\{\xi_1, \ldots, \xi_r\}$ in the space $(X_r, \boldsymbol{\xi}^r)$ but, thanks to the right shift of base point, $\boldsymbol{\gamma}$ reaches $\{\xi_2, \ldots, \xi_{r+1}\}$ in $(X_{r+1}, \boldsymbol{\xi}^{r+1})$. As the handle of $\underline{F}$ is attached to ξ_1, all the configuration point of $\boldsymbol{\xi}^{r+1}$ are reached by $(\boldsymbol{\gamma} \cup H(\underline{F}))$.

Proposition 3.2.41. *The morphisms F' and $F'^{(k)}$ are well defined. Moreover they pass to homology morphisms:*

$$F' : \mathcal{H}_r^{rel\,-} \to \mathcal{H}_{r+1}^{rel\,-} \text{ and } F'^{(k)} : \mathcal{H}_r^{rel\,-} \to \mathcal{H}_{r+k}^{rel\,-}.$$

Proof. The application of ret to c makes the chain $(ret(c), T)$ and the path $(\boldsymbol{\gamma} \cup H)$ well defined in X_r, as the configuration coordinates are disjoint. This shows that the chain morphisms are well defined, see Remark 3.2.16 that deals with product of chains with disjoint supports. They pass to homology thanks to the fact that we make the product of chains (as Remark 3.2.16) with $T(\underline{F})$ or $T(\underline{F^{(k)}})$ which are cycles in $C_1^{lf}(X_r, X_r^-, \mathbb{Z})$ and $C_k^{lf}(X_r, X_r^-, \mathbb{Z})$ respectively (as lines going to infinity points, see Remarks 5.1.3). □

Definition 3.2.42 (Action of F)**.** *Let $k > 2$ be an integer. We define the following family of operators:*

$$F^{(1)} = q^{\sum_{i=1}^n \alpha_i} F' \in \mathrm{Hom}_{\mathcal{R}} \left(\mathcal{H}_r^{rel\,-}, \mathcal{H}_{r+1}^{rel\,-} \right) \text{ and } F^{(k)} = q^{k(1-k)/2} q^{k \sum_{i=1}^n \alpha_i} F'^{(k)} \in \mathrm{Hom}_{\mathcal{R}} \left(\mathcal{H}_r^{rel\,-}, \mathcal{H}_{r+k}^{rel\,-} \right)$$

One last proposition that justifies the *divided powers* denomination.

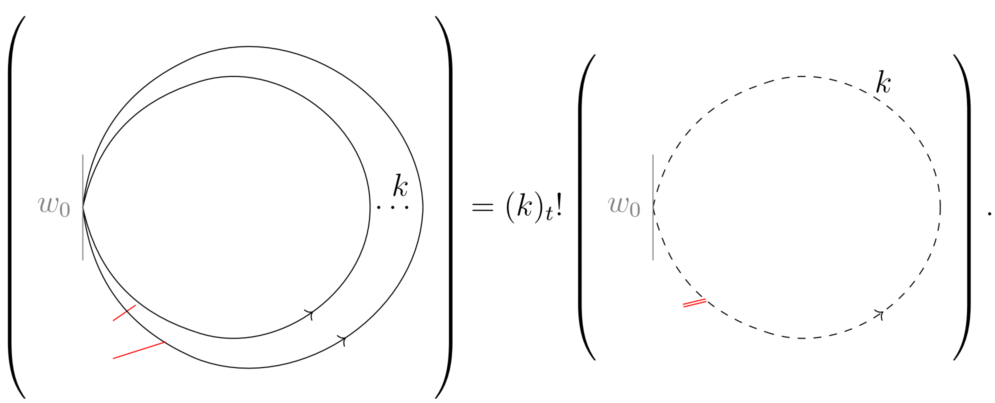
Proposition 3.2.43 (Divided powers of F)**.** *There is the following relation between elements of $\mathrm{Hom}_{\mathcal{R}} \left(\mathcal{H}_r^{rel\,-}, \mathcal{H}_{r+k}^{rel\,-} \right)$:*

$$(F^{(1)})^k = q^{k(k-1)/2} (k)_t! F^{(k)}.$$

Let $t = q^{-2}$, then:

$$(F^{(1)})^k = [k]_q! F^{(k)}$$

Proof. This is a direct consequence of the following equality of mixed class:



which can be proved as Corollary 3.2.27, and whatever stands inside the circles. This shows that $F'^k = (k)_t! F'^{(k)}$, and the first statement is immediate. To get the second equality, for $t = q^{-2}$ one uses directly Remark 3.2.37. □

3.2.3.2 Actions of E and K

We recall the definition of the isomorphism that adds a configuration point in w_0 before shifting it slightly to the right, see Remark 3.2.1.2:

$$\Phi^r : \quad \pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r) \quad \to \quad \pi_1(X_{r+1}^-, \boldsymbol{\xi}^{r+1}) \quad .$$

Lemma 3.2.44. *The morphism Φ^r lifted to the local system level:*

$$\Phi^r : L_r \upharpoonright_{X_r \setminus X_r^-} \to L_{r+1} \upharpoonright_{X_{r+1}^-}$$

is an isomorphism of local systems.

Proof. Let ρ_r be the representation of $\pi_1(X_r, \boldsymbol{\xi}^r)$ providing the local system L_r. The following diagram is commutative:

$$\begin{array}{ccc} \pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r) & \xrightarrow{\Phi^r} & \pi_1(X_{r+1}^-, \boldsymbol{\xi}^{r+1}) \\ \downarrow \rho_r & & \downarrow \rho_{r+1} \\ \mathbb{Z}^{n+1} = \bigoplus_{i \in \{1,\ldots,n\}} \mathbb{Z}\langle q^{\alpha_i} \rangle \oplus \mathbb{Z}\langle t \rangle & \xrightarrow{\mathrm{Id}} & \bigoplus_{i \in \{1,\ldots,n\}} \mathbb{Z}\langle q^{\alpha_i} \rangle \oplus \mathbb{Z}\langle t \rangle \end{array}$$

which proves the lemma. The commutation is easy to verify thinking of the presentation of $\pi_1(X_r \setminus X_r^-, \boldsymbol{\xi}^r)$ given in Remark 3.2.3. The morphism Φ^r simply adds a straight strands to the braid, not modifying its image by ρ_r. □

Remark 3.2.45. This remark is a recall. We have the following equality:

$$H_{r-1}(X_{r-1} \setminus X_{r-1}^-; L_{r-1}) = H_{r-1}(X_{r-1}(w_0); L_{r-1}) = \mathcal{H}_{r-1}^{\mathrm{rel}\,-} .$$

where $X_r(w_0)$ is the space of configurations of X_r with no coordinates in w_0. The first equality is the fact that $X_{r-1} \setminus X_{r-1}^-$ and $X_{r-1}(w_0)$ are canonically homeomorphic. The second one is Corollary 3.2.13.

From this identification one is able to define an operator E as in the following definition.

Definition 3.2.46 (Action of E)**.** *Let E be the operator defined as follows (we define its opposite $-E$):*

$$-E : \mathcal{H}_r^{rel\,-} \xrightarrow{\partial_*} H_{r-1}(X_r^-; L_r) \xrightarrow{(\Phi^r)^{-1}} H_{r-1}(X_{r-1} \setminus X_{r-1}^-; L_{r-1}) = \mathcal{H}_{r-1}^{rel\,-} .$$

The arrow ∂_* is the boundary map of the exact sequence of the pair (X_r, X_r^-). The arrow $(\Phi^r)^{-1}$ is the inverse of the isomorphism from Lemma 3.2.44 and the last equality is the above Remark 3.2.45.

The above definition states that E is (the opposite of) the boundary map of the relative exact sequence of the pair involved, everything else is just isomorphic identifications of homology modules. It means that E reads the part of the boundary that lives in X_r^-. We give a trivial example of computation with a standard code sequence.

Example 3.2.47 (Action of E on a code sequence)**.** Let $\mathbf{k} = (k_0, \ldots, k_{n-1}) \in E^0_{n,r}$, and $U_{\mathbf{k}}$ its associated standard code sequence. One can check the following property:

$$E \cdot U_{\mathbf{k}} = U(k_0 - 1, \ldots, k_{n-1}).$$

Consider first $U(k_0, 0, \ldots, 0)$ and let ϕ^{k_0} be the chain associated to the indexed k_0 dashed arc. We recall our definition of the standard simplex:

$$\Delta^{k_0} = \{0 \leq t_1 \leq \cdots \leq t_{k_0}\}$$

so that its only boundary part sent to configurations with one coordinate in w_0 is $\{t_1 = 0\} \in \Delta^k$. Remarking that ϕ^{k_0} restricted to $\{t_1 = 0\}$ is ϕ^{k_0-1}, one sees that the equality holds at the level of homology over $\mathbb{Z}$. To deal with the handle rule lifting process, we remark that only the leftmost configuration point embedded in $U(k_0, \ldots, k_{n-1})$ can join w_0. This is saying that the only part of the boundary of $U(k_0, \ldots, k_{n-1})$ lying in X_r^- corresponds to the leftmost point being in w_0. No local coefficient appears while appling $(\Phi^r)^{-1}$ (Lemma 3.2.44) thanks to the fact that the handle joining the leftmost configuration point is the leftmost handle, and it joins ξ_r, namely the leftmost base point's coordinate. Another way to say this is by remarking that the path following the leftmost handle, then going to w_0 along $U_{\mathbf{k}}$ then back to ξ_r along the boundary can be homotoped to w_0 without perturbating other handles.

The action of the operator K is a diagonal action encoding the value of r.

Definition 3.2.48 (Action of K)**.** *For $r \in \mathbb{N}^*$, the operator K is the following diagonal action over $\mathcal{H}_r^{rel\,-}$:*

$$K = q^{\sum_i \alpha_i} t^r \mathrm{Id}_{\mathcal{H}_r^{rel\,-}}.$$

We define the operator K^{-1} to be the inverse of K.

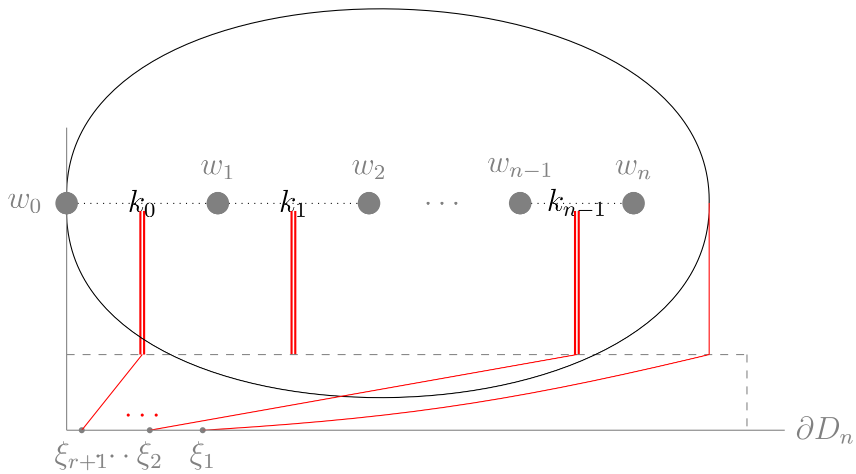
3.2.3.3 Homological $U_q\mathfrak{sl}(2)$ representation.

Let $\mathcal{H} = \bigoplus_{r \in \mathbb{N}^*} \mathcal{H}_r^{\mathrm{rel}\,-}$, the actions of $E, F^{(1)}$ and K are endomorphisms of $\mathcal{H}$. We have the following proposition.

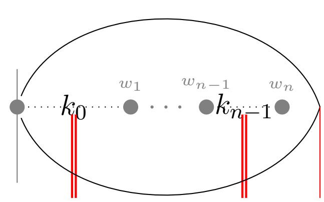
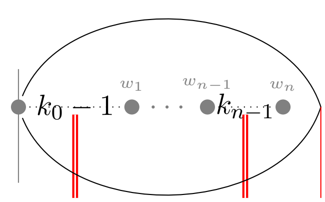
Proposition 3.2.49. *The operators $E, F^{(1)}$ and K satisfy the following relations:*

$$\begin{aligned} KE &= t^{-1}EK \\ KF^{(1)} &= tF^{(1)}K \\ \left[E, F^{(1)}\right] &= K - K^{-1}. \end{aligned}$$

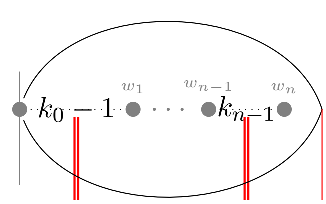
Proof. The first two relations are direct consequences of both facts that $F^{(1)}$ increases r by one, E decreases it by one, and of the definition of K. It remains to prove the last one. The proof can be performed without considering basis of $\mathcal{H}$, although we do it here using the basis of code sequences for an easier reading. Let $r \in \mathbb{N}^*$, we recall that $\mathcal{U} = (U_{\mathbf{k}})_{\mathbf{k} \in E^0_{n,r}}$ is a basis of $\mathcal{H}_r^{\mathrm{rel}\,-}$ as an $\mathcal{R}$-module. Let $\mathbf{k} = (k_0, \ldots, k_{n-1}) \in E^0_{n,r}$. First we compute the commutation between E and F' before renormalizing F' to $F^{(1)}$. The class $F' \cdot (U_{\mathbf{k}})$ corresponds to the following mixed one:



Applying $-E$ to this mixed class gives the part of its boundary lying in X_r^-. There are $r+1$ intervals embedded in this mixed class, r of them in the dashed arcs, and the last one in the plain arc. The part of the boundary lying in X_r^- is the sum of the leftmost point of the dashed arc going to w_0 and of the two boundary part of the plain arc that are in w_0. This corresponds to the following equality.

