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Symbolic Computation of Solutions for Three Generalized Nonlinear Partial Differential Equations by Using the Tanh Method

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Abstract

Three nonlinear partial differential equations, namely, the standard KdV equation, the Boussinesq equation and the generalized fifth-order KdV equation are considered here from of point the view of construct exact solutions for them. The equations that we consider here are in its most general form. New exact solutions which include periodic and soliton solutions are formally derived by using the tanh method. The programming language *Mathematica* is used.

Keywords: Nonlinear partial differential equation, KdV equation, Boussinesq equation, fKdV equation.

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1 Introduction

A great variety of physical, chemical and biological phenomena are governed by nonlinear partial differential equations, and the knowledge of closed form solutions for its is the great importance for many researches for several reasons: facilitates the testing of numerical solvers, help to physicists to better understand the mechanism that govern the physics models, provide knowledge to the physical problem, provide possible applications and aids to mathematicians in the stability analysis of solutions. It is well known that KdV equation

$$u_t + \alpha u u_x + \beta u_{xxx} = 0,$$

is one of the most important physic models which has been studied by many researches in the last decades. In the same way, the Boussinesq equation

$$u_{tt} + \alpha u_{xx} + \beta (u^2)_{xx} + \gamma u_{xxxx} = 0,$$

is considered as another important model as well the KdV. The two models are formed by completely integrable equations and therefore its posses an infinite number of conservations laws, so that give rise to N-soliton solutions for finite N, where $N \geq 1$. The generalized fifth-order KdV (fKdV) equation [1][2][3][4][5][6]

$$u_t + \omega u_{xxxxx} + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x = 0,$$

is a model which is integrable only in a few cases (see Sec. 5). A variety of direct and computational methods have been used to obtain exact solutions. Two of the most important direct methods are the Hirota method [7], and the inverse scattering method [8]. Some of the computational methods are the tanh method [9], generalized tanh method [10][11][12][13], tanh-coth method [1], generalized tanh-coth method [2][14][15][16][17], improved tanh-coth method [18], The Cole-Hopf transformation [19], Exp-function method [20][21], projective Riccati equations method [22], and the generalized projective Riccati equations method [23][24][25][26].

The principal objective of this paper is to obtain exact solutions for the KdV equation which include two arbitrary parameters, exact solutions for the Boussinesq equation with three arbitrary parameters and exact solutions for the generalized fKdV equations with four arbitrary parameters, by using the generalized tanh method [10][11][12][13]. This paper is organized as follows: In Sec. 2, we will be reviewed briefly of the generalized tanh method; In Sec. 3, we consider the KdV equation and we obtain exact solutions for it by using the method described in Sec. 2.; In Sec.4, we obtain exact solutions for the Boussinesq equation with three arbitrary parameters; In Sec.5, exact solutions for the fKdV in the most general form are obtained. Finally, some conclusions are given.

2 The tanh method

The wave transformation

$$u(x,t) = v(\xi)$$
$$\xi = x + \lambda t, \tag{2.1}$$

converts a PDE

$$G(t, x, u_x, u_t, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0 \tag{2.2}$$

to an ODE

$$H(\xi, v(\xi), v'(\xi), v''(\xi), \ldots) = 0. \tag{2.3}$$

The tanh method use as solution to (2.3) the expansion

$$v(\xi) = a_0 + \sum_{i=1}^{m} a_i \phi^i(\xi), \tag{2.4}$$

where

$$\phi'(\xi) = k + \phi^2(\xi), \tag{2.5}$$

and k is a parameter to be determined later. It is well known that the Riccati equation (2.5) has solutions given by

- If $k < 0$: $\phi(\xi) = -\sqrt{-k}\tanh(\sqrt{-k}\,\xi)$ and $\phi(\xi) = -\sqrt{-k}\coth(\sqrt{-k}\,\xi)$.
- If $k = 0$: $\phi(\xi) = -1\xi$.
- If $k > 0$: $\phi(\xi) = \sqrt{k}\tan(\sqrt{k}\,\xi)$ and $\phi(\xi) = \sqrt{k}\cot(\sqrt{k}\,\xi)$.

