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FRACTIONAL GROUP ITERATIVE METHODS FOR TWO DIMENSIONAL TIME-FRACTIONAL DIFFERENTIAL EQUATIONS

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FRACTIONAL GROUP ITERATIVE METHODS FOR TWO DIMENSIONAL TIME-FRACTIONAL DIFFERENTIAL EQUATIONS

by

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LIST OF ABBREVIATIONS

FPDE	Fractional partial differential equation
PDE	Partial differential equation
FDM	Finite difference method
1D	One dimension
2D	Two dimension
C-N	Crank-Nicolson
FI	Fully implicit
TFDE	Time fractional diffusion equation
FCE	Fractional cable equation
FADE	Fractional advection diffusion equation
TFCE	Time fractional cable equation
FEG	Fractional explicit group
FEDG	Fractional explicit decoupled group
FMEG	Fractional modified explicit group
FMEDG	Fractional modified explicit decoupled group
add	Addition
sub	Subtraction

div	Divisions
mult	Multiplication
R	Real number

KAEDAH-KAEDAH LELARAN KUMPULAN PECAHAN BAGI PERSAMAAN PEMBEZAAN PECAHAN-MASA DUA DIMENSI

ABSTRAK

Persamaan pembezaan separa pecahan (FPDE) merupakan alat yang amat berkesan dan kuat untuk pemodelan masalah dalam bidang kejuruteraan, fizik dan bidang-bidang lain disebabkan oleh sifat-sifat bukan tempatannya. Kebanyakan FPDE tidak boleh diselesaikan secara analisis. Oleh itu, membangunkan kaedah-kaedah yang tepat, cekap, stabil, dan yang mudah dilaksanakan merupakan tugas yang berat. Setakat ini, banyak kaedah-kaedah berangka telahpun dicadangkan untuk menyelesaikan FDPE masa dan/atau ruang, seperti kaedah perbezaan terhingga dan unsur terhingga. Satu sistem persamaan linear yang besar dan jarang akan timbul apabila FDPE didiskretkan dengan menggunakan skim pendiskretan tertentu berdasarkan kaedah unsur terhingga atau perbezaan terhingga, yang memerlukan masa pelaksanaan yang panjang, kerana penyelesaian yang terdahulu perlu disimpan jika ingin mengira penyelesaian semasa. Proses ini boleh merumitkan lagi pengiraan tersebut dan meningkatkan masa penggunaan CPU untuk kes-kes yang menggunakan kaedah-kaedah perbezaan terhingga titik standard. Oleh, satu teknik yang lebih pantas untuk menyelesaikan persamaan pembezaan pecahan diperlukan. Tesis ini memberi tumpuan kepada pembangunan kaedah-kaedah berangka yang baru, tepat, cekap dan pantas untuk menyelesaikan FDPE. Satu siri teknik kumpulan yang berasal dari dua skim perbezaan terhingga tersirat berdasarkan formula-formula penghampiran standard dan diputarkan dibina untuk menye-

lesaikan persamaan-persamaan penyebaran pecahan masa 2D, kabel, dan alir lintang-resapan. Penggunaan formula pembezaan putaran membawa kepada skim yang mempunyai kompleksiti pengiraan yang kurang kerana prosedur berlelaran hanya memerlukan penglibatan nod pada separuh titik mata lelaran dari seluruh grid dalam domain penyelesaian; oleh itu, sistem dengan bilangan persamaan linear yang dikurangkan dicapai. Keputusan eksperimen menunjukkan bahawa teknik lelaran kumpulan, yang berdasarkan kaedah anggaran perbezaan terhingga yang diputarkan adalah lebih pantas daripada teknik-teknik yang berdasarkan formula konvensional bertitik lima untuk menyelesaikan FPDE kerana kerumitan yang kurang dalam pengiraan teknik lelaran kumpulan. Penumpuan dan kestabilan kaedah kumpulan yang diubahsuai dianalisis dengan teknik matriks melalui induksi matematik.

FRACTIONAL GROUP ITERATIVE METHODS FOR TWO DIMENSIONAL TIME-FRACTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT

Fractional partial differential equations (FPDEs) are highly effective and powerful tools for modeling problems in engineering, physics, and other fields because of their non-local properties. Most FPDEs can not solved analytically. Therefore, developing accurate, efficient, stable, and easy-to-implement methods is an important task. To date, ample numerical methods have been proposed for solving the time and/or space FPDEs, such as the finite difference and finite element methods. A large and sparse system of linear equations will arise when FPDEs are discretised using certain discretisation schemes based on finite element or finite difference methods which requires a considerable amount of execution time since earlier solutions have to be saved if the current solution is to be computed. This process can further complicate the calculations and increase CPU usage time for cases that use standard point finite difference methods. Therefore, the requirement for faster techniques of solving fractional differential equations is necessary. This thesis focuses on the development of new, accurate, efficient, and fast numerical methods for solving FPDEs. A series of group techniques derived from two implicit finite difference schemes based on standard and rotated approximation formulas are constructed to solve 2D time fractional diffusion, cable, and advection-diffusion equations. The use of the rotated difference formula leads to schemes with less computational complexities because the iterative procedure

requires only the involvement of nodes on half of the total grid iterative points in the solution domain; thus, a reduced system of linear equations is attained. Experimental results show that the group iterative techniques, which are based on rotated finite difference approximation methods, are faster than the techniques based on the standard five-point formula for solving FPDEs because of the lower computational complexity of the former. The convergence and stability of the proposed methods are analyzed with the matrix eigenvalue technique via mathematical induction.