$-E \cdot \Bigg($  $\Bigg) = \Bigg($  $\Bigg) + C \times U(k_0, \ldots, k_{n-1})$

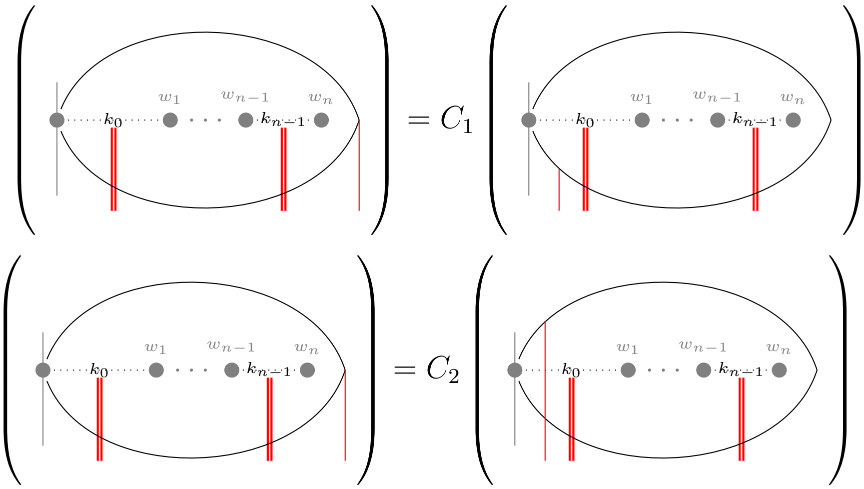
where the coefficient C is the computation of the boundary of the plain arc, $T(\underline{F})$. One can check easily that:

$\Bigg($  $\Bigg) = F' \cdot (E \cdot U(k_0, \ldots, k_{n-1}))$

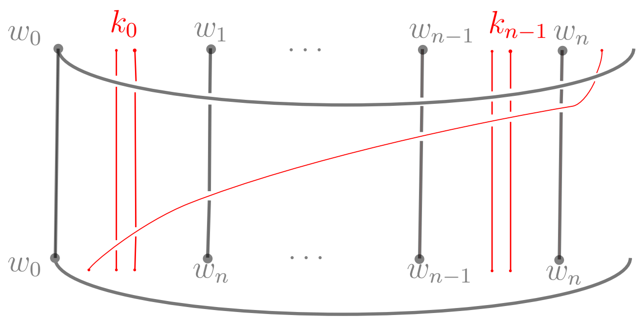
using Example 3.2.47. We also mention that this term is zero if $k_0 = 0$. This gives:

$$-[E, F'] \cdot U(k_0, \ldots, k_{n-1}) = C \times U(k_0, \ldots, k_{n-1})$$

so that it remains to compute the coefficient C. The coefficient C is the difference $C_2 - C_1$ where C_1 and C_2 satisfy the following equations:



We compute them using the handle rule. The coefficient C_1 corresponds to the local system coefficient of the following colored braid:



while C_2 corresponds to the colored braid with the red front strand passing in the back. We remark that in the braid picture we got rid of the parts of red handles that live outside the parenthesis. Outside the parenthesis, the paths consist in going to the base point without crossing each other staying in front of the w_i's, so that upper and lower the box, the contributions to the handle local system coefficient balance each other. Then it is straightforward to compute the local system coefficient of these braids, we get:

$$C_1 = t^{\sum_{i=0}^{n-1} k_i} = t^r \ , \ C_2 = t^{-r} q^{-2\sum_{i=1}^{n} \alpha_i}$$

so that:

$$-\left[E, F'\right] \cdot U(k_0, \ldots, k_{n-1}) = \left(t^{-r} q^{-2\sum_{i=1}^{n} \alpha_i} - t^r \right) \times U(k_0, \ldots, k_{n-1}).$$

We recall that:

$$\left[E, F^{(1)}\right] = q^{\sum \alpha_i} \left[E, F'\right]$$

which concludes the proof. $\square$

Theorem 4. *Let $q^{-2} = t$. The infinite module $\mathcal{H}$ together with the above described action of $E, F^{(1)}, K^{\pm 1}$ and $F^{(k)}$ for $k \geq 2$ yields a representation of the integral algebra $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ (see Section 1.4.3).*

Proof. The algebra $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ is presented in Section 1.4.3, Definition 1.4.13. We use same notations (from Section 1.4.3) for generators and we recover the same relations. Namely, the relations between E, $F^{(1)}$ and $K^{\pm 1}$ are recovered using Proposition 3.2.49, while the fact that $F^{(k)}$ are the so called divided powers of $F^{(1)}$, see Proposition 3.2.43, ensures that the relations involving them hold. The algebra $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ is presented in the literature in [Hab] and [J-K]. □

Remark 3.2.50. Even if it is not necessary to prove them knowing Proposition 3.2.43 (divided power property), we can check homologically the relations involving the divided powers of $F^{(1)}$ (relations introduced in Remark 1.4.14). Namely:

$$\left[E, F^{(n+1)}\right] = F^{(n)}\left(q^{-n}K - q^n K^{-1}\right)$$

is a simple computation of the relative boundary of a mixed class as in the proof of Proposition 3.2.49. While:

$$F^{(n)}F^{(m)} = \begin{bmatrix} n \\ n+m \end{bmatrix}_q F^{(n+m)}$$

is a direct consequence of the homological Corollary 3.2.30.

We have a complete homological description of the relations holding in $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$.

Remark 3.2.51. Using Proposition 3.2.49, one sees that we have a representation of the *simply connected* rational version of $U_q\mathfrak{sl}(2)$, for which are introduced generators that correspond to square roots of K and K^{-1}. See [DCP, § 9], Remark 2.2 of [Bas], or [C-P, § 9.1] for information about this version of $U_q\mathfrak{sl}(2)$.

The following corollary will be a key fact to recover T. Kohno's Theorem from [K1] and [K2] relating sub-modules of $U_q\mathfrak{sl}(2)$ corresponding to the kernel of the action of E (usually called *highest weight modules*) to homology modules.

Corollary 3.2.52. *Under the condition $q^{-2} = t$, the restriction of the braid representation to the kernel of the action of E yields a sub-module of $\mathcal{H}$ isomorphic to $\mathcal{H}^E = \bigoplus_{r\in\mathbb{N}^*} H_r^{BM}(X_r; L_r)$.*

Proof. For $r \in \mathbb{N}^*$, the relative long exact sequence of pairs gives this exact sequence of morphisms:

$$H_r(X_r^-; L_r) \longrightarrow H_r(X_r; L_r) \longrightarrow \mathcal{H}_r^{\text{rel}\,-} \xrightarrow{\partial_*} H_{r-1}(X_r^-; L_r) \longrightarrow H_{r-1}(X_r; L_r)$$

where we have avoided the notation BM as everything is Borel-Moore homology here. Using Lemma 3.1 of [Big1] one gets that $H_{r-1}(X_r; L_r)$ vanishes while Remark 3.2.45 implies that $H_r(X_r^-; L_r)$ vanishes. This provides a short exact sequence:

$$H_r(X_r; L_r) \longrightarrow \mathcal{H}_r^{\text{rel}\,-} \xrightarrow{\partial_*} H_{r-1}(X_r^-; L_r).$$

The kernel of the action of E is exactly the kernel of the map ∂_*. This implies the corollary, as the kernel of the action of E is isomorphic to the module of absolute homology. □

3.2.4 Computation of the $U_q\mathfrak{sl}(2)$ action

In this section we compute the action of the operators $E, F^{(1)}$ and K in the basis of multi arcs, in order to recognize the representation of $U_q\mathfrak{sl}(2)$ obtained over $\mathcal{H}$. First of all we define a normalized version of the multi-arc basis.

Definition 3.2.53 (Normalized multi-arcs.)**.** *Let* $\mathbf{k} \in E^0_{n,r}$, *let* $A(k_0, \ldots, k_{n-1})$ *be the following element of* $\mathcal{H}^{rel\,-}_r$*:*

$$A(\mathbf{k}) = q^{\alpha_1(k_1+\cdots+k_{n-1})+\alpha_2(k_2+\cdots+k_{n-1})+\cdots+\alpha_{n-1}k_{n-1}} A'(\mathbf{k}).$$

Let $\mathcal{A} = (A(\mathbf{k}))_{\mathbf{k}\in E^0_{n,r}}$ *be the corresponding family indexed by* $E^0_{n,r}$*. By convention,* $A(k_0, \ldots, k_{n-1})$ *is defined to be* $0 \in \mathcal{H}^{rel\,-}_r$ *when ever* $k_i = -1$ *for some* $i \in \{0, \ldots, n-1\}$*.*

Remark 3.2.54. The family $\mathcal{A}$ is obtained from $\mathcal{A}'$ by a diagonal matrix of invertible coefficients in $\mathcal{R}$ so that $\mathcal{A}$ is still a basis of $\mathcal{H}^{\text{rel}\,-}_r$ as an $\mathcal{R}$-module.

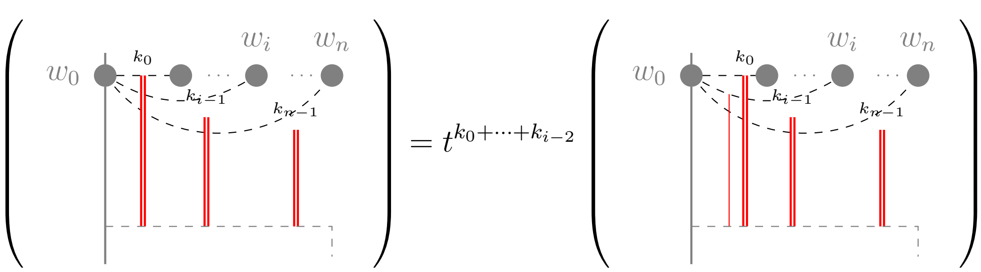
Remark 3.2.55. The same normalization allows one to pass from the r-loops family defined in [F-W] to the one defined in Section 3.2.2.2 of this work. This fact will allow us to deal with the homological conjectures of [F-W] using our normalization.

We are going to compute the action of operators in this basis, and will see that it recovers a well known basis of $U_q\mathfrak{sl}(2)$ Verma-modules.

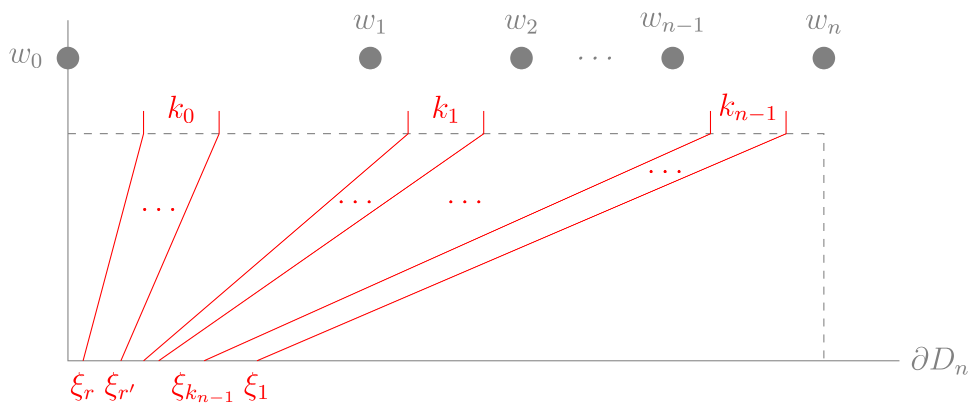
3.2.4.1 Action of E.

First we need a lemma to reorganize handles.

Lemma 3.2.56. *Let* $A'(k_0, \ldots, k_{n-1})$ *be the standard multi-arc associated with* $(k_0, \ldots, k_{n-1}) \in E^0_{n,r}$*. For* $i = 1, \ldots, n$*, it is subject to the following relation holding in* $\mathcal{H}^{rel\,-}_r$*:*



where, in the right term, only one component of the red tube indexed by k_i *had been moved to the extreme left of other red handles. Namely only the leftmost handle composing the* (k_i)*-handle (tube of* k_i *parallel handles) had been pushed to the left of the* (k_0)*-handle. Down* *the parenthesis, red handles are joining the base point following a usual dashed box, without crossing with each other. The left class follows this box:*



while the right one has the leftmost single handle following the leftmost path of the above dashed box. All other handles are right shifted.

Proof. It is a straightforward consequence of the handle rule. The braid involved is drawn in Figure 3.9, so that one sees the local system coefficient (we did not draw the the punctures as they don't play any role).

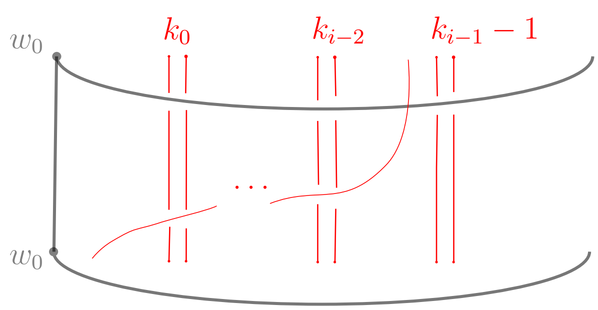


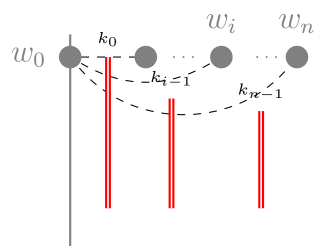
Figure 3.9 – Handle rule.

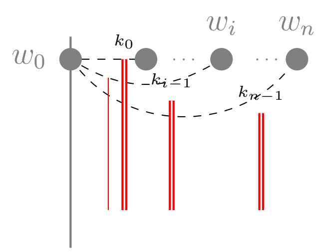
□

Lemma 3.2.57. *For any* $\mathbf{k} = (k_0, \ldots, k_{n-1}) \in E^0_{n,r}$*, the action of* E *over the standard multi arcs is the following:*

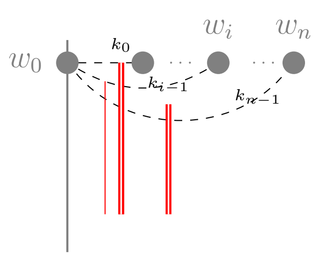
$$E \cdot A'(k_0, \ldots, k_{n-1}) = \sum_{i=0}^{n-1} t^{k_0+\cdots+k_{i-1}} A'(k_0, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_{n-1}).$$

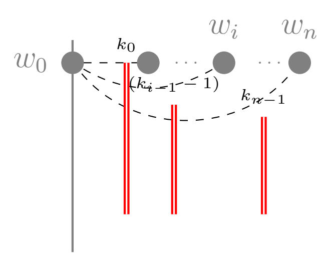
Proof. Every dashed component of $A'(k_0, \ldots, k_{n-1})$ has its leftmost component having one end in w_0. For $i = 1, \ldots, n-1$ we have from the above lemma:

$$A'(k_0, \ldots, k_{n-1}) =$$ 

$$= t^{k_0+\cdots+k_{i-2}}$$ 

Using exactly same arguments from Example 3.2.47, we have:

$$-\partial_*$$ 

$$=$$ 

$$+\cdots$$

where the rest of the terms concerns boundary terms coming from other arcs (different from the k_{i-1} indexed one). The minus sign is due to the fact that we oriented all the dashed arcs from left to right. Every dashed arc indexed by k_i for $i = 0, \ldots, n-1$ can be treated the same way. The boundary of $A(k_0, \ldots, k_{n-1})$ relative to w_0 is then the sum of these terms, and one gets the statement of the lemma. □

One has the following action over the normalized multi-arcs.

Proposition 3.2.58 (Action of E over multi-arcs)**.** *For any* $\mathbf{k} = (k_0, \ldots, k_{n-1}) \in E^0_{n,r}$*, the action of* E *over the (normalized) multi-arc is the following:*

$$E \cdot A(k_0, \ldots, k_{n-1}) = \sum_{i=0}^{n-1} q^{\alpha_1+\cdots+\alpha_i} t^{k_0+\cdots+k_{i-1}} A(k_0, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_{n-1}).$$

Proof. It is a simple computation:

$$\begin{aligned} E \cdot A(k_0, \ldots, k_{n-1}) &= q^{\alpha_1(k_1+\cdots+k_{n-1})+\cdots+\alpha_{n-1}k_{n-1}} \sum_{i=0}^{n-1} t^{k_0+\cdots+k_{i-1}} A'(k_0, \ldots, k_i - 1, \ldots, k_{n-1}) \\ &= \sum_{i=0}^{n-1} q^{\alpha_1+\cdots+\alpha_i} t^{k_0+\cdots+k_{i-1}} A(k_0, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_{n-1}). \end{aligned}$$

□

We emphasize the action in the case of one puncture.

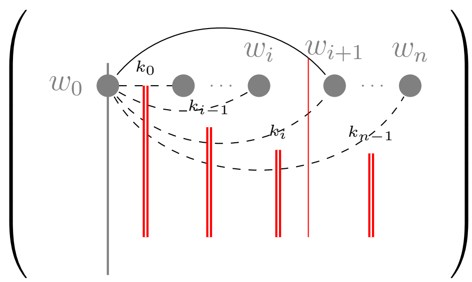
Corollary 3.2.59 ($n = 1$)**.** *Let $n = 1$, $k \in \mathbb{N}$, and $A(k)$ the associated element of $\mathcal{H}$. Then:*

$$E \cdot A(k) = A(k-1).$$

3.2.4.2 Action of $F^{(k)}$.

Let $i \in \{1, \dots, n\}$, and S_i be the following mixed class:

$$S_i =$$



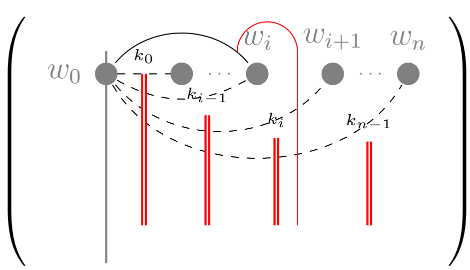
Namely one recognizes a standard $(k_0, \dots, k_{n-1})$-multi arc to which a plain arc as in the picture has been added. To compute the action of $F^{(1)}$ we need the following lemma allowing us to deal with S_i by recursion.