Substituting (2.4), along with (2.5) into (2.3) and collecting all terms with the same power in $\phi(\xi)$, we get a polynomial in the variable $\phi = \phi(\xi)$. The parameter m can be found by balancing the high-order linear term with the nonlinear terms in (2.3). Equating the coefficients of every power of ϕ to zero, we obtain an algebraic system in the variables k, a_0, a_1,..., a_m.. Solving the previous system, we obtain the values for k, a_0, a_1,..., a_m.. Lastly, we found solutions to (2.2) in the original variables.

3 The KdV equation

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation which has been derived in 1895 by the professor Diederik Johaness Koteweg (1848-1941) and its student Gustav De-Vries (1899-1934) with the aim to described a phenomenon observed by the naval engineering John Scott Russel (1808-1882) fifty years ago. More exactly, this phenomenon observed by Russel took place on the Union canal near to Edinburgh on 1834 and was called by Russel as the great wave of translation. In order to study and give an explication about of this observed phenomenon, Russell did several and extensive experiments in tanks of water, however this was a difficult task. Others researches as Airy, Stokes, Boussinesq and Rayleigh were done in a way independent, investigations of solitary waves with the aim to understand in a better way the structure of the great wave of translation observed by Rusell [27]. However , it was not until 1895 when Korteweg and De-Vries derived the following nonlinear equation (called Korteweg de-Vries equation (KdV)) which give a description about the propagation of waves of water with small amplitud

$$\frac{\partial\eta}{\partial\tau}=\frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial\zeta}\left(\frac{1}{2}\eta^2+\frac{2}{3}\alpha\eta+\frac{1}{3}\sigma\frac{\partial^2\eta}{\partial\zeta^2}\right) \tag{3.6}$$

$$\sigma=\frac{1}{3}h^3-\frac{Th}{\rho g}, \tag{3.7}$$

here η give the elevation of wave on surface of water, g is the gravity acceleration, T and ρ are constants associated to model, the independent variables τ and ζ are associate with time and space variables. Using the transformations

$$t=\frac{1}{2}\sqrt{\frac{g}{h\sigma}}\tau,\quad x=-\sigma^{-\frac{1}{2}}\zeta,\quad u=\frac{1}{2}\eta+\frac{1}{3}\alpha., \tag{3.8}$$

the equation (3.6) is transformed into equation

$$u_t+6uu_x+u_{3x}=0, \tag{3.9}$$

where subscripts denote partial derivatives. More details and applications on the KdV equation can be found in [28] [29][30][31][32][33][34][35][36] and [37].

We now give exact solutions for a more general than (3.9) equation

$$u_t + \alpha u u_x + \beta u_{3x} = 0. \qquad (3.10)$$

In this case Eq. (2.3) takes the form

$$\lambda v'(\xi) + \alpha v(\xi) v'(\xi) + \beta v^{(3)}(\xi) = 0. \qquad (3.11)$$

Substituting (2.4) into (3.11) we get an equation whose left hand side is a polynomial in the variable $\phi = \phi(\xi)$:

$$a\phi^{2m+1} + b\phi^{m+3} + \cdots = 0.$$

In this equation the highest order term of $v'(\xi)v(\xi)^2$ is $a\phi^{2m+1}$ while the highest order term of $v^{(3)}(\xi)$ is $b\phi^{m+3}$. Balancing these terms we obtain $2m+1 = m+3$ and then $m = 2$. Thus, we seek solutions of (3.11) in the form

$$v(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi), \qquad \xi = x + \lambda t.$$

