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On Stability Conditions for Filippov and Hamiltonian Systems

Sobre Condições de Estabilidade para Sistemas de Filippov e Sistemas Hamiltonianos

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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“ There’s only so much you can learn in one place,
The more that I wait, the more time that I waste

I haven’t got much time to waste, it’s time to make my way
I’m not afraid what I’ll face, but I’m afraid to stay
I’m going down my own road and I can make it alone
I’ll work and I’ll fight till I find a place of my own

Are you ready to jump?
Get ready to jump
Don’t ever look back, oh baby,
Yes, I’m ready to jump
Just take my hands
Get ready to jump

We learned our lesson from the start, my sisters and me
The only thing you can depend on is your family
And life’s gonna drop you down like the limbs of a tree
It sways and it swings and it bends until it makes you see ...”

(Madonna, *Jump*)

Resumo

Neste trabalho, abordamos aspectos qualitativos de vários fenômenos em sistemas de Filippov e em sistemas Hamiltonianos. No contexto de sistemas dinâmicos suaves por partes, concentramos nossa atenção em problemas em dimensões 2 e 3. No caso planar, desenvolvemos um mecanismo para analisar o desdobramento de policiclos que passam por certas singularidades de sistemas de Filippov (conhecidas como Σ-singularidades) em uma configuração típica, e o utilizamos para descrever completamente o diagrama de bifurcação de sistemas de Filippov em torno de alguns policiclos elementares. No caso tridimensional, obtivemos uma caracterização completa dos sistemas que são localmente estruturalmente estáveis em um ponto p da variedade de descontinuidade. Mais ainda, caracterizamos completamente os sistemas de Filippov robustos em uma vizinhança da variedade de descontinuidade, os quais são chamados de sistemas semi-localmente estruturalmente estáveis. Além disso, estudamos alguns fenômenos globais em sistemas de Filippov $3D$. Primeiramente, descrevemos o diagrama de bifurcação de um sistema em torno de um laço ("loop") do tipo homoclínico de codimensão um em uma singularidade genérica denominada dobra-regular, o qual não possui contrapartida no contexto suave. Em seguida, analisamos uma classe de sistemas que apresenta conexões robustas entre certas singularidades típicas, conhecidas como T-singularidades, as quais garantiram a existência de um comportamento caótico nas folheações associadas a tais sistemas de Filippov.

Em relação aos sistemas Hamiltonianos, estudamos alguns problemas que apresentam fenômenos exponencialmente pequenos. Mais especificamente, consideramos um modelo de interação kink-defect dado por um Hamiltoniano singularmente perturbado H_ε ($\varepsilon \geq 0$ representa o parâmetro perturbativo) com dois graus de liberdade, e determinamos condições sobre a energia do sistema para a existência de certas conexões heteroclínicas que surgem da quebra ($\varepsilon > 0$) de uma órbita heteroclínica contida no nível de energia zero do sistema limite H_0. Finalmente, investigamos a existência de soluções breather de equações diferenciais parciais reversíveis do tipo Klein-Gordon, as quais podem ser vistas como órbitas homoclínicas de um sistema Hamiltoniano de dimensão infinita.

Palavras-chave:

Sistemas de Filippov, Teoria da Bifurcação, Estabilidade Estrutural, Policiclos, Fenômenos Exponencialmente Pequenos

Abstract

In this work, we discussed qualitative aspects of several phenomena in Filippov and Hamiltonian systems. In the context of piecewise smooth dynamical systems, we have focused on problems in dimensions 2 and 3. In the planar case, we have provided a mechanism to analyze the unfolding of polycycles passing through certain singularities of Filippov systems (known as Σ-singularities) in a typical scenario and we have used it to completely describe the bifurcation diagram of Filippov systems around some elementary polycycles. In the three-dimensional case, we have obtained a complete characterization of the systems which are locally structurally stable at a point p in the switching manifold Σ. Moreover, we have completely characterized the Filippov systems which are robust in a neighborhood of the whole switching manifold, named semi-local structurally stable systems. In addition, we have studied some global phenomena in $3D$ Filippov systems. First we described the bifurcation diagram of a system around a codimension one homoclinic-like loop at a generic singularity named fold-regular singularity, which has no counterpart in the smooth context. Second, we analyzed a class of systems presenting robust connections between certain typical singularities, known as T-singularities, which have lead us to the existence of a chaotic behavior in the foliations associated to such Filippov systems.

Concerning to Hamiltonian Systems, we have studied some problems exhibiting exponentially small phenomena. More specifically, we considered a model of kink-defect interaction given by a singularly perturbed 2-dof Hamiltonian H_ε ($\varepsilon \geq 0$ stands for the perturbation parameter) and we have provided conditions on the energy of the system for the existence of certain heteroclinic connections arising from the breakdown ($\varepsilon > 0$) of a heteroclinic orbit lying in the zero energy level of the limit system H_0. Finally, we have investigated the existence of breathers of reversible Klein-Gordon partial differential equations, which can be seen as homoclinic orbits of an infinite-dimensional Hamiltonian system.

Keywords: Filippov Systems, Bifurcation Theory, Structural Stability, Polycycles, Exponentially Small Phenomena

Contents

Introduction

THIS work is devoted to the study of phenomena in Bifurcation Theory of Dynamical Systems. It is mainly divided into two parts. The first one is concerned about structural stability and generic bifurcation of Filippov systems in dimensions 2 and 3. The second one is dedicated to the study of problems of exponentially small splitting of separatrices in analytic Hamiltonian systems.

Piecewise Smooth Dynamical Systems

Discontinuities appear commonly in an extensive range of natural phenomena, as body collisions and systems having on/off switches (see [18, 32, 60] for more examples). In light of this, Mathematicians and Physicists have pursued ways to understand such intriguing aspects.

In the attempt to provide a mathematical description of the nonsmoothness inherent to the real world, the Piecewise Smooth Vector Fields (PSVF for short) have arisen. In fact, the Theory of Dynamical Systems have been essentially used to explain phenomena through differential equations, and thus, it seems reasonable to consider piecewise smooth differential equations to deal with such discontinuities.

Generally speaking, a PSVF is a system defined by smooth relations with different nature in some regions of the phase space. The separation set between these regions is referred as the *switching set* associated to the PSVF. In this case, a PSVF is multi-valued on the switching set, since we have two (or more) different rules governing the dynamics at these points. Typically, the switching set is a codimension one manifold, and for such a reason, it is also referred as *switching manifold*.

The Theory of Piecewise Smooth Dynamical Systems started to gain strength with the works of A. F. Filippov, which have provided the existence of solutions of PSVF through the method of differential inclusions (see [39]). Nevertheless, such an approach allows a PSVF to present several different solutions in the switching set. Therefore, a question was raised in the community:

What solution should be considered in the switching set?

Such a non-determinism of solutions has been extensively discussed over the years and it still remains without a final conclusion. Although, certain conventions of solutions have been highlighted due to their applicability to model real phenomena. Among all of them, we mention Utkin's convention and Filippov's convention (see [15, 55] an references therein).

Throughout the last decades, PSVF governed by Filippov's convention (known as Filippov systems) have been the most considered systems to model discontinuous phenomena and [39] seems to be unanimously accepted as an important contribution to the Theory of Dynamical Systems. Based on that, we were encouraged to develop a well established mathematical framework to deal with this kind of system.

In light of this discussion, we have contributed to the Theory of Piecewise Smooth Vector Fields by studying local and global aspects of structural stability and bifurcations of Filippov systems (in dimensions 2 and 3) having a codimension one switching manifold. We highlight that Structural Stability and Bifurcations of Smooth Vector Fields are matured topics which play a crucial role in the knowledge of the dynamics of a smooth system, and their counterparts in the nonsmooth context bring several complications which give rise to appealing problems.

Analytic Hamiltonian Systems

Since 1833, Hamiltonian systems are used to describe equations of motion of conservative mechanical systems, and thus they are frequently employed to study physical phenomena (see [76]). Special attention must be given to their applications in Celestial Mechanics which have been very fruitful over the years. The versatility of this class is one of the reasons which allows us to classify Hamiltonian systems as one of the most important research topics in Dynamical Systems. In addition, they exhibit a rich dynamics which may involve global instabilities, deterministic chaos and Arnold diffusion orbits (see [29] and references therein).

As a result of years of work, nowadays, the mathematical community has an extensive knowledge on Hamiltonian structures, nevertheless there are still many questions which remain open. Among the massive range of topics investigated on Hamiltonian system, the study of homoclinic and heteroclinic connections in this class is a classical problem which has been treated by many researchers. More specifically, one is interested to know what happens with a homoclinic/heteroclinic connection in nearly-integrable Hamiltonian systems.

In the study of the splitting of separatrices for regularly perturbed systems, Poincaré and later Melnikov (see [56, 75]) developed a general method which measures the distance between the invariant manifolds of hyperbolic critical points or periodic orbits. This method has been extended for general normally hyperbolic manifolds in [29].

However, in the case of rapidly forced systems and in singularly perturbed systems which are degenerate when the parameter vanishes, a difficult problem arises due to the fact that the Melnikov function depends on the perturbed parameter and, in fact, it turns out to be exponentially small with respect to this parameter. In [90], Henri Poincaré has considered the problem of exponentially small splitting of separatrices as the Fundamental Problem of Mechanics, nevertheless, there was a lack of rigorousness in most works regarding this topic until the end of the 80's and the beginning of the 90's.

Later on, rigorous approaches came out revealing the necessity of sophisticated techniques to obtain correct asymptotic formulas for the exponentially small splitting of separatrices as the complex parameterization of invariant manifolds, matching in the complex plane, Singular Perturbation Theory and Resurgence Theory(see [41, 52, 54, 80, 92] and references therein). Such phenomena are also known as *beyond all orders problems*. In fact, the breakdown of separatrices in the presence of exponentially small phenomena can not be seen for any truncated expansion of the system. In [8], one finds a detailed historical description of the mechanisms which were employed to succeed on asymptotics beyond all orders.

We emphasize that Melnikov Theory does not apply to problems of exponentially small splitting of separatrices. Nevertheless, in some specific cases, Melnikov function still is a first order for the distance between the invariant manifolds. On the other hand, there are also cases where Melnikov function is "too small" and does not give a first order for the formula of splitting, and in such cases, the first order can be obtained by the study of the so-called *inner equation* associated with the problem. See [8] for more details.

In this work, we analyze two singularly perturbed problems exhibiting exponentially small phenomena. More specifically, the first one concerns about a 2-degrees of freedom analytic Hamiltonian system for which we study the existence of certain heteroclinic connections, and the second one consists on the computation of an asymptotic formula for the splitting of separatrices of a infinite dimensional Hamiltonian system. It is worth mentioning that Melnikov function is a first order approximation for the splitting of separatrices in the first problem, nevertheless it does not work as a first order for the second one.

An Overall Description of the Main Results

In what follows, we roughly describe the problems treated in each chapter of this work, as well as the main results achieved.

First, we notice that **Chapter 1** is devoted to establish some basic concepts which are required to the reading of Chapters 2, 3, 4, 5, and 6. Apart of that, each chapter of this thesis is self-contained and can be read independently from the others. Also, except when explicitly mentioned, the notation assigned in each chapter does not apply to the remaining ones. It is worth saying that, even with our efforts to take into account the terminology previously used in the literature, we had to introduce several new concepts and notations in order to provide a rigorous treatment for this work.

Chapter 2: Polycycles of Planar Filippov Systems

Local bifurcations of planar Filippov systems at singularities contained in the switching manifold (Σ-singularities) have been extensively studied in the last years (see [55, 65] and references therein). In fact, local bifurcations of codimensions 0 and 1 are completely understood, and thus the interest on cycles (loops) passing through such singularities (which will be referred as **Σ-polycycles**) has recently grown (see [4, 40, 79]).

In light of the current works, we have observed the lack of a mechanism to deal with such global phenomena. In order to fill the absence of an approach to this problem, we developed a scheme to study the crossing orbits of Filippov systems $Z = (X, Y)$ around Σ-polycycles, which we called *Method of Displacement Functions*.

In **Theorems A** and **B**, we have provided some tools which can be used to characterize aspects of the mentioned methodology. It is worth mentioning that such results were heavily based on the analysis of the contact of the smooth components X and Y of a Filippov system $Z = (X, Y)$ with the switching manifold Σ.

The effectiveness of this approach has been shown in its application to obtain the complete description of bifurcation diagrams of Filippov systems around certain Σ-polycycles, which are elementary in some sense. In fact, **Theorems C, D** and **E** are devoted to describe the bifurcation diagrams of Filippov systems around three distinct codimension-two Σ-polycycles displayed in Figure 1.

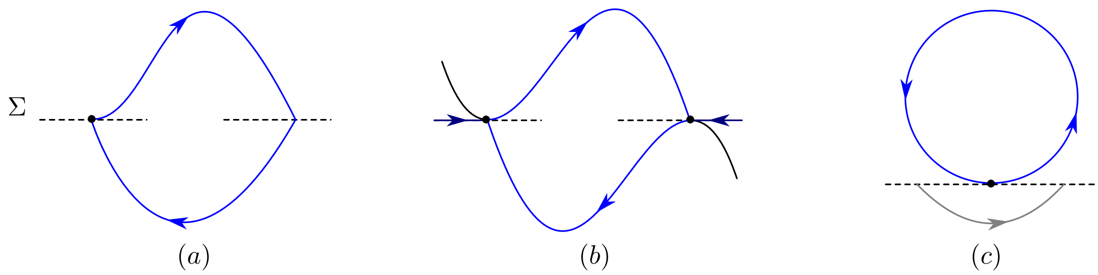


Figure 1: Σ-polycycles passing through: (a) a cusp-regular singularity, (b) two fold-regular singularities and (c) a visible-invisible fold-fold singularity.

It is worthwhile to mention that this chapter is based on [3].

Chapter 3: Generic Singularities of $3D$ Filippov Systems

On the contrary of the planar case, the understanding of the local structure of generic Σ-singularities of $3D$ Filippov systems has shown to be a challenging problem which has been considered by many researchers throughout the years. In particular, there are many works regarding the local structural stability (or instability) of a Filippov systems at a T-singularity, nevertheless the problem was still open (see [24, 25, 26, 38]).

In light of this, we offered a rigorous mathematical treatment of this problem. In **Theorem F**, we provide intrinsic conditions in Filippov systems which completely characterize the local structurally stability at a T-singularity.

It is worth mentioning that, the proof of Theorem F relied on the existence of the so-called *nonsmooth diabolo* at certain types of T-singularity. Such an object has already been studied for semi-linear Filippov systems (see [60] and references therein), nevertheless its existence was still not clear for general systems (without neglecting higher order terms). See Figure 2.

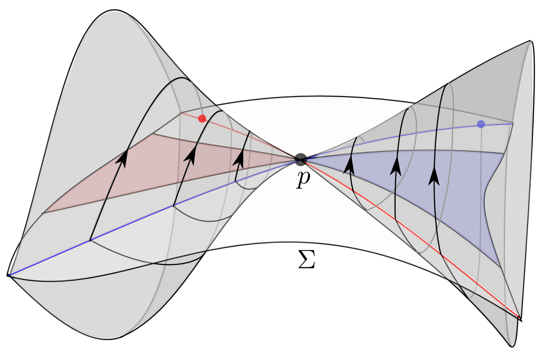


Figure 2: A nonsmooth diabolo at a T-singularity p.

In **Theorem G** we review the local structural stability of Filippov systems at the remaining flavors of fold-fold singularities (hyperbolic and parabolic). We emphasize that such a result was already known as a consequence of works [24, 25], notwithstanding we provided a new proof in this setting for completeness.

Finally, **Theorems H** and **I** have completely characterized the locally structurally stable systems and have shown that local structural stability is an open non-generic property of Filippov systems.

It is worthwhile to mention that this chapter is based on [44].

Chapter 4: Semi-Local Structural Stability

Taking into account the results on the local structural stability of Filippov systems at generic Σ-singularities, the next step towards the characterization of $3D$ structurally stable Filippov systems (from a global point of view) is to conceive a good description of the Filippov systems which are robust around the entire switching manifold (not only point-wisely).

In light of this, we have introduced two notions of topological equivalence on the space of all Filippov systems having a compact, connected and simply-connected switching manifold Σ (e.g. $\Sigma = \mathbb{S}^2$), denoted by Ω^r: the *sliding topological equivalence* and the *semi-local equivalence*. The concepts of *sliding structural stability* and *semi-local structural stability* were defined in the natural way.

Roughly speaking, the sliding topological equivalence identifies all elements of Ω^r with the same sliding dynamics (in the unstable and stable sliding regions) and the semi-local equivalence (at Σ) identifies Filippov systems having the same behavior in a small $3D$ neighborhood of the entire switching manifold Σ. The semi-local equivalence regards all orbits lying in an open set of $\mathbb{R}^3$ containing Σ, whereas the sliding equivalence concerns only with the features lying in Σ (2-dimensional).

In **Theorem J**, we provide a sliding version of the classical Peixoto's Theorem which has completely characterized the sliding structurally stable Filippov systems, and in **Theorem K**, a complete characterization of the semi-local structural stability in Ω^r is established. As a consequence, we have obtained that sliding structural stability is a generic property in Ω^r, nevertheless, the semi-local structural stability is an open non-generic property in Ω^r.

Also, in order to study structural stability from the semi-local point of view, we have provided an approach, called Σ-*blocks mechanism*, based on the definition of the isolating blocks introduced by Conley (see [27]), which can be used to analyze analogous problems in higher dimensions.

It is worthwhile to mention that this chapter is based on [45].