Lemma 3.2.60. *For $i \in \{2, \dots, n\}$, the following equality holds in $\mathcal{H}_r^{rel\,-}$:*

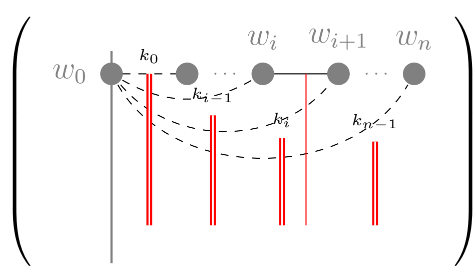
$$\begin{aligned} S_i =& (k_i + 1)_{t^{-1}} A'(k_0, \dots, k_i + 1, k_{i+1}, \dots, k_{n-1}) \\ & - t^{-k_i} (k_{i-1} + 1)_t A'(k_0, \dots, k_{i-1} + 1, k_i, \dots, k_{n-1}) \\ & + q^{-2\alpha_i} t^{-k_i} S_{i-1} \end{aligned}$$

Proof. By a breaking of a plain arc (see Example 3.2.23), one gets the following decomposition for S_i:

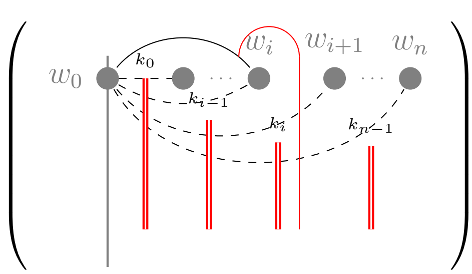
$$S_i =$$



$$+$$



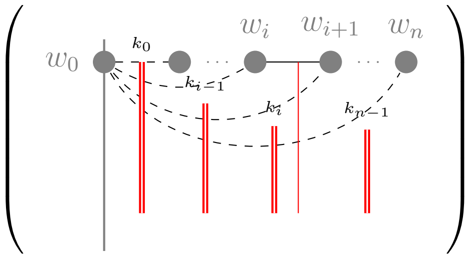
We treat both right terms independently. From the first one we get:



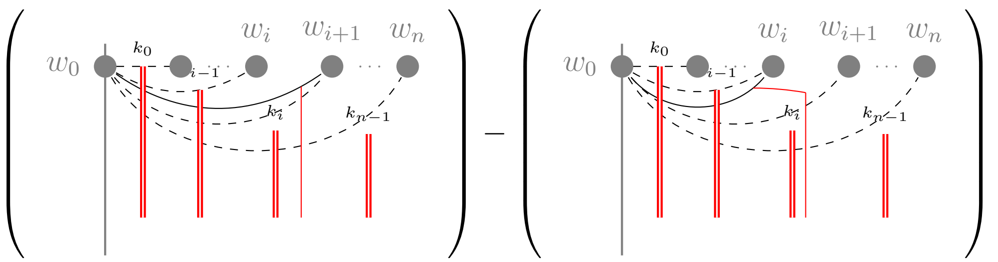
$$= q^{-2\alpha_i} t^{-k_i} S_{i-1}$$

which follows from the handle rule.

Again, breaking the plain arc, we treat the second term using the following equality:



=



To decompose these two terms in the standard multi-arc basis, one must apply Lemma 3.2.26 to crash a plain arc over a dashed one, after a simple application of the handle rule to reorganize the handles of the right term. This recovers the lemma. □

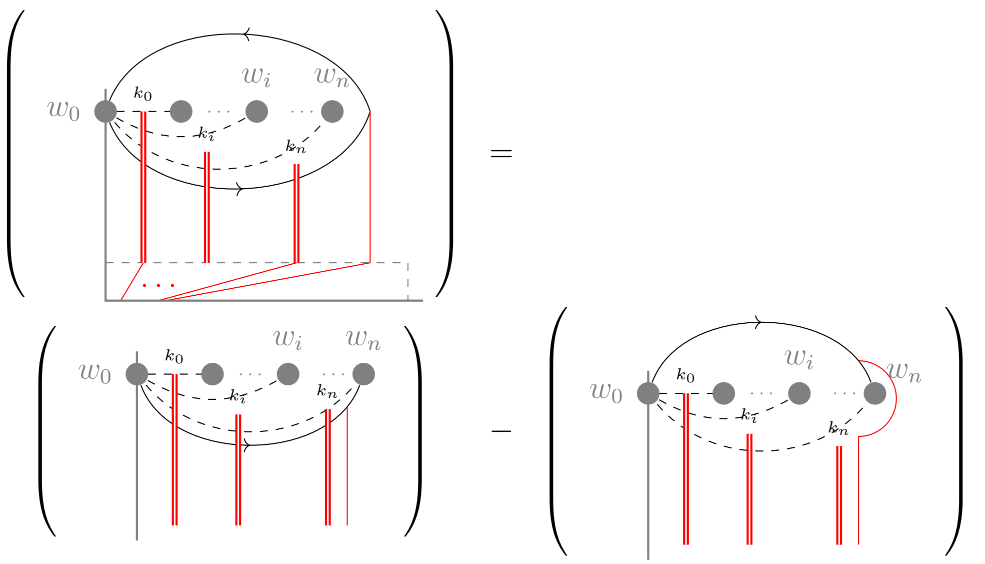
We use this lemma to compute the action of $F^{(1)}$ in the multi-arcs basis.

Lemma 3.2.61. *Let* $\mathbf{k} = (k_0, \ldots, k_{n-1}) \in E^0_{n,r}$*, the action of* $F^{(1)}$ *over the associated standard multi arc is the following:*

$$F^{(1)} \cdot A'(\mathbf{k}) = \sum_{i=0}^{n-1} q^{\sum_{j=1}^{i+1} \alpha_j} q^{-\sum_{j=i+2}^{n} \alpha_j} t^{-\sum_{j=i+1}^{n-1} k_j} (k_i + 1)_t (1 - q^{-2\alpha_{i+1}} t^{-k_i}) A'(\mathbf{k})_i$$

where $A'(\mathbf{k})_i$ *means* $A'(k_0, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_{n-1})$.

Proof. First, we compute the element $F' \cdot A'(k_0, \ldots, k_{n-1})$ of $\mathcal{H}^{\mathrm{rel}\,-}_r$. It corresponds to the following mixed class for which we give a decomposition in $\mathcal{H}_r^{\mathrm{rel}\,-}$:



This decomposition follows from a breaking of a plain arc (Example 3.2.23). The minus sign is due to the reverse of the orientation of right term's plain arc. The first term of the decomposition is in position to apply Lemma 3.2.26, while after a handle rule one recognizes S_{n-1} in the second term. Finally we get the following formula:

$$F' \cdot A'(k_0, \ldots, k_{n-1}) = (k_{n-1}+1)_t A'(k_0, \ldots, k_{n-1}+1) - q^{-2\alpha_n} S_{n-1}.$$

Thank to the recursive property of S_{n-1} the proof is achieved using Lemma 3.2.60, so that one gets:

$$F' \cdot A'(\mathbf{k}) = \sum_{i=0}^{n-1} q^{-2\sum_{j=i+2}^{n} \alpha_j} t^{-\sum_{j=i+1}^{n-1} k_j} (k_i+1)_t (1 - q^{-2\alpha_{i+1}} t^{-k_i}) A'(\mathbf{k})_i.$$

By multiplication of the action by $q^{\sum \alpha_i}$ one obtains the expected action for $F^{(1)}$ over the multi arc basis.

□

Proposition 3.2.62 (Action of $F^{(1)}$ over multi-arcs)**.** *Let $\mathbf{k} = (k_0, \ldots, k_{n-1}) \in E^0_{n,r}$, the action of $F^{(1)}$ over the associated standard (normalized) multi-arc is the following*

$$F^{(1)} \cdot A(\mathbf{k}) = \sum_{i=0}^{n-1} q^{-\sum_{j=i+2}^{n} \alpha_j} t^{-\sum_{j=i+1}^{n-1} k_j} q^{\alpha_{i+1}} (k_i+1)_t (1 - q^{-2\alpha_{i+1}} t^{-k_i}) A(\mathbf{k})_i.$$

Proof. It is a straightforward consequence of previous lemma and of the normalization sending the family $\mathcal{A}'$ to $\mathcal{A}$. □

We emphasize again the case $n = 1$.

Corollary 3.2.63 ($n = 1$)**.** *Let $n = 1$, $k \in \mathbb{N}$, and $A(k)$ the associated element of $\mathcal{H}$. Then:*

$$F^{(1)} \cdot A(k) = q^{\alpha_1}(k+1)_t (1 - q^{-2\alpha_1} t^{-k}) A(k+1)$$

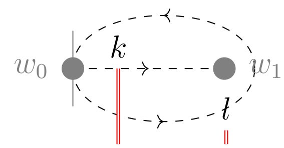
We end this section giving the action of the divided powers $F^{(l)}$ but only in the case of one puncture.

Proposition 3.2.64 (Action of $F^{(l)}$, $n = 1$)**.** *Let $n = 1$, $k \in \mathbb{N}$, and $A(k)$ the associated element of $\mathcal{H}$. Let $l \in \mathbb{N}^*$, then:*

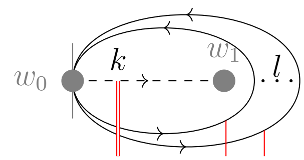
$$F^{(l)} \cdot A(k) = q^{\frac{-l(l-1)}{2}} q^{l\alpha_1} \binom{k+l}{k}_t \prod_{m=0}^{l} (1 - q^{-2\alpha_1} t^{-m}) A(k+l).$$

Proof.

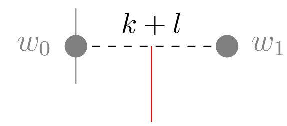
$$(l)_t! F'^{(l)} \cdot A(k) = (l)_t! \left(\quad \right)$$



$$= \left(\quad \right)$$



$$= \prod_{m=0}^{l} (k+m)_t! \, (1 - q^{-2\alpha_1} t^{-m}) \left(\quad \right)$$



The second equality comes from Corollary 3.2.27 and the last one is an iteration of the equality from Remark 3.2.34. Finally we have:

$$F'^{(l)} \cdot A(k) = \binom{k+l}{k}_t \prod_{m=0}^{l} (1 - q^{-2\alpha_1} t^{-m}) A(k+l).$$

The proposition is proved after the normalization passing from $F'^{(l)}$ to $F^{(l)}$. □

3.2.4.3 Recovering monoidality of Verma modules for $U_q\mathfrak{sl}(2)$.

Since in this section n (the number of punctures) is particularly important, we denote by $\mathcal{H}^{\alpha_1,\ldots,\alpha_n}$ the module $\mathcal{H}$ built from $X_r(w_0, \ldots, w_n)$ with coefficients in $\mathcal{R} = \mathbb{Z}[t^{\pm 1}, q^{\pm\alpha_1}, \ldots, q^{\pm\alpha_n}]$.

Remark 3.2.65. We recall the action of K. We distinguish the cases whether n is greater than 1 or not.

($n = 1$) Let $n = 1$ so that $\mathcal{R} = \mathbb{Z}[t^{\pm 1}, q^{\pm\alpha_1}]$. Let $k \in \mathbb{N}$, and $A(k)$ the associated element of $\mathcal{H}^{\alpha_1}$. Then:

$$K \cdot A(k) = q^{\alpha_1} t^k A(k).$$

($n > 1$) Let $n > 1$, $\mathbf{k} \in E^0_{n,r}$ and $A(\mathbf{k})$ the associated element of $\mathcal{H}^{\mathrm{rel}\,-}_r \in \mathcal{H}^{\alpha_1,\ldots,\alpha_n}$. Then:

$$K \cdot A(\mathbf{k}) = q^{\sum_{i=1}^n \alpha_i} t^r A(\mathbf{k}).$$

Proposition 3.2.66. *Let $t = q^{-2}$. The module $\mathcal{H}^{\alpha_1}$ is a Verma module for $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ of highest weight α_1.*

Proof. The presentation of the action over a Verma-module, is given in [J-K] (see relations (18)) and is recalled in Definition 1.4.17. Using Corollaries 3.2.59 and 3.2.63 and the above remark in the case $n = 1$, one recognizes the presentation of the Verma module. Namely, let $t = q^{-2}$, and let $s = q^{\alpha_1}$. Then:

$$K \cdot A(k) = q^{\alpha_1} t^k A(k) = sq^{-2k} A(k)$$

$$E \cdot A(k) = A(k-1)$$

and

$$F^{(1)} \cdot A(k) = q^{\alpha_1}(k+1)_t (1 - q^{-2\alpha_1} t^{-k}) A(k+1) = [k+1]_q\, (sq^{-k} - s^{-1} q^k) A(k+1).$$

The last equality uses Remark 3.2.37.

These expressions ensure that the isomorphism of $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$-modules:

$$\begin{cases} \mathcal{H}^{\alpha_1} & \to & V^{\alpha_1} \\ A(k) & \mapsto & v_k \text{ for } k \in \mathbb{N} \end{cases}$$

is $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ equivariant. ◻

Remark 3.2.67. There is an isomorphism of $\mathcal{R}$-modules:

$$\text{tens} : \begin{array}{ccc} \mathcal{H}^{\alpha_1,\ldots,\alpha_n} & \to & \mathcal{H}^{\alpha_1} \otimes \cdots \otimes \mathcal{H}^{\alpha_n} \\ A(k_0,\ldots,k_{n-1}) & \mapsto & A(k_0) \otimes \cdots \otimes A(k_{n-1}). \end{array}$$

Theorem 5 (Monoidality of Verma-modules.)**.** *The morphism:*

$$\text{tens} : \mathcal{H}^{\alpha_1,\ldots,\alpha_n} \to \mathcal{H}^{\alpha_1} \otimes \cdots \otimes \mathcal{H}^{\alpha_n}$$

is an isomorphism of $U_q\mathfrak{sl}(2)$-modules.

Proof. From Proposition 3.2.62 one remarks that the formulae satisfy:

$$\begin{aligned} \text{tens}\left(F^{(1)} \cdot A(\mathbf{k})\right) &= \text{tens}\left(\sum_{i=0}^{n-1} q^{-\sum_{j=i+2}^{n} \alpha_j} t^{-\sum_{j=i+1}^{n-1} k_j} q^{\alpha_{i+1}} (k_i+1)_t (1 - q^{-2\alpha_{i+1}} t^{-k_i}) A(\mathbf{k})_i\right) \\ &= \sum_{i=0}^{n-1} A(k_0) \otimes \cdots \otimes \left(q^{\alpha_{i+1}}(k_i+1)_t(1 - q^{-2\alpha_{i+1}} t^{-k_i})\right) A(k_i+1) \\ &\otimes q^{-\alpha_{i+2}} t^{-k_{i+1}} A(k_{i+1}) \otimes \cdots \otimes q^{-\alpha_n} t^{-k_{n-1}} A(k_{n-1}) \\ &= \sum_{i=0}^{n-1} \left(1 \otimes 1 \otimes \cdots \otimes F^{(1)} \otimes K^{-1} \otimes \cdots \otimes K^{-1}\right) A(k_0) \otimes \cdots \otimes A(k_{n-1}) \end{aligned}$$

where the $F^{(1)}$ in the sum is in the $(i+1)^{st}$ position, one recognizes the expression of $\Delta^n(F^{(1)})$.

