MAXIMAL IRREDUNDANT COVERINGS OF SOME FINITE GROUPS

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MAXIMAL IRREDUNDANT COVERINGS OF SOME FINITE GROUPS

by

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LIST OF ABBREVIATIONS

I-E Inclusion-Exclusion

LIST OF SYMBOLS AND NOTATIONS

$=$	equal to
$\neq$	not equal to
$\in$	belongs to
$\notin$	does not belong to
$\cup$	union
$\cap$	intersection
$\varnothing$	empty set
$\subseteq$	subset of
$\subset$	proper subset of
G	a group
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$N \trianglelefteq G$	N is a normal subgroup of G
$\cong$	isomorphic to
$\ncong$	not isomorphic to
C_n	cyclic group of order n
S_n	symmetric group of degree n
D_n	dihedral group of degree n
$\|G\|$	order of G
n-covering	covering consists of n members
$\sigma(G)$	minimal covering of G
$\mathfrak{C}_n$-covering	maximal irredundant n-covering with a core-free intersection
$\mathfrak{C}_n$-group	group with a $\mathfrak{C}_n$-covering
$(C_p)^n$	elementary abelian group of order p^n
G/H	factor group or quotient group of G by H
$\|G:H\|$	index of subgroup H in the group G
$N_G(H)$	normalizer of H in G
$\langle g \rangle$	cyclic group generated by g
#	size

LITUPAN MAKSIMAL TAK BERLEBIHAN BAGI BEBERAPA KUMPULAN TERHINGGA

ABSTRAK

Tujuan penyelidikan ini adalah untuk menyumbang keputusan lanjut tentang litupan bagi beberapa kumpulan terhingga. Hanya kumpulan bukan kitaran dipertimbangkan dalam kajian tentang litupan kumpulan. Memandangkan tidak ada kumpulan yang boleh dilitupi oleh hanya dua daripada subkumpulan wajarnya, suatu litupan harus mempunyai sekurang-kurangnya 3 daripada subkumpulan wajarnya. Jika suatu litupan mengandungi n subkumpulan (wajar), maka set bagi subkumpulan ini dipanggil litupan-n. Litupan untuk kumpulan G dikatakan minimal jika ia mengandungi bilangan subkumpulan wajar yang terkecil berbanding semua litupan yang lain; i.e. jika litupan minimal mengandungi m subkumpulan wajar maka notasi yang digunakan adalah $\sigma(G) = m$. Litupan bagi suatu kumpulan dikatakan tak berlebihan jika tiada subset wajar dari litupan tersebut yang juga melitupi kumpulan yang sama. Ternyata, semua litupan minimal adalah tak berlebihan tetapi akas pernyataan ini adalah tidak benar secara umum. Jika ahli litupan adalah kesemuanya subkumpulan normal maksimal dari kumpulan G, maka litupan tersebut dikenali sebagai litupan maksimal. Andaikan D sebagai tindanan kesemua ahli litupan. Maka n litupan dikatakan mempunyai tindanan bebas teras sekiranya teras bagi D merupakan subkumpulan remeh. Litupan maksimal yang tak berlebihan dengan tindanan bebas teras dikenali sebagai litupan-$\mathfrak{C}_n$ dan kumpulan yang mempunyai litupan jenis ini dikenali sebagai kumpulan-$\mathfrak{C}_n$. Kajian ini memfokuskan terhadap litupan minimal bagi kumpulan simetrik S_9 dan kumpulan

dwihedron D_n bagi $n \geq 3$ yang ganjil; pencirian terhadap kumpulan-p yang mempunyai litupan-$\mathfrak{C}_n$ bagi $n \in \{10, 11, 12\}$; dan pencirian terhadap kumpulan nilpoten yang mempunyai litupan-$\mathfrak{C}_n$ bagi $n \in \{9, 10, 11, 12\}$. Dalam tesis ini, batasan bawah dan batasan atas bagi $\sigma(S_9)$ juga telah ditentukan. (Walau bagaimanapun, nilai sebenar bagi $\sigma(S_9) = 256$ kemudiannya telah ditemui pada tahun 2016.) Bagi kumpulan dwihedron D_n, yang mana n ialah ganjil dan $n \geq 3$, hasil dibentangkan menerusi dua klasifikasi, i.e. n yang perdana dan n yang ganjil berbentuk gubahan. Bagi kumpulan-p, didapati bahawa kumpulan–p yang mempunyai litupan-$\mathfrak{C}_n$ bagi $n \in \{10, 11, 12\}$ hanyalah yang berisomorfisma dengan suatu kumpulan abelan permulaan dengan peringkat tertentu dan hasil kajian telah menunjukkan bukti yang kukuh terhadap kumpulan tersebut. Didapati juga bahawa sebilangan kumpulan-p mempunyai ketiga-tiga jenis litupan dan sebilangan lagi mempunyai dua daripada tiga jenis litupan. Bagi kumpulan nilpoten, didapati bahawa bagi $n \in \{10, 11, 12\}$, kumpulan nilpoten yang mempunyai litupan-$\mathfrak{C}_n$ adalah persis kumpulan-p yang diperoleh sebelumnya; tiada kumpulan nilpoten lain yang mempunyai litupan-$\mathfrak{C}_n$ bagi $n \in \{10, 11, 12\}$. Kumpulan nilpoten yang mempunyai litupan-$\mathfrak{C}_9$ juga berisomorfisma dengan suatu kumpulan abelan permulaan dengan peringkat tertentu.