Chapter 5: Quasi-Generic Loops in $3D$ Filippov Systems

Aiming to contribute to the development of Global Theory in three-dimensional Filippov systems, we devoted a chapter of this thesis to the study of an elementary $3D$ homoclinic-like connection. More specifically, we have considered $3D$ Filippov systems $Z = (X, Y)$ having a loop Γ passing through a fold-regular singularity p (X has a quadratic contact with Σ at p and Y is transverse to Σ at p, or vice-versa). See Figure 3.

In **Theorem L**, we show that, under some generic conditions, homoclinic-like loops passing through a fold-regular singularity are generic in one-parameter families, and in **Theorem M**, we describe the versal unfolding of some classes of such a global connection (see Figure 4).

In **Theorem N**, we provide a characterization of the basin of attraction of the considered loop, based on the study of the sliding dynamics and the first return map of Z.

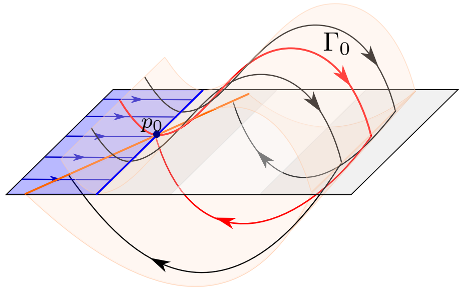


Figure 3: A homoclinic-like loop Γ_0 of Z_0 at a fold-regular singularity p_0.

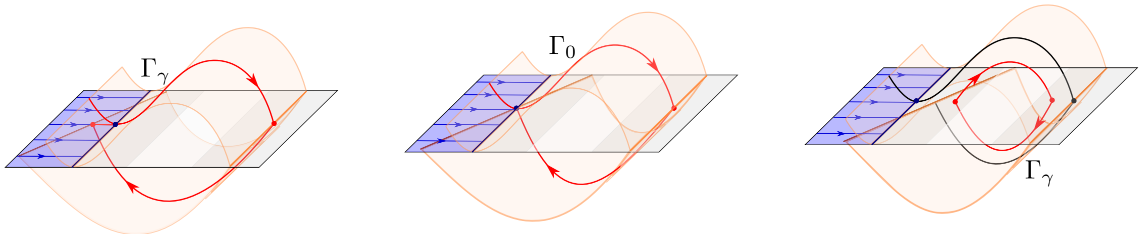


Figure 4: Versal unfolding of a homoclinic-like loop at a fold-regular singularity.

Aspects of modulus of stability inside this class were also discussed in **Theorem O** (for certain equivalence relation).

It is worthwhile to mention that this chapter is based on [46].

Chapter 6: T-Chains - A Chaotic $3D$ Foliation

Still with the purpose of a global understanding of nonsmooth phenomena in dimension 3, we have presented a robust global connection arising in $3D$ Filippov systems $Z = (X, Y)$, which brings a chaotic behavior in the foliation generated by the orbits of X and Y.

More specifically, we have considered that Z has a diabolo at a T-singularity p filled by crossing orbits, and we imposed some generic global assumptions in order to establish a communication between the two branches of such a diabolo (stable and unstable). In this scenario, Z has a robust homoclinic-like connection at p (see Figure 5).

In **Theorem P**, we have proved that the first return map associated with the foliation generated by Z has a Smale horseshoe, which induces chaos on the crossing orbits and pseudo-orbits (concatenation of orbits of X and Y) of Z.

Chapter 7: Critical Velocity in Kink-Defect Interaction Models

In [47], the authors have studied a toy-model which describes the interaction of kinks (solitons) of the sine-Gordon equation with a weak defect. More specifically, they considered a finite-dimensional reduction of the partial differential equation, which is given by a 2-degrees of freedom Hamiltonian H_ε, and they derived the so-called critical velocity v_c (or critical energy h_c), for which each solution with velocity greater than v_c is a kink. However, the computations given in [47] are not rigorous.

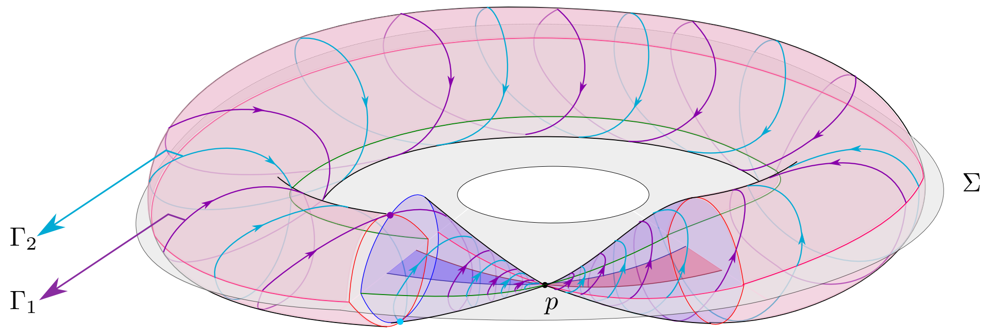


Figure 5: A Filippov system Z_0 satisfying hypotheses (TC) and (R) having two T-chains Γ_1 and Γ_2 passing through q_1 and q_2, respectively.

In light of this, we have presented a rigorous study on the Hamiltonian H_ε. The existence of kinks with small amplitude has been associated with the existence of heteroclinic connections of certain objects (critical points and periodic orbits) at infinity. We have provided a geometric approach to give conditions on the energy of the system to admit kinks. Generally speaking, the employed methods relied on computing the exponentially small transversality of invariant manifolds $W^{u,s}$ of critical points and periodic orbits at infinity.

Theorems Q, R, and **S** are devoted to show that the heteroclinic orbit in the energy level 0 of H_ε (with $\varepsilon = 0$) is destroyed giving rise to heteroclinic connections between certain elements (at infinity) for exponentially small (in ε) energy levels.

Finally, in **Theorem T**, we have obtained an asymptotic expression for the critical energy h_c such that the system admits kinks with small amplitude only for $h \geq h_c$.

It is worthwhile to mention that this chapter is based on [43].

Chapter 8: Breakdown of Breathers for Reversible Klein-Gordon Equations

Breathers are nontrivial time-periodic and spatially localized solutions of a wave equation which were introduced by [1] in the context of the sine-Gordon partial differential equation. Since then, the problem of existence of small breathers for classical partial differential equations has shown to be a hard subject to deal with.

The existence of small amplitude breathers for the Klein-Gordon equations have been considered in several works (see [31, 62, 71, 93, 94] and references therein), nevertheless it still remains as an open problem. In light of this, we considered reversible Klein-Gordon equations

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0,$$

where f is a real-analytic odd function which satisfies $f(u) = \mathcal{O}(u^5)$, and we have associated the existence of time-reversible breathers $u(x,t)$, with the existence of homoclinic orbits (with respect to the variable x) of the equation at the origin (which is a singular point).

From this problem, we derived a singular perturbed Hamiltonian $\mathcal{H}_\varepsilon$, with infinite degrees of freedom, having a homoclinic orbit for $\varepsilon = 0$. In **Theorem U**, we have computed an asymptotic formula for the distance between the invariant manifolds $W^{u,s}$

of $\mathcal{H}_\varepsilon$ in a transversal section, which turns out to be exponentially small with respect to the parameter ε.

Chapter 1

A Prelude on Filippov Systems

THIS chapter is devoted to shortly introduce Piecewise Smooth Vector Fields. More specifically, we present the concept of solution provided by Filippov's convention which will be adopted throughout this thesis. The definitions presented here are required to read Chapters 2, 3, 4, 5, and 6.

1.1 Basic Concepts

Let M be an open bounded connected set of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ be a smooth function having 0 as a regular value. Therefore, $\Sigma = f^{-1}(0)$ is an embedded codimension one submanifold of M which splits it in the sets $M^{\pm} = \{p \in M; \pm f(p) > 0\}$.

A **germ of vector field** of class $\mathcal{C}^r$ at a compact set $\Lambda \subset M$ is an equivalence class $\widetilde{X}$ of $\mathcal{C}^r$ vector fields defined in a neighborhood of Λ. More specifically, two $\mathcal{C}^r$ vector fields X_1 and X_2 are in the same equivalence class if:

- X_1 and X_2 are defined in neighborhoods U_1 and U_2 of Λ in M, respectively;
- There exists a neighborhood U_3 of Λ in M such that $U_3 \subset U_1 \cap U_2$;
- $X_1|_{U_3} = X_2|_{U_3}$.

In this case, if X is an element of the equivalence class $\widetilde{X}$, then X is said to be a representative of $\widetilde{X}$. The set of germs of vector fields of class $\mathcal{C}^r$ at Λ will be denoted by $\chi^r(\Lambda)$, or simply χ^r. For the sake of simplicity, a germ of vector field $\widetilde{X}$ will be referred simply by its representative X.

Analogously, a **germ of piecewise smooth vector field** of class $\mathcal{C}^r$ at a compact set $\Lambda \subset M$ is an equivalence class $\widetilde{Z} = (\widetilde{X}, \widetilde{Y})$ of pairwise $\mathcal{C}^r$ vector fields defined as follows: $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$ are in the same equivalence class if, and only if,

- X_i and Y_i are defined in neighborhoods U_i and V_i of Λ in M, respectively, $i = 1, 2$.
- There exist neighborhoods U_3 and V_3 of Λ in M such that $U_3 \subset U_1 \cap U_2$ and $V_3 \subset V_1 \cap V_2$.
- $X_1|_{U_3 \cap \overline{M^+}} = X_2|_{U_3 \cap \overline{M^+}}$ and $Y_1|_{V_3 \cap \overline{M^-}} = Y_2|_{V_3 \cap \overline{M^-}}$.

In this case, if $Z = (X, Y)$ is an element of the equivalence class $\widetilde{Z}$, then Z is said to be a representative of $\widetilde{Z}$. The set of germs of piecewise smooth vector fields of class $\mathcal{C}^r$ at Λ will be denoted by $\Omega^r(\Lambda)$, or simply Ω^r.

We emphasize that the germ language is used due to its effectiveness to describe local and semi-local phenomena.

If $Z = (X, Y) \in \Omega^r$ then a **piecewise smooth vector field** is defined in some neighborhood V of Λ in M as

$$Z(p) = F_1(p) + \operatorname{sgn}(f(p))F_2(p),$$

where $F_1(p) = \frac{X(p)+Y(p)}{2}$ and $F_2(p) = \frac{X(p)-Y(p)}{2}$.

The **Lie derivative** $Xf(p)$ of f in the direction of the vector field $X \in \chi^r$ at $p \in \Sigma$ is defined as $Xf(p) = \langle X(p), \nabla f(p)\rangle$. Accordingly, the **tangency set** between X and Σ is given by $S_X = \{p \in \Sigma;\ Xf(p) = 0\}$.

Remark 1.1.1. *Notice that the Lie derivative is well-defined for a germ $\widetilde{X} \in \chi^r$ since all the elements in this class coincide in Σ.*

For $X_1, \cdots, X_k \in \chi^r$, the higher order Lie derivatives of f are defined recurrently as

$$X_k \cdots X_1 f(p) = X_k(X_{k-1} \cdots X_1 f)(p),$$

i.e. $X_k \cdots X_1 f(p)$ is the Lie derivative of the smooth function $X_{k-1} \cdots X_1 f$ in the direction of the vector field X_k at p. In particular, $X^k f(p)$ denotes $X_k \cdots X_1 f(p)$, where $X_i = X$, for $i = 1, \cdots, k$.

For a piecewise smooth vector field $Z = (X, Y)$ the switching manifold Σ is generically the closure of the union of the following three distinct open regions:

- Crossing Region: $\Sigma^c(Z) = \{p \in \Sigma;\ Xf(p)Yf(p) > 0\}$.
- Stable Sliding Region: $\Sigma^{ss}(Z) = \{p \in \Sigma;\ Xf(p) < 0,\ Yf(p) > 0\}$.
- Unstable Sliding Region: $\Sigma^{us}(Z) = \{p \in \Sigma;\ Xf(p) > 0,\ Yf(p) < 0\}$.

Remark 1.1.2. *If there is no misunderstanding, the dependence of these regions on Z will be omitted. In addition, Σ can be denoted by $\Sigma(Z)$, in order to distinguish the regions of Σ corresponding to Z, when necessary.*

The tangency set of Z will be referred as $S_Z = S_X \cup S_Y$. Notice that Σ is the disjoint union $\Sigma^c \cup \Sigma^{ss} \cup \Sigma^{us} \cup S_Z$. Herein, $\Sigma^s = \Sigma^{ss} \cup \Sigma^{us}$ is called **sliding region** of Z. See Figure 1.1.

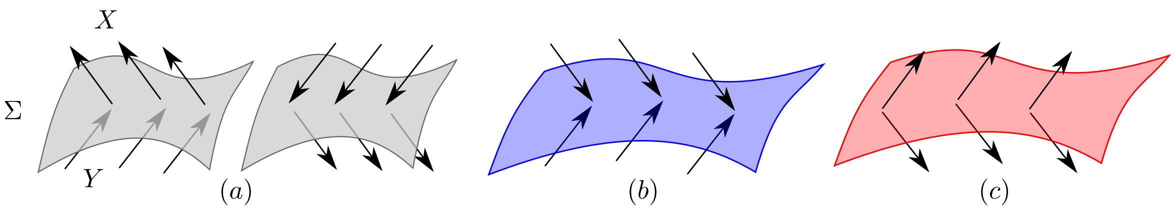


Figure 1.1: Regions in Σ: Σ^c in (a), Σ^{ss} in (b) and Σ^{us} in (c).

The concept of solution of Z follows the Filippov's convention (see, for instance, [39, 55, 103]). The local solution of $Z = (X, Y) \in \Omega^r$ at $p \in \Sigma^s$ is given by the **sliding vector field**

$$F_Z(p) = \frac{1}{Yf(p) - Xf(p)} \left(Yf(p)X(p) - Xf(p)Y(p) \right).$$

Notice that F_Z is a $\mathcal{C}^r$ vector field tangent to Σ^s. The critical points of F_Z in Σ^s are called **pseudo-equilibria** of Z.

Definition 1.1.3. *We defined the* ***normalized sliding vector field*** F_Z^N *of* Z *by*

$$F_Z^N(p) = Yf(p)X(p) - Xf(p)Y(p),$$

for every $p \in \Sigma^s$.

Notice that F_Z^N is also a $\mathcal{C}^r$ vector field tangent to Σ^s.

Remark 1.1.4. *The normalized sliding vector field can be* $\mathcal{C}^r$ *extended beyond the boundary of* Σ^s*. In addition, if* R *is a connected component of* Σ^{ss}*, then* F_Z^N *is a re-parameterization of* F_Z *in* R*, and so the phase portraits of both coincide. If* R *is a connected component of* Σ^{us}*, then* F_Z^N *is a (negative) re-parameterization of* F_Z *in* R*, then they have the same phase portrait, but the orbits are oriented in opposite direction.*

If $p \in \Sigma^c$, then the orbit of $Z = (X, Y) \in \Omega^r$ at p is defined as the concatenation of the orbits of X and Y at p. Nevertheless, if $p \in \Sigma \setminus \Sigma^c$, then it may occur a lack of uniqueness of solutions. In this case, the flow of Z is multivalued and any possible trajectory passing through p originated by the orbits of X, Y and F_Z is considered as a solution of Z. More details can be found in [39, 55].

In the following definition, we introduce the so-called Σ-singularities of a Filippov system.

Definition 1.1.5. *Let* $Z = (X, Y) \in \Omega^r$*, a point* $p \in \Sigma$ *is said to be:*

i) a ***tangential singularity*** *of* Z *provided that* $Xf(p)Yf(p) = 0$ *and* $X(p), Y(p) \neq 0$*;*

ii) a Σ***-singularity*** *of* Z *provided that* p *is either a tangential singularity, an equilibrium of* X *or* Y*, or a pseudo-equilibrium of* Z*.*

Remark 1.1.6. *A point* $p \in \Sigma$ *which is not a* Σ*-singularity of* Z *is also referred as a* ***regular-regular*** *point of* Z*.*

We say that γ is a **regular orbit** of $Z = (X, Y)$ if it is a piecewise smooth curve such that $\gamma \cap M^+$ and $\gamma \cap M^-$ are unions of regular orbits of X and Y, respectively, and $\gamma \cap \Sigma \subset \Sigma^c$.

Chapter 2

Polycycles of Planar Filippov Systems

In 1882, the concept of limit cycle was introduced by Henri Poincaré and since then, the detection of such an object has become one of the most interesting (and complicated) problems in Dynamical Systems. Over the years, other global structures were investigated, and the concept of polycycle have been established. Roughly speaking, a polycycle is a collection of certain singularities and some connections between them, having a first return map. This class of minimal sets has been extensively studied throughout the years, as in the so-called *Dulac's Problem*.

In Filippov systems, certain Σ-singularities present local invariant manifolds, and thus it leads us to study their global connections, which have no counterpart in the smooth context. Considering these new singularities appearing in the piecewise smooth context, the concept of polycycle is easily carried to Filippov systems.

In this chapter, we provide results on generic bifurcation of planar Filippov systems around polycycles. More specifically, we develop a mechanism to detect crossing phenomena bifurcating from a polycycle and we apply it to obtain the complete bifurcation diagram of vector fields around certain elementary polycycles.

2.1 Introduction

In the last years, the homoclinic-like connections through Σ-singularities of planar Filippov system have received attention of the mathematical community. In fact, since the local structure of many Σ-singularities is well established, it is reasonable to study such phenomena in order to contribute to the development of the Theory of Filippov Systems. Besides that, these objects frequently appear in applications (see [4, 11] and references therein), and thus the knowledge of bifurcations around global connections is also of interest to describe natural phenomena.