We do the same for E, from Proposition 3.2.58 we have:

$$\begin{aligned} \text{tens}\left(E \cdot A(k_0,\ldots,k_{n-1})\right) &= \text{tens}\left(\sum_{i=0}^{n-1} q^{\alpha_1+\cdots+\alpha_i} t^{k_0+\cdots+k_{i-1}} A(k_0,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_{n-1})\right) \\ &= \sum_{i=0}^{n-1} (K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1) A(k_0) \otimes \cdots \otimes A(k_{n-1}) \end{aligned}$$

which proves that the action of E over $\mathcal{H}^{\alpha_1,\ldots,\alpha_n}$ corresponds to the action of $\Delta^n(E)$ over the tensor product. The same proof works for the action of K so that the theorem holds. ◻

Remark 3.2.68. The above theorem suggests that their should exist a homological interpretation of the $U_q\mathfrak{sl}(2)$ coproduct. Probably in terms of gluing once punctured disks along arcs of their boundary. The morphism tens should then be the involved homological operator.

We remark that:

$$\operatorname{tens}(A'(\mathbf{k})) = q^{\alpha_1(k_1+\cdots+k_{n-1})+\alpha_2(k_2+\cdots+k_{n-1})+\cdots+\alpha_{n-1}k_{n-1}} A'(k_0) \otimes \cdots \otimes A'(k_{n-1})$$

so that multi-arcs are divided into tensor products of single arcs, with coefficients appearing from the gluing operation.

3.2.5 Homological braid action

3.2.5.1 Definition of the action.

The braid group $\mathcal{B}_n$ acts topologically over D_n by mapping class action. The action can be generalized to X_r as homeomorphisms extend to the configuration space coordinate by coordinate (as in Relation 2.1, Section 2.2). We show that this action passes to homology with local coefficients in L_r, treating separately the unicolored case from the general one. In the unicolored case, we get a representation of the standard braid group:

Lemma 3.2.69 (Representation of the braid group)**.** *Let $\alpha = \alpha_1 = \cdots = \alpha_n$ so that $\mathcal{R} = \mathbb{Z}\left[t^{\pm 1}, q^{\pm\alpha}\right]$. Let $\beta \in \mathcal{B}_n$ be a braid. The action of β by mapping class over X_r lifts to $\mathcal{H}_r^{rel\,-}$, so that it yields a homological representation of the braid group:*

$$\mathrm{R}^{hom}: \ \mathcal{B}_n \ \to \ \operatorname{End}_{\mathcal{R}}(\mathcal{H}^{\alpha,\dots,\alpha}) \ .$$

Proof. Let $\sigma \in \mathcal{B}_n$ be one of the standard braid generator, the lemma is a direct consequence of the invariance of the local system representation under the braid action. Namely, the commutativity of the following diagram:

$$\begin{array}{ccc} \pi_1(X_r, \boldsymbol{\xi}^r) & \xrightarrow{\widehat{\beta}_*} & \pi_1(X_r, \boldsymbol{\xi}^r) \\ \downarrow{\rho_r} & & \downarrow{\rho_r} \\ \mathbb{Z}^2 = \mathbb{Z}\langle q^\alpha \rangle \oplus \mathbb{Z}\langle t \rangle & \xrightarrow{\mathrm{Id}} & \mathbb{Z}\langle q^\alpha \rangle \oplus \mathbb{Z}\langle t \rangle \end{array}$$

where $\widehat{\beta}$ is the homeomorphism of X_r associated to β and $\widehat{\beta}_*$ the lift to the fundamental group. It is easy to see that for $l \in \{1, \dots, r-1\}$ and $k \in \{1, \dots, n\}$, the following equalities hold:

$$\rho_r\left(\mathrm{R}^{hom}(\sigma)_*(\sigma_l)\right) = \rho_r(\sigma_l) \text{ and } \rho_r\left(\mathrm{R}^{hom}(\sigma)_*(B_{r,k})\right) = \rho_r(B_{r,k})$$

considering generators of $\pi_1(X_r, \boldsymbol{\xi}^r)$ introduced in Remark 3.2.3. ☐

Remark 3.2.70. The above lemma has already been proved for the case $r = 2$ in Lemmas 1.2.20 and 2.2.2 (unicolored and general cases). The same proof can be adapted also for the above general case.

To deal with different colors, we need a morphism to follow the change of colors in $\mathcal{R}$.

Definition 3.2.71. *Let $s \in \mathfrak{S}_n$ be a permutation. We define the following morphism:*

$$\hat{s} : \begin{cases} \mathcal{R} & \rightarrow & \mathcal{R} \\ q^{\alpha_i} & \mapsto & q^{\alpha_{s(i)}} \\ t & \mapsto & t \end{cases} .$$

Lemma 3.2.72 (Representation of the colored braid groupoid)**.** *In the general case, let σ_i^s be a generator of the colored braid groupoid $\mathcal{B}_{1,\ldots,1}(D)$ (see Definition 3.1.4), with $i \in \{1, \ldots, n-1\}$ and $s \in \mathfrak{S}_n$. The action of σ_i^s by mapping class over X_r lifts to homology so that it yields a homological representation of the colored braid groupoid:*

$$\mathrm{R}^{hom}(\sigma_i^s) \in \mathrm{Hom}_{\mathcal{R}}\left(\mathcal{H}^{s(\alpha_1)} \otimes \cdots \otimes \mathcal{H}^{s(\alpha_n)}, \mathcal{H}^{s\tau_i(\alpha_1)} \otimes \cdots \otimes \mathcal{H}^{s\tau_i(\alpha_n)}\right)$$

where $\tau_i = (i, i+1) \in \mathfrak{S}_n$. This action commutes with the $\mathcal{R}$-structure.

Proof. The proof is almost the same as the one for Lemma 3.2.69. Namely, it is a consequence of the fact that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X_r, \boldsymbol{\xi}^r) & \xrightarrow{\mathrm{R}^{hom}(\sigma_i^s)_*} & \pi_1(X_r, \boldsymbol{\xi}^r) \\ \downarrow \rho_r & & \downarrow \rho_r \\ \mathbb{Z}\langle q^{\pm\alpha_i}, t^{\pm 1}\rangle_{i\in\{1,\ldots,n\}} & \xrightarrow{\widehat{\tau_k}} & \mathbb{Z}\langle q^{\pm\alpha_i}, t^{\pm 1}\rangle_{i\in\{1,\ldots,n\}} \end{array} .$$

The fact that this diagram commutes comes from the following remark:

$$\mathrm{R}^{hom}(\sigma_i^s)_* (B_{r,k}) = B_{r,k+1}$$

while other generators of $\pi_1(X_r, \boldsymbol{\xi}^r)$ are not perturbated by the action of σ_i^s (so that the proof is unchanged). ☐

The idea of this construction was originally due to R. Lawrence in [Law]. This is a generalization of her construction as we work in a relative case and obtain a larger representation.

3.2.5.2 Computation of the action.

In the case of two punctures w_1, w_2, we can perform the computation of the action of the single braid generator of $\mathcal{B}_2$, and first we recall classical operators necessary to define the R-matrix of $U_q\mathfrak{sl}(2)$.

Definition 3.2.73. *Let $q^{\frac{H\otimes H}{2}}$ be the following $\mathcal{R}$-linear maps:*

$$q^{\frac{H\otimes H}{2}} : \begin{cases} \mathcal{H}^{\alpha_1} \otimes \mathcal{H}^{\alpha_2} & \rightarrow & \mathcal{H}^{\alpha_1} \otimes \mathcal{H}^{\alpha_2} \\ A^{\alpha_1}(k) \otimes A^{\alpha_2}(k') & \mapsto & q^{(\alpha_1 - 2k)(\alpha_2 - 2k')/2} A^{\alpha_1}(k) \otimes A^{\alpha_2}(k') \end{cases}$$

and T the following one:

$$T : \begin{cases} \mathcal{H}^{\alpha_1} \otimes \mathcal{H}^{\alpha_2} & \rightarrow & \mathcal{H}^{\alpha_2} \otimes \mathcal{H}^{\alpha_1} \\ A^{\alpha_1}(k) \otimes A^{\alpha_2}(k') & \mapsto & A^{\alpha_2}(k') \otimes A^{\alpha_1}(k) \end{cases} .$$

where $A(k')^{\alpha_1}$, and $A(k)^{\alpha_2}$ are vectors of $\mathcal{H}^{\alpha_1}$, and $\mathcal{H}^{\alpha_2}$ respectively.

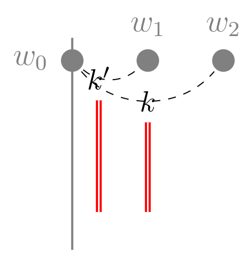
Lemma 3.2.74 (Braid action with two punctures)**.** *Let $k, k' \in \mathbb{N}$, σ_1 be the standard generator of the colored braid groupoid with two strands. Its action can be expressed as follows:*

$$\operatorname{tens}\left(\mathrm{R}^{hom}(\sigma_1)\left(A(k',k)^{\alpha_1,\alpha_2}\right)\right) = \left[q^{\frac{-\alpha_1\alpha_2}{2}} q^{\frac{H\otimes H}{2}} \circ \sum_{l=0}^{k} \left(q^{\frac{l(l-1)}{2}} E^l \otimes F^{(l)}\right) \circ T\right] A(k')^{\alpha_1} \otimes A(k)^{\alpha_2}.$$

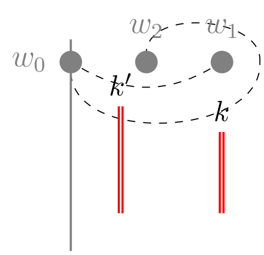
where $A(k',k)^{\alpha_1,\alpha_2}$, $A(k')^{\alpha_1}$, and $A(k)^{\alpha_2}$ are vectors of $\mathcal{H}^{\alpha_1,\alpha_2}$, $\mathcal{H}^{\alpha_1}$, and $\mathcal{H}^{\alpha_2}$ respectively.

Proof. We have the following equalities concerning homology classes:

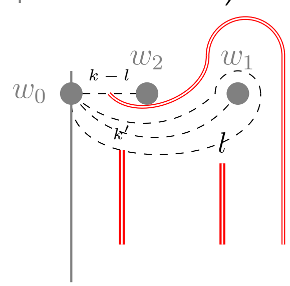
$$\mathrm{R}^{hom}(\sigma_1)\left(A'(k',k)^{\alpha_1,\alpha_2}\right) = \mathrm{R}^{hom}(\sigma_1)\left(\ \cdot\ \right)$$



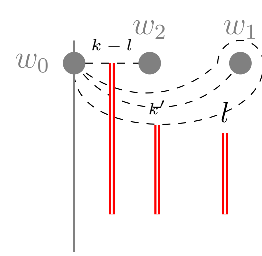
$$= \left(\ \cdot\ \right)$$



$$= \sum_{l=0}^{k} \left(\ \cdot\ \right)$$



$$= \sum_{l=0}^{k} t^{-k'(k-l)} t^{-l(k-l)} q^{-2(k-l)\alpha_1} \left(\ \cdot\ \right).$$



The second equality comes from a breaking of a dashed arc (Example 3.2.24), the last one is a handle rule, for which we draw the corresponding braid in Figure 3.10.

The bands represent a $(k-l)$-handle, a (l)-handle, and a (k') handle. On the top and on the bottom of this box there is the part of the path corresponding to the dashed box. Namely red arcs are going back to $\boldsymbol{\xi}$ without crossing themselves, and in the front of w_1 and w_2. As this braid has $(k-l)$ strands passing successively in the back of k' strands, l strands and

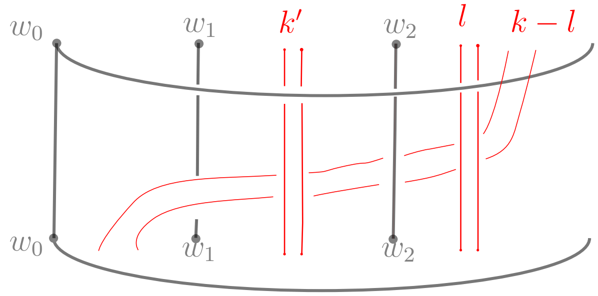


Figure 3.10 – Braided handle rule.

finally w_2, its local coefficient is $t^{-(k-l)(k'+l)}q^{-2\alpha_2}$. From the local coefficient of this braid we deduce the coefficient appearing in the last term.

Finally, applying the proof of Proposition 3.2.64 to crash a dashed loop on the indexed k dashed arc, we get:

$$\mathrm{R}^{hom}(\sigma_1)\left(A'(k',k)^{\alpha_1,\alpha_2}\right) = \sum_{l=0}^{k} t^{-(k'+l)(k-l)}q^{-2(k-l)\alpha_1}\binom{k'+l}{l}_t \prod_{m=0}^{l}(1-q^{-2\alpha_1}t^{-m})A'(k-l,k'+l)^{\alpha_2,\alpha_1}.$$

So that:

$$\mathrm{R}^{hom}(\sigma_1)\left(A(k',k)^{\alpha_1,\alpha_2}\right) = \sum_{l=0}^{k} t^{-(k'+l)(k-l)}q^{(-k+2l)\alpha_1}q^{-(k'+l)\alpha_2}\binom{k'+l}{l}_t \prod_{m=0}^{l}(1-q^{-2\alpha_1}t^{-m})A(k-l,k'+l)^{\alpha_2,\alpha_1}.$$

Let $t=q^{-2}$, passing the above expression to tens, we get for $\operatorname{tens}\left(\mathrm{R}^{hom}(\sigma_1^{\tau_1})\left(A(k',k)^{\alpha_1,\alpha_2}\right)\right)$ the following expression.

$$\sum_{l=0}^{k} q^{2(k'+l)(k-l)+(-k+2l)\alpha_1-(k'+l)\alpha_2}\binom{k'+l}{l}_t \prod_{m=0}^{l}(1-q^{-2\alpha_1}t^{-m})A(k-l)^{\alpha_2}\otimes A(k'+l)^{\alpha_1}.$$

By use of the expression of the action of $F^{(l)}$ in Proposition 3.2.64, one recognizes:

$$\left(\sum_{l=0}^{k} q^{2(k'+l)(k-l)+(-k+2l)\alpha_1-(k'+l)\alpha_2}E^l\otimes F'^{(l)}\right)A(k)^{\alpha_2}\otimes A(k')^{\alpha_1}$$

Finally, passing from $F'^{(l)}$ to $F^{(l)}$ (defined in Proposition 3.2.41 and Definition 3.2.42 resp.), we get:

$$\begin{aligned}\operatorname{tens}\left(\mathrm{R}^{hom}(\sigma_1)\left(A(k',k)^{\alpha_1,\alpha_2}\right)\right) &= \left(\sum_{l=0}^{k} q^{2(k'+l)(k-l)-(k-l)\alpha_1-(k'+l)\alpha_2}q^{\frac{l(l-1)}{2}}E^l\otimes F^{(l)}\right)A(k)^{\alpha_2}\otimes A(k')^{\alpha_1}\\ &= \left[q^{\frac{-\alpha_1\alpha_2}{2}}q^{\frac{H\otimes H}{2}}\circ\sum_{l=0}^{k}\left(q^{\frac{l(l-1)}{2}}E^l\otimes F^{(l)}\right)\circ T\right]A(k')^{\alpha_1}\otimes A(k)^{\alpha_2}\end{aligned}$$

□

Definition 3.2.75 (R-matrix)**.** *the R-matrix of the category of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ modules that was recalled in Definition 1.4.19 is the folloqing operator:*

$$R = q^{\frac{H\otimes H}{2}} \circ \sum_{l=0}^{k} \left(q^{\frac{l(l-1)}{2}} E^l \otimes F^{(l)} \right) \in \operatorname{Hom}_{\mathcal{R}} \left(\mathcal{H}^{\alpha_1} \otimes \mathcal{H}^{\alpha_2} \right).$$

Theorem 6 (Recovering the R-Matrix of $U_q\mathfrak{sl}(2)$)**.** *The representation of the colored braid groupoid over $\mathcal{H}^{\alpha_1,\dots,\alpha_n}$ is isomorphic to the R-matrix representation over the product of $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ Verma-modules $V^{\alpha_1} \otimes \cdots \otimes V^{\alpha_n}$ from Corollary 1.4.22.*

Proof. From Lemma 3.2.74, the following diagram:

$$\begin{array}{ccc} \mathcal{H}^{\alpha_1,\alpha_2} & \xrightarrow{\text{tens}} & \mathcal{H}^{\alpha_1} \otimes \mathcal{H}^{\alpha_2} \\ \Big\downarrow \mathrm{R}^{hom}(\sigma_1) & & \Big\downarrow q^{\frac{-\alpha_1\alpha_2}{2}} R\circ T \\ \mathcal{H}^{\alpha_2,\alpha_1} & \xrightarrow{\text{tens}} & \mathcal{H}^{\alpha_2} \otimes \mathcal{H}^{\alpha_1} \end{array}.$$

commutes. The action of a braid generator σ_i over a multi-arc is contained in a disk that contains the dashed arcs reaching w_i and w_{i+1} and no other so that the action does not perturbate the other arcs. This last fact shows that the proof with two punctures guarantees the general case, and that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}^{\alpha_1,\dots,\alpha_n} & \xrightarrow{\text{tens}} & \mathcal{H}^{\alpha_1} \otimes \cdots \otimes \mathcal{H}^{\alpha_n} \\ \Big\downarrow \mathrm{R}^{hom}(\sigma_i) & & \Big\downarrow Q(\sigma_i) \\ \mathcal{H}^{\tau_i\{\alpha_1,\dots,\alpha_n\}} & \xrightarrow{\text{tens}} & \mathcal{H}^{\alpha_{\tau_i(1)}} \otimes \cdots \otimes \mathcal{H}^{\alpha_{\tau_i(n)}} \end{array}.$$

where $Q(\sigma_i) = \mathrm{Id}^{\otimes i-1} \otimes q^{\frac{-\prod \alpha_i}{2}} R\circ T \otimes \mathrm{Id}^{\otimes n-i-1}$. Moreover all the morphisms involved commute with the $U_q\mathfrak{sl}(2)$ structure. This proves the theorem. □

3.2.6 Further directions

First we recall remarks about reducibility and faithfulness that replace the model developed in Section 3.2 in the Lawrence's representations theory.