MAXIMAL IRREDUNDANT COVERINGS OF SOME FINITE GROUPS

ABSTRACT

The aim of this research is to contribute further results on the coverings of some finite groups. Only non-cyclic groups are considered in the study of group coverings. Since no group can be covered by only two of its proper subgroups, a covering should consist of at least 3 of its proper subgroups. If a covering contains n (proper) subgroups, then the set of these subgroups is called an n-covering. The covering of a group G is called minimal if it consists of the least number of proper subgroups among all coverings for the group; i.e. if the minimal covering consists of m proper subgroups then the notation used is $\sigma(G) = m$. A covering of a group is called irredundant if no proper subset of the covering also covers the group. Obviously, every minimal covering is irredundant but the converse is not true in general. If the members of the covering are all maximal normal subgroups of a group G, then the covering is called a maximal covering. Let D be the intersection of all members in the covering. Then the covering is said to have core-free intersection if the core of D is the trivial subgroup. A maximal irredundant n-covering with core-free intersection is known as a $\mathfrak{C}_n$-covering and a group with this type of covering is known as a $\mathfrak{C}_n$-group. This study focuses only on the minimal covering of the symmetric group S_9 and the dihedral group D_n for odd $n \geq 3$; on the characterization of p-groups having a $\mathfrak{C}_n$-covering for $n \in \{10, 11, 12\}$; and the characterization of nilpotent groups having a $\mathfrak{C}_n$-covering for $n \in \{9, 10, 11, 12\}$. In this thesis, a lower bound and an upper bound for $\sigma(S_9)$ is established. (However, later it was found that the exact value for $\sigma(S_9) = 256$ has already been discovered in

2016.) For the dihedral groups D_n where n is odd and $n \geq 3$, results were presented in two classifications, i.e. the prime n and the odd composite n. For the p-groups, it was found that the only p-groups with $\mathfrak{C}_n$-coverings for $n \in \{10, 11, 12\}$ are those isomorphic to some elementary abelian groups of certain orders and the results established the concrete proof of the groups. It was also found that some p-groups have all three possible types of coverings and some others have two of the three types of coverings. For the nilpotent groups, it was found that for $n \in \{10, 11, 12\}$, the nilpotent groups having $\mathfrak{C}_n$-coverings are exactly the p-groups obtained earlier; no other nilpotent groups were found to have $\mathfrak{C}_n$-coverings for $n \in \{10, 11, 12\}$. The nilpotent groups having a $\mathfrak{C}_9$-covering are also isomorphic to some elementary abelian groups of certain orders.

CHAPTER 1

INTRODUCTION

1.1 Introduction

This chapter gives an introduction to the study. It starts with the background of the study and the literature review, followed by the problem statement, objectives, research questions, research methodology and lastly the organization of the thesis. Note that all groups described in this thesis are finite except if defined otherwise.

1.2 Background of the Study

A *group* is defined as a non-empty set G with a binary operation $*$ that satisfies three properties; namely the associativity, i.e. $a*(b*c) = (a*b)*c$ for all $a,b,c \in G$; the existence of an identity, i.e. there exists an element $e \in G$ such that for all $a \in G$, $e*a = a*e = a$; and the invertibility, i.e. for every $a \in G$ there exists $b \in G$ such that $a*b = b*a = e$. The notation $(G,*)$ is often used to mean that G is a group with a binary operation $*$. If, in addition, commutativity is satisfied, i.e. $a*b = b*a$ for all $a,b \in G$ then G is called an abelian group. The *order* (or *size*) of a group G is the cardinality of G or the number of elements in G which is usually denoted as $|G|$. If the order of G is finite, then G is called a finite group.

For a group G, the binary operation $a*a$ is denoted as a^2 for all $a \in G$. Thus $a^k = a*a*\cdots*a$ (k times). The order of an element in a group G is defined as the smallest positive integer n for which a^n is the identity element. Let p be a prime. A *p-group* is a group in which every element has a finite order and the order of every element is a power of p. The term p-group is typically used for a finite p-group, which is equivalent to a group of prime power order, p^n for $n \in \mathbb{N}$. A group G is an *elementary abelian group* if G is abelian and every non-trivial element has the same prime order. Thus by the definition, it asserts that every elementary abelian group is a p-group for prime p. However, not all p-groups are elementary abelian groups.

A *homomorphism* is a map between two groups such that the group operation is preserved. An *injective* homomorphism is a one-to-one mapping and a *surjective* homomorphism is an onto mapping. If a map is onto and one-to-one, it is called a *bijective* homomorphism. Two groups are *isomorphic* if there exists a bijective homomorphism between them (the homomorphism is called an isomorphism). Isomorphic groups have a matching correspondence in term of elements, subsets and group operations.

Let G be a group. A subset H of G is a *subgroup* of G, denoted $H \leq G$, if H is closed under the binary operation on G. A subset K of G is a *proper subgroup* of G, denoted $K < G$, if K is a subgroup of G which is not equal to G. A *maximal subgroup* is a proper subgroup of G which is not contained in any other proper subgroup of G.

Let G be a group and $\{H_i\}_{i \in I}$ where $I = \{1, 2, ..., n\}$ is an arbitrary collection of subgroups of G. Then the *intersection* of H_i for $i \in I$, i.e. $\cap_{i \in I} H_i$ is also a subgroup of G. Note that the *union* of H_i for $i \in I$, i.e. $\cup_{i \in I} H_i$ is not a subgroup of G in general. For example, if H and K are two subgroups of G then the union of H and K, $H \cup K$ is a subgroup of G if and only if either H is in K or K is in H. Furthermore, if a subgroup L of G is in $H \cup K$, then L must be either in H or in K.

From this point on, the binary operation $a*b$ for two elements a and b in a group G shall be denoted as ab for simplicity. An important property that certain subgroups may satisfy is the normality property. Before normality property is defined, the conjugacy concept will be introduced. Let G be a group and $x, y \in G$. Then, x is conjugate to y if and only if there exist an element $a \in G$ such that $ax = ya$. This relation is called *conjugacy* and usually expressed as $x \sim y := axa^{-1} = y$, where a^{-1} is the inverse of a. For an element $g \in G$, the *conjugacy class* of g is the set of elements conjugate to it, i.e. $\{xgx^{-1} | x \in G\}$.

