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HASTIR, Anthony; WINKIN, Joseph; Dochain, Denis

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On local exponential stability of equilibrium profiles of nonlinear distributed parameter systems ⋆

Hastir, A. * **Winkin, J.J.** * **Dochain, D.** **

* *University of Namur, Department of Mathematics and Namur Institute for Complex Systems (naXys), Rempart de la vierge, 8, B-5000 Namur, Belgium (e-mail: anthony.hastir@unamur.be, joseph.winkin@unamur.be).*
** *Université Catholique de Louvain, Institute of Information and Communication Technologies, Electronics and Applied Mathematics (ICTEAM), Avenue Georges Lemaitre 4-6, B-1348 Louvain-La-Neuve, Belgium (e-mail: denis.dochain@uclouvain.be)*

Abstract: Local exponential (exp.) stability of nonlinear distributed parameter, i.e. infinite–dimensional state space, systems is considered. A weakened concept of Fréchet differentiability ((Y, X)-Fréchet differentiability) for nonlinear operators defined on Banach spaces is proposed, including the introduction of an alternative space (Y) in the analysis. This allows more freedom in the manipulation of norm–inequalities leading to adapted Fréchet differentiability conditions that are easier to check. Then, provided that the nonlinear semigroup generated by the nonlinear dynamics is Fréchet–differentiable in the new sense, appropriate local exp. stability of the equilibria for the nonlinear system is established. In particular, the nonlinear semigroup has to be Fréchet differentiable on Y and (Y, X)–Fréchet differentiable in order to go back to the original state space X. This approach may be called "perturbation–based" since exp. stability is also deduced from exp. stability of a linearized version of the nonlinear semigroup. Under adapted Fréchet differentiability assumptions, the main result establishes that local exp. stability of an equilibrium for the nonlinear system is guaranteed as long as the exp. stability holds for the linearized semigroup. The same conclusion holds regarding instability. The theoretical results are illustrated on a convection-diffusion-reaction system.

Keywords: Distributed parameter systems – Nonlinear systems – Equilibrium – Exponential stability

1. INTRODUCTION

Deducing stability/instability of an equilibrium for a nonlinear distributed parameter, i.e. infinite-dimensional, system on the basis of the stability/instability properties of a linearization of it is not straightforward. For instance this is studied in (Al Jamal and Morris, 2018), (Al Jamal et al., 2014) or in (Kato, 1995) where exp. stability of the equilibrium for the linearization and also Fréchet differentiability of the nonlinear semigroup generated by the nonlinear dynamics are needed. This approach is often called linearized stability and has also been studied in (Henry, 1981; Smoller, 1983; Webb, 1985; Temam, 1997) among others. However, cheking Fréchet differentiability conditions for nonlinear operators is generally challenging or even impossible when these operators are unbounded. This is mainly due to the fact that the norms are not equivalent when working in infinite-dimensional spaces. For instance the theoretical framework that is proposed in Al Jamal and Morris (2018) is not directly applicable on their example and a case–by–case study has often to be performed by working directly on the nonlinear semigroup generated by the nonlinear dynamics instead of its generator. Other assumptions like the Lipschitz continuity (in the operators norm) of the Fréchet derivative of the nonlinear operator are also needed, which is a quite strong assumption since it depends critically on the considered topology.

The approach that is proposed here is an extension of the contribution of (Al Jamal and Morris, 2018) for deducing nonlinear local exp. stability or instability of an equilibrium of nonlinear distributed parameter systems. Some relaxed Fréchet differentiability conditions are required on the nonlinear semigroup generated by the nonlinear dynamics, by considering different spaces and different norms in the definition. This is called $(Y, X)-$Fréchet differentiability, where X is the state space and Y is an auxiliary space that has to be chosen according to the application. Since Y is chosen as involving more regularity than X, this allows more easily checkable Fréchet differentiability conditions, providing local exp. stability or instability of an equilibrium for nonlinear infinite-dimensional systems in a weaker sense, see (Hastir et al., 2019b).

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The paper is organized as follows. In Section 2 the class of considered nonlinear distributed parameter systems is introduced together with the related assumptions. The new concept of $(Y,X)-$Fréchet differentiability and the appropriate definitions of stability are given. Section 3 is dedicated to the main results of the paper. In particular, it is shown how to get adapted Fréchet differentiability of the nonlinear semigroup generated by the nonlinear dynamics. Then, stability of an equilibrium of the nonlinear model is deduced from the stability/instability properties of an appropriate linearization of it. The results are applied to a particular example of nonlinear distributed parameter system in Section 4.

2. PROBLEM STATEMENT

In this section, we aim at presenting the framework, the new concept of $(Y,X)-$Fréchet differentiability and the assumptions that will be under consideration in the sequel.

The nonlinear distributed parameter systems that are going to be studied are governed by the following abstract differential equation:

$$\begin{cases} \dot{\xi}(t) = \mathcal{A}\xi(t) + \mathcal{N}(\xi(t)), \\ \xi(0) = \xi_0, \end{cases} \tag{1}$$

where $\xi(t)$ is supposed to evolve in the (Hilbert) state space X. Moreover, the (unbounded) linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on X and $\mathcal{N} : \mathcal{D} \subset X \to X$ is a nonlinear operator defined on the closed convex subset $\mathcal{D}$. The initial condition ξ_0 lies in $D(\mathcal{A})\cap\mathcal{D}$. Such systems are studied e.g. in (Curtain and Zwart, 1995) and (Engel and Nagel, 2006) among others.

Note that, since $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup, it is dissipative under some additive perturbation, i.e. there exists $l_{\mathcal{A}} \geq 0$ such that $\mathcal{A} - l_{\mathcal{A}} I$ is dissipative.

The following assumption characterizes the well-posedness of (1). By well-posedness we mean the existence of a mild solution on $[0,\infty)$.