CHAPTER 1

PRELIMINARIES

1.1 Introduction

The idea of fractional calculus, which is the calculus of integrals and derivatives in a arbitrary order, dates back to 1695 in a discussion between L'Hôpital and Leibniz. Fractional calculus has elicited much interest over the past few decades, and its history and development were explored in detail by Miller and Ross (1993), Samko et al. (1993) and Podlubny (1999). Fractional partial differential equations (FPDEs) are defined as a type of equations that utilize fractional derivatives; which are considered as a powerful tools for describing the memory and hereditary characteristics of various materials (Yang, 2010). FPDEs can be used to model many problems in many applications, such as electron transportation (Scher and Montroll, 1975), high-frequency financial data (Mendes, 2009) and heat conduction (Sokolov et al., 2002). The FPDEs are complicated and are usually not amenable to analytical solution technique (Chen et al., 2010). Therefore, numerical techniques serve as alternative methods for analytical solutions and have received much interest from researchers who have been searching for efficient methods to solve FPDEs. Several numerical methods, such as finite element (Huang et al., 2008) and finite difference (Zhang, 2009), can be used to solve FPDEs. Among all approximation methods, the finite difference method is the oldest and most commonly used because of its simple and universal application (Mattheij et al., 2005). Various finite difference schemes, such as nine-point, five-point and group methods have been introduced over the past few years (Kimble and

White, 1990; Mohanty et al., 2006; Evans, 1987; Mohanty, 2010; Evans and Mohanty, 2002; Sam and Ali, 2014). In Evans (1997), the details of the point and group methods for solving elliptic, parabolic and hyperbolic PDEs were published. A classical finite difference method was regarded as the standard five-point method, and another rotated five-point method. The rotated five-point method was derived from the rotation of the $x-y$ axis at a clockwise angle of 45^o with regard to a standard mesh. In reference to the standard and rotated five-point discretisation techniques, a series of four-point explicit group techniques have been introduced. In an earlier work carried out by Yousif and Evans (1986), the authors developed an iterative scheme called the explicit group (EG) scheme by using a small fixed-size group strategy for standard grids that produce an economical computation rather than the standard point scheme for solving elliptic PDEs. Abdullah (1991) improved the EG technique by developing a scheme known as the explicit decoupled group (EDG), which was determined to be an effective Poisson solver for rotated grids using a small fixed-sized group strategy. This technique consumes less computation time than EG. Subsequently, Othman and Abdullah (2000) modified the EG method for solving Poisson equations by altering the order of the points on the grids used during the iterative process. The modified explicit group (MEG) method is more effective than the original EG and EDG techniques with regard to time consumption. Ali and Ng (2007) derived the fourth explicit group method for solving elliptic problems known as the modified explicit decoupled group (MEDG) method. MEDG was developed based on the discretisation of the skewed (rotated) $2h$-spaced five-point finite difference that forms a reduced system with a lower complexity than schemes developed using the standard $2h$-spaced five-point difference approximation. The MEDG method was shown to be more effective than the above

mentioned techniques (EG, EDG, and MEG methods) belonging to the explicit group series because it requires the least computational efforts compared to other group methods. Using explicit group methods has several advantages. These methods are easier to implement and involve less execution time than point iterative techniques. They are also more suitable for parallelism due to their explicitness. Later on, the formulation of these group methods were extended to solve more complex PDEs (Kew and Ali, 2015; Chew and Sulaiman, 2016; Tan et al., 2012; Ali and Kew, 2012).

The focus of this thesis is to develop a new series of explicit group iterative methods for solving FPDEs. The motivation of this study is presented in the following section together with the objectives and methodology used.

1.2 The Motivation of This Research

FPDEs are assumed to be the generalized form of classical PDEs, and they can provide good descriptions of several complex phenomena in system identification, signal processing, control and non-Brownian motion (Li et al., 2011). FPDEs play an important role in scientific and technological areas. Several problems in the fields of physics, mathematics, chemistry, and engineering with regard to time or space fractional derivatives can be solved with FPDEs (Carpinteri and Mainardi, 2014; Chaves, 1998; Srivastava and Trujillo, 2006; Basu and Acharya, 2002). Determination of effective methods for solving FPDEs has become an important problem, given the increase in the applications of these equations. Several approaches, including analytical and numerical techniques, are used to solve FPDEs. However, except for the simplified initial and boundary conditions (Chen et al., 2010), determining the analytical solutions for the majority of FPDEs are impossible. Therefore, it becomes very important to de-