In [65], Kuznetsov et al. provided a catalog of bifurcations occurring in one-parameter families of Filippov systems. Among them, they presented the *critical crossing cycle bifurcation* (CC-bifurcation), which consists in a one-parameter family Z_α of Filippov systems, for which Z_0 has a homoclinic-like connection at a fold-regular singularity. See Figure 2.1. Nevertheless, the authors have provided only one example of family presenting such a phenomenon and no study on generic bifurcation of this connection was done. We highlight that a fold-regular singularity is one of the simplest Σ-singularities in the planar case, and thus homoclinic-like connections through such a singularity are one of the most elementary global connections in this scenario.

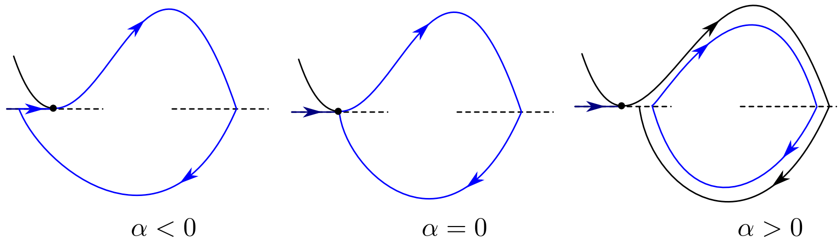


Figure 2.1: A one-parameter family Z_α presenting a crossing critical cycle bifurcation at $\alpha = 0$. For $\alpha < 0$, Z_α has a sliding cycle, and for $\alpha > 0$, Z_α has a crossing limit cycle.

In [55], Guardia et al. have approached the CC-bifurcation phenomena presented in [65] by means of Bifurcation Theory in a general setting. Finally, in [40], Freire et al. provided non-degeneracy conditions for which a Filippov system presents a CC-bifurcation. They have also shown, through a Poincaré map analysis, that the unfolding of a CC-bifurcation provided by [65] holds for this generic scenario. In [2], Andrade has provided a different proof for the results presented in [40]. It is worth mentioning that such a global phenomenon has already appeared in the local unfolding of Σ-singularities with higher degeneracies, as the fold-cusp singularity studied in [21, 22].

Recently, more complicated homoclinic-like connections through Σ-singularities were considered. In [79], Novaes et al. studied a codimension-two homoclinic-like connection at a visible-visible fold-fold singularity (see Figure 2.2-(a)) and its complete bifurcation diagram was provided. In [4], Andrade et al. have studied a class of systems presenting a homoclinic-like connection at a saddle-regular singularity (also know as *boundary-saddle singularity*), some bifurcations were described and a physical model realizing such a connection was given (see Figure 2.2-(b)). Other examples of global connections between Σ-singularities appear in [11, 67, 68, 70].

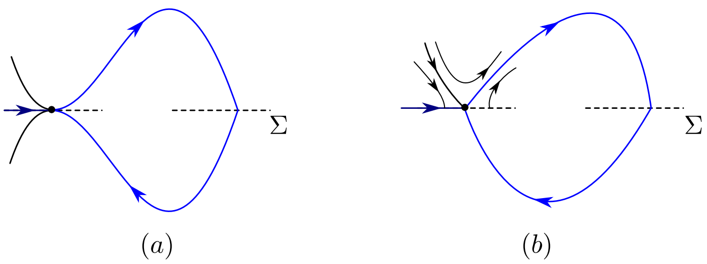


Figure 2.2: A homoclinic-like connection at a visible-visible fold-fold singularity (a) and at a saddle-regular singularity (b).

2.1.1 Description of the Results

Now we provide a briefly description of the results contained in this chapter. First, we establish a variation of the classical concept of polycycle for Filippov systems. In particular, we focus on Σ-polycycles, which are polycycles having all their singularities contained in the switching manifold Σ. It means that a Σ-polycycle is given by an oriented simple closed curve composed by a collection of Σ-singularities connected by regular orbits. The objects mentioned in [4, 11, 40, 55, 65, 67, 68, 70, 79] are examples of Σ-polycycles.

Following the techniques used in [4, 79], we develop a mechanism, named *Method of Displacement Functions* (see Section 2.3), to study the unfolding of Σ-polycycles in a typical scenario. It is worth mentioning that such a methodology presents certain novelty in comparison to the classical Melnikov theory and Lin's method (see [69]) commonly used to study global connections of smooth dynamical systems. Generally speaking, given a Filippov system Z_0 having a Σ-polycycle Γ_0, our method associates each Z near Z_0 to a system of nonlinear equations (depending smoothly on Z), which provides information on the crossing orbits of Z in a neighborhood of Γ_0.

In **Theorems A** and **B**, we provide some tools which can be used to characterize the system given by the method of displacement functions. Finally, in Theorems C, D, and E, we use such a mechanism to obtain a complete description of the bifurcation diagrams of certain Σ-polycycles.

In what follows, we discuss **Theorems C**, **D**, and **E**, in order to provide a smooth reading of the chapter.

Theorem C: Σ-Polycycles at a Regular-Cusp Singularity

Recall that $Z_0 = (X_0, Y_0)$ has a regular-cusp singularity at $p_0 \in \Sigma$ if X_0 has a contact of order 3 with Σ at p_0 and Y_0 is transverse to Σ at p_0, or vice-versa. Denoting the space of planar Filippov systems by Ω^r, we consider the class $\Omega_{RC} \subset \Omega^r$ of systems such that, $Z_0 = (X_0, Y_0) \in \Omega_{RC}$ if, and only if, Z_0 has a Σ-polycycle Γ_0 with a unique Σ-singularity $p_0 \in \Sigma$ contained in Γ_0, which is a regular-cusp singularity of Z_0.

Theorem C, which will be formally presented in Section 2.5.3, can be roughly stated as follows

Let $Z_0 \in \Omega_{RC}$. There exist neighborhoods $\mathcal{V}$ of Z_0 in Ω^r, V of the origin in $\mathbb{R}^2$, and a surjective function $(\beta, \lambda_1) : \mathcal{V} \to V$ with $(\beta, \lambda_1)(Z_0) = (0, 0)$, such that the parameters β, λ_1 completely describe the bifurcation diagram of Z_0 around its Σ-polycycle Γ_0, which is illustrated in Figure 2.3.

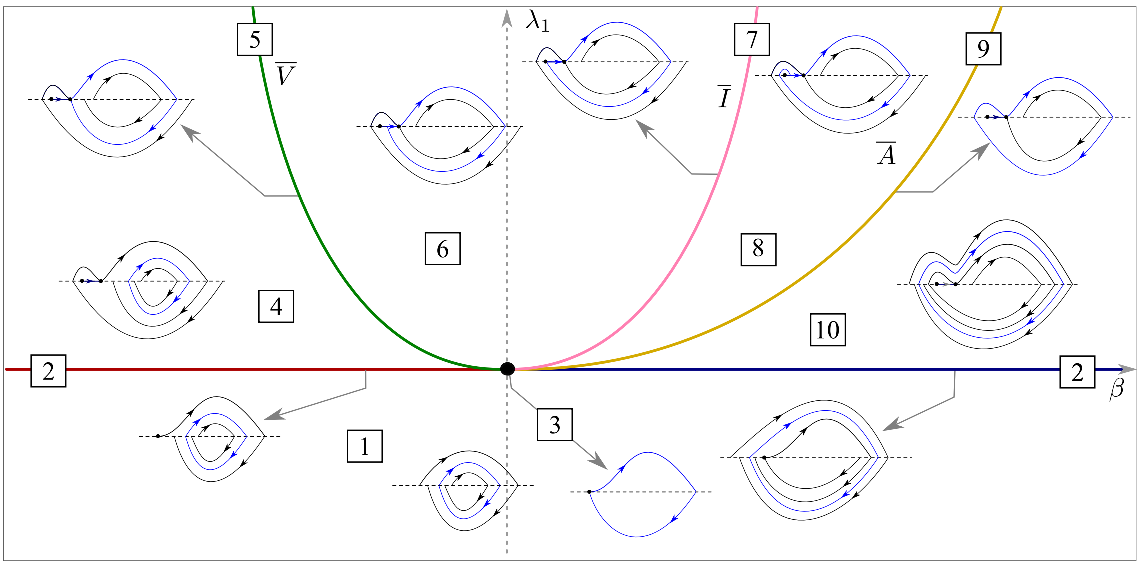


Figure 2.3: Bifurcation diagram of $Z_0 \in \Omega_{RC}$ around Γ_0. $\overline{V}$, $\overline{I}$, $\overline{A}$ and the β-axis are codimension-one bifurcation curves.

A formal and complete description of the bifurcation diagram in the Figure 2.3 is done in Theorem C.

Theorem D: Σ-Polycycles having Two Regular-Fold Singularities

In light of the extensively studied critical crossing cycle bifurcation, we consider a generalization of such Σ-polycycle. More specifically, we allow the Σ-polycycle to have two Σ-singularities of fold-regular type, instead of only one.

We consider the class $\Omega_{DRF} \subset \Omega^r$ of systems such that, $Z_0 = (X_0, Y_0) \in \Omega_{DRF}$ if, and only if, Z_0 has a Σ-polycycle Γ_0 with exactly two Σ-singularities, $p_1 \in \Sigma$ and $p_2 \in \Sigma$, contained in Γ_0 such that

i) p_1 and p_2 are regular-fold singularities of Z_0;

ii) there exist two curves γ_1 and γ_2 connecting p_1 and p_2, oriented from p_1 to p_2 and from p_2 to p_1, respectively, such that $\Gamma_0 = \gamma_1 \cup \gamma_2$, γ_1 is tangent to Σ at p_1 and transverse to Σ at p_2, and γ_2 is tangent to Σ at p_2 and transverse to Σ at p_1.

Theorem D, which will be formally presented in Section 2.5.5, can be roughly stated as follows

Let $Z_0 \in \Omega_{DRF}$. There exist neighborhoods $\mathcal{V}$ of Z_0 in Ω^r, V of the origin in $\mathbb{R}^2$, and a surjective function $(\beta_1, \beta_2) : \mathcal{V} \to V$ with $(\beta_1, \beta_2)(Z_0) = (0, 0)$, such that the parameters β_1, β_2 completely describe the bifurcation diagram of Z_0 around its Σ-polycycle Γ_0, which is illustrated in Figure 2.4.

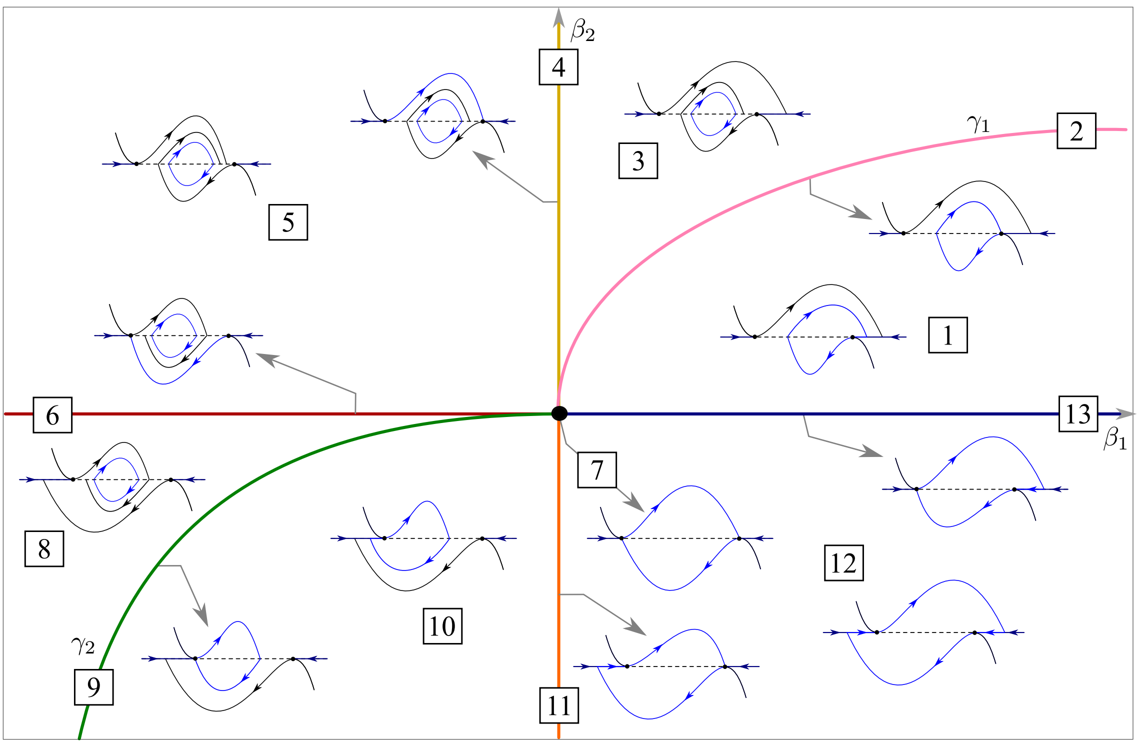


Figure 2.4: Bifurcation diagram of $Z_0 \in \Omega_{DRF}$ around Γ_0. In this case, γ_1, γ_2, the β_1-axis and the β_2-axis are codimension-one bifurcation curves.

A formal description of the bifurcation diagram in the Figure 2.4 is done in Theorem D.

Theorem E: Σ-polycycles at a Visible-Invisible Fold-Fold Singularity

In order to complete the description of Σ-polycycles having a unique fold-fold singularity, we consider the visible-invisible case (the visible-visible case was treated in [79] and the invisible-invisible case does not appear in Σ-polycycles).

Consider the class $\Omega_{FF} \subset \Omega^r$ of systems such that, $Z_0 = (X_0, Y_0) \in \Omega_{FF}$ if, and only if, Z_0 has a Σ-polycycle Γ_0 with a unique Σ-singularity, $p_0 \in \Sigma$, contained in Γ_0, such that

i) p_0 is a fold-fold singularity of visible-invisible type;

ii) Γ_0 is a hyperbolic limit cycle of X_0.

Theorem E (combined with Propositions 2.6.5, 2.6.6 and 2.6.7), which will be formally presented in Section 2.6.2, can be roughly stated as follows

Let $Z_0 \in \Omega_{FF}$. There exist neighborhoods $\mathcal{V}$ of Z_0 in Ω^r, V of the origin in $\mathbb{R}^2$, and a surjective function $(\alpha, \beta) : \mathcal{V} \to V$ with $(\alpha, \beta)(Z_0) = (0, 0)$, such that the parameters α, β completely describe the bifurcation diagram of Z_0 around its Σ-polycycle Γ_0, which is illustrated in Figure 2.5.

A formal description of the bifurcation diagram in the Figure 2.5 is done in Theorem D, and Propositions 2.6.5, 2.6.6 and 2.6.7.

We emphasize that a first return map (relative to Z_0) is defined in both sides of Γ_0 (see Figure 1-(c)). In this case, the stability of Γ_0 as a Σ-polycycle of Z_0 is totally determined by the stability of Γ_0 as a hyperbolic limit cycle of the smooth vector field X_0. In light of this, some situations presented in [33, 37] are not feasible, since the vector field Y_0 can not modify the stability of (the external side of) Γ_0.

2.1.2 Organization of the Chapter

The results of this chapter are organized as follows. Preliminary concepts are provided in Section 2.2.

In Section 2.3, we develop the method of displacement functions which makes use of transition maps, mirror maps and displacement functions introduced in Sections 2.3.1, 2.3.2 and 2.3.3, respectively.

Section 2.4 is devoted to the characterization of the transition maps and to state and prove Theorems A and B.

The Σ-polycycles containing only Σ-singularities of regular-tangential type are analyzed in Section 2.5. More specifically, in Section 2.5.1 we characterize the system of equations, given by the mechanism of displacement functions, for such class of Σ-polycycles. In Section 2.5.2, we prove general properties of Σ-polycycles containing a unique Σ-singularity of regular-tangential type and in Section 2.5.3, we state and prove Theorem C. Finally, Section 2.5.4 is devoted to extend the properties described in Section 2.5.2 to a wider class of systems and Theorem D is stated and proved in Section 2.5.5.

Finally, Σ-polycycles having a unique fold-fold singularity are considered in Section 2.6. In particular, Theorem E is stated and proved in Section 2.6.2.