Assumption 1. The closed convex subset $\mathcal{D}$ is $T(t)$-invariant, that is $T(t)\mathcal{D} \subset \mathcal{D}$ for all $t \geq 0$. In addition, it is supposed that the tangential condition $\lim_{h\to 0^+} \frac{1}{h} d(\xi + h\mathcal{N}(\xi)\mathcal{D}; \mathcal{D}) = 0$ holds for any $\xi \in \mathcal{D}$. We also assume that the nonlinear operator $\mathcal{N}$ is Lipschitz continuous on $\mathcal{D}$ and that there exists $l_{\mathcal{N}} \geq 0$ such that $\mathcal{N} - l_{\mathcal{N}} I$ is dissipative on $\mathcal{D}$.

Note that Assumption 1 implies that equation (1) has a unique mild solution on $[0,\infty)$ for all $\xi_0 \in \mathcal{D}$. By defining $S(t)\xi_0 := \xi(t)$ for all $t \geq 0, (S(t))_{t\geq 0}$ is a nonlinear semigroup on $\mathcal{D}$ whose infinitesimal generator is the operator $A+\mathcal{N}$. Note also that the tangential condition in Assumption 1 implies that $\mathcal{D}$ is $S(t)-$invariant.

In what follows, it is supposed that (1) possesses an equilibrium profile $\xi^e \in D(\mathcal{A}) \cap \mathcal{D}$, i.e. $\mathcal{A}\xi^e + \mathcal{N}(\xi^e) = 0$.

Assumption 2. The nonlinear operator $\mathcal{N}$ is Gâteaux differentiable at ξ^e, that is, there exists a linear operator $d\mathcal{N}(\xi^e) : X \to X$ such that

$$\lim_{\epsilon\to 0} \frac{\mathcal{N}(\xi^e + \epsilon h) - \mathcal{N}(\xi^e)}{\epsilon} = d\mathcal{N}(\xi^e)h,$$

where[1] $\xi^e, \xi^e + \epsilon h \in \mathcal{D}$. Moreover, it is assumed that $d\mathcal{N}(\xi^e)$ is bounded on X.

For the sake of simplicity, in the following, we shall write (1) around its equilibrium ξ^e. Hence by defining the variable $\hat{\xi} := \xi - \xi^e$ one has

$$\begin{cases} \dot{\hat{\xi}}(t) = \mathcal{A}\hat{\xi}(t) + \mathcal{N}(\hat{\xi}(t) + \xi^e) - \mathcal{N}(\xi^e), \\ \hat{\xi}(0) = \xi_0 - \xi^e =: \hat{\xi}_0. \end{cases} \tag{2}$$

The null function is an obvious equilibrium of (2) and since (1) is well-posed, (2) is also well-posed in the sense that the nonlinear operator $\mathcal{A}+\mathcal{N}(\cdot+\xi^e)-\mathcal{N}(\xi^e)$ is the infinitesimal generator of a nonlinear semigroup $(\hat{S}(t))_{t\geq 0}$ on $D(\mathcal{A})\cap\mathcal{D}^e$, with $\mathcal{D}^e := \mathcal{D} - \xi^e$.

Note that the shifted domain $\mathcal{D}^e$ is $\hat{S}(t)-$invariant, since $\mathcal{D}$ is $S(t)-$invariant.

Let us consider an auxiliary (possibly Banach) space Y that satisfies $D(\mathcal{A}) \cap \mathcal{D}^e \subset Y \subseteq X$ and $\|h\|_X \leq \|h\|_Y$ for all $h \in D(\mathcal{A}) \cap \mathcal{D}^e$. It will be of primary importance in the definition of the generalized concept of Fréchet differentiability, the $(Y,X)-$Fréchet differentiability, see e.g. (Hastir et al., 2019b). This new concept is in general more easily verifiable for nonlinear operators since it allows more freedom in the manipulation of norm inequalities for instance. This is mainly due to the fact that the space Y adds a degree of freedom in the analysis and is often chosen as a multiplicative algebra (L^∞ or Sobolev spaces $H^p, p \in \mathbb{N}$)[2].

Note that the nonlinear operator $\mathcal{N}$ is restricted to the domain $D(\mathcal{A}) \cap \mathcal{D}$ in what follows.

Definition 3. The nonlinear operator $\mathcal{N} : D(A) \cap \mathcal{D} \subset X \to X$ is called $(Y,X)-$Fréchet differentiable at ξ^e if there exists a bounded linear operator $d\mathcal{N}(\xi^e) : X \to X$ such that for all $h \in D(A) \cap \mathcal{D}^e, \mathcal{N}(\xi^e + h) - \mathcal{N}(\xi^e) = d\mathcal{N}(\xi^e)h + R(\xi^e, h)$ where

$$\lim_{\|h\|_Y\to 0} \frac{\|R(\xi^e,h)\|_X}{\|h\|_X} = 0,$$

or equivalently,

$$\lim_{\|h\|_Y\to 0} \frac{\|\mathcal{N}(\xi^e + h) - \mathcal{N}(\xi^e) - d\mathcal{N}(\xi^e)h\|_X}{\|h\|_X} = 0.$$

Taking Y to be identical to X entails that Definition 3 is equivalent to the standard definition of Fréchet differentiability, see e.g. (Al Jamal and Morris, 2018).

Since this new concept is introduced to link the stability properties of a linearized version of (2) with the stability properties of (2), we shall specify what we mean by exponential stability in this context.

Definition 4. The equilibrium ξ^e of (1) is said to be globally exponentially stable if there exist $\alpha, \beta > 0$ such that for all $\xi_0 \in D(A) \cap \mathcal{D}$, it holds $\|\xi(t) - \xi^e\|_X \leq \alpha e^{-\beta t}\|\xi_0 - \xi^e\|_X, t \geq 0$, or equivalently, $\|\hat{\xi}(t)\|_X \leq \alpha e^{-\beta t}\|\hat{\xi}_0\|_X$ for all $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$.