velop numerical methods for solving these equations. In the past few years, the finite difference technique has become important in this field, and several researchers have begun investigating the use of this technique in solving FPDEs, such as fractional diffusion, advection diffusion, wave and cable equations (Li and Li, 2015; Hu and Zhang, 2012; Chen, Deng and Wu, 2012; Wang and Vong, 2015; Pang and Sun, 2016; Wang et al., 2010; Hu and Zhang, 2016). The use of the finite difference method results in a large and sparse system of linear equations. Iterative methods require a smaller storage space compared with direct methods when the sparse matrix is involved (because several of their elements are zero). Hence, an iterative method is suitable for solving large and sparse linear system. A large linear system requires a large amount of execution time, particularly for fractional derivatives, because previous solutions need to be saved to compute the present solution. This process can further complicate the calculations and entails increased CPU usage time for cases that use standard point finite difference approaches. Since group methods have been found to reduce execution time in solving PDEs, it would be worth while to incorporate the group strategies in solving FPDEs. A preliminary work was done by Sunarto in solving $1D$ fractional differential equations using half and quarter sweep iterative methods (Sunarto et al., 2014, 2015, 2016). The finding and results showed that the quarter sweep iterative method is superior as compared with the half and full sweep methods. Due to this promising result and the fact that there has yet a published study analyzing the performance of group schemes for solving FPDEs, this study deal with the group finite difference schemes developed from standard and skewed discretisation formulas applied to obtain a fast numerical solution for 2D time FPDEs.

1.3 Research Objectives

The objectives of this study are as follows:

1. To develop two new series of the group methods derived from the standard and rotated (Crank-Nicolson (C-N) and fully implicit (FI)) iterative formulas in solving the $2D$ time fractional diffusion equations.

2. To extend the formulation of these group methods to solve the $2D$ time fractional cable and advection-diffusion equations.

3. To establish the convergence and stability properties of the developed group techniques derived from $2h$-spaced standard and skewed point methods.

4. To perform a comparative study between the proposed methods and point methods.

The results and findings from our experiments helped achieve the objectives of the study and provide motivation for further research in the area.

1.4 Methodology

The following methodology was used in this study:

1. In all problems, the fractional derivative is estimated using the Caputo fractional formula.

2. A new group methods formulation which are derived from two implicit schemes (C-N and FI) for solving the $2D$ time fractional diffusion, cable and advection-diffusion equations will be carried out.

3. Numerical experiments using the PC with 2.93 GHz Core 2 Duo, 2GB of RAM, of Windows 7 Professional and the (Mathematica) software are carried out to investigate the efficacy of the grouping methods.

4. Furthermore, the convergence and the stability analysis were analyzed using the

matrix eigenvalue with mathematical induction.

1.5 Organization of the Thesis

This thesis has been divided into the following chapters:

Chapter 2 comprises the mathematical background required in this thesis along with some iterative methods used for solving the linear system and the literature approach on numerical methods in solving fractional diffusion, cable and advection-diffusion equations. Chapter 3 considers the group iterative methods formula which have been developed from the standard and the skewed (C-N and FI) methods for solving the time fractional diffusion equations. Chapter 4 examines the stability and the convergence for the grouping techniques for solving the time fractional diffusion equations.

In Chapters 5 and 6, the grouping methods series are extended for solving the time fractional cable and the advection-diffusion equations. Chapter 7 presents the discussion and the conclusions of the study along with future work.

CHAPTER 2

BASIC CONCEPT AND LITERATURE REVIEW

2.1 Introduction

In the past few years, the FPDEs have been extensively studied. Comprehensive details regarding this topic have been published earlier (Oldham and Spanier, 1974; Podlubny, 1999; Herrmann, 2011). In this chapter, a basic concept and background needed for solving the following system of linear equations:

$$Au = b \tag{2.1}$$

where, $A = (a_{ij}) \in R^{n\times n}$ is $n \times n$ non-singular sparse matrix are reviewed with particular emphasis the studies published on the fractional diffusion, cable and advection-diffusion equations based on the finite difference method.

2.2 Fractional Calculus

The fractional calculus involves the integration and the differentiation to some arbitrary order. This area underwent a lot of progress when Leibniz invented a notation $\frac{d^n y}{dx^n}$ when asked by L'Hôpital in 1695 (what if n be $1/2$) at which he said that "It would cause a paradox.". Later, Leibniz stated that the differential calculus could have been applied for achieving the result. Several of the mathematicians have investigated this field further, which is called as fractional calculus. In the next sections (2.2.1-2.2.3(a)) the fractional integral and derivative with a Gamma function are given.

2.2.1 Eulers Gamma Function

$\Gamma(.)$ is called the Gamma function, which generalizes the factorial $n!$ and allows n to take also non-integer, and it is defined by the integral (Podlubny, 1999)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Gamma function have some properties as below:

1-$\Gamma(1+z) = z\Gamma(z)$

2-$\Gamma(n) = (n-1)!$

3-$\Gamma(1-z) = -z\Gamma(-z)$.