Conjugation can be extended from elements to subgroups. If H is a subgroup of a group G, $g \in G$ and the set $K = \{ghg^{-1} | h \in H\}$ is a subgroup of G, then K is called a conjugate subgroup to H. Any conjugate subgroup K to H is isomorphic to H. A subgroup N of a group G is called *normal* in G, denoted $N \trianglelefteq G$ if it is closed under conjugation, i.e. $gNg^{-1} = N$ for all $g \in G$. Proper subgroups and normal subgroups will turn out to be important in this study.

The set of elements g of a group G such that $g^{-1}Hg = H$ is called the normalizer of H in G, $N_G(H)$, where H is a subset of elements in G. If $H \leq G$, $N_G(H)$ is also a subgroup of G containing H, i.e. $H \leq N_G(H) \leq G$. A group G is a nilpotent group if $H \leq N_G(H)$ for every $H \leq G$.

A set of generators for a group G is a set of elements in G such that possible repeated application of the generators on themselves and each other is capable of producing all the elements in the group. This set is called a *generating set* of G.

Two groups, say G_1 and G_2 can form a new group by *direct product*, i.e., $G_1 \times G_2 =$

$\{(a,b)|a \in G_1, b \in G_2\}$. Generally, let $G_1, G_2, \cdots, G_n$ be groups. For $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ in $G_1 \times G_2 \times \cdots \times G_n$, define $(a_1, a_2, \cdots, a_n)(b_1, b_2, \cdots, b_n)$ to be the element $(a_1 b_1, a_2 b_2, \cdots, a_n b_n)$. Then, $G_1 \times G_2 \times \cdots \times G_n$ is a direct product of groups G_i for $i = 1, 2, \cdots, n$. The direct product of abelian groups is abelian, but if any one of the G_i is not abelian, then $G_1 \times G_2 \times \cdots \times G_n$ is also not abelian.

If H is a proper subgroup of a group G and a is an element of G, then $aH = \{ah|h \in H\}$ (resp. $Ha = \{ha|h \in H\}$) is called a left (resp. right) coset of H. If the left and right cosets are the same, then H is a normal subgroup and the cosets form a group called the factor (or quotient) group denoted G/H, under the new operation $*$ on G/H. The number of left (resp. right) cosets of H in G is called the *index* of H in G and is denoted as $|G : H|$.

A *permutation* of a set A is a function from A to A that is both one-to-one and onto. A *permutation group* is a group whose elements are permutations and in which the product of two permutations is the same permutation as obtained by performing them in succession.

A *cyclic group* is a permutation group whose generating set contains a single non-trivial element. The cyclic group of order n is denoted as C_n. Note that all cyclic groups are abelian. However, a group which is abelian is not necessarily cyclic.

A *symmetric group* of degree n, S_n, is a permutation group whose elements are permutations on n elements (n is called the degree of S_n). Let $X_n = \{1, 2, ..., n\}$ be a set of n elements. Then S_n is the set of all bijections (one-to-one and onto mappings) $\phi : X_n \to X_n$. The composition of two bijections is the binary operation defined on S_n.

The identity element of the group is the identity mapping $i : i(k) = k$ for all $k \in X_n$.

Elements of S_n are often written as product of cycles and each cycle in turn can be written as a product of transpositions. For example, in S_5, the element $(1\ 2)(3\ 4\ 5)$ contain cycles $(1\ 2)$ and $(3\ 4\ 5)$ where the cycle $(1\ 2)$ means that 1 and 2 are interchanged and the cycle $(3\ 4\ 5)$ means that 3 is mapped to 4, 4 is mapped to 5 and 5 is mapped to 3. The cycle $(1\ 2)$ is a transposition by itself but the cycle $(3\ 4\ 5)$ can be written as a product of transpositions $(3\ 4)(3\ 5)$. So the element $(1\ 2)(3\ 4\ 5) = (1\ 2)(3\ 4)(3\ 5)$ which means that it is a product of odd number of transpositions.

The subgroup of S_n generated by a rotation r and a reflection s satisfying the relations $r^n = s^2 = 1$ and $rs = sr^{-1}$ is called a *dihedral group* of degree n, denoted by D_n. It has order $2n$ and it is isomorphic to the group of all symmetries of a regular n-gon for $n \geq 3$. For an odd n , the normal subgroups are given by 1 and $\langle r^d \rangle$ for all divisors d of n, i.e. d/n. For an even n, the normal subgroups are 1, $\langle r^d \rangle$, $\langle r^2, s \rangle$ and $\langle r^2, rs \rangle$ for all d/n.

Group characterization is a description of a group by properties that are different from those mentioned in its definition. It should give an entirely new and useful description of the group containing a simpler formulation that can be verified more easily. In this study, the characterization means describing or determining the structure of groups having a certain property of finite covering. Specifically, minimal covering will be attempted for the symmetric group S_9, and also the dihedral group D_n for certain values of n; then for the p-groups and nilpotent groups, the characterization will be established by the maximal irredundant and core-free intersection n-covering in the case of $n \in \{10, 11, 12\}$ for p-groups and $n \in \{9, 10, 11, 12\}$ for nilpotent groups. All the terms in finite covering mentioned above will be introduced in the next section.

The Groups, Algorithms and Programming (GAP) software is an open source computer algebra system that can be used in describing the structures of elements and subgroups of the group concerned in relation to finding the coverings. In this study, GAP is used to find the minimal covering of the symmetric group S_9, and to characterize p-groups having a maximal irredundant n-covering for $n \in \{10, 11, 12\}$ with a core-free intersection. GAP is also used in describing some examples on dihedral groups and Symmetric group S_4.