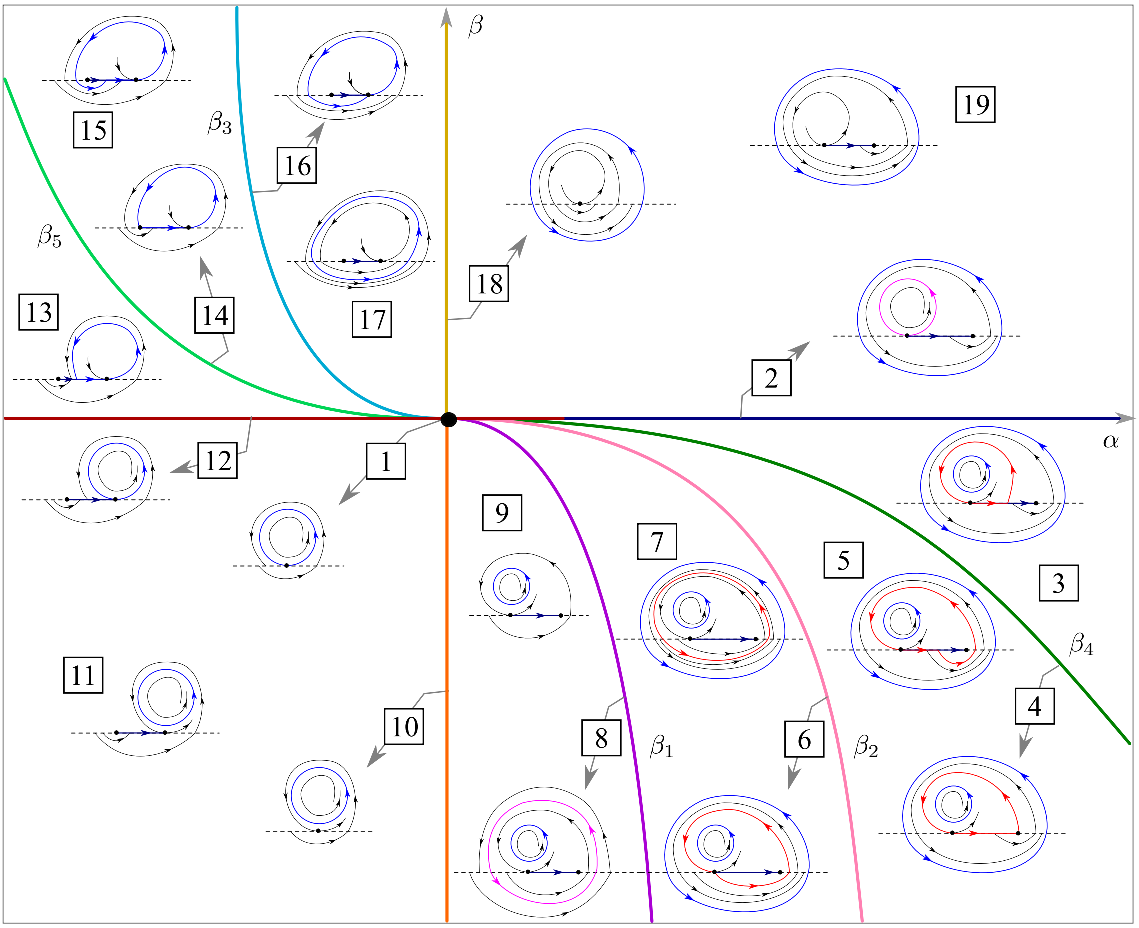


Figure 2.5: Bifurcation diagram of $Z_0 \in \Omega_{FF}$ around Γ_0. In this case, β_i, $1 \leq i \leq 5$, the α-axis and the β-axis are codimension-one bifurcation curves.

2.2 Preliminaries

In this section we introduce an overall description of basic concepts. Furthermore, we establish some definitions which will be used to study global closed connections of planar Filippov systems in a systematic way. Throughout this chapter, we consider piecewise smooth vector fields defined on an open bounded connected set $M \subset \mathbb{R}^2$ with switching manifold $\Sigma = h^{-1}(0)$ where $h : M \to \mathbb{R}$ is a smooth function having 0 as a regular value.

Remark 2.2.1. *Notice that, in this chapter, Ω^r and χ^r stand for the sets of planar piecewise smooth vector fields and planar smooth vector fields, respectively.*

Definition 2.2.2. *Let $Z = (X, Y) \in \Omega^r$, a point $p \in \Sigma$ is said to be:*

*i) a **tangential singularity** of Z provided that $Xh(p)Yh(p) = 0$ and $X(p), Y(p) \neq 0$;*

*ii) a **Σ-singularity** of Z provided that p is either a tangential singularity, an equilibrium of X or Y, or a pseudo-equilibrium of Z.*

Definition 2.2.3. *$X \in \chi^r$ has an **n-order contact** with Σ at p if $X^i h(p) = 0$, for $i = 1, \cdots, n-1$, and $X^n h(p) \neq 0$. In particular, for $n = 2, 3$, p is said to be a **fold point** and **cusp point** of X, respectively.*

Definition 2.2.4. *Let $p \in \Sigma$ be a tangential singularity of $Z = (X, Y)$, we say that p is:*

i) a **regular-tangential singularity of order** *n of Z provided that X (resp. Y) has a n-order contact with Σ at p and $Yh(p) \neq 0$ (resp. $Xh(p) \neq 0$);*

ii) a ***tangential-tangential singularity*** *of Z provided that $Xh(p) = Yh(p) = 0$.*

Remark 2.2.5. *We remark that, in the literature, it is common to distinguish regular-tangential singularities (when $Yh(p) \neq 0$) from tangential-regular singularities (when $Xh(p) \neq 0$), nevertheless, we make no difference between them throughout this chapter.*

In Definition 2.2.4 *i*, for $n = 2$ and $n = 3$, p is said to be a **regular-fold singularity** and **regular-cusp singularity** of Z, respectively. In Definition 2.2.4 *ii*, if p is a fold point of both X and Y, then p is said to be a **fold-fold singularity** of Z. In this case p is called

i) **visible-visible** if $X^2h(p) > 0$ and $Y^2h(p) < 0$.

ii) **visible-invisible** if $X^2h(p) > 0$ and $Y^2h(p) > 0$.

iii) **invisible-visible** if $X^2h(p) < 0$ and $Y^2h(p) > 0$.

iv) **invisible-invisible** if $X^2h(p) < 0$ and $Y^2h(p) > 0$.

Now, motivated by [55], we define the concept of local separatrix at a point $p \in \Sigma$, which will play an important role in this paper.

Definition 2.2.6. *If $p \in \Sigma$, the* ***stable (unstable) separatrix*** *$W^s_\pm(p)$ ($W^u_\pm(p)$) of $Z = (X_+, X_-)$ at a tangential singularity p in $\Sigma^\pm$ is defined as*

$$W^{s,u}_\pm(p) = \{q = \varphi_{X_\pm}(t(q), p);\ \varphi_{X_\pm}(I(q), p) \subset M^\pm \text{ and } \delta_{s,u} t(q) > 0\},$$

where, $\delta_u = 1$, $\delta_s = -1$, and $I(q)$ is the open interval with extrema 0 and $t(q)$.

If γ is a regular orbit of $Z = (X, Y)$, then $\partial\gamma$ is referred as the **ending points** of γ. Accordingly, a **cycle** is a closed regular orbit Γ of Z. If $\Gamma \cap \Sigma \neq \emptyset$, then Γ is called a **crossing cycle** of Z. Now, we define the concept of cycle for planar Filippov systems.

Definition 2.2.7. *A closed curve Γ is said to be a* ***polycycle*** *of $Z = (X, Y)$ if it is composed by a finite number of points, $p_1, p_2, \ldots, p_n$ and a finite number of regular orbits of Z, $\gamma_1, \gamma_2, \ldots, \gamma_n$, such that for each $1 \leq i \leq n$, γ_i has ending points p_i and p_{i+1}. Moreover:*

i) Γ is a S^1-immersion and it is oriented by increasing time along the regular orbits;

ii) if $p_i \in \Sigma$ then it is a Σ-singularity;

iii) if $p_i \in M^\pm$ then it is an equilibrium of either $X|_{M^+}$ or $Y|_{M^-}$;

iv) there exists a non-constant first return map defined, at least, in one side of Γ.

*In particular, if $p_i \in \Sigma$, for all $1 \leq i \leq n$, then Γ is said to be a Σ-**polycycle**.*

Remark 2.2.8. *Condition (i) in Definition 2.2.7 provides the minimality of polycycles of $Z \in \Omega^r$ (i.e. a polycycle Γ can not be written as union of two or more polycycles), avoiding connections as illustrated in Figure 2.6. This condition also establish the notion of sides of Γ, ext(Γ) and int(Γ), which is invoked in Condition (iv).*

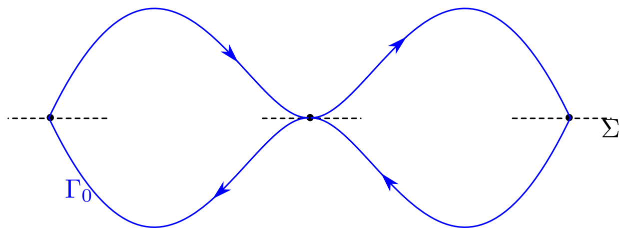


Figure 2.6: Example of a closed connection Γ_0 which is not an S^1-immersion.

The next example illustrates the importance of condition *(iv)* in Definition 2.2.7.

Example 2.2.9. *Let* $Z_0 = (X_0, Y_0)$ *be a nonsmooth vector field with* $h(x, y) = y$ *satisfying the following conditions:*

i) $(0, 0)$ *is a visible regular-fold singularity of* X_0 *and* $\pi_1 \circ X_0(0, 0) > 0$*;*

ii) $(a, 0)$*,* $a > 0$*, is a visible regular-fold singularity of* Y_0 *and* $\pi_1 \circ Y_0(a, 0) > 0$*;*

iii) $W^u_+(0, 0)$ *reaches* Σ *transversally at* $(a, 0)$*;*

iv) $W^u_-(a, 0)$ *reaches* Σ *transversally at* $(0, 0)$*.*

Therefore, Z_0 *presents a closed connection* Γ_0 *(see Figure 2.7). Nevertheless, there exists* $\varepsilon > 0$ *such that for each* $(x, 0) \in \Sigma$*, with* $0 < x < \varepsilon$*, the orbit of* X_0 *through* $(x, 0)$ *reaches the sliding region of* Σ *and slides to the regular-fold singularity* $(a, 0)$*, then it returns to* $(0, 0)$ *through the flow of* Y_0*. Hence, it is defined a first return map* $\mathcal{P} : [0, \varepsilon) \times \{0\} \to [0, \varepsilon) \times \{0\}$ *given by* $\mathcal{P}(x, 0) = (0, 0)$*. Consequently,* Γ_0 *is not a* Σ*-polycycle.*

We remark that there exist nonsmooth vector fields Z *sufficiently close to* Z_0 *which present* Σ*-polycycles and crossing limit cycles near* Γ_0 *(see Figure 2.7). However, the methods described in this paper cannot be applied to this kind of connection.*

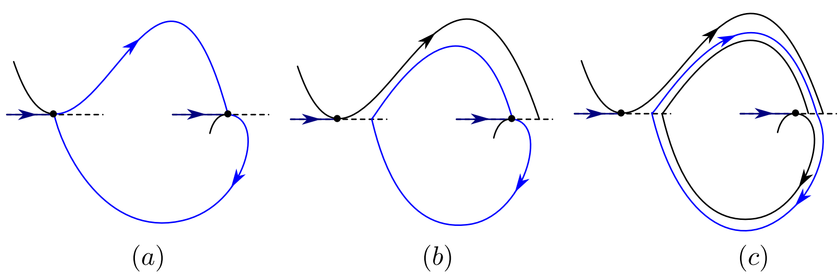


Figure 2.7: Example of (a) a closed connection Γ_0, which has a constant first-return map defined in the interior side of Γ_0, (b) a Σ-polycycle, and (c) a crossing limit cycle of Z close to Z_0.

Definition 2.2.10. *A* Σ*-polycycle* Γ *of* $Z = (X, Y) \in \Omega^r$ *is said to be a* ***regular-tangential*** Σ***-polycycle*** *or a* ***tangential-tangential*** Σ***-polycycle*** *provided that all the* Σ*-singularities of* Z *contained in* Γ *are regular-tangential singularities or tangential-tangential singularities, respectively.*

One of our main goals in this paper is to characterize qualitatively the systems in a neighborhood of a polycycle. To do this we introduce the following notions of equivalence and modulus of stability.

Definition 2.2.11. *Let $\mathcal{K}$ be a compact set of M. We say that Z and Z_0 are (topologically)* ***equivalent*** *at $\mathcal{K}$ if there exist neighborhoods U and V of $\mathcal{K}$ and an orientation preserving homeomorphism $h : U \to V$ which carries orbits of Z onto orbits of Z_0.*

Definition 2.2.12. *A compact invariant set $\mathcal{K}$ of Z is said to be* ***k-stable*** *if for any small neighborhood of Z in Ω^r there exists a k-parameter family of topologically distinct systems such that every system in this neighborhood of Z is equivalent at $\mathcal{K}$ to a system in this k-parameter family. If k is the smallest integer with this property, then we say that $\mathcal{K}$ has* ***modulus of stability*** *k. Define $\mathcal{S}(\mathcal{K}) := k$.*

2.3 Method of Displacement Functions

The aim of this section is to provide a systematic methodology for studying aspects of structural stability of Σ-polycycles in $2D$ nonsmooth vector fields via displacement functions as well as to describe the bifurcations of these objects.

In what follows, given a Σ-polycycle Γ_0 of $Z_0 \in \Omega^r$, we outline the method developed in this work for detecting all the crossing limit cycles with the same topological type of Γ_0 bifurcating from Γ_0. By "the same topological type" we understand the cycles which can be continuously deformed into Γ_0 inside a small annulus $\mathcal{A}$ around Γ_0. In general, our method consists in reducing the problem of finding crossing limit cycles to the study a system of nonlinear equations.

If Γ_0 contains k Σ-singularities p_i, $i = 1, \cdots, k+1$ $(p_1 = p_{k+1})$, then for each nonsmooth vector field $Z \in \Omega^r$ near Z_0, we associate the following system

$$\begin{cases} \Delta_1(Z)(x_1, x_2) = 0; \\ \Delta_2(Z)(x_2, x_3) = 0; \\ \quad\vdots \\ \Delta_{k-1}(Z)(x_{k-1}, x_k) = 0; \\ \Delta_k(Z)(x_k, x_1) = 0; \\ x_i \in \sigma_i(Z), \ i = 1, \cdots, k, \end{cases} \tag{2.3.1}$$

where $\sigma_i(Z)$ is a finite union of real intervals such that $\mathcal{A} \cap \Sigma^c \subset \cup_{i=1}^{k} \sigma_i(Z_0)$, and Δ_i is a displacement function which measures the splitting of the connection between p_i and p_{i+1} thorugh Γ_0, for $i = 1, \cdots, k$. In this case, (2.3.1) is referred as **crossing system**.

The remainder of this section is devoted to construct the displacement functions Δ_i in (2.3.1), which will be given via transition maps and mirror maps. We shall see that each solution $x(Z) = (x_1(Z), \cdots, x_k(Z))$ of (2.3.1) will correspond to a closed orbit $\Gamma(Z)$ of Z contained in $\mathcal{A}$ satisfying $x_i(Z) = \Gamma(Z) \cap \sigma_i(Z)$, $i = 1, \cdots, k$. In addition, if $x(Z)$ is an isolated solution of (2.3.1) such that $x_i(Z) \in \text{int}(\sigma_i(Z))$ for each $i = 1, \cdots, k$, then it corresponds to a crossing limit cycle of Z. On the other hand, if there exists $i \in \{1, \cdots, k\}$ such that $x_i(Z) \in \partial\sigma_i(Z)$ then this solution corresponds to a Σ-polycycle. Reciprocally, if Γ is a closed orbit of Z in $\mathcal{A}$ and $x_i = \Gamma \cap \sigma_i(Z)$ for $i = 1, \cdots, k$, then $(x_1, \ldots, x_k)$ is a solution of (2.3.1). Therefore, system (2.3.1) describes the whole crossing dynamics of Z in $\mathcal{A}$.

2.3.1 Transition Maps

In order to understand the behavior of the nonsmooth vector fields near Z_0 in $\mathcal{A}$ we shall study how the crossing trajectories of Z_0 behave near the Σ-singularities in Γ_0. With this purpose, we establish a precise definition for transition maps at points $p \in \Sigma$.

We shall see that a transition map is defined for each component, X and Y, of a nonsmooth vector field $Z = (X, Y)$. In light of this, we consider a smooth vector field $X_0 \in \chi^r$ on M and we study the behavior of its trajectories passing through the codimension one manifold $\Sigma \subset M$ given in Section 2.2.

Assume that X_0 satisfies the following set of hypotheses **(T)** at a point $p_0 \in \Sigma$:

(T_1) $X_0(p_0) \neq (0, 0)$;

(T_2) there exists $t_0 \in \mathbb{R}$ such that $q_0 = \varphi_{X_0}(t_0; p_0) \in M^{\pm}$,

where φ_{X_0} denotes the flow of X_0.

Let $\tau \subset M^{\pm}$ be a local transversal section of X_0 at q_0. From the Implicit Function Theorem for Banach Spaces there exist neighborhoods $\mathcal{U}_0 \subset \chi^r$ of X_0 and $V_0 \subset M$ of p_0, $\varepsilon > 0$, and a unique smooth function $s : \mathcal{U}_0 \times V_0 \to (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $s(X_0, p_0) = t_0$ and $\varphi_X(s(X, p); p) \in \tau$ for every $(X, p) \in \mathcal{U}_0 \times V_0$. Then, we define the **full transition map** of $X \in \mathcal{U}_0$ at p_0 as the map

$$\begin{array}{rccl} \overline{T^X_{p_0}}: & (\Sigma \cap V_0)_{p_0} & \longrightarrow & \tau \\ & p & \longmapsto & \varphi_X(s(X, p); p), \end{array}$$

where $(\Sigma \cap V_0)_{p_0}$ is the connected component of $\Sigma \cap V_0$ containing p_0.

Throughout this paper, when p_0 and p_1 belong to the same orbit of X, $\widehat{p_0 p_1}|_X$ will denote the **oriented arc-orbit** of X with extrema p_0 and p_1, i. e. $\widehat{p_0 p_1}|_X = \varphi_X(I; p_0)$ where $I = [0, t_1]$, $p_0 = \varphi_X(0, p_0)$, and $p_1 = \varphi_X(t_1, p_0)$. We shall omit the index X if there is no ambiguity. Since we are constructing transition maps for nonsmooth vector fields, it is only considered orbits of X which are contained in either $\overline{M^+}$ or $\overline{M^-}$. So, the domain of the full transition map has to be restricted to the following set

$$\sigma_X = \left\{ p \in (\Sigma \cap V_0)_{p_0};\ \widehat{p\overline{T^X_{p_0}}(p)} \text{ is contained in } \overline{M^{\pm}} \right\}.$$

Accordingly, the **transition map** of X at p_0 is defined as $T^X_{p_0} := \overline{T^X_{p_0}}|_{\sigma_X}$.