2.2.2 Fractional Integrals

The fractional integrals refer to the integrals of a random order (Podlubny, 1999). For a dependent function, $f(x)$ the fractional integer for the order, $\alpha > 0$, can be denoted as:

$$_cD_x^{-\alpha} f(x) \text{ or } _cI_x^{\alpha} f(x).$$

wherein, c and x represent the two limits of a fractional integral operator and these are generally called as the terminals of the fractional integral (Podlubny, 1999).

The Riemann-Liouville integral is defined as follows:

$$_cD_x^{-\alpha} f(x) = _cI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \ \text{Re}(\alpha) > 0. \tag{2.2}$$

The Riemann-Liouville integral (2.2) can also be derived in a different way after taking into consideration the n-fold integral for any function $f(x)$ in the following manner

(Dold and Eckmann, 1975):

$$ {}_cD_x^{-n} f(x) = {}_cI_x^n f(x) = \int_c^x dx_1 \int_c^{x_1} dx_2 \ldots \int_c^{x_{n-1}} f(x_n) dx_n. $$

Based on the Dirichlet approach, the n-fold integral is represented as the single integral as follows:

$$ {}_cI_x^n f(x) = \frac{1}{(n-1)!} \int_c^x \frac{f(x_n)}{(x-x_n)^{1-n}} dx_n \tag{2.3} $$

Equation (2.3) can be generalized after replacing the n by α and allowing $x_n = t$, which helps us arrive at the Eq. (2.2).

2.2.3 Fractional Derivatives

The fractional derivative can be described as the derivatives of an arbitrary order. On the other hand, the integer order derivatives refer to the order of derivatives that are restricted to the positive integers. Therefore, the fractional derivatives are known as the generalized form of the integer order derivatives. A notation of ${}_cD_x^{\alpha} f(x)$ is used for expressing the derivatives of the order α for the function $f(x)$, where α represents a random positive real number whereas c and x refer to the two limits connected to the fractional differentiation operation. The three major forms of the fractional derivatives which are commonly used are the Riemann-Liouville, Grünwald-Letnikov and the Caputo fractional derivatives.

2.2.3(a) Definitions of Fractional Derivatives

The fractional derivatives, ${}_cD_t^\alpha$, for the function, $f(t)$, in the Riemann-Liouville fractional derivatives of order α can be defined as follows (Klages et al. (2008)):

$$ {}_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}(\frac{d}{dt})\int_0^t \frac{f(\xi)}{(t-\xi)^\alpha}d\xi, 0<\alpha<1 \tag{2.4} $$

The general form of Eq. (2.4) is written in the following manner (Das, 2008)

$$ {}_cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_c^t \frac{f(\xi)}{(t-\xi)^{\alpha+1-n}}d\xi, (n-1)<\alpha<n \tag{2.5} $$

Caputo has also defined the fractional derivative as (Podlubny, 1999)

$$ {}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{(\frac{df(\xi)}{dt})}{(t-\xi)^\alpha}d\xi, 0<\alpha<1. \tag{2.6} $$

The Eqs. (2.4) and (2.6) are seen to be connected to the Riemann-Liouville fractional integral, wherein the relationship can be outlined as follows (Klages et al., 2008)

${}_0D_t^\alpha = {}_0D_t^1 {}_0I_t^{1-\alpha}, 0<\alpha<1$

${}_0^cD_t^\alpha = {}_0I_t^{1-\alpha} {}_0D_t^1, 0<\alpha<1.$

The Riemann-Liouville approach leads to initial conditions containing the limit values, of which have no known physical interpretation. While, for Caputo's approach, the initial conditions for fractional derivatives take on the same form as for integer-order differential equation, and this give better physical meaning (Podlubny, 1999).

The fractional derivatives can also be represented using the Grünwald-Letnikov

formula in the following manner (Sweilam et al., 2011):

$$ {}_0D_t^{\alpha} f(t) = \lim_{\tau \to 0} \frac{1}{\tau^{\alpha}} \sum_{k=0}^{[\frac{t}{\tau}]} w_k^{\alpha} f(t - k\tau),\ t \geq 0 \tag{2.7} $$

where $[\frac{t}{\tau}]$ is integer, $w_0^{\alpha} = 1, w_k^{\alpha} = (1 - \frac{\alpha+1}{k}) w_{k-1}^{\alpha}, k = 0, 1, 2..., \frac{t}{\tau}$. The formula is also known as the standard Grünwald-Letnikov formula, which is written as follows (Yuste and Acedo, 2005):

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\tau^{\alpha}} \sum_{k=0}^{\frac{t}{\tau}} w_k^{\alpha} f(t - k\tau) + O(\tau^p),\ t \geq 0 \tag{2.8} $$

2.3 Basic Mathematical Concepts

Usually, whenever finite difference methods are used to the numerical solution of FPDEs it involves a system containing n simultaneous equations, having n unknowns, for solving them. Here, we have outlined the different mathematical concepts containing the definitions from the matrix algebra and some theories which are important for studying the numerical methods.