3.2.6.1 A word about reducibility and faithfulness

Theorem 6 shows that the representation of the braid group over $\mathcal{H}_r^{\mathrm{rel}\,-}$ is the one over $W_{n,r}$ from Remark 1.4.24, notations taken from [J-K]. Theorem 1.4.25 states that these representations reduce to the kernel of the action of E, namely to the highest-weight modules $Y_{n,r}$.

Corollary 3.2.76. *The space $\mathcal{H}^E$ introduced in Corollary 3.2.52 is a graded representation of the braid group. To every $r \in \mathbb{N}^*$ corresponds a stratum of $\mathcal{H}^E$ isomorphic to $H_r^{BM}(X_r; L_r)$*

and to $Y_{n,r}$, and by $\mathcal{B}_n$-equivariant isomorphisms so that every stratum is irreducible. Moreover the set $\mathcal{U}^E = \{U(0,k_1,\ldots,k_{n-1}), (0,k_1,\ldots,k_{n-1}) \in E^0_{n,r}\}$ provides an $\mathcal{R}$-basis of the r-stratum and the set $\mathcal{F}^E = \{F(0,k_1,\ldots,k_{n-1}), (0,k_1,\ldots,k_{n-1}) \in E^0_{n,r}\}$ is generically a basis, under generic conditions of Corollary 3.2.29.

Proof. The fact that the r-stratum is isomorphic to the absolute homology module comes from Corollary 3.2.52 while the fact that it is isomorphic to $Y_{n,r}$ comes from the homological restriction to the kernel of E. The fact that the set $\mathcal{U}^E$ is a basis of the kernel of the action of E is easy to check as elements of $\mathcal{U}^E$ are exactly elements of $\mathcal{U}$ not having boundary joining w_0. The same holds for $\mathcal{F}^E$ with generic conditions from Corollary 3.2.29 conserved. □

Remark 3.2.77. The graded family $\mathcal{H}^E$ of braid representations is the so called family of Lawrence's representations, first introduced in [Law]. The fact that it recovers the $\mathcal{B}_n$-modules $Y_{n,r}$ was known as Kohno's theorem as it was proved in [K2] and [K1] (and in [J-K] for $r = 2$). In [Ito1], the basis of multiforks is stated to be generically a basis of the absolute homology module, and a $\mathcal{B}_n$-equivariant morphism between the multiforks basis and a suitable basis of the the highest weight module is presented in [Ito, Section 4.2]. All of this is recovered by the above corollary and links between basis are presented in details in Section 3.2.2.4.

Remark 3.2.78. The basis of code sequences and the one of multi-arcs are integral basis (i.e. as module over the integral Laurent polynomial ring) of $\mathcal{H}$. The basis of code sequences is a completion of the basis of the kernel of the action of E, while the multi-arcs one is the one recovering directly a usual basis of the tensor product of Verma-modules that fits with the monoidality of the $U_q\mathfrak{sl}(2)$ action over it, see Theorem 5.

From the fact that Lawrence's representations are generically faithful and that they are sub-representations of the ones over $\mathcal{H}$, the following is immediate.

Corollary 3.2.79. *The representations of the braid group over $\mathcal{H}$ are generically faithful.*

3.2.6.2 Homological interpretation of a natural pairing.

In [F-W, Section 3], they apply the transformation:

$$\theta : \begin{array}{ccc} D_n & \to & D_n \\ x+iy & \mapsto & -x+iy \end{array}$$

to the punctured disk and do so for all the construction of a local system over the configuration space. The same can be performed in our case, and one obtains a braid action over the $U_q\mathfrak{sl}(2)$ module $\bigoplus_{r\in\mathbb{N}^*} H_r^{BM}(X_r, X_r^+, L_r')$ where X_r^+ stands for configurations with one coordinate in $w_\infty = +1$ and L_r' the local system obtained by transport of L_r under θ.

Let R be a commutative ring, and M an n-manifold with boundary. Let $A \in \partial M$ be a closed manifold, there exists a *Lefschetz duality*:

$$D_M : H^k(M, A; R) \to H_{n-k}(M, B; R)$$

where $B = \partial M - A$ and for all k. See [Hat, Theorem 3.43] for a less general statement in the case of compact manifold, the proof can be adapted to the present case considering Borel-Moore homology. As $X_r^+ \subset \partial X_r - X_r^-$ and from this Lefschetz duality, there exists a well defined homological intersection pairing:

$$\bigoplus_{r\in\mathbb{N}^*} H_r^{BM}\left(X_r, X_r^-, L_r\right) \times \bigoplus_{r\in\mathbb{N}^*} H_r^{BM}\left(X_r, X_r^+, L_r'\right) \to \mathcal{R}$$

where $\mathcal{R}$ is still the Laurent polynomial ring defined in Subsection 3.2.1.2.

In [F-W, Theorem 4.3], although they do not provide a homological definition of the pairing, they suggest that this pairing is the Shapovalov linear form over Verma modules that is symmetric. They compute it between families of r-loops and get a pairing that is degenerated when the colors α_i's (local system inputs, see Definition 3.2.5) are integers. As the family of r-loops is not a basis of the homology as an $\mathcal{R}$-module (see Corollary 3.2.33) and particularly at integral colors, by computing the pairing in the $\mathcal{R}$-basis of multi-arcs based on w_0 on the left, and (their symmetric based) on w_∞ on the right of the pairing, one could hope to recover the scalar product defined in [Kas, Theorem VII.6.2]. The latter theorem is stated in the case of the simple finite dimensional modules of $U_q\mathfrak{sl}(2)$ but one could easily generalize it to the infinite Verma modules of $U_q\mathfrak{sl}(2)$.

Such a homological pairing was introduced by Bigelow, see Definition 1.2.24, and used to prove the faithfulness over the Laurent polynomial ring of BKL representations in [Big0]. Our integral version of the pairing would be adapted to be evaluated at roots of unity, so to study the faithfulness of the braid representations at roots of 1, and of the ADO representations, which is also related to the faithfulness of TQFT representations of punctured spheres, see Corollary 2.3.27.

3.2.6.3 Improving the local system.

We have defined a family of configuration spaces, to which we have associated a colored braid groupoid, which is a sub groupoid of the fundamental groupoid. We introduced two ways to obtain representations of this colored braid groupoid: one dimensional abelian representations or quantum representations - see Examples 3.1.12 and 3.1.13 - and these representations provide local systems. All this general framework was introduced in Section 3.1. Then we made used of this construction in a precise case and we applied to it homology with local coefficients. We ended up with homology modules naturally acted upon by the braid group. From this we obtained a homological model for some universal modules of $U_q\mathfrak{sl}(2)$, namely the Verma modules, together with their natural braiding. The latter was done in Section 3.2. It is natural to ask the following question.

Question. *By considering a different space of configuration points from the family presented in Section 3.1, and applying homology with local coefficients, does one recover Verma modules for all quantized semi-simple Lie algebra, or Kac-Moody algebra, together with their natural braiding?*

Some work has been done in the direction of this first question. Namely in the article [S-V], with different types of configuration space of points (fitting with the framework of Section 3.1) and the corresponding local system, the authors recover Verma modules for Kac-Moody algebra. To do so, the definition of multiforks introduced in Section 3.2 of this work, is generalized to different configuration spaces of points, so that one can hope for a generalization of the work done for $U_q\mathfrak{sl}(2)$ (Section 3.2) to different algebras.

Together with Verma-modules of $U_q\mathfrak{sl}(2)$, we recover in Section 3.2.5, the quantum braid group representations associated to it. From these braid group representations, one is close to get polynomial invariants of knots such as Alexander and Jones polynomials, see Chapter 4. These polynomials are known to contain topological classical invariants, more precisely the Alexander polynomial computes the abelian torsion while it's a conjecture that the family of Jones polynomials contains the simplicial volume of the complementary of the knot ([Mu-Mu]). We give hints in Chapter 4 about how to pass to the level of knots from this homological construction, and to get homological interpretations of these invariants. Some work has already been done in this direction in [Big3], [Big4], [Ito1], [Ito2], and [An], where the authors obtained homological interpretations of knot polynomials from homological braid group representations contained in our model. By giving the general construction in this section, one sees that it can be improved easily to get a non abelian model using the local system presented in Example 3.1.13. One could hope that this "non abelian" homological model recovers non abelian Reidemeister torsions of knots for instance. Again, some work has been done in this direction in [Z1] where the author constructs a generalization of Lawrence's representation with coefficients in $\mathbb{Z}[\mathcal{B}_r]$ the braid group ring, it recovers Lawrence's homological representation after some specialization, but no homology identification is made.

It is a natural and open question whether it exists a general model containing together classical and non-abelian Alexander torsion of knots. These homology modules with local coefficient system could be an answer to it, while passing from one dimensional representation of the colored braid groupoid to non-abelian quantum ones improves the model to a non-abelian one. In [H-L, Lemma 1.3] the authors give a formula to obtain the family of colored Jones polynomials of a knot from Verma-modules braid representations of $U_q\mathfrak{sl}(2)$. By a deformation of this method with Verma-modules, they also recover the colored Jones polynomials from a non-abelian construction and one could guess if this can be obtained from the homological model presented above in a non abelian version. Finally in their article they also guess what could be a Kashaev invariant for other Lie algebras, which can be related to the first question of this section.

3.2.6.4 Framed case.

All along this work we kept on normalizing braid representations, removing quadratic terms corresponding to framing information, see Remark 1.3.27. The theory should be improvable to the framed case. A first step has been done in [Ik], where the author constructs homological representations of the framed braid group in the spirit of Lawrence's representations and a family of monodromy representations coming from the *KZ-equation*. The classical KZ-representation of the unframed braid group is known to recover the quantum braid representations from a well known theorem proved in [Drin] and [K0]. The latter encouraged the author of [Ik] to formulate the conjecture that his Lawrence's and KZ type of representations of the framed braid group are related and recover quantum representations of the framed braid group. The conjecture is formulated in [Ik, Conjecture 1.3].

3.2.6.5 Enright representations.

In this work we deal with two big families of representations of the braid groups: the Lawrence's homological representations, and the $U_q\mathfrak{sl}(2)$ quantum representations over Verma modules. Both are graded, and they have in common that the first stratum of the grading is the Burau/Gassner representation (depending if you work -in the unicolored or multicolored case). There exists a third family of representation, namely *KZ representations* (defined in [Kas, Chapter XIX]) that is known to be isomorphic to the quantum ones ([K0] , [Drin]). The subtle point is that the traditional Lawrence's side recovers the reduced Burau representation. From Corollary 3.2.76, one sees that the representations over $\mathcal{H}$ constructed in Section 3.2 are an unreduced version of the traditional Lawrence's representations. The Enright representations of the braid group introduced in Remark 1.2.9 are graded representations built from the theory of higher Fox derivatives and recovers the unreduced Burau/Gassner representations. From our homological representations that correspond to an unreduced version of Lawrence's representations, the following natural question arises.

Open Question. *Does the graded family of Enright representations is recovered by the unreduced Lawrence one constructed in Section 3.2?*

From Theorem 6, we proved that the unreduced Lawrence's representations are isomorphic to the unreduced Verma module representations built from $U_q\mathfrak{sl}(2)$. A positive answer to the above question would be a unification of the four big families of braid representations, namely KZ, Lawrence, quantum and Enright representations.

The question was introduced earlier in this work, in Section 2.2.4. Colored BKL representations are the second stratum of Lawrence's representations, and we've built them using a Fox derivative method that may be a link with Enright representations. In Section 4.1 we prove that from these homological representations, one can obtain some information about the Nielsen number associated to a braid. In [FH], the authors suggest a way to compute these numbers using Fox derivative techniques. This may be a hint.

Chapter 4

Application to knots

In this section we apply Section 3.2 that provides a homological model for quantum braid representation to obtain homological interpretations for quantum knot polynomials. The word quantum stands for polynomials that can be obtained from quantum braid representations. This homological model leads to Theorem 7 where we prove that colored Jones polynomials of a braid closure can be expressed as some weighted sum of Lefschetz numbers associated to the action of the braid over configuration spaces of points in the punctured disk.

4.1 Colored Jones polynomials

In this first section we focus on a first family of quantum polynomials, namely *colored Jones polynomials*.

4.1.1 Colored Jones polynomials and Verma-modules

First, we construct finite dimensional $U_q\mathfrak{sl}(2)$-modules out of the Verma ones endowed by integral colors. We follow [J-K, Section 5] to do so. This will allow us to compute the colored Jones polynomial. Let $l \in \mathbb{N}^*$ be an integer, and consider the specialization ring morphism $\mathrm{aug}^1 : \mathbb{Z}\left[s^{\pm 1}, q^{\pm 1}\right] \to \mathbb{Z}\left[q^{\pm 1}\right]$ that sends $s \mapsto q^l$. Let V^s be the $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ Verma module defined in Definition 1.4.17, spanned by vectors $\{v_0, v_1, \ldots\}$, and:

$$\mathbf{V}^l = V^s \otimes_{s=q^l} \mathbb{Z}\left[q^{\pm 1}\right],$$

be the tensor product provided by morphism aug^1.

Remark 4.1.1 (Finite simple modules from Verma modules, [J-K, Section 5])**.** Let $S^l \in \mathbf{V}^l$ be the submodule spanned by vectors $\{v_0, \ldots, v_l\}$. These submodules ar equipped with an action of generators $F^{(k)}$ that truncates as [J-K, Relation (47)], which is straightforward from the formula in Definition 1.4.17. Namely:

$$F^{(k)} v_j = 0 \text{ if } k + j > l. \tag{4.1}$$

Let $n \in \mathbb{N}^*$, from the latter together with Relations (48) and (49) of [J-K], $(S^l)^{\otimes n} \subset \mathbf{V}^{l \otimes n}$ is a sub-representation of the braid group $\mathcal{B}_n$, see Definition 1.4.22 for the definition of the representation. Namely, the braid group representations can be specialized and restricted to finite dimensional modules. Moreover the obtained representations of the braid group $\mathcal{B}_n$ over $(S^l)^{\otimes n}$ are equivalent over $\mathbb{Q}(q)$ to the ones obtained from the standard construction for the $(l+1)$-dimensional simple representations of $U_q\mathfrak{sl}(2)$ (precisely defined in Section VIII.3 of [Kas]).