[1] Due to the convexity of the closed subset $\mathcal{D}$, any convex combinations of ξ^e and $\xi^e + \epsilon h$ is an element of $\mathcal{D}$, that is $a\xi^e + (1-a)(\xi^e + \epsilon h) \in \mathcal{D}$ for all $a \in [0,1]$.

[2] Note that there are no canonical choices for the space Y. It depends strongly on the application that is considered.

Definition 5. The equilibrium ξ^e of (1) is $(Y,X)-$locally exponentially stable if there exist $\delta,\alpha,\beta > 0$ such that, for all $\hat{\xi}_0 \in D(A)\cap\mathcal{D}^e$ with $\|\hat{\xi}_0\|_Y < \delta$, there holds $\|\hat{\xi}(t)\|_X \leq \alpha e^{-\beta t}\|\hat{\xi}_0\|_X, t \geq 0$.

Definition 6. The equilibrium ξ^e of (1) is said to be $(Y,X)-$locally stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $\hat{\xi}_0 \in D(A)\cap\mathcal{D}^e$, $\|\hat{\xi}_0\|_Y < \delta$ implies that $\|\hat{\xi}(t)\|_X < \epsilon, t \geq 0$. The equilibrium ξ^e is $(Y,X)-$(locally) unstable if it is not stable.

Let us consider the two following assumptions on (2), the nonlinearity $\mathcal{N}$ and the auxiliary space Y.

Assumption 7. The nonlinear abstract Cauchy problem (2) is well-posed on Y. Moreover, it is assumed that the Gâteaux derivative $d\mathcal{N}(\xi^e)$ of $\mathcal{N}$ is bounded on Y.

Assumption 8. The operator $\mathcal{N}$ is $(Y,X)-$Fréchet differentiable at ξ^e. Moreover, the nonlinear semigroup $(\hat{S}(t))_{t\geq 0}$ is supposed to be continuously dependent of the initial condition $\hat{\xi}_0$ in the sense that the inequality

$$\|\hat{S}(t)\hat{\xi}_0\|_X \leq \gamma_t\|\hat{\xi}_0\|_X \tag{3}$$

holds on the time interval $[0,t_0], t_0 \geq 0$, for some γ_t (that may depend on t).

We shall end this section by the following lemma that is a consequence of Assumption 8.

Lemma 9. Let us consider $\hat{\xi}(t)$, the solution of the abstract differential equation (2), where $t \in [0,t_0]$ for some nonnegative t_0. Then, under Assumption 8, the relation

$$\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|\mathcal{N}(\hat{\xi}+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} = 0$$

holds.

Proof. The function $\mathcal{N}(\hat{\xi}(\cdot)+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}(\cdot)$ is time–continuous on the interval $[0,t_0]$. Hence there exists $t^* \in [0,t_0]$ such that

$$\begin{aligned}&\sup_{t\in[0,t_0]}\|\mathcal{N}(\hat{\xi}(t)+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}(t)\|_X\\ &= \|\mathcal{N}(\hat{\xi}(t^*)+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}(t^*)\|_X.\end{aligned} \tag{4}$$

Moreover, according to (3),

$$\frac{1}{\|\hat{\xi}_0\|_X} \leq \gamma_{t^*}\frac{1}{\|\hat{\xi}(t^*)\|_X}. \tag{5}$$

Combining (4) and (5) yields

$$\begin{aligned}&\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|\mathcal{N}(\hat{\xi}+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X}\\ &= \lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|\mathcal{N}(\hat{\xi}(t^*)+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}(t^*)\|_X}{\|\hat{\xi}_0\|_X}\\ &\leq \gamma_{t^*}\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|\mathcal{N}(\hat{\xi}(t^*)+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}(t^*)\|_X}{\|\hat{\xi}(t^*)\|_X}.\end{aligned}$$

According to the properties of the auxiliary space Y, imposing that $\|\hat{\xi}_0\|_Y$ converges to 0 implies that so does $\|\hat{\xi}_0\|_X$. In view of (3), it follows that $\|\hat{\xi}(t^*)\|_X \to 0$ in this case. By the $(Y,X)-$Fréchet differentiability of $\mathcal{N}$, we have

$$\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\gamma_{t^*}\|\mathcal{N}(\hat{\xi}(t^*)+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}(t^*)\|_X}{\|\hat{\xi}(t^*)\|_X} = 0,$$

which entails that

$$\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|\mathcal{N}(\hat{\xi}+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} = 0.$$

□

This property will be of interest in the next section that is dedicated to the stability analysis of the equilibrium profile ξ^e and through which Assumptions 1, 2, 7 and 8 are assumed to hold.

3. NONLINEAR STABILITY ANALYSIS

In this section, the stability properties of a linearized model corresponding to (2) are used to deduce local (exp.) stability or instability of the semigroup generated by the dynamics of (2). First we linearize (2) around its equilibrium $\hat{\xi} = 0$ via a Gâteaux linearization. This yields the following linear abstract differential equation

$$\begin{cases}\dot{\overline{\xi}}(t) = \mathcal{A}\overline{\xi}(t) + d\mathcal{N}(\xi^e)\overline{\xi}(t),\\ \overline{\xi}(0) = \hat{\xi}_0 \in D(\mathcal{A})\cap\mathcal{D}^e.\end{cases} \tag{6}$$

Since the operator $d\mathcal{N}(\xi^e) - l_{d\mathcal{N}}I$ is dissipative by Assumption 2 and $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup, the operator $\mathcal{A}+d\mathcal{N}(\xi^e)$ is still the infinitesimal generator of a linear semigroup. We denote that semigroup by $(\overline{T}(t))_{t\geq 0}$. Moreover, since by Assumption 7, the abstract Cauchy problem 2 is well-posed on Y and that $d\mathcal{N}(\xi^e)-\tilde{l}_{d\mathcal{N}}I$ is dissipative on Y, the linearized dynamics (6) is well-posed on Y, see e.g. (Engel and Nagel, 2006, Bounded Perturbation Theorem).