1.3 Literature Review

A *covering* of a group G (also known as a component) is defined as a set of proper subgroups of G whose union is equal to the entire group. It is one of the fascinating topics in group theory. A group is said to be coverable if it has at least one covering. It is possible for the coverable group to have several distinct coverings. If the number of proper subgroups in a covering is n, then the covering is called an n-covering. The proper subgroups of a covering will at times be referred to as "members" of the covering.

A covering is called *irredundant* if whenever any subgroup in the collection is removed, the remaining subgroups fail to cover the entire group. In other words, the covering is irredundant if no proper subcollection (subset) is also a covering of the group.

The study of coverings of groups by its proper subgroups dates back from 1926, when Scorza proved that a group cannot be covered by two proper subgroups. He also proved that a group admits an irredundant covering by three subgroups if and only if it is isomorphic to a direct product of two cyclic groups of order 2, $C_2 \times C_2$ which is of order 4. No group can be covered by two proper subgroups since the union of two proper subgroups is not a group. The proof of irredundant covering by three subgroups mentioned above have been rediscovered by Haber and Rosenfeld (1959) and Bruckheimer et al. (1970).

A group is said to have a non-cyclic homomorphic image if the image under homomorphism is a non-cylic group. In 1954, Neumann investigated the covering of groups by cosets. He also characterized that a group has a *finite covering* if and only if it has a non-cyclic homomorphic image. He deduced that research on the covering of groups should involve only non-cyclic groups. Since then, several research on problems involving covering of groups in various perspectives have been done by researchers such as Brodie et al. (1988), and Brodie and Kappe (1993).

A covering of a group is called a *minimal covering* if it contains the least number of proper subgroups among all coverings of the group. Obviously, every minimal covering is irredundant, but the converse is not true. The study of minimal coverings was pioneered by Cohn (1994). He called a group that can be covered by n proper subgroups but no fewer, as an n-sum group. In this case, n is the minimal covering and denoted by $\sigma(G) = n$. He also classified a group G that has a minimal covering $\sigma(G) = 4, 5$ or 6 and conjectured that no group G has a minimal covering $\sigma(G) = 7$. Cohn also stated that if G is a solvable group, then $\sigma(G) = p^a + 1$ with p prime and

$a \in \mathbb{N}$. Both Cohn's conjectures were later proved by Tomkinson (1997). Zhang (2006) proved that there exists a group with $\sigma(G) = 15$.

The generalization for minimal coverings of symmetric and alternating groups was established by Maroti (2005). He proved that $\sigma(S_n) = 2^{n-1}$, if n is odd, except for $n = 9$, and $\sigma(S_n) \leq 2^{n-2}$ if n is even. He established the upper bound $\sigma(S_n) \leq 2^{n-1}$ and to find the lower bound, he defined a subset of S_n called Π containing all permutations in S_n which are products of at most two disjoint cycles. It was found that the lower bound of $\sigma(\Pi)$ equals the upper bound 2^{n-1}, however, he was unable to establish the equality for $n = 9$. Also, he deduced that if $n \notin \{7, 9\}$, then $\sigma(A_n) \geq 2^{n-2}$ with equality if and only if n is even but not divisible by 4.

Some researchers investigated the minimal coverings for symmetric and alternating groups. For example Cohn (1994) proved that $\sigma(A_5) = 10$ and $\sigma(S_5) = 16$. Then, Abdollahi et al. (2007) proved that $\sigma(S_6) = 13$. Then, Kappe and Radden (2010) determined the exact number of $\sigma(A_n)$ for $n \in \{7, 8, 9, 10\}$. Their results were obtained by the aid of GAP. Studies on minimal covering of other types of finite groups can be referred in other papers such as those by Lucido (2003), and Holmes (2006).

Sizemore (2013) studied on the covering of the dihedral group of degree n, D_n by a method of partitioning. He established that for $n \geq 3$ the minimal covering of D_n, $\sigma(D_n) = 3$ when n is even. He further investigated on the covering of D_n where the members of the covering has trivially pairwise intersection (also known as partition) and obtained the formula for the number of covering of this type as $\rho(D_n) = n + 1$. A further result is that $\sigma(D_n) < \rho(D_n)$ for a composite n. He eventually deduced that if

p is the smallest prime divisor such that p/n, then $\sigma(D_n) = \sigma(D_p) = p+1$.

If all members of a covering are maximal subgroups of the group, then the covering is called *maximal*. Let D be the intersection of all members of a covering. Then, D is called a core-free subgroup of G if $\bigcap_{g \in G} gDg^{-1} = 1$. Some covering of groups with a maximal, irredundant and core-free intersection n-covering are known precisely, for example Scorza in 1926 showed that if $n = 3$, then $D = 1$ and G is isomorphic to $C_2 \times C_2$ (Greco, 1953). Greco (1953) considered groups which could be covered by four proper subgroups with a core-free intersection and found that if G is a p-group, then $D = 1$ and $G \cong C_3 \times C_3$.

Bryce et al. (1997) completely characterized groups with a maximal irredundant core-free intersection 5-covering. They proved that G has a maximal irredundant 5-covering with a core-free intersection if and only if G is an elementary abelian group $C_2 \times C_2 \times C_2 \times C_2$ of order 16. Abdollahi (2005) completely characterized groups having a maximal irredundant 6-covering with a core-free intersection. Then, Abdollahi and Jafarian (2008) listed all groups having a maximal irredundant 7-covering with a core-free intersection.

Abdollahi et al. (2008) completely characterized p-groups with a maximal irredundant n-covering with a core-free intersection for $n \in \{7, 8, 9\}$. Ataei (2010) completely characterized nilpotent groups having a maximal irredundant 8-covering with a core-free intersection. Recently, Ataei and Sajjad (2011) characterized the 5-groups with a maximal irredundant 10-covering with a core-free intersection for the case of 5-group of order 5^3, 5^5 and 5^6. They also proved that if a p-group has a maximal irredundant

n-covering with a core-free intersection, then $D = 1$ and the p-group is an elementary abelian group.