Using the metanh program (see Apendix) with the instruction

```
metanh[∂t # + α * # ∂x # + β * ∂x,x,x # &, x + λ t, {a0, a1, a2, k, λ}, 2]
```

we get following solutions :

a) $k = 0$, $\lambda = -\alpha a_0$, $a_1 = 0$, $a_2 = -\frac{12\beta}{\alpha}$,

$$u_1(x,t) = a_0 - \frac{12\beta}{\alpha\left(x - \alpha a_0 t\right)^2}.$$

b) $\lambda = -8k\beta - \alpha a_0, a_1 = 0, a_2 = -\frac{12\beta}{\alpha}$,

$$u_2(x,t) = a_0 + \frac{12k\beta}{\alpha}\tanh^2\left(\sqrt{-k}\left(x - \left(8k\beta + \alpha a_0\right)t\right)\right).$$

$$u_3(x,t) = a_0 + \frac{12k\beta}{\alpha}\coth^2\left(\sqrt{-k}\left(x - \left(8k\beta + \alpha a_0\right)t\right)\right).$$

$$u_4(x,t) = a_0 - \frac{12k\beta}{\alpha}\tan^2\left(\sqrt{k}\left(x - \left(8k\beta + \alpha a_0\right)t\right)\right).$$

$$u_5(x,t) = a_0 - \frac{12k\beta}{\alpha}\cot^2\left(\sqrt{k}\left(x - \left(8k\beta + \alpha a_0\right)t\right)\right).$$

4 Exact solutions to the Boussinesq equation

The Boussinesq equation

$$u_{tt} + \alpha u_{xx} + \beta (u^2)_{xx} + \gamma u_{xxxx} = 0. \quad (4.12)$$

was proposed by Boussinesq for a model of nonlinear dispersive waves. In recent years, a lot of research work on Boussinesq equation has been invested. For example, its solitary wave solutions, shock wave solutions, periodic and other types of solutions were found in Refs. [38],[39],[40], and the relations between a nonlinear lattice, the Boussinesq equation and the KdV equation are studied in [41]. In this case, the Eq. (2.3) takes the form

$$v''(\xi)\lambda^2 + \alpha v''(\xi) + \beta\left(2v'(\xi)^2 + 2v(\xi)v''(\xi)\right) + \gamma v^{(4)} \quad (4.13)$$

Substituting (2.4) into (4.12) we get an equation whose left hand side is a polynomial in the variable $\varphi = \varphi(\xi)$:

$$a\varphi^{2m+2} + b\varphi^{m+4} + \cdots = 0.$$

In this equation the highest order term of $v(\xi)u''(\xi)$ is $a\varphi^{2m+2}$ while the highest order term of $v^{(4)}(\xi)$ is $b\varphi^{m+4}$. Balancing these terms we obtain $2m+2 = m+4$ and then $m = 2$. Thus, we seek solutions of (4.13) using the expansion

$$v(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi), \qquad \xi = x + \lambda t.$$

Using the metanh program (see Apendix) with the instruction

```
metanh[∂t,t # + α ∂x,x # + β * ∂x,x #^2 + γ * ∂x,x,x,x # &, x + λ t,
       {a0, a1, a2, k, λ}, 2]
```

we get following solutions :

a) $k = 0$, $a_0 = \frac{-\lambda^2-\alpha}{2\beta}$, $a_1 = 0$, $a_2 = -\frac{6\gamma}{\beta}$,

$$u_1(x,t) = -\frac{\lambda^2+\alpha}{2\beta} - \frac{6\gamma}{\beta(x+t\lambda)^2}.$$

b) $a_0 = \frac{-\lambda^2-\alpha-8k\gamma}{2\beta}$, $a_1 = 0$, $a_2 = -\frac{6\gamma}{\beta}$,

$$u_2(x,t) = -\frac{\lambda^2+\alpha+8k\gamma}{2\beta} + \frac{6k\gamma}{\beta}\tanh^2\left(\sqrt{-k}(x+t\lambda)\right).$$

$$u_3(x,t) = -\frac{\lambda^2+\alpha+8k\gamma}{2\beta} + \frac{6k\gamma}{\beta}\coth^2\left(\sqrt{-k}(x+t\lambda)\right).$$

$$u_4(x,t) = -\frac{\lambda^2+\alpha+8k\gamma}{2\beta} - \frac{6k\gamma}{\beta}\tan^2\left(\sqrt{k}(x+t\lambda)\right).$$

$$u_5(x,t) = -\frac{\lambda^2+\alpha+8k\gamma}{2\beta} - \frac{6k\gamma}{\beta}\cot^2\left(\sqrt{k}(x+t\lambda)\right).$$