2.3.1 Matrix Algebra

An arbitrary system of n linear equations in n unknowns can be written as:

$$ \begin{aligned} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n &= b_1 \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n &= b_2 \\ \vdots \qquad\qquad & \vdots \\ a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n &= b_n \end{aligned} \tag{2.9} $$

where $u_1, u_2, u_3, \cdots, u_n$ are the unknowns and the subscripted $a's$ and $b's$ denote constants. This system can be rewritten in the matrix form as (2.1)

$$Au = b$$

here A is the matrix of order $n \times n$ while u and b are row vectors of n order such that:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{11} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \; u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

All entries in A can be represented as a_{ij},where i and j represent the rows and the columns, respectively. When the values of A and b are already known, then the system solution (2.1) involves the vector u. The system possesses an exclusive solution of $u = A^{-1}b$, only when A is non-singular $(detA \neq 0)$. However, for larger matrices, it is difficult to apply this definition to solving the system. In such cases, the coefficient matrix A properties like the positive definiteness, diagonal dominance, and consistently ordered help in deciding the solvability of the system. In our study, we have assumed all the matrices as square matrices having an order n, unless otherwise stated. If the matrices, A and B, have the same size, then, they are said to be equal, if all their corresponding entries as equal. Mathematically, it is represented as $a_{ij} = b_{ij}$ when $1 \leq i, j \leq n$.

Definition 2.1

A matrix $A = [a_{ij}]$ is said to be positive $(A > 0)$ if $a_{ij} > 0$ for $1 \leq i, j \leq n$. However, the matrix A is non negative $(A \geq 0)$ if $a_{ij} \geq 0$ for $1 \leq i, j \leq n$ (Berman and Plemmons,

1979).

Definition 2.2

i) A matrix A is called a zero (null) matrix if all the entries are zero.

ii) A matrix $A = [a_{ij}]$ is called an identity matrix if $a_{ii} = 1$for all $1 \leq i \leq n$ and $a_{ij} = 0$ for all $1 \leq i, j \leq n$ where $i \neq j$.

The following discusses several useful properties of a matrix due to Golub and Van Loan (1983) and Mitchell (1969). The matrix $A = [a_{ij}]$ of order n is:

i) Symmetric, if $A = A^T$.

ii) Diagonal, if $a_{ij} = 0$ for all $1 \leq i, j \leq n$ where $i \neq j$.

iii) Diagonally dominant, if $|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$for all $1 \leq i \leq n$.

iv) Tridiagonal matrix, which it has the form as following

$$A = \begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & & \\ & & & & c & a \end{bmatrix}.$$

v) Lower triangular, if $a_{ij} = 0$ for $i \leq j$ and strictly lower triangular if $a_{ij} = 0$ for $i < j$.

vii) Upper triangular, if $a_{ij} = 0$ for $i \geq j$ and strictly upper triangular if $a_{ij} = 0$ for $i > j$.

viii) Sparse matrix, if most of the entries elements are zeroes.

ix) Dense matrix, if most of the entries elements are non-zeroes.

The determinant of a matrix A is denoted as $det(A)$ or $|A|$. For a matrix A with only a single entry, the determinant of A is the value of the single entry itself. If matrix A

is of order 2, for example $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$. Minor of an element a_{ik} is the determinant of the sub matrix in matrix A. It is denoted as M_{ik}. The cofactor of the element a_{ik} can be obtained from $C_{ik} = (-1)^{i+k} M_{ik}$. Therefore the determinant of A is given by

$$|A| = \sum_{k=1}^{n} M_{ik}, 1 \leq i \leq n.$$

Definition 2.3

A matrix A is said to be

1) Block Diagonal, if

$$A = \begin{bmatrix} D_1 & & & & \\ & D_1 & & & \\ & & D_1 & & \\ & & & \ddots & \\ & & & & D_1 \end{bmatrix}$$

2) Block Tridiagonal, if

$$A = \begin{bmatrix} D_1 & U_1 & & & & \\ L_2 & D_2 & U_2 & & & \\ & L_3 & D_3 & U_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & L_{n-1} & D_{n-1} & U_{n-1} \\ & & & & L_n & D_n \end{bmatrix}$$

where D_i ,$1 \leq i \leq n$ are square matrices, whereas $U_i's$ and $L_i's$ are rectangular matrices (Evans, 1997). If the $D_i's$ are square diagonal matrices, it is known as T-matrix (Young, 1971).