Let us consider the following assumptions that will constitute the basis in making the link between (exp.) stability or instability of $(\overline{T}(t))_{t\geq 0}$ and $(\hat{S}(t))_{t\geq 0}$, see (Hastir et al., 2019b, Section 3).

Assumption 10. The nonlinear C_0-semigroup $(\hat{S}(t))_{t\geq 0}$ is $Y-$Fréchet differentiable at 0. For the case where $(\overline{T}(t))_{t\geq 0}$ is exponentially stable on X, it is also assumed that $(\overline{T}(t))_{t\geq 0}$ satisfies

$$\|\overline{T}(t)\hat{\xi}_0\|_Y \leq \eta\|\hat{\xi}_0\|_Y, t\geq 0, \forall\hat{\xi}_0 \in Y \text{ s.t. } \|\hat{\xi}_0\|_Y < \delta^*,$$

for some $\eta > 0$ and $\delta^* > 0$ that may depend on η.

Remark 11.
1) Assumption 10 implies Lyapunov stability of the equilibrium ξ^e of system (1) on the space Y.
2) Under Assumptions 7 and 10, the estimate

$$\|\hat{S}(t)\hat{\xi}_0\|_Y \leq M\|\hat{\xi}_0\|_Y, t\geq 0, \text{ for } \|\hat{\xi}_0\| < \delta \tag{7}$$

holds for some $M > 0$ and $\delta > 0$ that may depend on M. Indeed, the $Y-$Fréchet differentiability of $(\hat{S}(t))_{t\geq 0}$ in Assumption 10 yields the identity

$$\hat{S}(t)\hat{\xi}_0 = \overline{T}(t)\hat{\xi}_0 + r(\xi^e,\hat{\xi}_0),$$

where

$$\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|r(\xi^e,\hat{\xi}_0)\|_Y}{\|\hat{\xi}_0\|_Y} = 0,$$

that is, for all $\epsilon > 0$, there exists $\tilde{\delta} > 0$ such that $\|r(\xi^e,\hat{\xi}_0)\|_Y < \epsilon\|\hat{\xi}_0\|_Y$ for $\|\hat{\xi}_0\|_Y < \tilde{\delta}$. Let us pick any $\epsilon > 0$. Then $\delta := \min(\delta^*,\tilde{\delta}) > 0$ is such that

$$\begin{aligned}\|\hat{S}(t)\hat{\xi}_0\|_Y &\leq \|\overline{T}(t)\hat{\xi}_0\|_Y + \|r(\xi^e,\hat{\xi}_0)\|_Y\\ &\leq \eta\|\hat{\xi}_0\|_Y + \epsilon\|\hat{\xi}_0\|_Y =: M\|\hat{\xi}_0\|_Y,\end{aligned}$$

for $\|\hat{\xi}_0\|_Y \leq \delta$ and $M := \eta + \epsilon$.

3) Let us consider the norm $|||x||| := \sup_{t\geq 0} \|\hat{S}(t)x\|_Y$ for $x \in Y, \|x\|_Y < \delta$. The norms $|||\cdot|||$ and $\|\cdot\|_Y$ are locally equivalent around the equilibrium ξ^e, that is
$$\|x\|_Y \leq |||x||| \leq M\|x\|_Y,$$
for some $M > 0$ and $\delta > 0$ such that $\|x\|_Y < \delta$. This is valid since (7) is satisfied. Moreover, $(\hat{S}(t))_{t\geq 0}$ is a contraction C_0-semigroup on $(Y, |||\cdot|||)$.

Under Assumptions 1, 2, 7, 8 and 10, the $(Y, X)-$Fréchet differentiability of $(\hat{S}(t))_{t\geq 0}$ at 0 with $(\overline{T}(t))_{t\geq 0}$ as Fréchet derivative is established in the following lemma, see (Hastir et al., 2019b).

Lemma 12. Let us consider a space $(Y, \|\cdot\|_Y)$ satisfying $D(A)\cap\mathcal{D}^e \subset Y \subseteq X$. Under Assumptions 1, 2, 7, 8 and 10, the nonlinear C_0-semigroup $(\hat{S}(t))_{t\geq 0}$ is $(Y, X)-$Fréchet differentiable at 0 and its Fréchet derivative is given by the linear C_0-semigroup $(\overline{T}(t))_{t\geq 0}$ whose infinitesimal generator is $A + d\mathcal{N}(\xi^e)$, that corresponds to the Gâteaux derivative of $A + \mathcal{N}(\cdot + \xi^e) - \mathcal{N}(\xi^e)$ at 0.