5 The general fifth-order KdV equation and its exact solutions

The nonlinear generalized KdV equation of fifth order (fKdV equation) reads

$$u_t + \omega u_{xxxxx} + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x = 0, \tag{5.14}$$

where α, β, γ and ω are arbitrary and real parameters with $\gamma \neq 0$. This equation describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice and it is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. The authors in [8] implemented the inverse scattering transform method to handle the nonlinear equations of physical significance where soliton solutions and rational solutions were developed. Some important particular cases of the Eq. (5.14) are determined as follows [1][2][3]:

- Kaup-Kupershmidt equation (KK equation)

$$u_t + u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x = 0. \tag{5.15}$$

- Sawada-Kotera equation (SK equation)

$$u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x = 0. \tag{5.16}$$

- Caudrey-Dodd-Gibbon equation

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0. \tag{5.17}$$

- Lax equation

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0. \tag{5.18}$$

- Ito equation

$$u_t + u_{xxxxx} + 3uu_{xxx} + 6u_x u_{xx} + 2u^2 u_x = 0. \tag{5.19}$$

As the constants α, β and γ change, the properties of the equation (5.14) drastically change. For instance, the Lax equation with $\alpha = 10$, $\beta = 20$, and $\gamma = 30$, and the SK equation where $\alpha = \beta = \gamma = 5$, are completely integrable. These two equations have N-soliton solutions and an infinite set of conserved laws. Another example is the KK equation with $\alpha = 10$, $\beta = 25$, and $\gamma = 20$, which is known to be integrable, and has bilinear representations, but for which the explicit form of the N-soliton solutions is not known. A fourth equation in this class is the Ito equation, with $\alpha = 3$, $\beta = 6$, and $\gamma = 2$, which is not completely integrable, but has a limited number of special conserved laws. Exact solutions for the previous particular cases of the fKdV, and other most general cases have been obtained by the authors in [1][2][3][4][5][6].

Using the wave transformation, we have that the equation (5.14) is reduced to the nonlinear ordinary differential equation (ODE) of fifth order

$$\gamma v'(\xi)v(\xi)^2 + \alpha v^{(3)}(\xi)v(\xi) + \lambda v'(\xi) + \beta v'(\xi)v''(\xi) + \omega v^{(5)}(\xi) = 0 \tag{5.20}$$

Substituting (2.4), along with (2.5) into (5.20) and collecting all terms with the same power in $\varphi(\xi)$, we get a polynomial in the variable $\varphi = \varphi(\xi)$. This polynomial has the form

$$a\,\varphi(\xi)^{3m+1} + b\,\varphi(\xi)^{2m+3} + c\,\varphi(\xi)^{m+5} + \text{lower degree terms} \tag{5.21}$$

The parameter m can be found by balancing the high-order linear term with the nonlinear terms in (5.21). We assume that $m \geq 1$ to avoid trivial solutions. The degrees of the highest terms are $m+5$ (the degree of the term $c\,\varphi(\xi)^{m+5}$), $2m+3$ (the degree of the term $b\,\varphi(\xi)^{2m+3}$) and $3m+1$ (the degree of the term $a\,\varphi(\xi)^{3m+1}$). The only integer value of m for which $3m+1 = 2m+3$ or $3m+1 = m+5$ or $2m+3 = m+5$ is $m = 2$. Thus, we seek solutions of (5.20) in the form

$$v(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi), \qquad \xi = x + \lambda t.$$