Notations. Let $l, k \in \mathbb{N}^*$, and $S_{nl} = \bigoplus_{i=0}^{nl} Ker\left(K - q^{nl-2i}1\right) = \bigoplus_{r=0}^{nl} W_{n,r} \in (\mathbf{V}^l)^{\otimes n}$ be the direct sum of the first k weight spaces. The latter is an $\mathcal{R}$-module spanned by the following set:

$$\mathcal{B}_{S_{nl}} = \{v_{i_1} \otimes \cdots \otimes v_{i_n} \text{ s.t. } \sum i_j \leq nl\}$$

and is a sub representation of $\mathcal{B}_n$ (but not a $U_q\mathfrak{sl}(2)$-submodule), see Remark 1.4.24. Let $\mathcal{S}^l = (S^l)^{\otimes n} \in S_{n,l}$. It is the $\mathcal{R}$-module spanned by the following set:

$$\mathcal{B}_{\mathcal{S}^l} = \{v_{i_1} \otimes \cdots \otimes v_{i_n} \in S_{n,l} \text{ s.t. } i_k \leq l, \forall k\}$$

and is a sub representation of $\mathcal{B}_n$ from previous remark (and a $U_q\mathfrak{sl}(2)$ submodule).

The following lemma is an adaptation of [H-L, Lemma 1.3].

Lemma 4.1.2. *Let $\beta \in \mathcal{B}_n$ be such that its closure is a knot. Let $Q(\beta)$ be its quantum representation over $\mathbf{V}^l$ and given by the $U_q^{\frac{L}{2}}\mathfrak{sl}(2)$ R-matrix (see Definition 1.4.22), then:*

$$\operatorname{Tr}(Q(\beta)K^{-1}, S_{nl}) = \operatorname{Tr}(Q(\beta)K^{-1}, \mathcal{S}^l)$$

where $\operatorname{Tr}(Q(\beta), Z)$ means the trace of the action $Q(\beta)$ restricted to the $\mathcal{B}_n$-subrepresentation Z.

Proof. Let β be a braid and $\tau = \operatorname{perm}(\beta)$. The fact that the closure of β is a knot guarantees that τ is an n-cycle of $\mathfrak{S}_n$ (it permutes all the punctures). Let $(Q(\beta)K^{-1})_{i_1,\ldots,i_n}^{s_1,\ldots,s_n}$ be the matrix of $Q(\beta)K^{-1}$ in the basis $\mathcal{B}(\mathcal{S}_{nl})$.

Remark 4.1.3. One has $i_k \leq l$ implies $s_{\tau(k)} \leq l$ for a term not to be 0.

The latter is due to: $F^{(k)}v_j = 0$ whenever $k + j > l$, $Ev_j = v_{j-1}$ and from the expression of the R-matrix and of its inverse that can be found in [H-L, Section 1.1.2]. These are the same reasons why S^l is stable under the $\mathcal{B}_n$-action. As we want to compute the trace we only have to concern with entries verifying $i_k = s_k$.

Suppose one i_k is less than l, then $s_{\tau(k)} \leq l$ (Remark 4.1.3), so $i_{\tau(k)} \leq l$ (as we only treat diagonal terms), so that $s_{\tau^2(k)} \leq l$ (Remark 4.1.3) and so on. Finally, all the $i_{\tau^m(k)}$ for $m \in \mathbb{N}$ must be lower than l. As the set $\{\tau^m(1), 1 \leq m \leq n\}$ is the whole set $\{1, \ldots, n\}$, whenever one i_k is lower than l in a vector $v_{i_1} \otimes \cdots \otimes v_{i_n}$, all the i_k's must be lower than l for the corresponding diagonal term not to be 0. Similarly whenever one i_k is strictly greater than l in a vector $v_{i_1} \otimes \cdots \otimes v_{i_n}$, all the i_k's must be strictly greater than l for the corresponding diagonal term not to be 0.

It remains two types of vectors for the diagonal term not to be zero. If all the i_k's are strictly greater than l, then $\sum_k i_k \geq n(l+1) > nl$ and $v_{i_1} \otimes \cdots \otimes v_{i_n}$ is not in S_{nl}. The only vectors in S_{nl} corresponding to non-zero diagonal terms are the ones such that $i_k \leq l$ for all k, namely the ones of $\mathcal{S}^l$. This concludes the proof. □

The finite dimensional representations $\mathcal{S}^l$ of the braid group constructed above are the one used to define the colored Jones polynomial. The following definition is given at the beginning of Subsection 1.1.4 of [H-L].

Definition 4.1.4 (Colored Jones polynomial). *Let $n, l \in \mathbb{N}^*$. If the closure of an n-strand braid β is the knot K, then the $(l+1)$-colored Jones polynomial of K is the following:*

$$\mathrm{J}_K(l+1) = q^{w(\beta)\frac{(l+1)^2-1}{2}} \mathrm{Tr}\left(Q(\beta)K^{-1}, \mathcal{S}^l\right)$$

where $w(\beta)$ is the writhe of the braid β, namely the sum of crossings' signs. The function $\mathrm{Tr}\left(Q(\beta)K^{-1}, \mathcal{S}^l\right)$ means the trace of the operator $Q(\beta)K^{-1}$ (see Definition 1.4.22 for Q, while K is one of the $U_q\mathfrak{sl}(2)$ generator) restricted to $\mathcal{S}^l$.

Remark 4.1.5. We gave the definition of the $(l+1)$-Jones polynomial to fit with our choice of $\mathcal{S}^l$. It differs from [H-L] where their W_N is the N-dimensional finite simple module of $U_q\mathfrak{sl}(2)$, while our S^l is the $(l+1)$-dimensional one.

4.1.2 Homological trace formula for colored Jones polynomials

Definition 4.1.6 (Homological trace). *Let X be a topological space and R be a commutative ring of coefficients such that and $H = H_\bullet(X, R)$, the finite homological complex associated to X with coefficients in R, is a complex of free R-modules. Let f be a continuous self-map of X. The H-*Lefschetz number* of f is the following number:*

$$\mathcal{L}\left(f, H\right) = \sum_i (-1)^i \mathrm{Tr}\left(f_*, H_i\left(X, R\right)\right) \in R$$

where $\mathrm{Tr}\left(f_*, H_i\left(X, R\right)\right)$ means the trace of the action of f over the i^{th} homology module of H.

We consider the local system L_r defined over $X_r(w_0, \ldots, w_n)$ in Section 3.2 under the condition $t = q^{-2}$. The coefficients ring is $\mathcal{R} = \mathbb{Z}\left[q^{\pm\alpha_i}, q^{\pm 1}\right]_{i=1,\ldots,n}$ or equivalently $\mathcal{R} = \mathbb{Z}\left[s_i^{\pm 1}, q^{\pm 1}\right]_{i=1,\ldots,n}$ (setting $q^{\alpha_i} = s_i$). We recall that the braid group $\mathcal{B}_n$ acts by R^{hom} over the complex $\mathcal{H}_\bullet^{\mathrm{rel}\,-}(X_r) := H_\bullet^{BM}\left(X_r, X_r^-; L_r\right)$ from Lemma 3.2.69.

Proposition 4.1.7 (A homological trace formula for colored Jones). *Let $l \in \mathbb{N}$. If the closure of an n-strand braid β is the knot K, then the $(l+1)$-colored Jones polynomial satisfies the following homological formula:*

$$\mathrm{J}_K(l+1) = (q^{w(\beta)\frac{(l+1)^2-1}{2}}) q^{-nl} \sum_{r=0}^{nl} (-1)^r \left[\mathcal{L}\left(\mathrm{R}^{hom}(\beta), \mathcal{H}_\bullet^{rel\,-}(X_r)\right)\right]_{\alpha_i = l} q^{2r}.$$

Traces considered in the sum (namely in the Lefschetz numbers) are elements of $\mathcal{R}$, and we make the specialization $\alpha_i = l$ (for $i = 1, \ldots, n$), corresponding to the specialization ring morphism:

$$\mathrm{aug}^n : \begin{array}{ccc} \mathbb{Z}\left[s_i^{\pm 1}, q^{\pm 1}\right] & \rightarrow & \mathbb{Z}\left[q^{\pm 1}\right] \\ s_i & \mapsto & q^l \end{array}$$

already introduced at the beginning of this section.

Proof. From Definition 4.1.4, the aim of the proof is to compute:

$$\operatorname{Tr}\left(Q(\beta)K^{-1}, \mathcal{S}^l\right).$$

From Lemma 4.1.2, we have:

$$\operatorname{Tr}\left(Q(\beta)K^{-1}, \mathcal{S}^l\right) = \operatorname{Tr}(Q(\beta)K^{-1}, S_{nl}).$$

The following lemma computes $\operatorname{Tr}(Q(\beta)K^{-1}, S_{nl})$ and concludes the proof.

Lemma 4.1.8. *There is the following formula:*

$$\operatorname{Tr}(Q(\beta)K^{-1}, S_{nl}) = q^{-nl} \sum_{r=0}^{nl} (-1)^r \left[\mathcal{L}\left(\mathrm{R}^{hom}(\beta), \mathcal{H}_\bullet^{rel\,-}(X_r)\right)\right]_{\alpha_i = l} q^{2r}$$

where the traces considered in the sum are elements of the ring $\mathcal{R}$, and we make the specialization $\alpha_i = l$, corresponding to the specialization ring morphism aug^n.

Proof. We recall from Proposition 3.2.8 that $\mathcal{H}_r^{\mathrm{rel}\,-}$ is the only non vanishing module of the complex $H_\bullet^{BM}(X_r, X_r^-; L_r)$. We also know from Theorem 6 that the $\mathcal{R}$-modules $W_{n,r}$ are $\mathcal{B}_n$-equivalent to $\mathcal{H}_r^{\mathrm{rel}\,-}$. The lemma is an immediate consequence of:

$$S_{nl} = \bigoplus_{r=0}^{nl} W_{n,r}$$

and the fact that pre-composing by K^{-1} provides a coefficient $q^{-nl}q^{2r}$, corresponding to the action of the diagonal operator K^{-1} over $W_{n,r}$. □

□

The above formula suggests that colored Jones polynomials compute the beginning of a generating series of Lefschetz numbers indexed by r over first complexes $\mathcal{H}_\bullet^{\mathrm{rel}\,-}(X_r)$. The latter homology complexes are the ones of X_r with local system L_r and relative to X_r^-. The rest of this section is devoted to the interpretation of this homological trace formula in terms of fixed point theory.

4.1.3 Nielsen fixed point theory and Lefschetz numbers

We make recalls about the *generalized Lefschetz number* (first introduced in [Hu]), the *abelianized Nielsen number* ([H-J, Section 1.(B)]) and conclude with the *mod K Nielsen numbers* (for K a normal subgroup of the fundamental group) ([J, Chapter III.2] or [RFB, Part 1]).

We follow [H-J, Section 1] for the definitions concerning generalized and abelianized Nielsen theories. Let X be a compact connected topological space admitting a universal covering space, and f be a continuous self-map of X. The fixed point set $Fix(f) = \{x \in X \text{ s.t. } f(x) = x\}$ splits into the union of *Nielsen fixed point classes*.

Definition 4.1.9. *Two fixed points x and y of f are in the same Nielsen class if there exists a path λ joining them and such that λ is homotopic (rel. endpoints) to its image under f. Nielsen classes are isolated so that their fixed point indexes are well defined.*

Suppose f fixes the base point of X so that we can define f_π to be the lift of f to $\pi_1(X)$ (otherwise the choice of a path w from x_0 to its f-image is necessary to define f_π).

Definition 4.1.10. *We say that $\alpha, \beta \in \pi_1(X)$ are in the same f-Reidemeister class if there exists $\gamma \in \pi_1(X)$ such that $\alpha = f_\pi(\gamma)\beta\gamma^{-1}$. We call π_R the group of such f*-conjugacy classes *and $\mathbb{Z}\pi_R$ its group ring.*

Let x be a fixed point and c be a path from x_0 to x. The Reidemeister class of the loop $(f \circ c)c^{-1}$ is independent of the choice of c and called the coordinate of x.

Proposition 4.1.11 ([H-J, Section 1.(A)])**.** *Two fixed points are in the same Nielsen class iff they have the same coordinates. The involved Reidemeister class is thus the* Nielsen class coordinate*.*

Definition 4.1.12. *The* generalized Lefschetz number *of f is defined to be:*

$$\mathcal{L}_R(f) = \sum_{[\alpha] \in \pi_R} i_{[\alpha]} \, [\alpha] \in \mathbb{Z}\pi_R$$

where $[\alpha]$ is a class in π_R and $i_{[\alpha]}$ the index of the corresponding Nielsen class.

The following fact makes the link with the traditional notions of Lefschetz numbers and Nielsen numbers and can be taken as definition for the present work.

Fact 4.1.13. *The usual Nielsen number and Lefschetz number are the following numbers:*

$$N(f) = \sharp\{[\alpha] \ \ s.t. \ \ i_{[\alpha]} \neq 0\}$$
$$L(f) = \sum_{[\alpha]} i_{[\alpha]}.$$

Theorem 4.1.14 ([Hu, Theorem 1.13],[H-J, Section 1.(A)])**.** *Suppose X is a CW complex. A cellular decomposition of X lifts to cells of $\widetilde{X}$, its universal covering space. These cells* *constitute a $\mathbb{Z}\pi_1(X)$ basis of the cellular chain complex of $\widetilde{X}$. Let f be a chain map and $\widetilde{f}$ be a lift of f to $\widetilde{X}$ (considering a lift of the base point). Then:*

$$\mathcal{L}_R(f) = \sum_q (-1)^q \left[\mathrm{Tr}(\widetilde{f}, C_q(\widetilde{X}, \mathbb{Z}\pi_R))\right]_R \in \mathbb{Z}\pi_R.$$

where $\left[\mathrm{Tr}(\widetilde{f}, C_q(\widetilde{X}, \mathbb{Z}\pi_R))\right]_R$ is the π_R-class of the trace of the action of $\widetilde{f}$ over the $\mathbb{Z}\pi_1(X)$-module $C_q(\widetilde{X}, \mathbb{Z}\pi_R)$.

The proof of the above theorem relies on the lifting property of maps to the universal cover. See a sketch of proof in [Ha, Theeorem 2.2]. This proof can be adapted to Borel-Moore chain complexes over punctured manifold if f is a proper map, as Borel-Moore corresponds to an inductive limit of chain complexes over compact topological spaces. One can also easily adapt the theory to other covering spaces having the good lifting properties, namely those which correspond to normal subgroups of the fundamental group. Indeed, the theory of generalized Lefschetz number corresponds to working at the level of the universal covering space but has analogs working in other covering spaces. For instance the maximal abelian cover level gives rise to the notion of abelianized Nielsen number.

Definition 4.1.15 ([H-J, Section 1.(A)])**.** *Two fixed points are in the same* homological Nielsen class *if there exists a path λ joining them and such that λ is homologous (rel. endpoints) to its image under f. The* abelianized Nielsen number $\mathcal{N}^{ab}(f)$ *is the number of homological Nielsen classes with non-zero index.*

We consider the following morphism:

$$\mu : H_1(X) \to H = Coker\left(1 - f_* \in \mathrm{End}(H_1(X))\right).$$

The coordinate of a homological Nielsen class is the H-class of the loop $(f \circ c)c^{-1}$ where c is a path from x_0 to a fixed point of the class. We recall that if $\widehat{X}$ is the maximal abelian cover of X, the chain complex $C_\bullet(\widehat{X})$ is endowed with a $\mathbb{Z}H_1(X)$-action. Again, there is a trace formula computing the abelianized Nielsen number.