It is worth to notice that p_0 may not be contained in the domain σ_X of the transition map $T^X_{p_0}$ (see Figure 2.8). However, if X is defined in $\overline{M^{\pm}}$ and $q_0 \in M^{\pm}$ (recall that q_0 defines the local transversal section τ), then $p_0 \in \sigma_X$ provided that the arc-orbit $\widehat{p_0 q_0}$ of X is contained in $M^{\pm}$.

In Section 2.4, we characterize the full transition map $\overline{T^{X_0}_{p_0}}$ for vector fields $X_0 \in \chi^r$ having a n-order contact with Σ at p_0. Moreover, we describe how $\overline{T^X_{p_0}}$ behaves for X in a small neighborhood of X_0 in χ^r.

2.3.2 Mirror Maps

Assume that X_0 has a $2n$-order contact with Σ at p_0 for some $n \in \mathbb{N}$. We shall see that, for each $p \in \Sigma$ near p_0, with $p \neq p_0$, there exists a time $t(p)$ such that $t(p_0) = 0$ and $\varphi_{X_0}(t(p); p) \in \Sigma$. Moreover, the flow of X_0 will define a germ of diffeomorphism at p_0,

$$\begin{array}{rccl} \rho: & (\Sigma, p_0) & \longrightarrow & (\Sigma, p_0) \\ & p & \longmapsto & \varphi_{X_0}(t(p); p). \end{array}$$

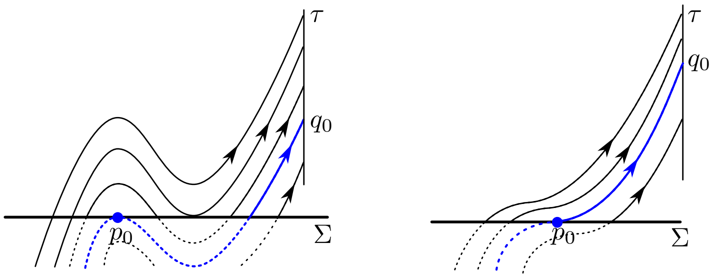


Figure 2.8: Unfolding of a transition map at a cusp point. Left: $p_0 \notin \sigma_X$. Right: $p_0 \in \sigma_X$.

In this case, $\rho(p_0) = p_0$ and we say that ρ is the **involution** associated with X_0 at p_0.

Through a local change of coordinates and a rescaling of time, we can assume that $p_0 = (0,0)$, $h(x,y) = y$, and

$$X_0(x,y) = \begin{pmatrix} 1 \\ \ell_0 x^{2n-1} + \mathcal{O}(x^{2n}, y) \end{pmatrix}, \tag{2.3.2}$$

where $\ell_0 > 0$. In this case, for each $p \in \Sigma$ the orbit connecting p and $\rho(p)$ will be contained in $\overline{M^-}$ (see Figure 2.9).

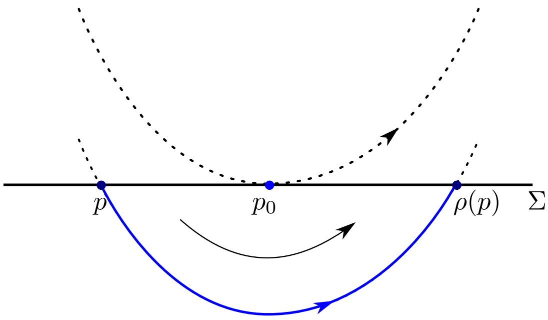


Figure 2.9: Involution ρ of X_0 at p_0.

Notice that $\varphi_{X_0}(t(x);(x,0)) \in \Sigma$ if, and only if, $\pi_2 \circ \varphi_{X_0}(t(x);(x,0)) = 0$. In this case, $\rho(x) = x + t(x)$. Expanding φ_{X_0} around $t = 0$ we get

$$\pi_2 \circ \varphi_{X_0}(t;(x,y)) = y + \sum_{i=1}^{2n} \frac{X_0^i h(x,y)}{i!} t^i + \mathcal{O}(t^{2n+1}). \tag{2.3.3}$$

From (2.3.2), we see that

$$X_0^i h(x,y) = \ell_0 \frac{(2n-1)!}{(2n-i)!} x^{2n-i} + \mathcal{O}(x^{2n-i+1}, y). \tag{2.3.4}$$

Now, define the map

$$S(s,x) = \frac{2n}{\ell_0 x^{2n}} \pi_2 \circ \varphi_{X_0}(sx;(x,0)).$$

Notice that, if $S(s,x) = 0$, $x \neq 0$, and $s \neq 0$, then $\pi_2 \circ \varphi_{X_0}(sx,(x,0)) = 0$. From (2.3.3) and (2.3.4) we obtain that

$$S(s,x) = \frac{(1+s)^{2n} - 1}{s} + \mathcal{O}(x).$$

Since $S(-2,0) = 0$ and $\partial_s S(-2,0) = n > 0$, it follows from the Implicit Function Theorem that there exists $s(x) = -2 + \mathcal{O}(x)$ such that $S(s(x), x) = 0$. From the definition of S, for $t(x) = xs(x)$, we have that $\varphi_{X_0}(t(x); (x,0)) \in \Sigma$, and then the involution ρ is straighly defined.

From the construction above, it follows that there exists a compact neighborhood $V_0 \subset M$ of p_0 such that the involution $\rho : (\Sigma \cap V_0)_{p_0} \to (\Sigma \cap V_0)_{p_0}$ is well defined and characterized as

$$\rho(x) = x + t(x) = -x + \mathcal{O}(x^2). \tag{2.3.5}$$

Now, we show that the a vector field $X \in \chi^r$ sufficiently near X_0 still induces an involution in $(\Sigma \cap V_0)_{p_0}$ but a finite set of points. In what follows we also characterize it. For simplicity, identify $(\Sigma \cap V_0)_{p_0}$ with the interval $[-\varepsilon_0, \varepsilon_0]$ and p_0 with 0.

From definition of ρ, there exists $\varepsilon_0^* > 0$ such that the intervals $I = [-\varepsilon_0, -\varepsilon_0/2]$ and $\rho(I) = [\varepsilon_0^*, \varepsilon_0]$ are connected by orbits of X_0 contained in M^-, and X_0 is transverse to Σ at every point of $I \cup \rho(I)$. Since I is compact, given $\varepsilon > 0$, there exists a small neighborhood $\mathcal{U}_1 \subset \chi^r$ of X_0 such that, for each $X \in \mathcal{U}_1$, there exist $\varepsilon_X^*, \varepsilon_X > 0$ satisfying

i) $|\varepsilon_X^* - \varepsilon_0^*|, |\varepsilon_X - \varepsilon_0| < \varepsilon$;

ii) each point of I is connected to a unique point of $[\varepsilon_X^*, \varepsilon_X]$ through an orbit of X contained in M^-;

iii) X is transverse to Σ at each point of $I \cup [\varepsilon_X^*, \varepsilon_X]$.

Notice that $[-\varepsilon_0/2, \varepsilon_X^*]$ and the orbit connecting $-\varepsilon_0/2$ and ε_X^* give rise to a compact region K^- of M^- such that X is regular at every point of K^- (see Figure 2.10). Thus, each orbit of X entering in K^- must leave it through another point. It allows us to see that Xh has at least one zero in $(-\varepsilon_0/2, \varepsilon_X^*)$ and it has to be an even order contact of X with Σ having the same concavity of p_0. Throughout this section, an even order contact of a vector field X with Σ having the same concavity of p_0 will be called invisible, otherwise it will be called visible.

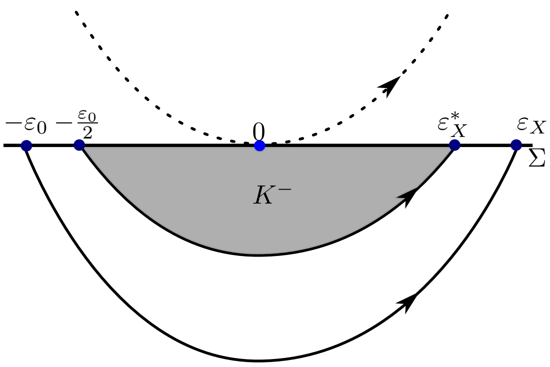


Figure 2.10: Compact region K^- for $X \in \mathcal{U}_1$.

Since $X_0 h(x) = \ell_0 x^{2n-1} + \mathcal{O}(x^{2n})$, there exist a neighborhood $\mathcal{U}_0 \subset \mathcal{U}_1$ of X_0, $\mathcal{C}^r$ functions $a_i : \mathcal{U}_0 \to (-\varepsilon, \varepsilon)$ such that $a_i(X_0) = 0$, $i = 0, \cdots, 2n-2$, and a positive function $\ell : \mathcal{U}_0 \to (\ell_0 - \varepsilon, \ell_0 + \varepsilon)$ with $\ell(X_0) = \ell_0$ satisfying

$$Xh(x) = P_X(x) + \mathcal{O}(x^{2n}),$$

where $P_X(x) = \sum_{i=0}^{2n-2} a_i(X) x^i + \ell(X) x^{2n-1}$. Furthermore, we can take the initial neighborhood V_0 sufficiently small such that the zeroes of Xh in $[-\varepsilon_0, \varepsilon_0]$ are controlled by the polynomial P_X. Hence, it follows that there exist exactly N_X points $r_i \in (-\varepsilon_0/2, \varepsilon_X^*)$, with $1 \leq N_X \leq 2n-1$, such that X has a n_i-order contact with Σ at r_i for some $n_i \geq 2$, $i = 1, \cdots, N_X$. In this case, $n_i \leq 2n$. Accordingly, let $\mathcal{E}_X$ be the finite subset of $(-\varepsilon_0/2, \varepsilon_X^*)$ containing

i) r_i, $i = 1, \cdots, N_X$, such that either n_i is odd or n_i is even and X has a visible contact with Σ at r_i;

ii) $p \in (-\varepsilon_0/2, \varepsilon_X^*)$, such that p and r_i belong to the same orbit of X, for some $i = 1, \cdots, N_X$, and the arc-orbit of X with extrema p and r_i is contained in $\overline{M^-}$ (see Figure 2.11).

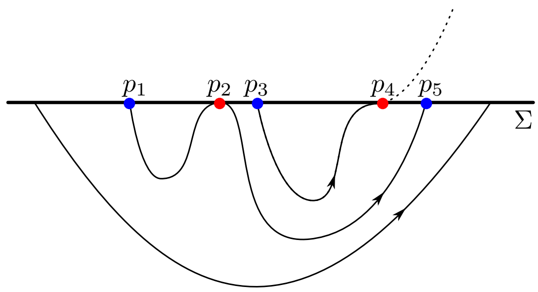


Figure 2.11: Example of some points in $\mathcal{E}_X$, p_1, p_3, p_5 satisfy condition (ii) and p_2, p_4 satisfy condition (i).

If $r_i \in (-\varepsilon_0, \varepsilon_X) \setminus \mathcal{E}_X$, for some $i = 1, \cdots, N_X$, then X has an invisible even order contact with Σ at r_i. So, applying the same process above we find $\varepsilon_i^-, \varepsilon_i^+ > 0$ sufficiently small and an involution $\rho_X^i : (r_i - \varepsilon_i^-, r_i + \varepsilon_i^+) \to (r_i - \varepsilon_i^-, r_i + \varepsilon_i^+)$ induced by the flow of X at r_i. In this case, ρ_X^i is a diffeomorphism with a unique fixed point at r_i, and

$$\rho_X^i(x) = r_i - (x - r_i) + \mathcal{O}((x - r_i)^2). \tag{2.3.6}$$

Now, if $p \in [-\varepsilon_0, \varepsilon_X] \setminus (\mathcal{E}_X \cup \{r_1, \cdots, r_{N_X}\})$, then X is transverse to Σ at p and there exists a unique point $p^* \in (-\varepsilon_0/2, \varepsilon_X^*) \setminus (\mathcal{E}_X \cup \{r_1, \cdots, r_{N_X}\})$ such that X is transverse to Σ at p^*, p and p^* belong to the same orbit of X, and the arc-orbit of X with extrema p and p^* is contained in $\overline{M^-}$. It allows us to extend the involutions ρ_X^i to an involution

$$\overline{\rho_X} : [-\varepsilon_0, \varepsilon_X] \setminus \mathcal{E}_X \to [-\varepsilon_0, \varepsilon_X] \setminus \mathcal{E}_X,$$

induced by the flow of X. We refer $\overline{\rho_X}$ as the **involution** of X at p_0.

Notice that $\overline{\rho_X}$ is a diffeomorphism for which r_i, $i = 1, \cdots, N_X$, are its only fixed points. Moreover, these points are invisible ever order contact of X with Σ and the expansion of $\overline{\rho_X}$ at these points is given by (2.3.6). Thus $\overline{\rho_X}$ is completely characterized and $\overline{\rho_{X_0}} = \rho$, where ρ is given by (2.3.5).

Remark 2.3.1. *Consider the points r_i for which X has a visible even contact with Σ and the points p such that p and r_i belong to the same orbit of X and the arc-orbit of X with extrema p and r_i is contained in $\overline{M^-}$. Notice that they are connected with two or more distinct points of Σ through a unique orbit of X contained in $\overline{M^-}$. Thus, $\overline{\rho_X}$ cannot be uniquely extended to such points. Consequently, they had to be included in the set $\mathcal{E}_X$.*

On the other hand, the involution $\overline{\rho_X}$ could be extended to points r for which X has an odd order contact with Σ. Indeed, the orbit of X through r enters in K^- and leaves it through a unique point of Σ. Nevertheless, in this case, $\overline{\rho_X}$ would not be differentiable at $\overline{\rho_X}(r)$ as we can see in the following example: Assume that X has a cusp point at r, so the orbit of X through r reaches Σ at $\tilde{r}$ transversally, and the arc-orbit of X with extrema r and $\tilde{r}$ is contained in $\overline{M^-}$. Then, we define a germ of involution $\overline{\rho_X} : (\Sigma, r) \to (\Sigma, \tilde{r})$ induced by X (see Figure 2.11 with $p_3 = \tilde{r}$ and $p_4 = r$). Combining the results of Section 2.4 with transversality arguments, we shall see that $\overline{\rho_X}$ is differentiable at r and, moreover, $\overline{\rho_X}(x) = \tilde{r} + A(x-r)^3 + \mathcal{O}((x-r)^4)$, with $A \neq 0$. Consequently, $\overline{\rho_X}$ has the following expansion at $\tilde{r}$

$$\overline{\rho_X}(x) = r + \left(\frac{x-\tilde{r}}{A}\right)^{1/3} + \mathcal{O}((x-\tilde{r})^{2/3}),$$

which implies that ρ_X is not differentiable at $\tilde{r}$.

We aim to use these involutions for detecting closed connections of nonsmooth vector fields. Thus, in order to avoid pseudo-connections (see [55] for more details), we restrict $\overline{\rho_X}$ to the set

$$\sigma_X^{inv} = \{p \in [-\varepsilon_0, \varepsilon_X] \setminus \mathcal{E}_X;\ Xh(p) \leq 0\}.$$

Accordingly, the restriction $\rho_X := \overline{\rho_X}|_{\sigma_X^{inv}}$ is referred as **mirror map** of X at p_0. The condition $Xh(p) \leq 0$ on the domain σ_X^{inv} comes from the initial assumptions which imply that the orbit connecting p and $\rho_X(p)$ is contained in $\overline{M^-}$ for every $p \in \sigma_X^{inv}$. When considering nonsmooth systems these orbits could be contained in $\overline{M^+}$. In this case, the condition on σ_X^{inv} is changed to $Xh(p) \geq 0$.

Example 2.3.2. *Consider the family of vector fields $X_\lambda(x,y) = (1, x^3 - \lambda x)$, for $\lambda \geq 0$, and $\Sigma = \{y = 0\}$. Notice that the orbits of X_λ are given by the level curves of $H_\lambda(x,y) = y - x^4/4 + \lambda x^2/2$. If $\lambda = 0$, then $\overline{\rho_{X_0}}(x,0) = (-x,0)$ is the involution associated with X_0 at the origin. Now, for $\lambda > 0$, the orbit passing through the origin, which is a visible fold point, splits Σ into three sets: $D_1 = (-\infty, -\sqrt{2\lambda}) \cup (\sqrt{2\lambda}, +\infty)$, $D_2 = (-\sqrt{2\lambda}, 0)$, and $D_3 = (0, \sqrt{2\lambda})$ (see Figure 2.12). Hence, X_λ defines the following involution on $D_\lambda = D_1 \cup D_2 \cup D_3$:*

$$\overline{\rho_{X_\lambda}}(x) = \begin{cases} -x, & \text{for } x \in D_1, \\ -\sqrt{2\lambda - x^2}, & \text{for } x \in D_2, \\ \sqrt{2\lambda - x^2}, & \text{for } x \in D_3. \end{cases}$$

In this case, the mirror map ρ_{X_λ} of X_λ is the restriction of $\overline{\rho_{X_\lambda}}$ to $\sigma_{X_\lambda}^{inv} = (-\infty, -\sqrt{2\lambda}) \cup (-\sqrt{2\lambda}, -\sqrt{\lambda}] \cup (0, \sqrt{\lambda}]$.