Using the metanh program (see Apendix) in the form

```
metanh[
 ∂t # + ω * ∂x,x,x,x,x # + α * # * ∂x,x,x # + β * ∂x # * ∂x,x # + γ * #^2 * ∂x # &,
 x + λ t, {a0, a1, a2, k, λ}, 2]
```

and defining

$$A=\sqrt{(2\alpha+\beta)^2-40\omega\gamma},\quad B=\beta^2+A\beta+2\alpha\beta-12\gamma\omega$$
$$\text{and}\quad C=-\beta^2+A\beta-2\alpha\beta+12\gamma\omega,$$

we get the following solutions of the general fifth KdV equation Eq. (5.14)

a) $\lambda=-\frac{2Bk^2}{\gamma}$, $a_0=-\frac{2k}{\gamma}(A+2\alpha+\beta)$, $a_1=0$, $a_2=-\frac{3}{\gamma}(A+2\alpha+\beta)$,

$$u_1(x,t)=\frac{k(A+2\alpha+\beta)}{\gamma}\left(3\tanh^2\left(\frac{\sqrt{-k}}{\gamma}\left(\gamma x-2Bk^2t\right)\right)-2\right).$$

$$u_2(x,t)=\frac{k(A+2\alpha+\beta)}{\gamma}\left(3\coth^2\left(\frac{\sqrt{-k}}{\gamma}\left(\gamma x-2Bk^2t\right)\right)-2\right).$$

$$u_3(x,t)=-\frac{k(A+2\alpha+\beta)}{\gamma}\left(3\tan^2\left(\frac{\sqrt{k}}{\gamma}\left(\gamma x-2Bk^2t\right)\right)+2\right).$$

$$u_4(x,t)=-\frac{k(A+2\alpha+\beta)}{\gamma}\left(3\cot^2\left(\frac{\sqrt{k}}{\gamma}\left(\gamma x-2Bk^2t\right)\right)+2\right).$$

$\lambda=\frac{2Ck^2}{\gamma}$, $a_0=\frac{2k}{\gamma}(A-2\alpha-\beta)$, $a_1=0$, $a_2=\frac{3}{\gamma}(A-2\alpha-\beta)$,

$$u_5(x,t)=-\frac{k(A-2\alpha-\beta)}{\gamma}\left(3\tanh^2\left(\frac{\sqrt{-k}}{\gamma}\left(\gamma x+2Ck^2t\right)\right)-2\right).$$

$$u_6(x,t)=-\frac{k(A-2\alpha-\beta)}{\gamma}\left(3\coth^2\left(\frac{\sqrt{-k}}{\gamma}\left(\gamma x+2Ck^2t\right)\right)-2\right).$$

$$u_7(x,t)=\frac{k(A-2\alpha-\beta)}{\gamma}\left(3\tan^2\left(\frac{\sqrt{k}}{\gamma}\left(\gamma x+2Ck^2t\right)\right)+2\right).$$

$$u_8(x,t)=\frac{k(A-2\alpha-\beta)}{\gamma}\left(3\cot^2\left(\frac{\sqrt{k}}{\gamma}\left(\gamma x+2Ck^2t\right)\right)+2\right).$$

5.1. Special cases of the fKdV.

The program metanh gives the following solutions in the special cases corresponding to equations (5.15)-(5.19).