Other interesting studies of coverings are coverings of a group by normal subgroups done by Goranzi and Lucchini (2015), coverings of group by conjugate of proper subgroups done by Britnell and Maroti (2013), classification of groups having a unique covering by proper subgroups done by Brodie (2003), investigation on the maximal number of subgroups in an irredundant covering of finite groups by Rogério (2014), and many more. In this study, the focus is on finding the minimal covering of the symmetric group S_9 and minimal covering of dihedral group, as well as characterizing p-groups having a maximal irredundunt n-covering for $n \in \{10, 11, 12\}$ with a core-free intersection and nilpotent groups having a maximal irredundant n-covering for $n \in \{9, 10, 11, 12\}$ with a core-free intersection.

1.4 Problem Statement

Minimal coverings for symmetric groups have been investigated by some previous researchers. It was first studied by Cohn (1994) who proved that the exact number of minimal covering for the symmetric group S_5 is equal to 16, i.e. $\sigma(S_5) = 16$. Then, Maroti (2005) established the general formula for finding minimal covering of S_n, i.e. $\sigma(S_n) \leq 2^{n-2}$ if n is even and $\sigma(S_n) = 2^{n-1}$ if n is odd. Abdollahi et al. (2007) determined that the exact number of minimal covering for the symmetric group S_6 is equal to 13, i.e. $\sigma(S_6) = 13$. A recent study done by Swartz (2014) investigated the exact value for S_n when n is divisible by 6. For the case when n is odd, the formula $\sigma(S_n) = 2^{n-1}$ is not applicable for the symmetric group S_9. Thus, further studies are needed to obtain the minimal covering for S_9 in order to establish the result for all symmetric groups.

Dihedral groups are among the simplest examples and one of the important class of finite groups. At the point of this research no study was found specifically on finding the minimal coverings for D_n. The aim was to find the general formula for determining the minimal coverings of D_n. The result presented will take into consideration the work done by Sizemore (2013).

The complete characterization for groups with a maximal irredundant n-covering with a core-free intersection for $n \in \{3,4,5,6,7\}$ have been done before by Scorza in 1926, Greco (1953), Bryce et al. (1997), Abdollahi et al. (2005) and Abdollahi and Jafarian (2008), respectively. Then, Abdollahi et al. (2008) characterized p-groups with a maximal irredundant n-covering with a core-free intersection for $n \in \{7,8,9\}$. This is then followed by Ataei (2010), who characterized nilpotent groups having a maximal irredundant 8-covering with a core-free intersection. A recent study done by Ataei and Sajjad (2011) resulted to the characterization of 5-groups having a maximal irredundant 10-covering with a core-free intersection, except for the 5-group of order 5^4. Therefore, the study of coverings for nilpotent groups with such property needs to be extended to $n \geq 9$.

1.5 Objectives of the Study

The objectives of this study are as the following:

(i) to determine the minimal covering of the Symmetric group S_9.

(ii) to determine the minimal covering of dihedral groups D_n for odd n.

(iii) to characterize p-groups having a maximal irredundant n-covering for $n \in \{10, 11, 12\}$ with a core-free intersection.

(iv) to characterize all nilpotent groups with a maximal irredundant n-covering for $n \in \{9, 10, 11, 12\}$ with a core-free intersection.

1.6 Research Questions

With respect to the research objectives stated above, this study will therefore address the following research questions:

1. What is the range of values for $\sigma(S_9)$, i.e. the minimal covering of symmetric group S_9?

2. What is the value for $\sigma(D_n)$, i.e. the minimal covering fof D_n for odd $n \geq 3$?

3. Which p-groups have a

   (i) $\mathfrak{C}_{10}$-covering?

   (ii) $\mathfrak{C}_{11}$-covering?

   (iii) $\mathfrak{C}_{12}$-covering?

4. Which nilpotent groups have a

   (i) $\mathfrak{C}_9$-covering?

   (ii) $\mathfrak{C}_{10}$-covering?

   (iii) $\mathfrak{C}_{11}$-covering?

   (iv) $\mathfrak{C}_{12}$-covering?

1.7 Research Methodology

In this study, the minimal covering of S_9 is determined by utilizing the method used by Holmes (2006) in obtaining the minimal covering of some sporadic groups and by Kappe and Radden (2010) in determining the minimal covering of alternating groups of degree $n \in \{7, 8, 9, 10\}$. GAP is utilized to describe the structure of S_9, in particular to find the maximal subgroups, their respective conjugacy classes and identifying the cycle structure of the elements in S_9. With the result, a mapping is set up manually to discover which conjugacy classes must be chosen to produce the maximal subgroups that can cover S_9. For characterizing groups with a maximal irredundant core-free intersection n-covering, some techniques developed by Abdollahi et al. (2008), Ataei (2010) and Ataei and Sajjad (2011) are adapted. For the p-groups, certain orders require the assistance of GAP to list the maximal subgroups and choosing the right collection of n maximal subgroups (corresponding to the n-covering) that can cover the p-group irredundantly and with a core-free intersection.

1.8 Organization of the Thesis

This thesis consists of six chapters. In **Chapter 1**, some background of the study is given which includes relevant definitions and terminologies. This is followed by the literature review, problem statement, objectives and research questions. The methodology used is also explained. Finally, the organisation of this thesis is presented in this section.

Chapter 2 contains preliminaries and important concepts in the theory of groups and subgroups relevant to the study. It also includes notations and basic results that are needed for the whole thesis. This chapter also contains some results on the minimal covering and a maximal irredundant n-covering with a core-free intersection to show the significance of this research based on the previous studies done on these types of coverings.