2.3.3 Displacement Functions

Now, we are able to define the displacement functions associated with a Σ-polycycle Γ_0 of $Z_0 = (X_0, Y_0)$. Assume that Γ_0 has k tangential singularities p_i of order $n_i \in \mathbb{N}$, $1 \leq i \leq k$. Let γ_i be the regular orbit of Z_0 connecting p_i to p_{i+1}, $i = 1, \dots, k-1$, γ_k be the regular orbit of Z_0 connecting p_k and p_1, and consider sufficiently small neighborhoods U_i of p_i, $1 \leq i \leq k$. Notice that for each p_i, $i \in \{1, \dots, k\}$, one of the following statements hold:

(**E**) $\Gamma_0 \cap U_i \setminus \{p_i\}$ is contained in either M^+ or M^-;

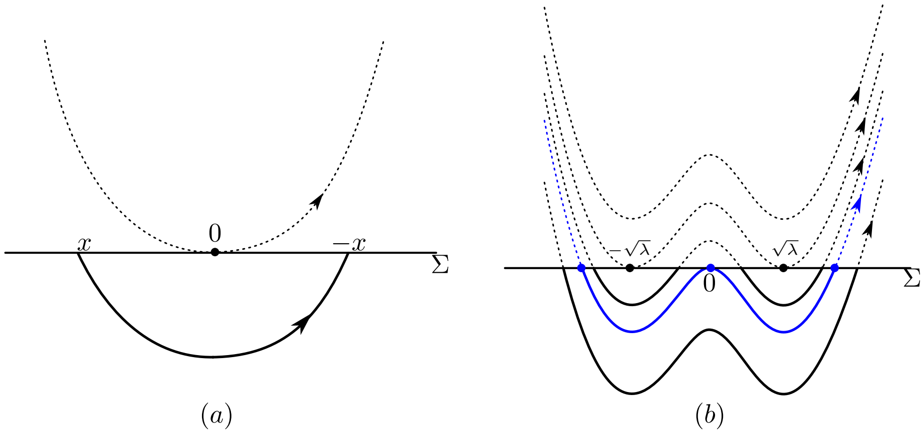


Figure 2.12: Involution of X_λ in Example 2.3.2 for: (a) $\lambda = 0$ and (b) $\lambda > 0$.

(**O**) $\Gamma_0 \cap U_i \setminus \{p_i\}$ has one connected component in M^+ and the other one in M^-.

Suppose that **(O)** holds for p_i and assume, without loss of generality, that $W^u(p_i) \cap \Gamma_0 \cap U_i \subset M^+$ and $W^s(p_i) \cap \Gamma_0 \cap U_i \subset M^-$. Let τ_i^u and τ_i^s be transversal sections of X_0 and Y_0 at the points $q_i^u \in W_+^u(p_i)$ and $q_i^s \in W_-^s(p_i)$, which are contained in U_i, respectively. From the construction performed in Section 2.3.1 there exist transition maps of X_0 and Y_0 at p_0, $T_i^u : \sigma_i(X_0) \to \tau_i^u$ and $T_i^s : \sigma_i(Y_0) \to \tau_i^s$, respectively.

Now, suppose that **(E)** holds for p_i and assume, without loss of generality, that $\Gamma_0 \cap U_i \subset \overline{M^+}$. Let τ_i^s and τ_i^u be transversal sections of X_0 at the points $q_i^s \in W_+^s(p_i)$ and $q_i^u \in W_+^u(p_i)$, which are contained in U_i, respectively. In this case, we have two distinguished situations:

(I) If $\Sigma \cap U_i \setminus \{p_i\}$ has one connected component in the sliding region of Z_0, then let $\sigma_i^{\pitchfork}(X_0)$ be the restriction to $\overline{M^+}$ of a local transversal section of X_0 at p_i. Clearly, the flow of X_0 induces maps $T_i^u : \sigma_i^{\pitchfork}(X_0) \to \tau_i^u$ and $T_i^s : \sigma_i^{\pitchfork}(X_0) \to \tau_i^s$, which are restrictions of diffeomorphisms.

(II) If $\Sigma \cap U_i \setminus \{p_i\} \subset \Sigma^c$, then besides the maps $T_i^u : \sigma_i^{\pitchfork}(X_0) \to \tau_i^u$ and $T_i^s : \sigma_i^{\pitchfork}(X_0) \to \tau_i^s$, induced by the flow of X_0, we can also define other maps in the following way: first, notice that this situation is only possible when Y_0 has an invisible even order contact with Σ at p_i, and thus, we consider the mirror map $\rho_i : \sigma_i^{inv}(Y_0) \to \Sigma \cap U_i$ of Y_0 at p_i (see Section 2.3.2). Now, let $T_-^{X_0} : \sigma_i^-(X_0) \to \tau_i^s$ and $T_+^{X_0} : \sigma_i^+(X_0) \to \tau_i^u$ be the transition maps of X_0 at p_0 with respect to the transversal sections τ_i^s and τ_i^u, respectively. Now, define the section

$$\sigma_i^t(Z_0) = \rho_i^{-1}(\sigma_i^+(X_0) \cap \rho_i(\sigma_i^{inv}(Y_0))),$$

and the maps

$$\begin{aligned} T_i^s &: \sigma_i^-(X_0) \to \tau_i^s, \quad T_i^s = T_-^{X_0}, \\ T_i^u &: \sigma_i^t(Z_0) \to \tau_i^u, \quad T_i^u = T_+^{X_0} \circ \rho_i. \end{aligned} \tag{2.3.7}$$

Thus, in this case, we have maps $T_i^{u,s}$ induced by crossing orbits of Z_0.

Summarizing, if p_i has type **(O)**, **(E-I)** or **(E-II)**, then we define $\sigma_i(Z_0)$ as $\sigma_i(X_0) \cap \sigma_i(Y_0)$, $\sigma_i^{\pitchfork}(X_0)$ or $\sigma_i^{\pitchfork}(X_0) \cup (\sigma_i^t(Z_0) \cap \sigma_i^-(X_0))$, respectively. So, in any case, we construct maps $T_i^{u,s} : \sigma_i(Z_0) \to \tau_i^{u,s}$ induced by crossing orbits of Z_0. We refer the maps $T_i^{u,s}$ as **transfer functions** (see Figure 2.13).

Now, the regular orbit γ_i connecting p_i to p_{i+1}, $i = 1, \cdots, k$, induces a diffeomorphism $D_i : \tau_i^u \to \tau_{i+1}^s$ such that $D_i(p_i) = p_{i+1}$.

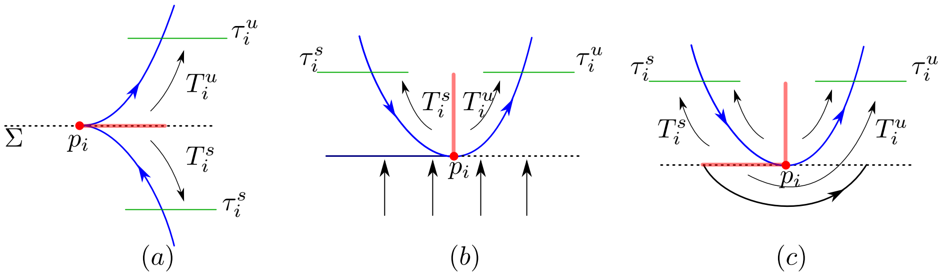


Figure 2.13: Transfer functions of types (a)-**(O)**, (b)-**(E-I)** and (c)-**(E-II)**.

For a sufficiently small neighborhood $\mathcal{V} \subset \Omega^r$ of Z_0 in Ω^r, we see that all the maps used to construct the transfer functions $T_i^{u,s}$ above are also defined for each $Z \in \mathcal{V}$ (see Sections 2.3.1 and 2.3.2). Thus, for each $Z \in \mathcal{V}$, the transfer functions $T_i^{u,s}(Z) : \sigma_i(Z) \to \tau_i^{u,s}$ and the diffeomorphisms $D_i(Z) : \tau_i^u \to \tau_{i+1}^s$ can be constructed in the same way as described above. In particular, the domain $\sigma_i^{\pitchfork}(X_0)$ is perturbed into

$$\sigma_i^{\pitchfork}(X) = \{p \in \sigma_i^{\pitchfork}(X_0);\ \widehat{pT_i^u(Z)(p)}|_X \text{ and } \widehat{T_i^s(Z)(p)p}|_X \text{ are contained in } \overline{M^+}\}.$$

We now relate all these informations through displacement functions.

Definition 2.3.3. *The i-**th displacement function** of Z is defined as*

$$\begin{array}{rccl} \Delta_i(Z): & \sigma_i(Z) \times \sigma_{i+1}(Z) & \longrightarrow & \mathbb{R} \\ & (x_i, x_{i+1}) & \longmapsto & \phi \circ T_i^u(Z)(x_i) - \phi \circ D_i^{-1} \circ T_{i+1}^s(Z)(x_{i+1}), \end{array}$$

where $\phi : \tau_i^u \to \mathbb{R}$ is a parameterization of τ_i^u.

Clearly, the zeroes of the $i-$th displacement function of Z does not depend on the parameterization of τ_i. It is straightforward to see that two points, $x_i \in \sigma_i(Z)$ and $x_{i+1} \in \sigma_{i+1}(Z)$, are connected through an orbit of Z if, and only if, $\Delta_i(Z)(x_i, x_{i+1}) = 0$.

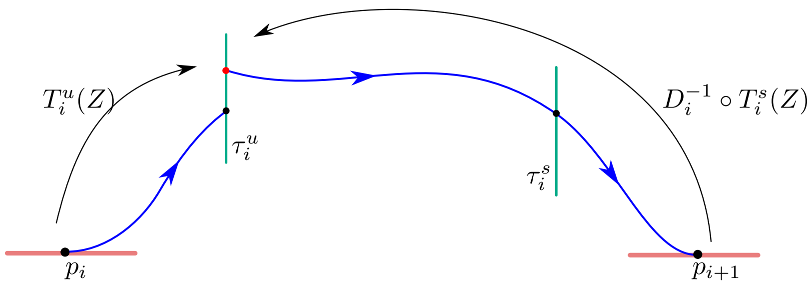


Figure 2.14: Construction of the $i-$th displacement function of $Z \in \mathcal{V}$.

Remark 2.3.4. *We emphasize that the construction of displacement functions as in Definition* (2.3.3) *allows us to describe the complete bifurcation diagrams of a vector field in Ω^r around many different types of Σ-polycycles, in particular the ones analyzed later on in this paper. We highlight that in all the cases all the bifurcating crossing limit cycles with the same topological type of Γ_0 are detected by this method. However, there exist tangential singularities which admit bifurcation of global connections in their local unfoldings, for instance the cusp-cusp singularity. In these cases, such global connections would not be detected by our method for Σ-polycycles through these singularities.*

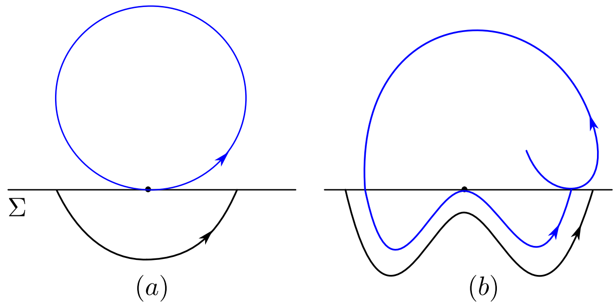


Figure 2.15: In (a) we a have a Σ-polycycle Γ_0 of Z_0 through a tangential-tangential singularity p_0 which is a visible fold for X_0 and an invisible $2n$-order contact of Y_0 with Σ, $n \geq 1$. In (b) we have a Σ-polycycle Γ in a neighborhood Γ_0 occurring for $Z \in \Omega^r$ near Z_0 which is not detected by the proposed method.

2.4 Characterization of Transition Maps

In this section we characterize the transition maps of $Z_0 = (X_0, Y_0)$ at $p \in \Sigma$ and we also study how they typically change for unfoldings of Z_0.

Firstly, notice that if $X \in \chi^r$ is transversal to $\Sigma = h^{-1}(0)$ at p, then the transition map $T_p^X|_\sigma$ is a diffeomorphism at p and σ is an open set of Σ containing p.

Now, assume that $X \in \chi^r$ has a $n-$order contact with Σ at p. Consider coordinates (x, y) at p (i.e. $x(p) = y(p) = 0$) such that $h(x, y) = y$ and write $X = (X_1, X_2)$ in this coordinate system. In this case $X_1(0, 0) \neq 0$, and thus $X_1(x, y) \neq 0$, for every (x, y) in some neighborhood U of the origin. By performing a time rescaling, we obtain that $X(x, y)$ and $\widetilde{X}(x, y) = (\text{sgn}(X_1(0, 0)), f(x, y))$, with $f(x, y) = X_2(x, y)/|X_1(x, y)|$, have the same integral curves in U. It is easy to see that $Xh(x, y) = |X_1(x, y)|\widetilde{X}h(x, y)$. In general, $X^i h(0, 0) = 0$ if, and only if, $\widetilde{X}^i h(0, 0) = 0$. Moreover, one can prove that $X^i h(0, 0)$ and $\widetilde{X}^i h(0, 0)$ have the same sign. In what follows, without loss of generality, we take $X(x, y) = (\delta, f(x, y))$, with $\delta = \pm 1$.

Lemma 2.4.1. *Assume that $X = (\delta, f(x, y))$, with $\delta = \pm 1$, has a n-order contact with Σ at $(0, 0)$, i.e. $X^i h(p) = 0$, $i = 0, 1, \ldots, n - 1$, and $X^n h(p) \neq 0$. Then:*

(a) $\dfrac{\partial^{i-1} f}{\partial x^{i-1}}(0, 0) = 0$, *for* $i = 1, 2, \ldots, n - 1$, *and* $\dfrac{\partial^{n-1} f}{\partial x^{n-1}}(0, 0) \neq 0$.

(b) $Xh(x, 0) = \alpha x^{n-1} + \mathcal{O}(x^n)$, *where* $sgn(\alpha) = \delta^{n-1} sgn(X^n h(0, 0))$.

Proof. Firstly, the statement (a) follows by noticing that $0 = Xh(0, 0) = f(0, 0)$ and

$$X^i h(0, 0) = \delta^{i-1} \frac{\partial^{i-1} f}{\partial x^{i-1}}(0, 0).$$

Now, since $Xh(x, 0) = \langle X(x, 0), (0, 1)\rangle = f(x, 0)$, expanding $Xh(x, 0)$ in Taylor series around $x = 0$, we obtain that

$$Xh(x, 0) = \frac{\partial^{n-1} f}{\partial x^{n-1}}(0, 0) x^{n-1} + \mathcal{O}(x^n).$$

Hence, the statement (b) follows by taking $\alpha = \delta^{n-1} X^n h(0, 0)$. □

From Lemma 2.4.1 it follows that X is transversal to Σ for every $(x,0) \in V \cap \Sigma \setminus \{(0,0)\}$, where V is a small neighborhood of the origin O. Let X be defined in $\overline{M^{\pm}}$ and assume that

(A) either the oriented arc-orbit $\widehat{Oq_0}|_X$ or $\widehat{q_0O}|_X$ is contained in $M^{\pm}$.

In the first case $q_0 = \varphi_X(T_0;0,0) \neq (0,0)$, and in the second one $q_0 = \varphi_X(-T_0;0,0) \neq (0,0)$, for some $T_0 > 0$.

Let $q_0 = (x_0, y_0)$. Since $\pi_1(X(q_0)) = \delta \neq 0$, it follows that

$$\tau = \{(x_0, y);\ y \in (y_0 - \varepsilon, y_0 + \varepsilon)\} \tag{2.4.1}$$

is a transversal section of X at q_0, for ε sufficiently small. Take $\varepsilon > 0$ such that $\tau \subset V \cap M^{\pm}$. Therefore, the full transition map of X at $(0,0)$ is $T_X : (V \cap \Sigma)_0 \to \tau$ given by

$$T_X(x,0) = (x_0, \pi_2(\varphi_X(\delta x_0 - \delta x; x, 0))).$$

Now, we use Lemma 2.4.1 to determine the domain σ of the transition map of X at $(0,0)$.

Corollary 2.4.2. *Assume that X has a n-order contact with Σ at $(0,0)$. Then, the following statements hold:*

i) if n is odd, then $\sigma = (-\varepsilon, \varepsilon) \times \{0\}$, for $\varepsilon > 0$ sufficiently small;

ii) if n is even, then $\sigma = I \times \{0\}$, where I is either $[0, \varepsilon)$ or $(-\varepsilon, 0]$, for $\varepsilon > 0$ sufficiently small.

Proof. If n odd, then $Xh(x,0) = \alpha x^k + \mathcal{O}(x^{k+1})$, where $k = n-1$ is even. It means that $\text{sgn}(\alpha)Xh(x,0) > 0$ for $x \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and $\varepsilon > 0$ sufficiently small. So all the orbits of X passing through $(-\varepsilon, \varepsilon) \times \{0\}$ enter (or leave) $M^{\pm}$. If n is even, then $Xh(x,0) = \alpha x^k + \mathcal{O}(x^{k+1})$, where k is odd. It means that $\text{sgn}(\alpha)Xh(x,0) > 0$, for $x \in (0, \varepsilon)$, and $\text{sgn}(\alpha)Xh(x,0) < 0$, for $x \in (-\varepsilon, 0)$, where $\varepsilon > 0$ is sufficiently small. We conclude the proof by observing that the transition map is defined in the unique domain where $Xh(x,0)$ has the same sign of $X_2(q_0)$. □

In what follows we describe the expression of the full transition map T_X of X at $(0,0)$, when the origin is a n-order contact.