Proposition 4.1.16 ([H-J, Section 1.(B)])**.** *The abelianized Reidemeister trace $\mathcal{L}_H(f)$ is an element of $\mathbb{Z}H$ such that the coefficient of $h \in H$ is the index of the homological class with coordinate h. Thus $\mathcal{L}_H(f) = \mathcal{L}_R(f)^{ab}$ (the generalized Lefschetz number abelianized), and it satisfies the following trace formula:*

$$\mathcal{L}_H(f) = \sum_q (-1)^q \mu \left[\mathrm{Tr}(\widetilde{f}, C_q(\widehat{X}, \mathbb{Z}H_1(X)))\right] \in \mathbb{Z}H.$$

The number of non-zero terms is $\mathcal{N}^{ab}(f)$.

Remark 4.1.17. Although the abelianized Nielsen number is less refined than the Nielsen number, it is useful for computation. Namely it enjoys the following properties:

- $\mathcal{N}^{ab}(f) \leq N(f)$.
- $L(f) = \sum i_{[\alpha]_H}$ where the terms in the seum are the indexes of homological classes associated to α.

One could work with a covering space corresponding to a normal subgroup of the π_1 and obtain the same kind of result. Namely, the abelianized Nielsen theory corresponds to the the *mod K* Nielsen theory ([J, Chapter III.2]) for $K = [\pi_1(X), \pi_1(X)]$. We will precise the group K of interest to us in the context of the following subsection.

Theorem 4.1.18 ([J, Theorem 2.5])**.** *The Nielsen number, abelianized Nielsen number, and the mod K Nielsen numbers are homotopy invariants.*

4.1.4 Colored Jones polynomials and Lefschetz numbers

For $r \in \mathbb{N}^*$, we recall that the space $X_r(w_0) \in X_r$ consists in configurations with no coordinate in w_0. Let $\beta \in \mathcal{B}_n$, $\hat{\beta} \in Homeo^+(D_n)$, stabilizing w_0 and such that the isotopy class of $\hat{\beta}$ is represented by β. We make the remark that $\hat{\beta}$ can be chosen to fix w_0 as half-Dehn twists corresponding to braid generators can be chosen so to be supported in the interior of the punctured disk. Let:

$$\widehat{\beta}^r : \begin{array}{ccc} X_r(w_0) & \to & X_r(w_0) \\ \{z_1, \ldots, z_r\} & \mapsto & \{\widehat{\beta}(z_1), \ldots, \widehat{\beta}(z_r)\} \end{array}$$

be the corresponding self homeomorphism of $X_r(w_0)$.

Theorem 7. *Let $\beta \in \mathcal{B}_n$, and $l \in \mathbb{N}^*$. Then:*

$$\mathrm{J}_K(l+1) = (q^{w(\beta)\frac{(l+1)^2-1}{2}})q^{-nl} \sum_{r=0}^{nl} (-1)^r \left[\mathcal{L}_H \left(\widehat{\beta}^r \right) \right]_{\alpha_i = l} q^{2r}.$$

Here, $\mathcal{L}_H(\widehat{\beta}^r) \in \mathbb{Z}H$ (the abelianized Lefschetz number of $\widehat{\beta}^r$) where:

$$H = Coker \left(1 - \beta^r_{1,\mathbb{Z}} \in \mathrm{End}(H_1(X_r(w_0))) \right)$$

is then specialized by the augmentation morphism aug^n.

Proof. First we need the specialization $\mathbb{Z}H \to \mathbb{Z}[q^{\pm 1}]$ to be well defined, which is the following lemma.

Lemma 4.1.19. *Let $\beta \in \mathcal{B}_n$ be a braid and $\beta^r_{1,\mathbb{Z}}$ its associated action on $H_1(X_r, \mathbb{Z})$. Then the augmentation morphism: $\mathrm{aug}^n : \mathbb{Z}H_1(X_r, \mathbb{Z}) \to \mathbb{Z}[q^{\pm 1}]$ given by $\alpha_1 = \cdots = \alpha_n = l$ factors through $Coker(1 - \beta^r_{1,\mathbb{Z}})$. Namely there exists a morphism $\mathbb{Z}H \to \mathbb{Z}[q^{\pm 1}]$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{Z}H_1 & \xrightarrow{coker} & \mathbb{Z}H \\ & \searrow^{\mathrm{aug}} & \downarrow \\ & & \mathbb{Z}[q^{\pm 1}] \end{array} .$$

Proof of Lemma 4.1.19. The proof is immediate from the commutation of the diagram in the proof of Lemma 3.2.69. □

Then we recall from Proposition 3.2.13 that the complexes $\mathrm{H}^{lf}_{\bullet}(X_r(w_0); L_r)$ and $\mathrm{H}^{lf}_{\bullet}(X_r, X_r^-; L_r)$ are isomorphic. From Proposition 5.2.2 in the Appendix we know that the alternated sum of traces over a finite dimensional chain complex is computable at the level of homology. Hence, in Proposition 4.1.7, the Lefschetz numbers involved correspond to an alternated sum of traces of homology actions. They are equal to alternated sum of traces of the corresponding chain complex actions and this fits with Proposition 4.1.16 that introduces the trace formula for abelianized Lefschetz numbers. □

Remark 4.1.20. These Lefschetz numbers are the abelianized ones but specialized by the morphism aug. They contain the following information.

- The specialization under aug corresponds to a *mod* K Lefschetz number ([J, Chapter III.2]), such that K is the kernel of $\text{aug} \circ \text{ab} : \pi_1(X_r(w_0)) \to \mathbb{Z}\left[q^{\pm 1}\right]$. This kernel is a normal subgroup of $\pi_1(X_r(w_0))$ that corresponds to a lower abelian covering space such that the covering deck transformations group is $\mathbb{Z} = \mathbb{Z}\langle q \rangle$. The number of non-zero terms corresponds to the number of the corresponding *mod* K Nielsen number that we denote by $\mathcal{N}^q(f)$. The latter counts the *mod* K Nielsen classes, where two fixed points are in the same class iff there exists a path joining them and which differs from its $\widehat{\beta}^r$-image by an element of K.

- From Remark 4.1.17 we conclude that if one succeed in counting non-zero terms in the weighted expression for the colored Jones polynomials, one would get lower bounds for some classical Nielsen numbers.

- They may contain a precise evaluation of classical Lefschetz numbers (under the specialization $q \mapsto 1$).

4.1.5 Further directions.

In Theorem 7, we showed that the colored Jones polynomials are related to abelianized Lefschetz numbers. One could expect to refine the result and extract topological information out of it in the following directions.

- There exists a generalized relative Lefschetz theory (see [NO-W]), and one could easily imagine an abelianized version of it in the spirit of Definition 4.1.16. The relative theory consists in the computation of homotopical invariants for self-maps of pairs. In Nielsen fixed point theory one is interested in the following question:

  Question. *Does there exist a self-map in a chosen isotopy class that realizes the Nielsen number, namely that has $N(f)$ fixed points.*

As we discussed the loss of information while computing lower bounds for $N(f)$, this question is a minimization problem. It was shown to have a negative answer in general. One very simple counterexample is due to Jiang ([Sch, Example 1.1]). He proved that the Nielsen number of some application is 1 while any self map in its class must have at least 2 fixed points. By consideration of an appropriate relative Nielsen number, one finds a Nielsen number equal to two and providing a positive answer to the above question. Then, in the context of the above question, a relative Nielsen number should be finer than an absolute one.

In Proposition 4.1.7 we see that the colored Jones polynomial computes a homological trace formula over relative homology modules. In Theorem 7 we used an identification with absolute homology module to fit with absolute Nielsen theory. As far as we know, there does not exist a trace formula over relative homology modules computing the relative Nielsen number. Still, as the generalized relative Nielsen theory exists and the colored Jones polynomial computes a relative homological trace formula we make the following conjecture:

Conjecture. *The Lefschetz numbers involved in the colored Jones formula of Theorem 7 are specializations of abelianized relative Lefschetz numbers.*

- One information that we didn't use is the fact that we get Lefschetz numbers over configuration spaces of the punctured disk. We make the following immediate remark.

  Remark 4.1.21. Let f be a self homeomorphism of the punctured disk D_n, and $\hat{f}$ the associated homeomorphism of X_r. If $Z = \{z_1, \ldots z_r\}$ is a fixed point of $\hat{f}$, then:

  - Z is a set of periodic points of f.
  - One could associate a braid to Z and the study of its braid type gives information about periods, see [Mat, Section 2.7].

  There exists a general Lefschetz fixed point theory associated to the study of periodic points and it constitutes a consequent literature. From the above remark, one could expect the colored Jones polynomial to compute Lefschetz numbers associated to periodic point classes over the punctured disk. The author paid attention to the notion of *twisted Lefschetz zeta function*, which is by definition (in its abelianized version, see [H-J, Section 1.(D)]):

$$\zeta_H(f) = exp\left(\sum_{p=0}^{\infty} \frac{\mathcal{L}_H(f^p)}{p} t^p\right)$$

  a formal power series in the variable t, namely living in $\mathbb{Z}H\,[[t]]$. If one finds a relation between the Lefschetz number of a self-homeomorphism $\hat{f}$ of X_r and periodic points of f, then one may extract information about Lefschetz zeta functions out of the colored Jones polynomials. This claim relies on the fact that the weighted sum inside the exponential would present similarities with the weighted sum for colored Jones of Theorem 7. The zeta functions provide information about the growth rate of periodic points.

- We mention also [GG1] and [GG2] where the authors relate fixed point theory over configuration spaces to fixed point theory of multivalued maps.

- In [J-W], a twisted Lefschetz number theory is developed. Namely, by considering a ring of coefficients twisted by a representation of the fundamental group, the authors obtained a twisted version of the generalized Lefschetz numbers. By modifying our local system L_r, replacing it by a non-abelian one in the spirit of Example 3.1.13, one could expect to recover twisted Lefschetz numbers.

- In [FH] the authors provide a way to compute generalized Lefschetz numbers over surfaces from Fox derivatives computation. In Section 3.2.6.5 we asked the question whether Enright representations of the braid groups defined from higher Fox derivatives could be recovered by the homological representation defined over $\mathcal{H}_\bullet^{\mathrm{rel}\,-}$. The link between the trace of these representations and the generalized Lefschetz number (Theorem 7) computable using Fox calculus ([FH]) is a positive sign. A first step was already done in this direction. Namely, it was shown [Mat, Theorem (3.3)] that Lefschetz numbers are contained in the trace of Burau/Gassner representations using Fox theory of derivation. Our result is a generalization of the latter in some sense, as the family of homological representation we are working on contains Burau and computes Lefschetz numbers.

- It could be interesting to succeed in re-proving that the colored Joned polynomial is a knot invariant using the dynamical trace formula. Periodic points can be interpreted in terms of mapping torus. Indeed, let x be a periodic point of a self-map f and T_f be the mapping torus of f. there is a natural flow over f consisting in running around the torus. By iteration of this flow starting at x one would get a link in T_f as x is periodic. Two knots corresponding to two points in the same periodic homological Nielsen class are then homologous. This remark should be an idea for proving the knot invariance.

- Finally, one would be interested in the Nielsen-Thurston classification of mapping classes. Namely, there are strong links between fixed point theory and Nielsen-Thurston classification of homeomorphisms that are encoded in Lefschetz numbers. The latter could bring information about pseudo-Anosov homeomorphism for instance, from the computation of colored Jones polynomials.

Chapter 5

Appendix: Homology features

5.1 A homology adapted to non-compact spaces

All along this work we worked on topological spaces with punctures, that are non-compact spaces. The good homology theory to work with these kinds of topological spaces is the locally finite homology isomorphic in our case to the Borel-Moore homology. By "good homology" theory we mean that many properties holding for singular homology of compact spaces can be generalized to Borel-Moore's, as it controls the non-compact phenomena arising at punctures. We give general ideas and definitions of these homologies in this section. Let X be a locally compact topological space.

Definition 5.1.1 (Locally finite homology)**.** *The* locally finite chain complex *associated to X is the chain complex for which we allow infinite sums of singular chains under the condition that their geometrical realization in X is locally finite (for the topology of X). The latter guarantees that the boundary map is well defined.*

Let $Y \subset X$. The relative to Y *locally finite chain complex corresponds to the locally finite chain complex of X mod out by the one of Y.*

The homology of locally finite chains is the homology complex corresponding to these definition of chain complexes. We use the notation $H_\bullet^{lf}(X)$ to denote the locally finite homology complex.

Recalls. The homology of locally finite chains is isomorphic to the Borel-Moore homology that can be defined as follows:

$$H_\bullet^{BM}(X) = \varprojlim H_\bullet (X, X \setminus A)$$

where the inverse limit is taken over all compact subsets A of X. The relative case is then the following:

$$H_\bullet^{BM}(X, Y) = \varprojlim H_\bullet (X, (X \setminus A) \cup Y)$$

for $Y \subset \partial X$.

The above fact that Borel-Moore homology consists in a limit of homology complex over compact spaces allows generalizations of many compact singular homology properties.

Remark 5.1.2. All these definitions are identical in the case of homology with local coefficients.

Locally finite homology have very different properties than the usual ones when the space is non compact. We emphasize this point in the following remark.

Remark 5.1.3. First property of the Borel-Moore homology that should help the reader feeling differences.

(Compact space) If X is compact, then the singular and locally finite homology are identical.

(Real line) Any 0-chain is null homologous (so that the 0-homology does not encode connectedness). Let p be a point, the chain:

$$\sigma = \sum_{i=0}^{\infty} [p+i, p+i+1)$$

has p as Borel-Moore boundary (the Borel Moore homology does not see points at infinity in some sense). While the chain:

$$\sum_{-\infty<k<\infty} [k, k+1)$$

has no boundary and hence is a cycle. The latter shows that $H_k^{BM}(\mathbb{R}) = \mathbb{Z}$ if $k = 1$ and is 0 otherwise and can be generalized to $H_k^{BM}(\mathbb{R}^n) = \mathbb{Z}$ if $k = n$ and is 0 otherwise.

(Sphere) The latter allows to compute the homology of spheres and balls using $H_k^{BM}(\mathbb{R}^n) = H_k^{BM}(\mathbb{S}^{n-1})$.

(Punctures) Let D_n be the punctured disk, and c be a small circle running once around a puncture p. Then c is a cycle using same kind of telescopic infinite chain as in the previous point.

(Submanifold) More generally, any closed oriented submanifold defines a class in Borel–Moore homology, but not in ordinary homology unless the submanifold is compact.

5.2 Some homological algebra

We recall standard homological algebra properties used in this work. First we recall the *excision theorem* that is one of the required Eilenberg-Steenrod axioms for a homology theory.

Theorem 5.2.1 (Excision Theorem,[Hat, Theorem 2.20])**.** *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X - A, A - Z) \to (X, A)$ induces isomorphism:*

$$H_n(X - Z, A - Z) \to H_n(X, A)$$

for all n. Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms:

$$H_n(B, A \cap B) \to H_n(X, A)$$

for all n.

Of course the above theorem applied to relative Borel-Moore homology with local coefficients. We end this appendix with a well known property of trace formulae.

Proposition 5.2.2 (Hopf trace formula)**.** *Suppose $C_\bullet$ is a finite chain complex of finitely generated abelian group and Φ is a chain self-map. Then:*

$$\sum_{i \in \mathbb{N}} (-1)^n \mathrm{Tr}(\Phi : C_n \to C_n) = \sum_{i \in \mathbb{N}} (-1)^n \mathrm{Tr}(\Phi_* : H_n \to H_n).$$

□

Bibliography

[ADO] Y. Akutsu, T. Deguchi, and T. Ohtsuki *Invariants of colored links*, J. Knot Theory Ramifications, (1992), no. 2, 161–184.