Proof. Pick any $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$ and $t \in [0, t_0]$. Let us define $\phi(t) = \hat{\xi}(t) - \overline{\xi}(t)$, where $\hat{\xi}(t)$ and $\overline{\xi}(t)$ are the solutions to (2) and (6) at time t, respectively, with the same initial conditions $\hat{\xi}_0$. It follows that
$$\begin{aligned}
\dot{\phi}(t) &= \dot{\hat{\xi}}(t) - \dot{\overline{\xi}}(t) \\
&= \mathcal{A}\hat{\xi}(t) + \mathcal{N}(\hat{\xi}(t) + \xi^e) - \mathcal{N}(\xi^e) - \mathcal{A}\overline{\xi}(t) - d\mathcal{N}(\xi^e)\overline{\xi}(t) \\
&= \mathcal{A}\phi(t) + d\mathcal{N}(\xi^e)\hat{\xi}(t) - d\mathcal{N}(\xi^e)\hat{\xi}(t) + \mathcal{N}(\hat{\xi}(t) + \xi^e) \\
&\quad - \mathcal{N}(\xi^e) - d\mathcal{N}(\xi^e)\overline{\xi}(t) \\
&= \mathcal{A}\phi(t) + d\mathcal{N}(\xi^e)\phi(t) + R(\xi^e, \hat{\xi}(t)),
\end{aligned}$$
where $R(\xi^e, \hat{\xi}(t)) = \mathcal{N}(\hat{\xi}(t) + \xi^e) - \mathcal{N}(\xi^e) - d\mathcal{N}(\xi^e)\hat{\xi}(t)$. Obviously, $\phi(0) = 0$. Taking the inner product of $\phi(t)$ and $\dot{\phi}(t)$ yields
$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi(t)\|_X^2 &= \langle\dot{\phi}(t), \phi(t)\rangle_X \\
&= \langle\mathcal{A}\phi(t) + d\mathcal{N}(\xi^e)\phi(t) + R(\xi^e, \hat{\xi}(t)), \phi(t)\rangle_X \\
&= \langle(\mathcal{A} - l_{\mathcal{A}}I)\phi(t), \phi(t)\rangle_X + \langle l_{\mathcal{A}}\phi(t), \phi(t)\rangle_X \\
&\quad + \langle d\mathcal{N}(\xi^e)\phi(t), \phi(t)\rangle_X + \langle R(\xi^e, \hat{\xi}(t)), \phi(t)\rangle_X \\
&\leq (l_{\mathcal{A}} + \|d\mathcal{N}(\xi^e)\|_{op})\|\phi(t)\|_X^2 + \|R(\xi^e, \hat{\xi}(t))\|_X\|\phi(t)\|_X,
\end{aligned}$$
where Assumption 2 and the Cauchy-Schwarz have been used. The notation $l_{\mathcal{A}} + \|d\mathcal{N}(\xi^e)\|_{op} =: k$ is adopted in what follows. By applying Young's inequality to the previous inequality, see e.g. (Krstic and Smyshlyaev, 2008), it follows that
$$\frac{1}{2}\frac{d}{dt}\|\phi(t)\|_X^2 \leq (k + \frac{1}{2})\|\phi(t)\|_X^2 + \frac{1}{2}\|R(\xi^e, \hat{\xi}(t))\|_X^2.$$
Applying Grönwall's inequality with $\phi(0) = 0$ (see for instance (Curtain and Zwart, 1995, Lemma A.6.7)), gives
$$\begin{aligned}
\|\phi(t)\|_X^2 &\leq e^{(2k+1)t}\int_0^t e^{-(2k+1)s}\|R(\xi^e, \hat{\xi}(s))\|_X^2 ds \\
&\leq e^{(2k+1)t_0}\int_0^{t_0}\|R(\xi^e, \hat{\xi}(s))\|_X^2 ds.
\end{aligned}$$
Equivalently,
$$\begin{aligned}
\|\phi(t)\|_X &\leq e^{\frac{(2k+1)t_0}{2}}\left(\int_0^{t_0}\|R(\xi^e, \hat{\xi}(s))\|_X^2 ds\right)^{\frac{1}{2}} \\
&= e^{\frac{(2k+1)t_0}{2}}\|R(\xi^e, \hat{\xi})\|_{L^2([0,t_0];X)} \\
&\leq t_0^{\frac{1}{2}} e^{\frac{(2k+1)t_0}{2}}\|R(\xi^e, \hat{\xi})\|_{L^\infty([0,t_0];X)}.
\end{aligned}$$
The positive constant $t_0^{1/2}e^{\frac{(2k+1)t_0}{2}}$ is denoted by λ_{t_0} in the following. Lemma 9 implies that
$$\begin{aligned}
&\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|\phi(t)\|_X}{\|\hat{\xi}_0\|_X} = \lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|\hat{S}(t)\hat{\xi}_0 - \overline{T}(t)\hat{\xi}_0\|_X}{\|\hat{\xi}_0\|_X} \\
&\leq \lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\lambda_{t_0}\|R(\xi^e, \hat{\xi})\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} \\
&= \lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\lambda_{t_0}\|\mathcal{N}(\hat{\xi} + \xi^e) - \mathcal{N}(\xi^e) - d\mathcal{N}(\xi^e)\hat{\xi}\|_{L^\infty([0,t_0];X)}}{\|\hat{\xi}_0\|_X} \\
&= 0.
\end{aligned}$$
Hence, $\hat{S}(t)$ is $(Y, X)-$Fréchet differentiable at 0 with $\overline{T}(t)$ as Fréchet derivative. □

Note that the $Y-$Fréchet differentiability of $(\hat{S}(t))_{t\geq 0}$ in Assumption 10 can be obtained by using similar arguments as those in Lemma 9. That is, $\mathcal{N}$ has notably to be $Y-$Fréchet differentiable at ξ^e and the continuous dependence of $(\hat{S}(t))_{t\geq 0}$ on the initial condition $\hat{\xi}_0$ has to hold by using $Y-$norms.

All the facts established so far lead us to the following theorem that allows to make the connection between (exp.) stability or instability of $(\overline{T}(t))_{t\geq 0}$ and local (exp.) stability or instability of $(\hat{S}(t))_{t\geq 0}$. Here we mean "local" in the sense of Definition 5. The proof of the following theorem goes along the lines of (Al Jamal et al., 2014, Theorem 3.3), wherein the concepts have been adapted and re–worked to fit our specific framework, see (Hastir et al., 2019b, Theorem 3.1).

Theorem 13. Let Assumptions 1, 2, 7, 8 and 10 hold. If 0 is a globally exponentially stable equilibrium of the linearized system (6), then it is a $(Y, X)-$locally exponentially stable equilibrium of (2). Conversely, if 0 is a $(Y, X)-$unstable equilibrium of (6), it is $(Y, X)-$locally unstable for the nonlinear system (2).