Theorem A. *Suppose that $X \in \chi^r$ has a n-order contact with Σ at $p = (0,0)$. In addition, assume that X satisfies condition* **(A)**. *Then the full transition map $T_X : (V \cap \Sigma)_0 \to \tau$ (where τ is given in 2.4.1) is given by:*

$$T_X(x,0) = (x_0, y_0 + \kappa x^n + \mathcal{O}(x^{n+1})),$$

where $sgn(\kappa) = -\delta^n sgn(X^n h(0,0))$.

Proof. As we have seen before, we can assume that $X = (\delta, f(x,y))$. Consider the change of coordinates $\phi(u,v) = (x(u,v), y(u,v))$, where $x(u,v) = \delta u$ and $y(u,v) = \varphi_X^2(u;0,v)$ (φ_X^2 denotes $\pi_2 \circ \varphi_X$). Notice that

$$\begin{aligned} &\frac{\partial x}{\partial u}(0,0) = \delta,\ \frac{\partial x}{\partial v}(0,0) = 0,\ \frac{\partial y}{\partial u}(0,0) = f(0,0) = 0, \text{ and} \\ &\frac{\partial y}{\partial v}(0,0) = \frac{\partial \varphi_X^2}{\partial y}(0;0,0) = 1. \end{aligned} \tag{2.4.2}$$

Therefore, ϕ is a diffeomorphism around the origin. In addition, it can be proved that ϕ is a conjugation between X and $\mathcal{S}(u,v) = (1,0)$ (see [57]). In this new coordinate system, (u,v), Σ and τ becomes, respectively,

$$\widetilde{\Sigma} = \phi^{-1}(\Sigma) = \{(u,v) \in \mathbb{R}^2; \varphi_X^2(u;0,v) = 0\} \text{ and } \widetilde{\tau} = \{(\delta x_0, v);\ v \in (-\varepsilon,\varepsilon)\}.$$

See Figure 2.16.

Since $\varphi_X^2(0;0,0) = 0$ and from (2.4.2), the Implicit Function Theorem implies the existence of $\gamma : (-\eta,\eta) \to \mathbb{R}$ such that $\gamma(0) = 0$ and $\widetilde{\Sigma} = \{(u,\gamma(u)) \in \mathbb{R}^2;\ u \in (-\eta,\eta)\}$.

Notice that $\varphi_{\mathcal{S}}(t;u,v) = (t+u,v)$, so the full transition map $T_{\mathcal{S}} : \widetilde{\Sigma} \to \widetilde{\tau}$ is given by

$$T_{\mathcal{S}}(u,\gamma(u)) = \varphi_{\mathcal{S}}(\delta x_0 - u, u, \gamma(u)) = (\delta x_0, \gamma(u)).$$

Now, we must characterize the function γ around $u = 0$. Computing the k-th derivative of $\varphi_X^2(u;0,\gamma(u)) = 0$ in the variable u, and using that $\varphi_X(u;0,\gamma(u)) = (\delta u, \varphi_X^2(u,0,\gamma(u))) = (\delta u, 0)$, we get

$$\gamma^{(k)}(u) = -\delta^{k-1}\frac{\partial^{k-1} f}{\partial x^{k-1}}(\delta u, 0)\left(\frac{\partial \varphi_X^2}{\partial y}(u;0,\gamma(u))\right)^{-1} + \sum_{i=1}^{k-1} P_i^k(u)\gamma^{(i)}(u), \tag{2.4.3}$$

where P_i^j are continuous functions. From Lemma 2.4.1 (a) and equation (2.4.3) we obtain that $\gamma^{(k)}(0) = 0$, for every $1 \leq k \leq n-1$ and

$$\gamma^{(n)}(0) = -\delta^{n-1}\frac{\partial^{n-1} f}{\partial x^{n-1}}(0,0) = -X^n h(0,0).$$

Consequently, $T_{\mathcal{S}}(u,\gamma(u)) = (\delta x_0, \alpha u^n + \mathcal{O}(|u|^{n+1}))$, where $\alpha = -X^n h(0,0)$.

From the above construction, the following diagram is commutative.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\quad T_X \quad} & \tau \\ {\scriptstyle \phi^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \phi^{-1}} \\ \widetilde{\Sigma} & \xrightarrow[\quad T_{\mathcal{S}} \quad]{} & \widetilde{\tau} \end{array}$$

Since $\pi_1 \circ \phi^{-1}(x,0) = \delta x$ and $\phi^{-1}(x,0) \in \widetilde{\Sigma}$, it follows that $\phi^{-1}(x,0) = (\delta x, \gamma(\delta x))$. Also, observe that $(x_0,y_0) = \varphi_X(T_0,0,0) = (\delta T_0, \varphi_X^2(T_0,0,0))$. So, $\delta x_0 = T_0$. Hence,

$$\begin{aligned}
T_X(x,0) &= \phi \circ T_{\mathcal{S}} \circ \phi^{-1}(x,0) \\
&= \phi \circ T_{\mathcal{S}}(\delta x, \gamma(x)) \\
&= \phi(\delta x_0, \alpha\delta^n x^n + \mathcal{O}(x^{n+1})) \\
&= (x_0, \varphi_X^2(\delta x_0; 0, \alpha\delta^n x^n + \mathcal{O}(x^{n+1}))) \\
&= (x_0, \varphi_X^2(T_0; 0, \alpha\delta^n x^n + \mathcal{O}(x^{n+1}))) \\
&= \left(x_0, \varphi_X^2(T_0;0,0) + \frac{\partial \varphi_X^2}{\partial y}(T_0;0,0)(\alpha\delta^n x^n + \mathcal{O}(x^{n+1})) + \mathcal{O}(x^{2n})\right) \\
&= \left(x_0, y_0 + \frac{\partial \varphi_X^2}{\partial y}(T_0,0,0)\alpha\delta^n x^n + \mathcal{O}(x^{n+1})\right) \\
&= (x_0, y_0 + \kappa x^n + \mathcal{O}(x^{n+1})),
\end{aligned}$$

where

$$\kappa = -\frac{\partial \varphi_X^2}{\partial y}(T_0, 0, 0)X^n h(0,0)\delta^n.$$

Finally , we can take $|T_0|$ small enough such that $\dfrac{\partial \varphi_X^2}{\partial y}(T_0; 0, 0) > 0$ since $\dfrac{\partial \varphi_X^2}{\partial y}(0; 0, 0) = 1 > 0$. Therefore, $\text{sgn}(\kappa) = -\delta^n \text{sgn}(X^n h(0,0))$. □

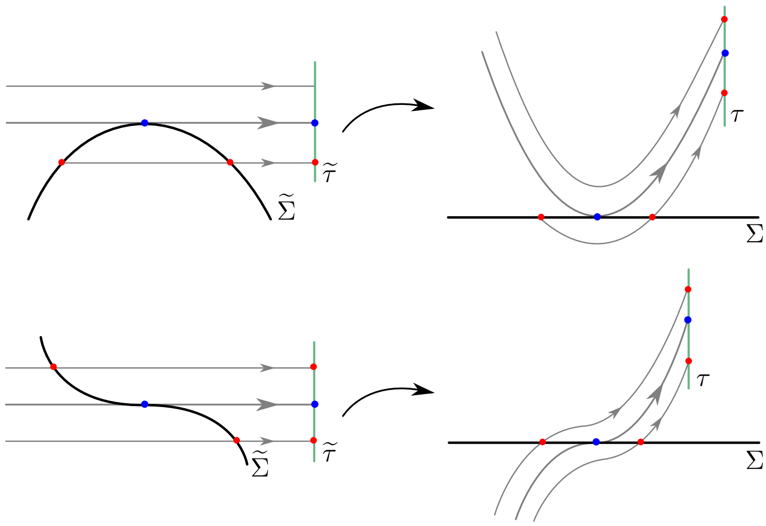


Figure 2.16: Illustration of the change of coordinates $\phi^{-1}(u, v)$ at a fold and a cusp point.

Now, let $X_0 \in \chi^r$ satisfy the assumptions of Theorem A. We know that there exist $\varepsilon > 0$ and a neighborhood $\mathcal{U}$ of X_0 such that a full transition map $T_X : (-\varepsilon, \varepsilon) \to (y_0 - \varepsilon, y_0 + \varepsilon)$ is defined for each $X \in \mathcal{U}$ (see Section 2.3.1). In what follows we shall characterize this map.

Theorem B. *Suppose that $X_0 \in \chi^r$ has a n-order contact with Σ at $p = (0, 0)$, with $n \geq 2$. In addition, assume that X_0 satisfies condition* **(A)**. *Then, there exist a neighborhood $\mathcal{U}_0$ of X_0 in χ^r, $n-2$ surjective functions $\lambda_i : \mathcal{U}_0 \to (-\delta, \delta)$, $i = 1, \cdots, n-2$, depending continuously on X, such that for each $X \in \mathcal{U}_0$ there exists a diffeomorphism $h_X : (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon) \times \{0\}$ for which the full transition map $T_X : (-\varepsilon, \varepsilon) \times \{0\} \to \tau$ is given by:*

$$T_X(h_X(x)) = \Big(x_0, \lambda_0(X) + \kappa(X)x^n + \sum_{i=1}^{n-2} \lambda_i(X)x^i + \mathcal{O}(x^{n+1})\Big),$$

where $\lambda_0 = \pi_2 \circ T_X(0, 0)$, $sgn(\kappa) = -\delta^n sgn(X^n h(0, 0))$ and $\delta = \pm 1$.

Proof. In what follows, for the sake of simplicity, we shall identify $(-\varepsilon, \varepsilon) \times \{0\}$ and τ with the intervals $(-\varepsilon, \varepsilon)$ and $(y_0 - \varepsilon, y_0 + \varepsilon)$, respectively.

From the discussion above, define the continuous map

$$\begin{array}{rccl} T: & \mathcal{U} & \longrightarrow & \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R})/\sim \\ & X & \longmapsto & [T_X - T_X(0)], \end{array}$$

where $\mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})/\!\sim$ is the space of germs of $\mathcal{C}^\infty$ functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(0)=0$, with the equivalence relation

$$f\sim g \text{ if, and only if, } f-g=\mathcal{O}(x^{n+1}).$$

As usual, $[f]$ denotes the equivalence class of $\mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})/\!\sim$ which contains $f\in\mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})$.

Denote $T(X_0)$ by T_0 and notice that T is surjective onto an open neighborhood of T_0 in $\mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})/\!\sim$. In fact, consider the vector field X_0 in the straightened form $\mathcal{S}=(1,0)$, then Σ is the graph $\{(x,h(x));\ x\in(-\varepsilon,\varepsilon)\}$ in these coordinates, for some $\varepsilon>0$ sufficiently small, and $T_0(x)=h(x)$ (see proof of Theorem A). Therefore, any sufficiently small perturbation of h in the space of functions corresponds to the transition map of a vector field X in $\mathcal{U}$ by considering a small change in the coordinate system.

From Theorem A it follows that $T_0=[f_0]$, where $f_0(x)=\kappa x^n$. Now, since the stable unfolding of f_0 is given by $F_\lambda(x)=\kappa x^n+\sum_{i=1}^{n-2}\lambda_i x^i$, there exists a neighborhood $\mathcal{W}$ of T_0 in $\mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})/\!\sim$ such that, for each $f\in\mathcal{W}$, there exist $n-2$ parameters $\lambda_i=\lambda_i(f)$ and a diffeomorphism $h_f:\mathbb{R}\to\mathbb{R}$, such that

$$f(h_f(x))=\kappa x^n+\sum_{i=1}^{n-2}\lambda_i x^i+\mathcal{O}(x^{n+1}).$$

In addition, the parameters λ_i and h_f depend continuously on f.

Taking $\mathcal{U}_0=T^{-1}(\mathcal{W})$, we have that for each $X\in\mathcal{U}_0$

$$T_X(h_X(x))=\lambda_0+\kappa x^n+\sum_{i=1}^{n-2}\lambda_i x^i+\mathcal{O}(x^{n+1}),$$

where $\lambda_i:\mathcal{U}_0\to(-\delta,\delta)$, for $i=1,\cdots,n-2$, are surjective functions depending continuously on X and $\lambda_0=\pi_2\circ T_X(0)$. □

2.5 Regular-Tangential Σ-Polycycles

This section is devoted to apply the method of displacement functions, described in Section 2.3, for obtaining bifurcation diagrams of nonsmooth vector fields around some regular-tangential Σ-polycycles (see Definition 2.2.10). More specifically, in Section 2.5.1, we describe the displacement functions appearing in the crossing system (2.3.1) for such Σ-polycycles. In Section 2.5.2, we prove that at most one crossing limit cycle bifurcates from Σ-polycycles having a unique regular-tangential singularity. Then, in Section 2.5.4 we generalize the previous result for Σ-polycycles having several regular-tangential singularities. In particular, the bifurcation diagrams of Σ-polycycles having either a unique Σ-singularity of regular-cusp type or only two singularities of regular-fold type are completely described in Sections 2.5.3 and 2.5.5, respectively.

2.5.1 Description of the Crossing System

Assume that $Z_0=(X_0,Y_0)\in\Omega^r$ has a Σ-polycycle Γ_0 containing k regular-tangential singularities p_i of order $n_i\in\mathbb{N}$, $1\le i\le k$. Consider a coordinate system (x,y) satisfying that, for each $i\in\{1,2,\dots,k\}$, $x(p_i)=a_i$, $y(p_i)=0$, and $h(x,y)=y$ near p_i.

Firstly, we shall characterize Γ_0 locally around each point p_i, $i=1,\dots,k$. Assume that, for a given $i\in\{1,\dots,k\}$, p_i satisfies $Y_0h(p_i)\neq0$ and consider a small neighborhood U_i of p_i. Accordingly, p_i has one of the following types

(R_1) $\Sigma \cap U_i \setminus \{p_i\}$ has a connected component contained in Σ^c and another in Σ^s, and $\Gamma_0 \cap W_+^{s,u}(p_i) \neq \emptyset$ (see Figure 2.17 (a));

(R_2) $\Sigma \cap U_i \setminus \{p_i\}$ has a connected component contained in Σ^c and another in Σ^s and either $\Gamma_0 \cap W_+^s(p_i) = \emptyset$ or $\Gamma_0 \cap W_+^u(p_i) = \emptyset$ (see Figure 2.17 (b,c));

(R_3) $\Sigma \cap U_i \setminus \{p_i\} \subset \Sigma^c$ (see Figure 2.17 (d)).

The points p_i satisfying $Xh(p_i) \neq 0$ are classified analogously.

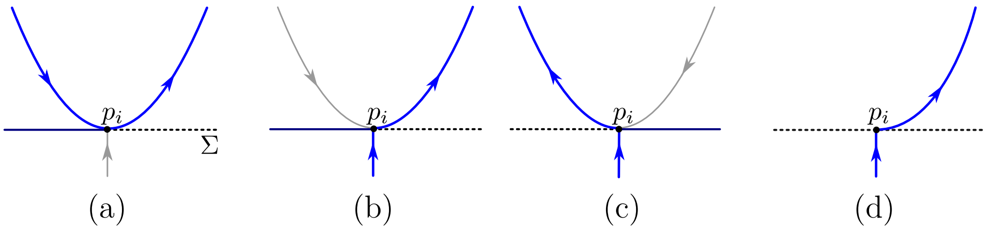


Figure 2.17: Types of local characterization of Γ_0 around the regular-tangential singularity p_i: Figure (a) and its time reversing illustrate type R_1; Figures (b,c) and their time reversing illustrate type R_2; Figure (d) and its time reversing illustrate type R_3. Bold lines represent the intersection $\Gamma_0 \cap W_+^{s,u}(p_i)$. Dashed lines represents Σ^c.

If p_i is of type R_1, then we consider $\sigma_i(Z_0) = \{a_i\} \times (-\varepsilon_i, +\varepsilon_i) \cap M^+$. So, we can follow the case **(E-I)** from Section 2.3.3 to construct the transfer functions $T_i^{u,s} : \sigma_i(X_0) \to \tau_i^{u,s}$ defined by the flow of X_0. Recall that T_i^s and T_i^u are restrictions of germs of diffeomorphisms (see Figure 2.18).

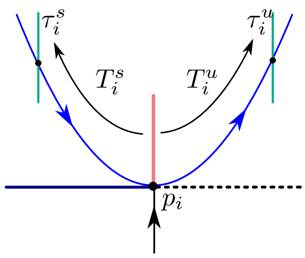


Figure 2.18: Construction of the maps $T_i^{u,s}$: type R_1.

If p_i is of type R_2 or R_3, we consider the tangential section $\sigma_i(Z_0) = (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \{0\} \cap \Sigma^c$, where ε_i is sufficiently small. So, we can follow the case **(O)** from Section 2.3.3 to construct the transfer functions $T_i^u : \sigma_i(Z_0) \to \tau_i^u$ and $T_i^s : \sigma_i(Z_0) \to \tau_i^s$ induced by the flows of X_0 and Y_0, respectively. Notice that T_i^s is the restriction of a germ of diffeomorphism and Theorem A is applied to characterize T_i^u (see Figure 2.19).

Now, in order to describe the displacement functions associated with Γ_0, we characterize the unfolding of each tangential singularity.

If p_i is of type R_1, then T_i^s and T_i^u are germs of diffeomorphisms at p_i. So, as described in Section 2.3.3, for any $Z = (X, Y) \in \Omega^r$ in a small neighborhood $\mathcal{V}_i$ of Z_0, there exist transfer functions $T_i^s(Z) : \sigma_i(Z) \to \tau_i^s$ and $T_i^u(Z) : \sigma_i(Z) \to \tau_i^s$ which are also germs of diffeomorphisms at p_i.