.Definition 2.4

Let the vector u be given by $u^T = [u_1, u_2, ..., u_n]$, the following scalars are defined as the $1, 2$, and ∞ norm of a vector u:

$$\|u\|_1 = |u_1| + |u_2| + \cdots + |u_n|$$

$$\|u\|_2 = \left(\sum_{i=1}^{n} |u_i|^2\right)^{\frac{1}{2}}, \ \|u\|_\infty = \sup_{1 \leq i \leq n} |u_i|.$$

In general, L_k-norms are given by

$$\|u\|_k = \left(\sum_{i=1}^{n} |u_i|^k\right)^{\frac{1}{k}}, \ 1 \leq k \leq \infty.$$

2.4 Finite Difference Method

A finite difference method (FDM) is a universally applicable and efficient numerical method that employed to solve PDEs. In this method, derivatives in PDEs are replaced through finite difference approximations. The solution domain is divided into discrete points prior to applying any numerical methods. The solution domain is segmented into squares through grid lines that are parallel with the x-axis (having a uniform length Δx) and the y-axis (having a uniform length Δy) such that: $\Delta x = \Delta y = h$. Figure 2.1 shows the description. The $u(x_i, y_j, t_k)$ is approximated through a notation of $u^k_{i,j}$, which is then calculated with the help of the finite difference method. The grid point is referred to as the point that consists of the form $u(x_i, y_j, t_k)$. Estimating the ap-

proximate solution values for a continuous function $u(x, y, t)$ present in the grid points (Atkinson and Han, 2001) is of interest. It has been observed that the finite difference approximation techniques are based on the Taylor series expansion. The Taylor series expansion in the case of any function, $u(x,y,t)$, which is expanded (x,y,t) at (x_i+h) and (x_i-h) respectively, are as follows,

$$u(x+h,y,t) = u(x,y,t) + \frac{h}{1!}u_x(x,y,t) + \frac{h^2}{2!}u_{xx}(x,y,t) + \frac{h^3}{3!}u_{xxx}(x,y,t) + \ldots \quad (2.10)$$

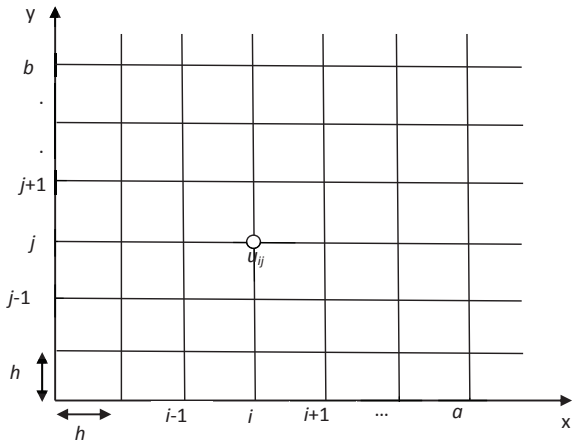

Figure 2.1: Discretisation of the solution domain.

$$u(x-h,y,t) = u(x,y,t) - \frac{h}{1!}u_x(x,y,t) + \frac{h^2}{2!}u_{xx}(x,y,) - \frac{h^3}{3!}u_{xxx}(x,y,t) + \ldots \quad (2.11)$$

We can rewrite Eqs. (2.10) and (2.11) by using the double subscript notation as the following

$$u^k_{i+1,j} = u^k_{i,j} + h(u_x)^k_{i,j} + \frac{h^2}{2!}(u_{xx})^k_{i,j} + \frac{h^3}{3!}(u_{xxx})^k_{i,j} + \ldots \quad (2.12)$$

$$u^k_{i-1,j} = u^k_{i,j} - h(u_x)^k_{i,j} + \frac{h^2}{2!}(u_{xx})^k_{i,j} - \frac{h^3}{3!}(u_{xxx})^k_{i,j} + \ldots \quad (2.13)$$

Eq. (2.12) can be written as,

$$\begin{aligned}\frac{\partial u_{i,j}^k}{\partial x} &= \frac{u_{i+1,j}^k - u_{i,j}^k}{h} - \frac{h}{2!}\frac{\partial^2 u_{i,j}^k}{\partial x^2} - \frac{h^2}{3!}\frac{\partial^3 u_{i,j}^k}{\partial x} + \ldots \\ &= \frac{u_{i+1,j}^k - u_{i,j}^k}{h} + O(h)\end{aligned} \tag{2.14}$$

Similarly from Eq. (2.13), we can get,

$$\begin{aligned}\frac{\partial u_{i,j}^k}{\partial x} &= \frac{u_{i,j}^k - u_{i-1,j}^k}{h} - \frac{h}{2!}\frac{\partial^2 u_{i,j}^k}{\partial x^2} - \frac{h^2}{3!}\frac{\partial^3 u_{i,j}^k}{\partial x} + \ldots \\ &= \frac{u_{i,j}^k - u_{i-1,j}^k}{h} + O(h).\end{aligned} \tag{2.15}$$

After truncating the 2^{nd} and the higher order derivatives, we can obtain the Eq. (2.16), that refers to a forward standard difference approximation formula for $\frac{\partial u}{\partial x}$ or u_x for the grid points (x,y,t) having a 1^{st} order accurate or $O(h)$ accurate. We can also obtain the Eq. (2.17) that is a backwards standard difference formula of the $O(h)$ accurate.

$$\frac{\partial u_{i,j}^k}{\partial x} \approx \frac{u_{i+1,j}^k - u_{i,j}^k}{h}, \tag{2.16}$$

$$\frac{\partial u_{i,j}^k}{\partial x} \approx \frac{u_{i,j}^k - u_{i-1,j}^k}{h}. \tag{2.17}$$