Proof. Let us choose $\hat{\xi}_0 \in D(A)\cap\mathcal{D}^e$. First, observe that, by Lemma 12, $\hat{S}(t)$ is $(Y, X)-$Fréchet differentiable at 0, i.e. $\hat{S}(t)\hat{\xi}_0 = \overline{T}(t)\hat{\xi}_0 + r(\xi^e, \hat{\xi}_0)$, where
$$\lim_{\|\hat{\xi}_0\|_Y\to 0}\frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} = 0. \tag{8}$$
According to Remark 11 3, the norms $|||\cdot|||$ and $\|\cdot\|_Y$ are locally equivalent. Hence it holds that
$$\lim_{|||\hat{\xi}_0|||\to 0}\frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} = 0.$$
That is, for any $t > 0$ and $\epsilon > 0$, there exists $\delta(t, \epsilon) > 0$ such that, if $|||\hat{\xi}_0||| < \delta(t, \epsilon)$,
$$\frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} < \epsilon.$$
By the strong continuity in t of the semigroups $(\hat{S}(t))_{t\geq 0}$ and $(\overline{T}(t))_{t\geq 0}$, the function $r(\xi^e, \hat{\xi}_0)$ is also continuous in

t. Since 0 is a globally exponentially stable equilibrium of (6), there exist $\alpha \geq 1$ and $\beta > 0$ such that for all $\hat{\xi}_0 \in D(A) \cap \mathcal{D}^e$

$$\|\overline{T}(t)\hat{\xi}_0\|_X \leq \alpha e^{-\beta t}\|\hat{\xi}_0\|_X,\ t \geq 0. \quad (9)$$

Hence there exist $\epsilon > 0$ and $t_0 < +\infty$ such that, for $\tau \in [0, t_0]$,

$$\begin{aligned} \|\hat{S}(\tau)\hat{\xi}_0\|_X &\leq \|\overline{T}(t)\hat{\xi}_0\|_X + \|r(\xi^e, \hat{\xi}_0)\|_X \\ &\leq \alpha e^{-\beta\tau}\|\hat{\xi}_0\|_X + \epsilon\|\hat{\xi}_0\|_X \leq C\|\hat{\xi}_0\|_X, \end{aligned} \quad (10)$$

where $C = \alpha + \epsilon$. Let us choose $t_0 = \frac{\ln(4\alpha)}{\beta} > 0$. Writing (9) with t replaced by t_0 gives

$$\|\overline{T}(t_0)\hat{\xi}_0\|_X \leq \frac{1}{4}\|\hat{\xi}_0\|_X.$$

In addition,

$$\lim_{|||\hat{\xi}_0||| \to 0} \frac{\|\hat{S}(t_0)\hat{\xi}_0 - \overline{T}(t_0)\hat{\xi}_0\|_X}{\|\hat{\xi}_0\|_X} = 0,$$

that is, there exists $\delta > 0$ such that, if $|||\hat{\xi}_0||| < \delta$, then

$$\|\hat{S}(t_0)\hat{\xi}_0 - \overline{T}(t_0)\hat{\xi}_0\|_X \leq \frac{1}{4}\|\hat{\xi}_0\|_X.$$

Hence,

$$\begin{aligned} \|\hat{S}(t_0)\hat{\xi}_0\|_X &= \|\hat{S}(t_0)\hat{\xi}_0 - \overline{T}(t_0)\hat{\xi}_0 + \overline{T}(t_0)\hat{\xi}_0\|_X \\ &\leq \|\hat{S}(t_0)\hat{\xi}_0 - \overline{T}(t_0)\hat{\xi}_0\|_X + \|\overline{T}(t_0)\hat{\xi}_0\|_X \\ &\leq \frac{1}{2}\|\hat{\xi}_0\|_X = e^{-\ln 2}\|\hat{\xi}_0\|_X. \end{aligned}$$

Let $k > 0$ be an integer. By using the semigroup property and the fact that $\hat{S}^k(t_0)$ maps $D(A) \cap \mathcal{D}^e$ into $D(A) \cap \mathcal{D}^e$ for every $k \in \mathbb{N}$, for every $t_0 \geq 0$, one gets

$$\|\hat{S}(kt_0)\hat{\xi}_0\|_X = \|\hat{S}^k(t_0)\hat{\xi}_0\|_X \leq e^{-(\ln 2)k}\|\hat{\xi}_0\|_X, \quad (11)$$

where we have been using recursively the fact that if $|||\hat{\xi}_0||| < \delta$, then $|||S(t_0)\hat{\xi}_0||| < \delta$. For $t > 0$, let[3] $k = \lfloor \frac{t}{t_0} \rfloor$ and $\tau = t - kt_0 \in [0, t_0]$. By using the semigroup property, (10) and (11),

$$\|\hat{S}(t)\hat{\xi}_0\|_X \leq C\|\hat{S}(kt_0)\hat{\xi}_0\|_X \leq Ce^{-\gamma t}\|\hat{\xi}_0\|_X$$

for $\gamma \leq \frac{\ln 2}{t_0}$. This implies that 0 is a $(Y, X)-$locally exponentially stable equilibrium for (2).

In order to prove the second part of the theorem, let 0 be a $(Y, X)-$locally stable equilibrium to the nonlinear system (2). One has

$$\hat{S}(t)\hat{\xi}_0 = \overline{T}(t)\hat{\xi}_0 + r(\xi^e, \hat{\xi}_0). \quad (12)$$

Since 0 is $(Y, X)-$locally stable, it follows that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\hat{\xi}_0\|_Y < \delta$, then

$$\|\hat{S}(t)\hat{\xi}_0\|_X \leq \frac{\epsilon}{2},$$

for all $t \geq 0$. Since $\|\hat{\xi}_0\|_X \leq \|\hat{\xi}_0\|_Y$, it follows from (8) that

$$\lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_Y} \leq \lim_{\|\hat{\xi}_0\|_Y \to 0} \frac{\|r(\xi^e, \hat{\xi}_0)\|_X}{\|\hat{\xi}_0\|_X} = 0.$$

Hence, $\|r(\xi^e, \hat{\xi}_0)\|_X$ has to converge to 0 when so does $\|\hat{\xi}_0\|_Y$. Consequently, there exists δ^* with $0 < \delta^* < \delta$ such that, if $\|\hat{\xi}_0\|_Y < \delta^*$, then $\|r(\xi^e, \hat{\xi}_0)\|_X \leq \frac{\epsilon}{2}$. Since $\|\hat{\xi}_0\|_Y < \delta^* < \delta$, it follows from (12) and from the last inequality that

$$\|\overline{T}(t)\hat{\xi}_0\|_X \leq \|r(\xi^e, \hat{\xi}_0)\|_X + \|\hat{S}(t)\hat{\xi}_0\|_X \leq \epsilon. \qquad \square$$

[3] The symbol $\lfloor \cdot \rfloor$ is used to denote the integer part of a real number.