Chapter 3 contains the result on the minimal covering of the symmetric group S_9. The structure of S_9 in terms of the maximal subgroups, the conjugacy classes of maximal subgroups and the cycle structure of the elements are identified with the aid of GAP. With the information obtained from GAP, the bounded number of the minimal covering for S_9 is obtained. In addition, the minimal covering for dihedral groups D_n for $n \geq 3$ are investigated for the case when n is odd.

The n-coverings of p-groups for $n \in \{10, 11, 12\}$ with the property that the coverings are maximal, irredundant and core-free intersection are classified in **Chapter 4**.

Chapter 5 classifies all nilpotent groups having a maximal irredundant core-free intersection of n-coverings for $n \in \{9, 10, 11, 12\}$.

The study is then concluded and summarized in **Chapter 6**. Some suggestions and open problems for future research in this area are also provided.

1.9 Summary

This chapter briefly explained the background of the study and literature review. It also stated the problem statement to emphasize the reason why the study is significant. This is followed by the objectives of the study as well as research questions and methodology. Lastly, it describes the contents of each chapter of the thesis.

CHAPTER 2

PRELIMINARIES

2.1 Introduction

This chapter provides some definitions and facts on the theory of groups and covering of groups that are used in the study. Section 2.2 recalls some facts that were mentioned in Chapter 1 and then Section 2.3 continues with some further theories on groups and subgroups. In Section 2.4, the cycle structure and conjugacy classes are discussed to clarify terms and methods that will be used in the next chapters. The preliminaries and relevant results on minimal covering and maximal irredundant n-covering is provided in Section 2.5. In Section 2.6, the motivation from coverings of symmetric groups, dihedral groups, p-groups and nilpotent groups are discussed.

2.2 Definitions and Notations

This section shall review some definitions and terms already mentioned in Chapter 1 but stated here as proper statements. Some are accompanied by an example for clarification. These are well-known definitions and terms and can be referred to from many books on group theory. Note that the definition of a group and subgroup as well as some facts related to them shall be skipped as these are well-known and very basic.

Definition 2.1. (MAXIMAL SUBGROUP) Let M be a subgroup of a group G. Then, M is a maximal subgroup of G if $M < G$ such that there is no subgroup H with $M < H < G$.

Definition 2.2. (NORMAL SUBGROUP) A subgroup H of a group G is normal in G denoted by $H \trianglelefteq G$ if its left and right cosets coincide, i.e., $gH = Hg$ for all $g \in G$.

Definition 2.3. (DIRECT PRODUCT) Let $G_1, G_2, \cdots, G_n$ be groups with binary operations $*_1$, $*_2$, $\cdots$, $*_n$ respectively. The direct product of $(G_1, *_1)$, $(G_2, *_2)$, $\cdots$, $(G_n, *_n)$ is defined as $(G_1 \times G_2 \times \cdots \times G_n, *) = (\{(a_1, a_2, \cdots, a_n) | a_i \in G_i\}, *)$ with $(a_1, a_2, \cdots, a_n) * (b_1, b_2, \cdots, b_n) = (a_1b_1, a_2b_2, \cdots, a_nb_n)$ for all $(a_1, a_2, \cdots, a_n)$, $(b_1, b_2, \cdots, b_n) \in G_1 \times G_2 \times \cdots \times G_n$.

It can be proven that $(G_1 \times G_2 \times \cdots \times G_n, *)$ defined in Definition 2.3 is a group. In this thesis, discussions are mainly on direct products of cyclic groups, i.e. $C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$ for a finite integer $n \geq 1$ where each p_i is a prime number.

The p-groups shall be discussed in Chapter 4. The following are definitions related to p-groups.

Definition 2.4. (p-GROUP) A group G is a p-group for a prime p if every element in G has order a power of p.

Remark 2.1. A subgroup H of a group G is a p-subgroup of G for a prime p if H itself is a p-group.

Example 2.1. A cyclic group of order 2, C_2, is an example of a 2-group.

Nilpotent groups shall be discussed in Chapter 5. The following is the relevant definition from Kurzweil (2004).

Definition 2.5. (NILPOTENT) Let G be a group. If $H < N_G(H)$ for all $H < G$, then G is nilpotent.

Definition 2.6. (ELEMENTARY ABELIAN GROUP) Let G be a finite abelian group. Then G is called elementary abelian if every non-identity element has order a prime p. Thus, every elementary abelian group is a p-group for some prime p.

Remark 2.2. The elementary abelian groups are groups G of the form $\overbrace{C_p \times C_p \times \ldots \times C_p}^{n \text{ times}}$ where C_p is the cyclic group of order a prime p and G has order p^n, hence said to have rank n. The notation for this group is $(C_p)^n$.

Example 2.2. The elementary abelian group $C_2 \times C_2$ defined as $C_2 \times C_2 = \{(a,b) | a \in C_2, b \in C_2\}$ is a group of order 4, The elements of $C_2 \times C_2$ are $(1,1)$, $(1,b)$, $(a,1)$ and (a,b). Thus, $C_2 \times C_2$ is a non-cylic group of order 4.

Definition 2.7. (ISOMORPHIC GROUPS) Two groups G_1 and G_2 are isomorphic groups, denoted $G_1 \cong G_2$, if there exists an isomorphism (bijective homomorphism) between them.