[AMU] J.E. Andersen, G. Masbaum, and K.Ueno, *Topological Quantum Field Theory and the Nielsen-Thurston classification of M(0,4)*, Mathematical Proceedings of the Cambridge Philosophical Society **141** (2006), 477-488.

[An] C. Anghel, *A Homological model for the coloured Jones polynomials*, arXiv:1712.04873,(2017).

[B-B] G. Band, P. Boyland *The Burau estimate for the entropy of a braid*, Algebraic & Geometric Topology **7**, (2007), 1345–1378.

[B-N] D.Bar-Natan, *A Note on the Unitarity Property of the Gassner Invariant*, Bulletin of Chelyabinsk State University (Mathematics, Mechanics, Informatics) **3-358-17** (2015), 22–25.

[Bas] S. Baseilhac, *Quantum coadjoint action and the 6j-symbols of Uq sl(2)*, in Interactions between Hyperbolic Geometry, Quantum Topology and Number Theory, AMS Cont. Math. Proc. Ser. **541**, (2011) 103–144.

[Big0] S. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. **14** (2000), 471–486.

[Big1] S. Bigelow, *Homological representations of the Iwahori-Hecke algebra*, Geom. Topol. Monographs **7**, (2004), 493–507.

[Big2] S. Bigelow, *The Lawrence–Krammer representation*, Proc. Sympos. Pure Math. **71**, Amer.Math. Soc. Providence, (2003), 51–68.

[Big3] S. Bigelow, *A homological definition of the Jones polynomial*, Invariants of knots and 3-manifolds (Kyoto, 2001), Geom. Topol. Monogr., volume **4** (2002), 29–41.

[Big4] S. Bigelow, *A homological definition of the HOMFLY polynomial*, Algebraic and Geometric Topology, (2007).

[Bir] J. Birman, *Braids, Links and Mapping Class Groups.*, Annals of Mathematics Studies **82** Princeton University Press, Princeton, New Jersey, (1975).

[BCGP2] C. Blanchet, F. Costantino, N. Geer, B. Patureau, *Non semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants*, Advances in Mathematics **301** (2016), 1–78.

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Three-manifold invariants derived from the Kauffman bracket*, Topology **31** (1992), no. 4, 685–699.

[Br] R. Brown, *Topology and Groupoids*, Booksurge, (2006).

[RFB] R.F. Brown, *More about Nielsen Theories and Their Applications*, Handbook of Topological Fixed Point Theory, (2005), Springer, Dordrecht, 433–462.

[BFGT] P. Bushlanov, B. Feigin, A. Gainutdinov, I. Tipunin, *Lusztig limit of quantum* $\mathfrak{sl}(2)$ *at root of unity and fusion of* $(1,p)$ *Virasoro logarithmic minimal models*, Nuclear Physics, B **818** (3), 179–195.

[C-P] V. Chari, A. N. Pressley, *A guide to Quantum Groups*, Cambridge University Press, (1995).

[CGP1] F. Costantino, N. Geer, B. Patureau, *Some remarks on the unrolled quantum group of* $\mathfrak{sl}(2)$, J. Pure Appl. Algebra **219**, no. 8 (2015), 3238–3262.

[CGP2] F. Costantino, N. Geer, B. Patureau, *Quantum invariants of* 3*-manifolds via link surgery presentations and non-semi-simple categories*, Journal of Topology **7** (2014), 1005–1053.

[CGP3] F.Costantino, N.Geer, B.Patureau, *Relations between Witten-Reshetikhin-Turaev and non semi-simple* $\mathfrak{sl}(2)$ 3*-manifold invariants*, Algebraic Geometry and Topology, **15** (2015), 1363–1386.

[C-M] F. Costantino, J. Murakami, *On SL(2,C) quantum 6j-symbol and its relation to the hyperbolic volume*, Quantum Topology, **4** (2013), no.3, 303–351.

[C-J] E. Creath, D. Jakelic, *Highest-weight vectors in tensor products of Verma modules for* $U_q\mathfrak{sl}(2)$, Boletim da Sociedade Paranaense de Matemática, Vol **36**, No 4, (2018).

[DCP] C. De Concini, C. Procesi *Quantum Groups*, in "D-modules, Representation Theory and Quantum Groups (Venice 1992)", Springer Verlag, Lect. Notes Math. 1565 (1993) 31–140.

[DR] M. De Renzi, *Non-Semisimple Extended Topological Quantum Field Theories* To Appear in Memoirs of the American Mathematical Society.

[Drin] V. G. Drinfel'd, *Quasi-Hopf algebras*, Leningrad Math. **J.1** (1990), 1419–1457.

[FH] E. Fadell, S. Husseini, *The Nielsen number on surface*, Topological Methods in Nonlinear Functional Analysis, Contemp. Math. **21**, (1983), 59–98.

[F-M] B. Farb, D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press (2012).

[F-W] G. Felder, C. Wieczerkowski, *Topological Representations of the Quantum Group* $U_q\mathfrak{sl}(2)$, Commun. Math. Phys. **138** (1991), 583–605.

[GG1] D. Goncalvez, J. Guaschi, *Fixed points of n-valued maps on surfaces and the Wecken property - a configuration space approach*, Science China Mathematics **60**, (2017), 1561–1574.

[GG2] D. Goncalvez, J. Guaschi, *Fixed points of n-valued maps, the fixed point property and the case of surfaces - a braid approach*, Indag. Math. **29**, (2018), 91–124.

[Gor] L. Gorniewicz, *On the Lefschetz Fixed Point Theorem*, Handbook of Topological Fixed Point Theory, Springer, Dordrecht, 43–82.

[Hab] K. Habiro, *An integral form of the quantized enveloping algebra of sl2 and its completions*, J. Pure Appl. Alg. **211** (2007), 265–292.

[Ha] E. Hart, *Algebraic Techniques for Calculating the Nielsen Number on Hyperbolic Surfaces*, Handbook of Topological Fixed Point Theory, (2005), Springer, Dordrecht 463–487.

[Hat] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002).

[H-J] H.-H. Huang, B. Jiang, *Braids and periodic solutions*, Topological Fixed Point Theory and Applications, Lecture Notes in Math. **1411**, Springer-Verlag, Berlin, Heidelberg, New York, (1989), 107–123.

[Hu] S. Husseini, *Generalized Lefschetz numbers*, Trans. Amer. Math. **272**, (1982), 247–274.

[H-L] V. Huynh and T. T. Q. Lê, *On the Colored Jones Polynomial and the Kashaev invariant*, Fundam. Prikl. Mat. **11** (2005), no. 5, 57–78.

[Ik] A. Ikeda, *Homological and Monodromy Representations of Framed Braid Groups*, Communications in Mathematical Physics (2018), **359**, 1091–1121.

[Ito] T. Ito, *Reading the dual Garside length of braids from homological and quantum representations*, Comm. Math. Phys. *335* (2015) 345–367.

[Ito1] T. Ito, *A homological representation formula of colored Alexander invariant*, Adv. Math. **289** (2016), 142–160.

[Ito2] T. Ito, *Topological formula of the loop expansion of the colored Jones polynomials*, Trans. Amer. Math. Soc., (2019).

[J-K] C. Jackson and T. Kerler, *The Lawrence-Krammer-Bigelow representations of the braid groups via $U_q\mathfrak{sl}(2)$*, Adv. Math, **228**, (2011), 1689–1717.

[J] B. Jiang, Lectures on Nielsen Fixed Point Theory, Contemp. Math., vol. 14, Amer. Math Soc., Providence, Rhode Island, (1983).

[J2] B. Jiang, *A primer of Nielsen fixed point theory*, Handbook of Topological Fixed Point Theory, Springer, Dordrecht, 617–645.

[J-W] B. Jiang, S. Wang *Twisted Topological Invariants Associated with Representations* Topics in Knot Theory., (1993), **399** Springer, Dordrecht, 211–227.

[Kas] C. Kassel, *Quantum Groups.*, Graduate Texts in Mathematics **155** Springer-Verlag, New York, (1995).

[K-T] C. Kassel, V. Turaev, *Braid Groups*, Springer (2008).

[Kar] K. Karvounis *Braid group action on projective quantum* $\mathfrak{sl}(2)$ *modules*, arXiv:1909.11162 [math.GT].

[Knu] K. Knudson, *On the kernel of the Gassner representation*, Archiv der Mathematik **85**, (2005), 108–117.

[K0] T. Kohno, *Monodromy representations of braid groups and Yang-Baxter equations*, Ann. Inst.Fourier **37** (1987), 139–160.

[K1] T. Kohno, *Hyperplane arrangements, local system homology and iterated integrals*,Arrangements of Hyperplanes—Sapporo 2009, 157–174, Mathematical Society of Japan, Tokyo, Japan, 2012.

[K2] T. Kohno, *Quantum and homological representations of braid groups*, Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale (2012), 355–372.

[Kra] D. Krammer, *Braid groups are linear*, Ann. Math. **155** (2002), 131–156.

[Kra1] D.Krammer, *Braid groups are linear*, Annals of Mathematics **155** (2002), 451–486.

[Law] R. Lawrence, *Homological representations of the Hecke algebra*, Comm. Math. Phys. **135** (1990), 141–191.

[Len] S. Lentner, *The unrolled quantum group inside Lusztig's quantum group of divided powers*, Preprint (2017), arXiv:1702.05164.

[Lus] G. Lusztig, *Finite dimensional Hopf algebras arising from quantum groups* , J. Am. Mat. Soc. **3** (1990) , 257–296.

[Lus2] G. Lusztig, *Quantum groups at roots of* 1, Geom. Ded., **35** (1990), 89–114.

[Mac] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Notes, **213**, (2005).

[Mat] T. Matsuoka, *Periodic Points and Braid Theory*, Handbook of Topological Fixed Point Theory, (2005), Springer, Dordrecht, 171–216.

[Mu-Mu] H. Murakami, J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math., **186**, (2001), 85–104.

[NO-W] B. Norton-Odenthal, P. Wong *A relative generalized Lefschetz number*, Topology and its Applications **56(2)**, (1994), 141–157.

[Oht] T. Ohtsuki, *Quantum invariants. A study of knots, 3–manifolds, and their sets*, Series on Knots and Everything **29** World Scientific Publishing Co., Inc., River Edge, NJ, (2002).

[P-P] L. Paoluzzi, L. Paris, *A note on the Lawrence-Krammer-Bigelow representation*, Algebr. Geom. Topol., **2** (2002), 499–518.

[R-T] N. Reshetikhin, V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys., **127** (1990), no. 1, 1–26.

[RT2] N.Yu. Reshetikhin, V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

[Sal] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^N$, Invent. Math. **88**, (1987), no. 3, 603–618.

[S-V] V. Schechtman and A. Varchenko, *Quantum Groups and Homology of Local Systems*, ICM-90 Satellite Conference Proceedings, Algebraic Geometry and Analytic Geometry, 182–197.

[Sch] H. Schirmer, *A Survey of Relative Nielsen Fixed Point Theory*, Proceedings of the Conference on Nielsen Theory and Dynamical Systems, Contemp. Math. **152**, (1993), 291–309.

[Z1] H. Zheng, *A reflexive representation of braid groups*, J. Knot Theory Ramifications, **14** (2005), 467–477.

[Z2] H. Zheng, *Faithfulness of the Lawrence representation of braid groups*, arXiv:math/0509074v1 .

Abstract: We provide homological interpretations for some quantum invariants. We recall basic notions involved in this work: topological ones on one hand (braids, mapping class groups, and homological representations of the latter) and algebaic ones on the other hand (Hopf algebra, quantum groups, categories of modules, braiding).

Then, we study “small cases”: We show that the Gassner representation is contained in quantum representations of the braid group. We build Bigelow-Krammer-Lawrence representations in a colored version and we give matrices for the action. Finally we study the non semi-simple TQFT (built by Blanchet – Costantino – Geer – Patureau) representation of the mapping class group of the sphere with 4 punctures. We recognize homological representation inside of it, and this leads to the faithfulness of the representation.

In the last chapter, we study modules of relative and locally finite homology modules with coefficients in an abelian local system, over configuration spaces of punctured disks. We endow them with an algebra representation of the quantized algébra of $\mathfrak{sl}(2)$ in an integral version. We recognize a tensor product of integral Verma modules. We identify the natural braid group representation induced on this homology (by mapping class) with the ones obtain by the braiding of the quantized algebra of $\mathfrak{sl}(2)$. This work extends Kohno’s theorem (recovered via a nice homological operation) in several directions:

- it relates homological representations to the entire tensor product of Verma modules (and not only to the highest weight vectors)
- it includes a homological interpretation for the action of $U_q\mathfrak{sl}(2)$, whose definitions are inspired by a work of Felder - Wieczerkowski for which the homological aspect remained conjectural.
- it is an integral version of Kohno’s theorem, namely it preserves the integral structure on Laurent polynomials, thus exposes conditions of genericity previously required by Kohno’s theorem.

We finally reach the level of knot invariants: in Chapter 4, this homological model for quantum braid representations allows us to show that colored Jones polynomials compute some weighted sum of abelianized Lefschetz numbers.

Résumé : Cette thèse comporte des interprétations homologiques de certains invariants quantiques, plus particulièrement ceux associés aux groupes de tresses. Le Chapitre 3 étudie des groupes d'homologie localement finie, relative et à coefficients dans un système local abélien sur des espaces de configurations de points dans le disque épointé. Nous munissons ces complexes d'une action du groupe quantique $U_q\mathfrak{sl}(2)$ dans une version entière, et nous reconnaissons un produit tensoriel de modules de Verma entiers. Enfin, nous retrouvons une action naturelle du groupe des tresses (par homéomorphisme) sur ces modules homologiques, et nous montrons qu'il s'agit de la représentation obtenue par la R-matrice de la catégorie de modules de $U_q\mathfrak{sl}(2)$. Les représentations homologiques obtenues sont une généralisation des représentations de Lawrence, donc elles sont fidèles. Elles permettent de retrouver homologiquement plusieurs propriétés de la catégorie de modules sur $U_q\mathfrak{sl}(2)$. Nous donnons des bases entières de l'homologie (i.e. des bases en tant que module sur un anneau entier de polynômes de Laurent). L'action de $U_q\mathfrak{sl}(2)$, ainsi que celle du groupe des tresses, respectent cette structure, tout comme l'isomorphisme vers le produit tensoriel de modules de Verma. Ce travail étend le théorème de Kohno (retrouvé via une jolie opération homologique) dans plusieurs directions :

- il relie les représentations homologiques à tout le produit tensoriel de modules de Verma (et plus seulement aux vecteurs de plus haut poids)
- il inclue une interprétation homologique de l'action de $U_q\mathfrak{sl}(2)$, dont les définitions sont inspirées par un travail de Felder – Wieczerkowski dans lequel l'aspect homologique restait jusqu'ici conjectural.
- il en est une version entière, c'est à dire qu'il préserve la structure d'anneau entier sur les polynômes de Laurent, exhibant ainsi précisément les conditions de généricité précédemment requises par le théorème de Kohno.

Ce modèle homologique (pour les représentations quantiques de tresses) est ensuite appliqué aux nœuds vus comme des clôtures de tresses dans le Chapitre 4, et permet d'obtenir une formule des traces (homologiques) pour les polynômes de Jones coloriés, qui s'apparente à une somme pondérée de nombres de Lefschetz abélianisés.

Le manuscrit contient également un chapitre (Chapitre 2) d'étude concrète de "petits cas" (car les représentations homologiques sont une famille graduée de représentations). Nous montrons explicitement que les représentations de Gassner du groupe des tresses sont des représentations quantiques, et nous donnons des matrices pour une version colorée des représentations de Bigelow–Krammer–Lawrence - construites au préalable. Nous étudions également le premier niveau de graduation de la représentation du groupe modulaire de la sphère à quatre pointes obtenue via la TQFT non semi-simple (construite par Blanchet – Costantino – Geer – Patureau), nous retrouvons une représentation de nature homologique, ce qui aboutit à la fidélité de cette représentation.