Remark 14. As introduced with Definition 5 the main difference in our approach is that $\|\hat{\xi}_0\|_Y$ is assumed to converge to 0 instead of $\|\hat{\xi}_0\|_X$, because of the $(Y, X)-$Fréchet differentiability of the nonlinear semigroup. Note that this leads to additional technical difficulties, notably when we need to apply successively the property that $\|\hat{S}(t_0)\hat{\xi}_0\|_X \leq e^{-\ln 2}\|\hat{\xi}_0\|_X$ whenever $\|\hat{\xi}_0\|_Y < \delta$, on higher composition orders of the nonlinear semigroup, i.e. on $\hat{S}^k(t_0)\hat{\xi}_0, k \in \mathbb{N}, k > 1$. This is possible due to Assumptions 7 and 10 that allow to use a locally equivalent norm to $\|\cdot\|_Y$, namely $|||\cdot|||$. For instance this technical detail is not needed in (Al Jamal et al., 2014) because $\|\hat{S}(t_0)\hat{\xi}_0\|_X \leq e^{-\ln 2}\|\hat{\xi}_0\|_X$ implies automatically that $\|\hat{S}(t_0)\hat{\xi}_0\|_X \leq \delta$ whenever $\|\hat{\xi}_0\|_X \leq \delta$.

Note that one of the most important features of the analysis is the choice of the auxiliary space Y. It has e.g. to be chosen in order to avoid limitations in the manipulations of norm-inequalities. Good choices are in general L^∞ or Sobolev spaces ($H^p, p \in \mathbb{N}_0$) which are all multiplicative algebras. Hence, they allow for example to split the norm of the product of two functions into the product of the norms, which is not permitted in L^p-spaces, $1 \leq p < \infty$, in which Hölder inequality has to be applied. To give intuition about the auxiliary space Y it may be noted that the condition $D(\mathcal{A}) \cap \mathcal{D}^e \subset Y \subseteq X$ is quite natural depending on the application. Indeed, the choice of Y is induced by the domain of the linear operator $\mathcal{A}$ which takes more regularity into account.

Hereafter are some guidelines and some intuition in order to apply the new method to a specific nonlinear distributed parameter systems whose (exp.) stability or instability has to be analyzed. The method is summarized in Figure 1. The objective is to deduce exponential stability or instability of an equilibrium profile for a nonlinear distributed parameter system, where the state space is called X. First, a Gâteaux linearized version of the nonlinear system is built and its exponential stability is studied. Then, after the choice of the auxiliary space Y, the nonlinear semigroup is proved to be $Y-$Fréchet differentiable and its linearization has to be "Lyapunov stable" on Y. The new concept of $(Y, X)-$Fréchet differentiability plays its role now to make the connection between Y and X to deduce exponential stability or instability of the equilibrium for the nonlinear system (by using $X-$norms). What is specific here is that local means that the $Y-$norm of the initial condition is small instead of its $X-$norm.

4. APPLICATION

In this section, the previous results are applied to a particular nonlinear convection–diffusion–reaction model, which is governed by the following partial differential equation (PDE):

$$\begin{cases} \dfrac{\partial x}{\partial t} = \dfrac{1}{Pe}\dfrac{\partial^2 x}{\partial z^2} - \dfrac{\partial x}{\partial z} + \alpha(\delta - x)e^{\frac{-\mu}{1+x}}, \\ \dfrac{\partial x}{\partial z}(0) = Pe\, x(0), \dfrac{\partial x}{\partial z}(1) = 0. \end{cases} \quad (13)$$

The state variable $x(z, t)$ represents the temperature in a nonisothermal axial dispersion tubular reactor, at time

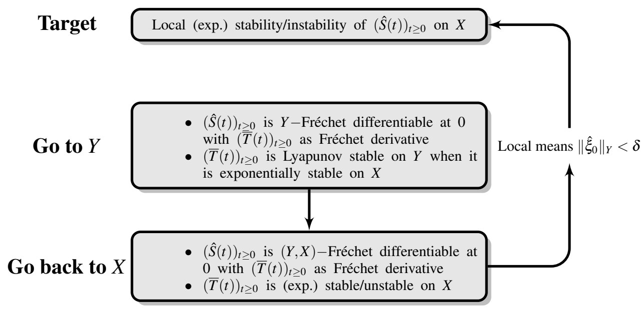

Fig. 1. Guidelines in order to apply the new method.

$t \geq 0$ and at position $z \in [0,1]$. It is supposed to evolve in the state space $X := L^2(0,1)$ and it is constrained to remain above -1 (this is imposed by the physical constraints of the problem). The number Pe is called the thermal Peclet number which is equal to v/D where v is the superficial velocity of the fluid inside the reactor and D is the diffusion coefficient. The constants α, δ and μ depend on the parameters of the system, see e.g. (Dochain, 2016; Hastir et al., 2019a). The nonlinear function $\alpha(\delta - x)e^{-\mu/1+x}$ in the dynamics models the variation of the reaction rate as a function of the temperature and is due to the Arrhenius law, see e.g. (Aksikas et al., 2007).