Example 2.3. The symmetric group S_3 and dihedral group D_3 are isomorphic groups. This can be shown below. Consider the set $X_3 = \{1,2,3\}$. Acting on the set X_3, S_3 can be defined as $S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. Now, consider D_3 acting on the vertices of the equilateral triangle below:

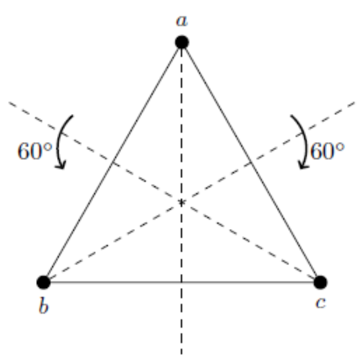

Figure 2.1: D_3 Acting on the Vertices of the Equilateral Triangle

The elements of D_3 are:

$$
\begin{aligned}
(a) &= \text{ identity},\\
(a\ b) &= \text{ reflection about the vertical line through } c,\\
(a\ c) &= \text{ reflection about the vertical line through } b,\\
(b\ c) &= \text{ reflection about the vertical line through } a,\\
(a\ b\ c) &= \text{ rotation } 60^\circ \text{ anti-clockwise},\\
(a\ c\ b) &= \text{ rotation } 60^\circ \text{ clockwise}.
\end{aligned}
$$

Hence, there is one-to-one and onto correspondence between elements of S_3 and elements of D_3.

The number of partitions of a group G by cosets of a subgroup H is equally important and is defined below.

Definition 2.8. (INDEX) Let H be a proper subgroup of a group G. Then the number of left (resp. right) cosets of H in G is called the index of H in G and is denoted as $|G:H|$.

An example of the index of a subgroup is given below.

Example 2.4. The set of integers modulo 12, $\mathbb{Z}_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$ has subgroup $\langle 4\rangle = \{0,4,8\}$. The cosets of $\langle 4\rangle$ are $0+\langle 4\rangle = \{0,4,8\}$, $1+\langle 4\rangle = \{1,5,9\}$, $2+\langle 4\rangle = \{2,6,10\}$ and $3+\langle 4\rangle = \{3,7,11\}$. These are the only cosets of $\langle 4\rangle$ in $\mathbb{Z}_{12}$. So, the index of $\langle 4\rangle$ in $\mathbb{Z}_{12}$ is 4.

Remark 2.3. The index of a subgroup H of a group G divides the order of G.

The following defines a subgroup of G called a Sylow p-subgroup.

Definition 2.9. (SYLOW p-SUBGROUP) If G is a group of order $p^k m$ where p does not divide m, then a subgroup of order p^k is called a Sylow p-subgroup of G.

In other words, a Sylow p-subgroup is a maximal p-subgroup of a group G and it is not a proper subgroup of any other p-subgroup of G.

Example 2.5. The set of integers modulo 12, $\mathbb{Z}_{12} = \{0, 1, 2, \cdots, 11\}$ has order $2^2 \cdot 3$. The only Sylow 2-subgroup is $\{0, 3, 6, 9\} = \langle 3 \rangle$ and the only Sylow 3-subgroup is $\{0, 4, 8\} = \langle 4 \rangle$.

The next section presents some propositions and lemmas from group theory that is required for the later sections.

2.3 Related Facts on Groups and Subgroups

This section gives some further facts about groups and subgroups including terms that will be used in Chapter 3, 4 and 5. Some familiar and generally known facts are obtained from Robinson (1996) and Fraleigh (2003). Others will be properly introduced with citations.

In studies involving order of a group and its subgroups, the following proposition states an important and relevant fact.

Proposition 2.1. *(Lagrange's Theorem) If G is a finite group and H is a subgroup of G then the order of H divides the order of G.*

By Lagrange's theorem, if G is a group of order p^{α} for an integer $\alpha \in \mathbb{Z}^{+}$, all proper nontrivial subgroups of G must have order p^k for some $k \in \{1, 2, \ldots, \alpha - 1\}$.

According to the following proposition, a group G has a subgroup of order p if p divides the order of G. This will be used in Chapter 5.

Proposition 2.2. *(Cauchy's Theorem) If G is a finite group of order n and p is a prime dividing n then G has an element of order p.*

Proposition 2.3. *Let G be a finite group. Then G is a p-group if and only if $|G|$ is a power of p.*

Proposition 2.4. *Any group of order p, where p is a prime, is isomorphic to the cyclic group C_p.*

Lemma 2.1. *(Adkins and Weintraub, 1999) If p is a prime and G is a group of order p^2, then $G \cong C_{p^2}$ and $G \cong C_p \times C_p$.*

The following defines when a group is a product of two of its subgroups.

Definition 2.10. (Kirtland, 2017) A group G is a product of two subgroups H_1 and H_2 if $G = H_1H_2$, where $H_1H_2 = \{ab | a \in H_1 \text{ and } b \in H_2\}$.

The following proposition and lemma are useful for the work in Chapter 4.

Proposition 2.5. *(Cox, 2004) Suppose that N is a normal subgroup of a finite group G and let $g \in G$. If the order of g is relatively prime to $|G : N|$, then $g \in N$.*

Lemma 2.2. *(Hungerford, 1974) Let H_1 and H_2 be subgroups of finite index of a group G. Then $|G : H_1 \cap H_2| \leq |G : H_1||G : H_2|$. Furthermore, $|G : H_1 \cap H_2| = |G : H_1||G : H_2|$ if and only if $G = H_1H_2$.*

Nilpotent groups are considered in Chapter 5. The following lemmas are needed for the discussion in that chapter.

Lemma 2.3. *(Rotman (1995) and Grillet (2011))*

(i) Every finite p-group is nilpotent.

(ii) A group G is nilpotent if and only if it is isomorphic to a direct product of p-groups (for various primes p).

(iii) A group G is nilpotent if and only if it is the direct product of its Sylow p-subgroups.

(iv) A group G is nilpotent if and only if all its Sylow p-subgroups (for primes p) are normal.

(v) Every maximal subgroup of a nilpotent group G is normal and has prime index.

Proposition 2.6. *(Grillet, 2011) Let P be a Sylow p-subgroup of G. Then the following are equivalent:*

(i) P is the unique Sylow p-subgroup of G.

(ii) P is normal in G.

(iii) All subgroups generated by elements with order a power of p, for some prime p, are p-groups.