5.1.1. Kaup-Kupershmidt equation :

$u_t+u_{xxxxx}+10uu_{xxx}+25u_xu_{xx}+20u^2u_x=0.$

- $u_1(x,t)=4k\left(3\tanh^2\left(\sqrt{-k}\left(x-176k^2t\right)\right)-2\right).$
- $u_2(x,t)=4k\left(3\coth^2\left(\sqrt{-k}\left(x-176k^2t\right)\right)-2\right).$
- $u_3(x,t)=-4k\left(3\tan^2\left(\sqrt{k}\left(x-176k^2t\right)\right)+2\right).$
- $u_4(x,t)=-4k\left(3\cot^2\left(\sqrt{k}\left(x-176k^2t\right)\right)+2\right).$
- $u_5(x,t)=\frac{1}{2}k\left(3\tanh^2\left(\sqrt{-k}\left(x-k^2t\right)\right)-2\right).$
- $u_6(x,t)=\frac{1}{2}k\left(3\coth^2\left(\sqrt{-k}\left(x-k^2t\right)\right)-2\right).$
- $u_7(x,t)=-\frac{1}{2}k\left(3\tan^2\left(\sqrt{k}\left(x-k^2t\right)\right)+2\right).$
- $u_8(x,t)=-\frac{1}{2}k\left(3\cot^2\left(\sqrt{k}\left(x-k^2t\right)\right)+2\right).$

5.12. Sawada-Kotera equation (SK equation) :

$u_t+u_{xxxxx}+5uu_{xxx}+5u_xu_{xx}+5u^2u_x=0.$

- $u_1(x,t)=6k\tanh^2\left(\sqrt{-k}\left(x-\frac{124k^2t}{5}\right)\right)-\frac{32k}{5}.$
- $u_2(x,t)=6k\coth^2\left(\sqrt{-k}\left(x-\frac{124k^2t}{5}\right)\right)-\frac{32k}{5}.$
- $u_3(x,t)=-\frac{2}{5}k\left(15\tan^2\left(\sqrt{k}\left(x-\frac{124k^2t}{5}\right)\right)+16\right).$
- $u_4(x,t)=-\frac{2}{5}k\left(15\cot^2\left(\sqrt{k}\left(x-\frac{124k^2t}{5}\right)\right)+16\right).$
- $u_5(x,t)=4k\left(3\tanh^2\left(\sqrt{-k}\left(x-16k^2t\right)\right)-2\right).$

- $u_6(x,t)=4k\left(3\coth^2\left(\sqrt{-k}\left(x-16k^2t\right)\right)-2\right).$
- $u_7(x,t)=-4k\left(3\tan^2\left(\sqrt{k}\left(x-16k^2t\right)\right)+2\right).$
- $u_8(x,t)=-4k\left(3\cot^2\left(\sqrt{k}\left(x-16k^2t\right)\right)+2\right).$
- $u_9(x,t)=k\left(6\tanh^2\left(\sqrt{-k}\left(x-k^2t\right)\right)-5\right).$
- $u_{10}(x,t)=k\left(6\coth^2\left(\sqrt{-k}\left(x-k^2t\right)\right)-5\right).$
- $u_{11}(x,t)=-k\left(6\tan^2\left(\sqrt{k}\left(x-k^2t\right)\right)+5\right).$
- $u_{12}(x,t)=-k\left(6\cot^2\left(\sqrt{k}\left(x-k^2t\right)\right)+5\right).$
- $u_{13}(x,t)=2k\left(3\tanh^2\left(\sqrt{-k}\left(x+4k^2t\right)\right)-2\right).$
- $u_{14}(x,t)=2k\left(3\coth^2\left(\sqrt{-k}\left(x+4k^2t\right)\right)-2\right).$
- $u_{15}(x,t)=-2k\left(3\tan^2\left(\sqrt{k}\left(x+4k^2t\right)\right)+2\right).$
- $u_{16}(x,t)=-2k\left(3\cot^2\left(\sqrt{k}\left(x+4k^2t\right)\right)+2\right).$
- $u_{17}(x,t)=6k\tanh^2\left(\sqrt{-k}\left(x-(76k^2+40a_0k+5a_0^2)t\right)\right)+a_0.$
- $u_{18}(x,t)=6k\coth^2\left(\sqrt{-k}\left(x-(76k^2+40a_0k+5a_0^2)t\right)\right)+a_0.$
- $u_{19}(x,t)=a_0-6k\tan^2\left(\sqrt{k}\left(x-(76k^2+40a_0k+5a_0^2)t\right)\right).$
- $u_{20}(x,t)=a_0-6k\cot^2\left(\sqrt{k}\left(x-(76k^2+40a_0k+5a_0^2)t\right)\right).$
- $u_{21}(x,t)=a_0-\frac{6}{\left(x-5a_0^2t\right)^2}.$