System (13) can be described by using the abstract formulation (1) where the (unbounded) linear operator $\mathcal{A}$ is defined by $\mathcal{A}x = \frac{1}{Pe}\frac{d^2x}{dz^2} - \frac{dx}{dz}$ for $x \in D(\mathcal{A}) := \left\{x \in H^2(0,1), \frac{dx}{dz}(0) = Pe\, x(0), \frac{dx}{dz}(1) = 0\right\}$. The nonlinear operator $\mathcal{N}$ is given by $\mathcal{N}(x) = \alpha(\delta - x)e^{-\mu/(1+x)}$ where $x \in \mathcal{D} := \left\{x \in X, -1 \leq x, 0 \leq \frac{1}{\delta}(x - \chi) \leq 1 \text{ a.e. on}[0,1]\right\}$, where χ is called the asymptotic reaction invariant and is subject to the PDE

$$\begin{cases} \dfrac{\partial \chi}{\partial t} = \dfrac{1}{Pe}\dfrac{\partial^2 \chi}{\partial z^2} - \dfrac{\partial \chi}{\partial z}, \\ \dfrac{\partial \chi}{\partial z}(0) = Pe\, \chi(0), \dfrac{\partial \chi}{\partial z}(1) = 0. \end{cases}$$

From (Laabissi et al., 2001) system (13) is known to be well-posed. Recent researches concentrate on the analysis and the existence of equilibria of (13), see e.g. (Hastir et al., 2019a). In particular, in (Hastir et al., 2019a), it is shown that systems like (13) exhibit different numbers of equilibria, depending on the parameters of the model, especially the diffusion coefficient. For some specific values of the parameters μ and δ and above a sufficiently high value of D, system (13) switches from one to three equilibria, see (Hastir et al., 2019a, Section IV); let us denote by x^e any equilibrium of (13). Many years ago, asymptotic stability of the equilibria of (13) has been studied in (Varma and Aris, 1977). More recently, the question of asymptotic stability of the equilibria has been discussed in (Dochain, 2016). Exponential stability of the equilibria of (13) for a Gâteaux linearization of it has been studied for the first time in (Hastir et al., 2019c) where linear bistability is proved. That is, in the case of only one equilibrium profile, the latter has been shown to be exponentially stable for a Gâteaux linearized model corresponding to (13) and in the case where the reactor exhibits three equilibria, the pattern "stable–unstable–stable" is highlighted (for the linearized model). In the same reference it is also shown that the nonlinear operator $\mathcal{N}$ is Gâteaux differentiable at the equilibrium x^e and its Gâteaux derivative is given by the bounded linear operator $d\mathcal{N}(x^e) : X \to X$,

$$d\mathcal{N}(x^e)h := \left(\alpha \frac{\mu\,(\delta - x^e)}{(1+x^e)^2} e^{\frac{-\mu}{1+x^e}} - \alpha e^{\frac{-\mu}{1+x^e}}\right) h.$$

Also in (Hastir et al., 2019c) limitations in the analysis of the stability of the equilibria are discussed. In particular, the nonlinear operator that models the Arrhenius law has been shown to be not Fréchet differentiable. Hence, the standard approaches of (Kato, 1995) or (Al Jamal and Morris, 2018) about nonlinear exponential stability could not be used in this context. Here comes the new method that is presented above, see (Hastir et al., 2019b, Section 4). In this study the auxiliary space Y has been chosen as $C(0,1)$ equipped with the $L^\infty(0,1)-$norm. In (Hastir et al., 2019b), it is shown that, with this smaller space, the nonlinear operator in the dynamics of (13) is $(Y,X)-$Fréchet differentiable and also $Y-$Fréchet differentiable. This together with continuous dependence of the solutions of (13) on the initial conditions on X and on Y lead us to the following result, see (Hastir et al., 2019b, Section 3).

Lemma 15. Let us denote by $(S(t))_{t\geq 0}$ the nonlinear semigroup generated by the operator $\mathcal{A}+\mathcal{N}$. Moreover, let the linear operator $\mathcal{A} + d\mathcal{N}(x^e)$ be the infinitesimal generator of the semigroup $(\overline{T}(t))_{t\geq 0}$. By choosing $X := L^2(0,1)$ and $Y := C(0,1)$ equipped with the $L^\infty(0,1)-$norm, the nonlinear semigroup $(S(t))_{t\geq 0}$ is (Y,X) and $Y-$Fréchet differentiable at x^e with $(\overline{T}(t))_{t\geq 0}$ as Fréchet derivative.

Proof. This is a direct consequence of (Hastir et al., 2019b, Lemmas 3.1 and 4.4). □

This allows to consider the following theorem that concludes on the nonlinear stability properties of the equilibria of (13), see (Hastir et al., 2019b, Theorem 4.1).

Theorem 16. Consider the nonlinear PDE (13) that describes the time evolution of the temperature in a non-isothermal axial dispersion tubular reactor. In the case where the reactor exhibits one equilibrium profile, the latter is $(C(0,1), L^2(0,1))-$locally exponentially stable for the nonlinear system (13). In the case of three equilibria the pattern $(C(0,1), L^2(0,1))-$"locally exponentially stable - locally unstable - locally exponentially stable" holds, which is called bistability.

5. CONCLUSION/PERSPECTIVES

In this paper, an extended concept of Fréchet differentiability, based on an auxiliary space for nonlinear operator defined on infinite-dimensional spaces, was proposed. This is in general more easily manipulable and allows more freedom in the handling of norm-inequalities since the alternative space can be chosen depending on the considered problem. This new concept was shown as being the keypoint in showing more easily adapted Fréchet differentiability conditions for nonlinear semigroups. On the basis of the stability properties of a corresponding linearized semigroup, appropriate local (exp.) stability or instability of the original nonlinear semigroup was deduced.

Perspectives aim at considering linear feedback stabilization of systems like (13). Indeed, since the nonlinear part of (13) does not change by adding bounded linear feedback and since continuous dependence of the nonlinear semigroup on the initial conditions is preserved as long as bounded feedbacks are considered, adapted Fréchet differentiability of the "closed–loop" semigroup is guaranteed [4], see Lemmas 9 and 12. A direct consequence is that any exponentially stabilizing state feedback for the linearized dynamics remains locally (exp.) stabilizing for the nonlinear system. This allows to design a stabilizing feedback on a linear system instead of a nonlinear one, in order to apply it to the latter.

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[4] By "closed–loop" semigroup, we mean the semigroup generated by the closed–loop dynamics.