Remark 2.4. A finite abelian group G has a unique Sylow p-subgroup for each prime p that divides the order of G.

The following lemma is used to compute the number of maximal subgroups for the elementary abelian p-groups.

Lemma 2.4. *(Newton, 2011) If G is a finite elementary abelian p-group for some prime p, then the number of maximal subgroups is equal to* $\dfrac{|G|-1}{p-1}$.

2.4 Cycle Structure and Conjugacy Class

This section introduces what is called the cycle structure of elements in a permutation group and the conjugacy classes. Permutation groups which will be discussed in this study are the symmetric group S_9 and dihedral groups of degree n, D_n.

Consider a smaller symmetric group, say $G = S_5$. Recall that an element of G can be written as a product of cycles. An order of a cycle is the number of multiplication by itself to get the identity (1). It also refers to the number of distinct object in the cycle. For example, the cycle $(1\ 2)$ has order 2 and clearly $(1\ 2)(1\ 2) = (1)$. For this G, $(1\ 2)(3\ 4\ 5)$, $(2\ 4)(1\ 3\ 5)$, $(1\ 5)(2\ 3\ 4)$ for example, are all of the same *cycle structure* (or type). This is because each of the elements is a product of a cycle of order 2 and a cycle of order 3.

Definition 2.11. (CONJUGACY CLASS OF ELEMENTS) Let G be a group and $x \in g$. The conjugacy class of x in G is denoted as x^G where $x^G = \{y \in G | gxg^{-1} = y$ for some $g \in G\}$.

The following proposition relates the conjugacy of elements and the cycle structure specifically for the symmetric group S_n.

Proposition 2.7. *(Smith and Romanowska, 1999) Elements of the symmetric group* S_n *are conjugate if and only if they have the same cycle structure.*

An example of the conjugacy classes elements will be given following the definition of presentation below. Often, a group is identified by a presentation. This means that a set of generators X is given for the group G together with a set of defining relations R. Defining relations are equations involving the generators and their inverses, which are required to hold in G. Note that a "word" over G is a finite string $w = a_1a_2...a_l$ with each $a_i \in X$. The length of w is $l = l(w) = |w|$ and $l = 0$ refers to the empty word.

Definition 2.12. (PRESENTATION) A group presentation is of the form $\langle X|R\rangle$ where X is the set of generators of the group and R is the set of relations which are equations between words in G.

As an example, the dihedral group D_3 could be defined as the group with presentation $\langle r,s|r^3 = s^2 = 1, rs = sr^{-1}\rangle$. Another example, the symmetric group S_3 could be defined as the group with presentation $\langle x,y|x^2 = y^2 = 1,(xy)^3 = 1\rangle$. A presentation $\langle X|R\rangle$ defines a group, which is roughly the largest group generated by X such that all equations in R holds in G. Group presentations are used to systematically enumerate small groups.

Example 2.6. The dihedral group $D_4 = \langle r,s|r^4 = s^2 = 1, rs = sr^{-1}\rangle$ has 8 elements but has 5 different conjugacy classes. The conjugacy classes are $\{1\}$, $\{r^2\}$, $\{s,r^2s\}$, $\{r,r^3\}$ and $\{rs,r^3s\}$. The distinct members of each conjugacy class have the same criteria in terms of action on the vertices of a square, for example if the vertices are $1,2,3$ and 4, and $r = (1\ 2\ 3\ 4)$ and $s = (2\ 4)$ are defined, then r and r^3 are 90 degree rotations clockwise/anti-clockwise, s and r^2s are reflections about the diagonals, and rs and r^3s are reflections about an edge bisector.

Similar to the conjugacy classes of elements, the set of all subgroups of a group G

can be partitioned into conjugacy classes of subgroups. The following defines conjugate subgroups.

Definition 2.13. (CONJUGATE SUBGROUPS)(Clement et al., 2017) Two subgroups H and K of a group G are called conjugate if $gHg^{-1} = K$ for some $g \in G$.

The following is an example of conjugacy classes of subgroups.

Example 2.7. Refer to the dihedral group D_4 again (Example 2.6). Since D_4 has order 8, by Lagrange's theorem, possible subgroups of D_4 have order 1, 2, 4 and 8. Clearly D_4 has a subgroup of order 1 which is $\langle 1 \rangle$ and a subgroup of order 8 which is D_4 (itself). Subgroups of order 2 are $\langle s \rangle$, $\langle r^2 \rangle$,$\langle r^2 s \rangle$, $\langle rs \rangle$ and $\langle r^3 s \rangle$. Subgroups of order 4 are $\langle r \rangle$, $\langle r^2, s \rangle$ and $\langle r^2, rs \rangle$. The subgroups $\langle s \rangle$ and $\langle r^2 s \rangle$ are conjugate, as are $\langle rs \rangle$ and $\langle r^3 s \rangle$. All other subgroups are conjugate only to themselves. Therefore D_4 has eight conjugacy classes of subgroups.

In Chapter 3 , conjugacy classes of maximal subgroups are considered. Chapter 3 also discusses the covering for the dihedral group D_n which requires the following lemmas.

Lemma 2.5. *(Roman, 2011) For dihedral group D_n of order $2n$, D_n has $\sum_{d \backslash n}(d+1)$ subgroups where d is a divisor of n ($d \backslash n$), and each subgroup is either of the form;*

(i) $\langle r^d \rangle$, *where* $d \backslash n$

or

(ii) $\langle r^d, sr^i \rangle$ *where* $d \backslash n$ *and* $0 \leq i < d$.

Lemma 2.6. *(Nagpoul and Jain, 2005) Let G be a group, and N a normal subgroup in G. Then, there exists a surjective homomorphism $\phi : G \to G/N$ such that $\phi(g) = gN$ for all $g \in G$ and $Ker\phi = N$.*