5.13. Lax equation :

$$u_t+u_{xxxxx}+10uu_{xxx}+20u_xu_{xx}+30u^2u_x=0.$$

- $u_1(x,t)=\frac{2}{3}k\left(3\tanh^2\left(\sqrt{-k}\left(x-\frac{368k^2t}{3}\right)\right)-5\right).$

- $u_2(x,t) = \frac{2}{3}k\left(3\coth^2\left(\sqrt{-k}\left(x - \frac{368k^2t}{3}\right)\right) - 5\right).$
- $u_3(x,t) = -\frac{2}{3}k\left(3\tan^2\left(\sqrt{k}\left(x - \frac{368k^2t}{3}\right)\right) + 5\right).$
- $u_4(x,t) = -\frac{2}{3}k\left(3\cot^2\left(\sqrt{k}\left(x - \frac{368k^2t}{3}\right)\right) + 5\right).$
- $u_5(x,t) = 2k\left(3\tanh^2\left(\sqrt{-k}\left(x - 56k^2t\right)\right) - 2\right).$
- $u_6(x,t) = 2k\left(3\coth^2\left(\sqrt{-k}\left(x - 56k^2t\right)\right) - 2\right).$
- $u_7(x,t) = -2k\left(3\tan^2\left(\sqrt{k}\left(x - 56k^2t\right)\right) + 2\right).$
- $u_8(x,t) = -2k\left(3\cot^2\left(\sqrt{k}\left(x - 56k^2t\right)\right) + 2\right).$
- $u_9(x,t) = \frac{1}{3}k\left(6\tanh^2\left(\sqrt{-k}\left(x - \frac{98k^2t}{3}\right)\right) - 7\right).$
- $u_{10}(x,t) = \frac{1}{3}k\left(6\coth^2\left(\sqrt{-k}\left(x - \frac{98k^2t}{3}\right)\right) - 7\right).$
- $u_{11}(x,t) = -\frac{1}{3}k\left(6\tan^2\left(\sqrt{k}\left(x - \frac{98k^2t}{3}\right)\right) + 7\right).$
- $u_{12}(x,t) = -\frac{1}{3}k\left(6\cot^2\left(\sqrt{k}\left(x - \frac{98k^2t}{3}\right)\right) + 7\right).$
- $u_{13}(x,t) = -2k\,\text{sech}^2\left(\sqrt{-k}\left(x - 16k^2t\right)\right).$
- $u_{14}(x,t) = 2k\,\text{csch}^2\left(\sqrt{-k}\left(x - 16k^2t\right)\right).$
- $u_{15}(x,t) = -2k\sec^2\left(\sqrt{k}\left(x - 16k^2t\right)\right).$
- $u_{16}(x,t) = -2k\csc^2\left(\sqrt{k}\left(x - 16k^2t\right)\right).$
- $u_{17}(x,t) = 2k\tanh^2\left(\sqrt{-k}\left(x - 2\left(28k^2 + 40a_0k + 15a_0^2\right)t\right)\right) + a_0.$
- $u_{18}(x,t) = 2k\coth^2\left(\sqrt{-k}\left(x - 2\left(28k^2 + 40a_0k + 15a_0^2\right)t\right)\right) + a_0.$
- $u_{19}(x,t) = a_0 - 2k\tan^2\left(\sqrt{k}\left(x - 2\left(28k^2 + 40a_0k + 15a_0^2\right)t\right)\right).$
- $u_{20}(x,t) = a_0 - 2k\cot^2\left(\sqrt{k}\left(x - 2\left(28k^2 + 40a_0k + 15a_0^2\right)t\right)\right).$

- $u_{21}(x,t) = a_0 - \frac{2}{\left(x-30a_0^2 t\right)^2}$.