If subtracting (2.13) from (2.12) and rearranging it, we can get the central standard difference formula

$$\begin{aligned}\frac{\partial u_{i,j}^k}{\partial x} &= \frac{u_{i+1,j}^k - u_{i-1,j}^k}{2h} - \frac{h^2}{3!}\frac{\partial^3 u_{i,j}^k}{\partial x^3} + \ldots \\ &= \frac{u_{i+1,j}^k - u_{i-1,j}^k}{2h} + O(h^2).\end{aligned} \tag{2.18}$$

The central standard difference formula is more accurate compared to the forward and backward difference formula because the truncation errors have a higher order and thus result in smaller error. By adding (2.12) and (2.13) and rearranging it, we get the

central standard difference formula for the second order derivative:

$$\begin{aligned}\frac{\partial^2 u^k_{i,j}}{\partial x^2} &= \frac{u^k_{i+1,j} - 2u^k_{i,j} + u^k_{i-1,j}}{h^2} + \frac{2h^2}{4!}\frac{\partial^4 u^k_{i,j}}{\partial x^4} + ... \\ &= \frac{u^k_{i+1,j} - 2u^k_{i,j} + u^k_{i-1,j}}{h^2} + O(h^2).\end{aligned} \tag{2.19}$$

Similarly, the following difference formulas for $\frac{\partial u^k_{i,j}}{\partial y}$ can be obtained:

Forward standard difference formula

$$\frac{\partial u^k_{i,j}}{\partial y} \approx \frac{u^k_{i,j+1} - u^k_{i,j}}{h}. \tag{2.20}$$

Backward standard difference formula

$$\frac{\partial u^k_{i,j}}{\partial y} \approx \frac{u^k_{i,j} - u^k_{i,j-1}}{h}. \tag{2.21}$$

Central standard difference formula

$$\frac{\partial u^k_{i,j}}{\partial y} \approx \frac{u^k_{i,j+1} - u^k_{i,j-1}}{2h}. \tag{2.22}$$

Central standard difference formula for the second order derivative:

$$\frac{\partial^2 u^k_{i,j}}{\partial y^2} = \frac{u^k_{i,j+1} - 2u^k_{i,j} + u^k_{i,j-1}}{h^2} + O(h^2). \tag{2.23}$$

2.5 Convergence

One of the most important topics to be studied includes the convergence of the approximation method. Let us assume $u^k_{i,j}$ to be the computed solution for an approx-

imation method, while $U(x,y,t)$ is an exact solution for the PDE. Any approximation method that approximates the given PDE, is referred to as convergent, when the numerical solution, $u_{i,j}^k$, is seen to approach the exact solution, $U(x,y,t)$, for the PDE, for every value of the independent variable when the grid space (i.e., h, τ in the approximation above) tends to zero (Fletcher, 1991). The difference noted between the computed solution $u_{i,j}^k$ and exact solution $U(x,y,t)$ is known as the solution error, which is denoted as $e_{i,j}^k$ in the following manner:

$$e_{i,j}^k = U(x,y,t) - u_{ij}^k.$$

2.6 Stability and Consistency

The idea of the stability is related to the decay or growth of the errors that have been introduced during any stage of computing. The method is stable if the error does not grow with time, but gets negligible as there is advancement in the computational process. The two most commonly used techniques for analyzing the stability of the method are the matrix and the Fourier methods.

Let us assume that the vector for the solution values of $U^{n+1} = [u_1^{n+1}, u_2^{n+1}, ..., u_m^{n+1}]$ for the PDEs at the $(n+1)th$ time-level is connected to the solution value vector at the n^{th} time-level by (Smith, 1985)

$$U^{n+1} = AU^n + b^n \tag{2.24}$$

wherein b^n refers to the column vector of the zeroes and unknown boundary values, and matrix A is the $n \times n$ matrix containing known elements. If the matrix is stable,

(2.24), then the norm of matrix A must be compatible with the norm of u, and has to satisfy the following formula:

$$\|A\| \leq 1,$$

when the solution shows no increase with an increase in t. For determining the stability of the FDM scheme, using the Fourier series (i.e., Von Neumann process), the primary line of errors can be expressed as the finite Fourier series. Thereafter, the stability (or the instability) of a method can be determined after considering if the different Fourier components of the error distribution amplify or decay while advancing to the subsequent time level. The a finite difference method is said to be consistent with the PDEs if the finite difference equation *F(u)* tends to the actual PDEs *F(U)* as the grid spacings are small.

Lax Equivalence Theorem:

Given a properly posed linear initial-value problem and a linear finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence (Smith, 1985).

This theorem also hold for FPDEs as explained by Lubich (1986) .

2.7 Iterative Methods For Sparse Linear System

All the iterative methods begin by initially guessing the solution values for the mesh points and thereafter, using the difference equation as the basis for calculating the new improved values. The process is continuously repeated till it reaches convergence for all the mesh points. However, the shortage of such methods basically lies in the choice of a proper initial guess, which is used for beginning the iterative process.