5.14. Ito equation :
$u_t + u_{xxxxx} + 3uu_{xxx} + 6u_x u_{xx} + 2u^2 u_x = 0.$

- $u_1(x,t) = 2k\left(3\tanh^2\left(\sqrt{-k}x\right) - 2\right)$.
- $u_2(x,t) = 2k\left(3\coth^2\left(\sqrt{-k}x\right) - 2\right)$.
- $u_3(x,t) = -2k\left(3\tan^2\left(\sqrt{k}x\right) + 2\right)$.
- $u_4(x,t) = -2k\left(3\cot^2\left(\sqrt{k}x\right) + 2\right)$.
- $u_5(x,t) = 10k\left(3\tanh^2\left(\sqrt{-k}\left(x - 96k^2 t\right)\right) - 2\right)$.
- $u_6(x,t) = 10k\left(3\coth^2\left(\sqrt{-k}\left(x - 96k^2 t\right)\right) - 2\right)$.
- $u_7(x,t) = -10k\left(3\tan^2\left(\sqrt{k}\left(x - 96k^2 t\right)\right) + 2\right)$.
- $u_8(x,t) = -10k\left(3\cot^2\left(\sqrt{k}\left(x - 96k^2 t\right)\right) + 2\right)$.

6 Conclusions

In this paper, by means of the tanh method and the use of the symbolic computation package *Mathematica*, we obtain exact solutions for several kinds of important nonlinear evolution partial differential equations. The tanh method is straightforward and effective. We also may apply the mentioned method to a coupled system of nonlinear PDE's. This is the main objective of a forthcoming work. Other recent works related to exact solutions of nonlinear PDE's may be found in [42]-[56].

7 Appendix : The program metanh.

```
metanh[op_, xi_, vars_, m0_] :=
  Module[{φ1, φ2, φ3, φ4, φ5, u1, u2, u3, u4, u5, u, xp1, n,
    rules, xp2, system, sols, sols0, sols1, sols2, sols3,
    sols4},
   {Clear[ξ, k];
    φ1 = -√-k * Tanh[√-k * xi]; φ2 = -√-k * Coth[√-k * xi];
    φ3 = √k * Tan[√k * xi]; φ4 = √k * Cot[√k * xi]; φ5 = -1 / xi;
    u1 = Sum[a_i φ1^i, {i, 0, m0}]; u2 = Sum[a_i φ2^i, {i, 0, m0}];
    u3 = Sum[a_i φ3^i, {i, 0, m0}]; u4 = Sum[a_i φ4^i, {i, 0, m0}];
    u5 = Sum[a_i φ5^i, {i, 0, m0}];
    u = Sum[a_i φ[xi]^i, {i, 0, m0}];
    xp1 = Numerator[Factor[op[u] /. {xi → ξ}]];
    n = Max[Select[Level[xp1, Infinity, Heads → True],
           Head[#] == Derivative &] //. {Derivative[x_] :→ x}];
    rules = Table[D[φ[ξ], {ξ, j}] → D[k + φ[ξ]^2,
        {ξ, j - 1}], {j, 1, n}];
    xp2 = CoefficientList[xp1 //. rules, φ[ξ]];
    system = Complement[Map[# == 0 &, xp2], {True, False}];
    sols = Union[Solve[system, vars]];
    sols0 = Union[Select[sols, MemberQ[#, k → 0] &]];
    sols1 = Complement[sols, sols0];
    sols2 = Map[{Sort[#], u5 //. #} &, sols0];
    sols3 = Map[{Sort[#], u1 //. #, u2 //. #, u3 //. #, u4 //. #} &,
             sols1];
    sols4 = Join[sols2, sols3]
   }; Union[sols4]];
```

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