For the system of Eq. (2.1), wherein A is seen to be a non-singular matrix, b refers to the constant vector, while u is an unknown vector which has to be solved. It is seen that the general linear iterative methods are of the form:

$$u^{(k+1)} = Gu^{(k)} + c, \quad k = 0,\ 1,\ 2,\ \ldots \tag{2.25}$$

where G refers to the iteration matrix that depends on the A and b in the (2.1). After selecting the initial value of $u^{(0)}$, we use the (2.25) for generating $u^1, u^2, \ldots$. If the process is seen to be convergent, then, the successive values for $u^k$are seen to be near the exact solution u (Watkins, 2002). The coefficient matrix from (2.1) is written as follows:

$$A = D - U - L \tag{2.26}$$

wherein D refers to the diagonal matrix $A, -U$ and $-L$ are seen to be strict upper and lower triangular elements for A, respectively. The three commonly used iterative methods, which are presented are Jacobi, Gauss-Seidel and the Successive Over-Relaxation (SOR) iterative techniques.

For the Jacobi method, the Equation (2.1) can be rewritten in the following manner:

$$Du = (L+U)u + b. \tag{2.27}$$

Assuming D^{-1} exists, (2.27) can be replaced by

$$u = D^{-1}(L+U)u + D^{-1}b.$$

Thus, the Jacobi iterative method is defined by

$$u^{(k+1)} = Ju^{(k)} + d,$$

where the iteration matrix J is given by $J = D^{-1}(L+U)$ and $d = D^{-1}b$.

The above technique is also called as the Simultaneous Displacement technique.

The Gauss-Seidel technique is a modified Jacobi technique, or known as the Successive Displacement technique. In this technique, as soon as the updated values become available, they are applied. The Equation (2.1) is rewritten in the matrix form in the following manner:

$$(D-L)u = Uu + b.$$

Then, the Gauss-Seidel method is defined by

$$Du^{(k+1)} = Lu^{(k+1)} + Uu^{(k)} + b. \tag{2.28}$$

By multiplying both sides by D^{-1}, we obtain the following equation

$$u^{(k+1)} = Eu^{(k+1)} + Fu^{(k)} + g, \tag{2.29}$$

where $E = D^{-1}L$, $F = D^{-1}U$ and $g = D^{-1}b$. Equation (2.29) can be written as

$$(I-E)u^{(k+1)} = Fu^{(k)} + g. \tag{2.30}$$

Since L is a strictly lower triangular matrix, then the $det(I-E) = 1$. The $(I-E)$ matrix is a non-singular matrix, therefore $(I-E)^{-1}$ exists and the Gauss-Seidel iterative

method can be written in the form:

$$u^{(k+1)} = Gu^{(k)} + t,$$

where G is the Gauss-Seidel iterative matrix and is defined by

$$G = (I - E)^{-1}F$$

and

$$t = (I - E)^{-1}g.$$

The accelerated version of the above Gauss-Seidel iterative method is the Successive Over-Relaxation (SOR) iterative method which involves acceleration parameter ω which is used in its iterative formula to accelerate the rate of convergence (Hadjidimos, 2000). Let $u^{(k+1)}$ be the Gauss-Seidel iterative formula as in (2.29). We introduce the acceleration factor ω into Equation (2.29) as:

$$\bar{u}^{(k+1)} = \omega u^{(k+1)} + (1 - \omega)\bar{u}^{(k)}. \tag{2.31}$$

Substitute the values of $u^{(k+1)}$ into (2.31) to obtain

$$\bar{u}^{(k+1)} = \omega(E\bar{u}^{(k+1)} + F\bar{u}^{(k)} + g) + (1 - \omega)\bar{u}^{(k)}$$

which may be rewritten as

$$(I - \omega E)\bar{u}^{(k+1)} = [\omega F + (1 - \omega)I]\bar{u}^{(k)} + \omega g.$$

Since $(I - \omega E)$ is non-singular for any choice of ω, we have

$$\bar{u}^{(k+1)} = L_\omega \bar{u}^{(k)} + (I - \omega E)^{-1} \omega g \tag{2.32}$$

where L_ω is the SOR iteration matrix given as

$$L_\omega = (I - \omega E)^{-1}[\omega F + (1 - \omega)I].$$

As $\omega = 1$, equation (2.32) will become the Gauss-Seidel iterative method.

2.8 Literature Review

The literature review is arranged based on the equation type. First, the fractional diffusion equation will be considered. Secondly, the literature regarding the fractional cable equation was investigated, followed by the fractional advection-diffusion equation. We wish to provide sufficient information regarding the FDM used for solving all the three equations and establish the contribution of our study.

2.8.1 Finite Difference Methods for Solving Time Fractional Diffusion Equation (TFDE)

The fractional differential equations involving the fractional differential operators in space and/or time are seen to be a general form of the classical form of differential equations. Various finite difference schemes such as explicit, implicit and Crank-Nicolson have been developed in solving 1D TFDE (Yuste and Acedo, 2005; Langlands and Henry, 2005; Karatay et al., 2013). Recently, a lot of interest has been