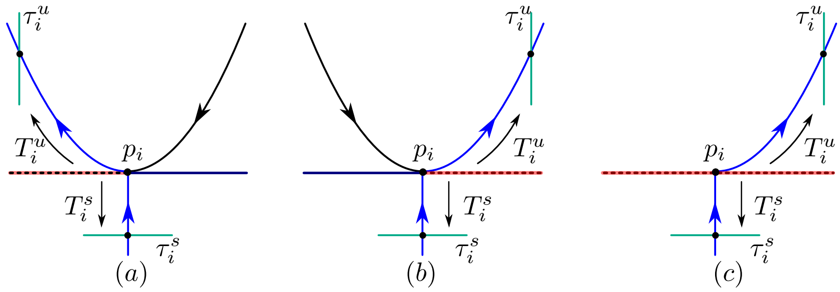


Figure 2.19: Construction of the maps $T_i^{u,s}$: types R_2 ((a) and (b)) and R_3 (c).

From Theorem B there exists a neighborhood $\mathcal{V}$ of Z_0 such that for each $Z = (X, Y) \in \mathcal{V}$ the transfer function corresponding to p_i, for $i \in \{1, 2, \ldots, k\}$, is given by

$$T_i^u(Z)(h_Z^i(x)) = \kappa_i(Z)(x - a_i)^n + \sum_{j=0}^{n_i-2} \lambda_j^i(Z)(x - a_i)^j + \mathcal{O}((x - a_i)^{n+1}), \qquad (2.5.1)$$

where $h_Z^i : (a_i - \varepsilon, a_i + \varepsilon) \to (a_i - \varepsilon, a_i + \varepsilon) \times \{0\}$ is a diffeomorphism, with $h_{Z_0}^i(a_i) = a_i$, $\operatorname{sgn}(\kappa_i(Z)) = \operatorname{sgn}(\kappa_i(Z_0))$, and $\lambda_j^i(Z)$, for $j \in \{0, 1, \ldots, n_i - 2\}$, are parameters.

Notice that $T_i^s(Z)$ is a germ of diffeomorphism on $\sigma_i(Z)$. Thus, for $Z \in \mathcal{V}$ and for each $i = 1, \cdots, k$ we have obtained two maps $T_i^{s,u}(Z)$ defined in a neighborhood of p_i which describes the behavior of the orbits contained in M^+ connecting points of $\tau_i^{s,u}$ and $\sigma_i(Z)$. In addition, each transversal section τ_{i-1}^u is connected to τ_i^s via a diffeomorphism $D_i(Z)$ satisfying:

$$[D_{i-1}(Z)]^{-1} \circ T_i^s(Z)(h_Z^i(x)) = \widetilde{c}_{i-1}(Z) + \widetilde{d}_{i-1}(Z)(x - a_i) + \mathcal{O}_2(x - a_i), \qquad (2.5.2)$$

where $\widetilde{c}_{i-1}(Z_0) = q_{i-1}^u$, and $\operatorname{sgn}(\widetilde{d}_{i-1}(Z)) = \operatorname{sgn}(\widetilde{d}_{i-1}(Z_0))$ (see Figure 2.14). Recall that, in the above expression, we are assuming that $Y_0 h(p_i) \neq 0$. The case $X_0 h(p_i) \neq 0$ follows analogously.

Now, let $\mathcal{A}$ be an open annulus around Γ_0 containing the sections $\sigma_i(Z_0)$. Using the above characterization of the transfer functions and their unfoldings and Definition 2.3.3 we obtain that:

$$\Delta_i(Z)(h_Z^i(x_i), h_Z^{i+1}(x_{i+1})) = \overline{\Delta_i(Z)}(x_i^u, x_{i+1}^s) + \mathcal{O}_{N_i+1}(x_i^u) + \mathcal{O}_{M_i+1}(x_{i+1}^s),$$

where $x_i^u = x_i - a_i$, $x_{i+1}^s = x_{i+1} - a_{i+1}$, and

$$\overline{\Delta_i(Z)}(x_i^u, x_{i+1}^s) = \beta_i(Z) + P_i^{N_i}(x_i^u) + Q_i^{M_i}(x_{i+1}^s).$$

Here, $\beta_i(Z) = \Delta_i(Z)(h_Z^i(a_i), h_Z^{i+1}(a_{i+1}))$ and satisfies $\beta_i(Z_0) = 0$. In addition, $P_i^{N_i}$ and $Q_i^{M_i}$ are non-vanishing polynomials of degree $N_i \leq \max\{2,\ n_i - 2\}$ and $M_i \leq \max\{2,\ n_{i+1} - 2\}$ with coefficients depending on Z and satisfying $P_i^{N_i}(0) = Q_i^{M_i}(0) = 0$.

Finally, the crossing system (2.3.1) is equivalent to the **auxiliary crossing system**:

$$\begin{cases} \overline{\Delta_1(Z)}(x_1^u, x_2^s) + \mathcal{O}_{N_1+1}(x_1^u) + \mathcal{O}_{M_1+1}(x_2^s) = 0, \\ \overline{\Delta_2(Z)}(x_2^u, x_3^s) + \mathcal{O}_{N_2+1}(x_i^u) + \mathcal{O}_{M_2+1}(x_3^s) = 0, \\ \qquad\vdots \\ \overline{\Delta_{k-1}(Z)}(x_{k-1}^u, x_k^s) + \mathcal{O}_{N_{k-1}+1}(x_{k-1}^u) + \mathcal{O}_{M_{k-1}+1}(x_k^s) = 0, \\ \overline{\Delta_k(Z)}(x_k^u, x_1^s) + \mathcal{O}_{N_k+1}(x_k^u) + \mathcal{O}_{M_1+1}(x_1^s) = 0, \\ x_i^{s,u} = f_i^{s,u}(x_i) = x_i - a_i, \ i = 1, \cdots, k, \\ h_Z^i(x_i) \in \sigma_i(Z), \ i = 1, \cdots, k. \end{cases} \tag{2.5.3}$$

2.5.2 Σ-Polycycles having a unique regular-tangential singularity

Without loss of generality, the following conditions characterize the nonsmooth vector fields $Z_0 = (X_0, Y_0)$ which admit a Σ-polycycle having a unique regular-tangential singularity of order n (see Figure 2.20):

i) There exists $p \in \Sigma$ such that X_0 has a n-order contact with Σ at p, $n \geq 2$ and $Y_0 h(p) \neq 0$.

ii) $W_+^u(p)$ intersects Σ^c at $q \neq p$ and the arc-orbit $\widehat{pq}|_{X_0}$ is contained in M^+;

iii) $W_-^s(p)$ intersects Σ^c at $r \neq p$ and the arc-orbit $\widehat{pr}|_{Y_0}$ of Y_0 is contained in M^-;

iv) If $r \neq q$, there exists a regular orbit of Z_0 connecting r and q.

Accordingly, consider Γ_0 as the union of the arc-orbits $\widehat{pr}|_{Z_0}$, $\widehat{rq}|_{Z_0}$, and $\widehat{qp}|_{Z_0}$.

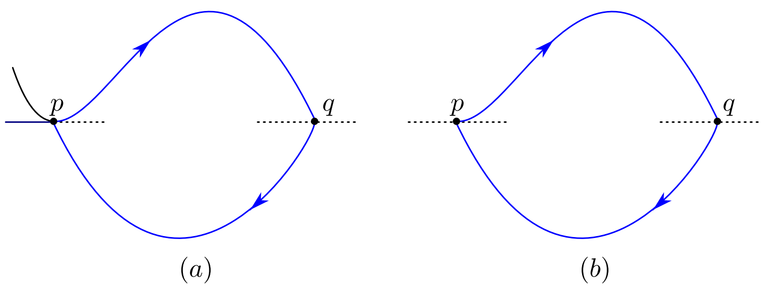


Figure 2.20: An example of Σ-polycycle Γ_0 having a unique regular-tangential singularity of order n, when $q = r$, (a) n is even and when (n) n is odd.

Following the previous section for $k = 1$, $a_1 = 0$, and $x_1 = x_2 = x$ the displacement function $\Delta(Z) : \sigma(Z) \to \mathbb{R}$ writes

$$\begin{aligned} \Delta(Z)(h_Z(x)) = \ & T^u(Z)(h_Z(x)) - [D(Z)]^{-1} \circ T^s(Z)(h_Z(x)) \\ = \ & \lambda_0(Z) + \kappa(Z)x^n + \sum_{j=1}^{n-2} \lambda_j(Z)x^j + \mathcal{O}(x^{n+1}) \\ & - \widetilde{c}(Z) - \widetilde{d}(Z)x + \mathcal{O}_2(x), \end{aligned}$$

where $\text{sgn}(\widetilde{d}(Z)) = \text{sgn}(\widetilde{d}(Z_0))$. Here, it is easy to see that assumptions (i)-(iv) imply that $\widetilde{d}(Z_0) < 0$. Taking

$$\beta(Z) = \lambda_0(Z) - \widetilde{c}(Z), \quad \lambda(Z) = \lambda_1(Z) - \widetilde{d}(Z), \quad \text{and} \quad \eta(Z) = (\beta(Z), \lambda(Z)), \tag{2.5.4}$$

the displacement function $\Delta(Z)(h_Z(x))$ writes

$$\Delta(Z)(h_Z(x)) = \beta(Z) + \lambda(Z)x + \mathcal{O}_2(x). \tag{2.5.5}$$

Notice that $\eta : \mathcal{V} \to V$ is a surjective function onto a small neighborhood V of $(0, -\widetilde{d}(Z_0))$ satisfying $\beta(Z_0) = 0$ and $\lambda(Z_0) = -\widetilde{d}(Z_0) \neq 0$. In this case, the auxiliary crossing system (2.5.3) is reduced to the equation $\beta(Z) + \lambda(Z)x + \mathcal{O}_2(x) = 0$, $h_Z(x) \in \sigma(Z)$.

As a first result on the Σ-polycycle Γ_0 we have the following proposition.

Proposition 2.5.1. *Let Γ_0 be a Σ-polycycle having a unique regular-tangential singularity of order n satisfying (i)-(iv). Then, Γ_0 attracts the orbits passing through the section $\sigma(Z_0)$ (domain of $T^u(Z_0)$). In this case, we say that Γ_0 is C-attractive.*

Proof. Notice that the first return map associated with the Σ-polycycle Γ_0 of Z_0 is given by $\mathcal{P}_0(x) = ([D(Z_0)]^{-1} \circ T^s(Z_0))^{-1} \circ T^u(Z_0)(x)$, where, from (2.5.1) and (2.5.2) (recall that $h_{Z_0} = Id$),

$$T^u(Z_0)(x) = \kappa(Z_0)x^n + \mathcal{O}_{n+1}(x) \quad \text{and} \quad [D(Z_0)]^{-1} \circ T^s(Z_0)(x) = \widetilde{d}(Z_0)\, x + \mathcal{O}_2(x).$$

Hence,

$$\mathcal{P}_0(x) = \frac{\kappa(Z_0)}{\widetilde{d}(Z_0)}\, x^n + \mathcal{O}_{n+1}(x).$$

Therefore, for x small enough, $|\mathcal{P}_0(x)| < |x|$, which means that Γ_0 attracts the orbits passing through the section $\sigma(Z_0)$ (domain of $T^u(Z_0)$). □

In what follows we state the main result of this section.

Proposition 2.5.2. *Let Z_0 be a nonsmooth vector field having a Σ-polycycle Γ_0 containing a unique regular-tangential singularity of order n satisfying (i)-(iv). Then, the following statements hold.*

i) There exist an annulus $\mathcal{A}_0$ at Γ_0 and a neighborhood $\mathcal{V}$ of Z_0 such that each $Z \in \mathcal{V}$ has at most one crossing limit cycle bifurcating from Γ_0 in $\mathcal{A}_0$, which is hyperbolic and attracting.

ii) Let $Z_{\beta,\lambda}$ be a continuous 2-parameter family in $\mathcal{V}$ such that $Z_{0,-\widetilde{d}(Z_0)} = Z_0$ and satisfying $Z_{\beta,\lambda} \in \eta^{-1}(\beta, \lambda)$, for every $(\beta, \lambda) \in V$. Then, for each λ near $-\widetilde{d}(Z_0)$ and for each connected component C of $\sigma(Z_{\beta,\lambda}) \cap \mathcal{A}_0$, there exists a non-empty open interval $I_{\lambda,C}$, satisfying $I_{\lambda,C} \times \{\lambda\} \subset V$, such that $Z_{\beta,\lambda}$ has a hyperbolic attracting crossing limit cycle passing through C, for each $\beta \in I_{\lambda,C}$.

Proof. Consider the function $\eta : \mathcal{V} \to V$ given by (2.5.4). For each $Z = (X, Y) \in \mathcal{V}$, we associate the displacement function $\Delta(Z)$ given in (2.5.5). From Section 2.3.3, we have that there exists $\varepsilon > 0$ such that, for each $Z \in \mathcal{V}$, there exists a function $\widetilde{\Delta}(Z) : (-\varepsilon, \varepsilon) \to \mathbb{R}$ which is an extension of $\Delta(Z)$.

Define the $\mathcal{C}^r$ function $\mathcal{F} : \mathcal{V} \times V \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ as

$$\mathcal{F}(Z, \beta, \lambda, x) = \widetilde{\Delta}(Z)(h_Z(x)) - \beta(Z) - \lambda(Z)x + \beta + \lambda x,$$

and notice that

$$\mathcal{F}(Z_0, 0, -\widetilde{d}(Z_0), 0) = 0, \quad \text{and} \quad \partial_x \mathcal{F}(Z_0, 0, -\widetilde{d}(Z_0), 0) = -\widetilde{d}(Z_0) \neq 0.$$

From the Implicit Function Theorem for Banach Spaces and reducing $\mathcal{V}$ and V if necessary, there exists a unique $\mathcal{C}^r$ function $\mathcal{X} : \mathcal{V} \times V \to (-\varepsilon, \varepsilon)$ such that $\mathcal{F}(Z, \beta, \lambda, x) = 0$ if, and only if, $x = \mathcal{X}(Z, \beta, \lambda)$.

Since

$$\mathcal{F}(Z, \beta, \lambda, x) = \beta + \lambda x + \mathcal{O}_2(x),$$

it follows that $\mathcal{X}(Z, 0, \lambda) = 0$, for every $(Z, 0, \lambda) \in \mathcal{V} \times V$. Consequently, we can see that

$$\mathcal{X}(Z, \beta, \lambda) = -\frac{\beta}{\lambda} + \mathcal{O}_2(\beta). \tag{2.5.6}$$

It follows from the definition of the function $\mathcal{F}$ that

$$\mathcal{X}^*(Z) = \mathcal{X}(Z, \beta(Z), \lambda(Z)) \tag{2.5.7}$$

is the unique zero of $\widetilde{\Delta}(Z)$ in $(-\varepsilon, \varepsilon)$. Hence, $\Delta(Z)$ has at most one zero in $\sigma(Z)$. Moreover, since

$$\frac{\partial \widetilde{\Delta}(Z_0)}{\partial x}(\mathcal{X}^*(Z_0)) = -\widetilde{d}(Z_0) > 0,$$

it follows from (2.5.5) that

$$\frac{\partial \widetilde{\Delta}(Z)}{\partial x}(\mathcal{X}^*(Z)) = \lambda(Z) + \mathcal{O}_2(\mathcal{X}^*(Z)) > 0,$$

for Z sufficiently near Z_0. Therefore, the crossing limit cycle is hyperbolic and attracting (from construction). The proof of item (i) follows by taking $\mathcal{A}_0 = \{p \in M; d(p, \Gamma_0) < \varepsilon\}$, where d denotes the Hausdorff distance.

Now, consider the family $Z_{\beta,\lambda}$ given in item (ii). The unique zero of $\widetilde{\Delta}(Z_{\beta,\lambda})$ is given by

$$x^*(\beta, \lambda) = \mathcal{X}^*(Z_{\beta,\lambda}) = -\frac{\beta}{\lambda} + \mathcal{O}_2(\beta). \tag{2.5.8}$$

Recall that each isolated zero, x_0, of $\Delta(Z_{\beta,\lambda})$ is either a crossing limit cycle (if $x_0 \in \text{int}(\sigma(Z_{\beta,\lambda})))$ or a Σ-polycycle (if $x_0 \in \partial\sigma(Z_{\beta,\lambda})$). So, let $C = (a, b)$ be a connected component of $\sigma(Z_{\beta,\lambda}) \subset (-\varepsilon, \varepsilon)$ for some fixed parameter $\lambda \in \pi_2(V)$. Hence, from (2.5.8), there exists a non-empty open interval $I_{\lambda,C}$ such that $I_{\lambda,C} \times \{\lambda\} \subset V$ and $x^*(\beta, \lambda) \in \text{int}(C)$ whenever $\beta \in I_{\lambda,C}$. □

Remark 2.5.3. *If we change the roles of s and u in the assumptions (i)-(iv) in order to reverse the orientation of the cycle, all the results remain the same reversing the stability.*

Let Z be a nonsmooth vector field sufficiently near Z_0, and consider $\mathcal{X}^*(Z)$ given by (2.5.7). Propositions 2.5.1 and 2.5.2 provides the following possibilities for the crossing dynamics in a small annulus $\mathcal{A}_0$ of Γ_0:

i) if $\mathcal{X}^*(Z) \notin \sigma(Z)$, then Z has no crossing limit cycles or Σ-polycycles;

ii) if $\mathcal{X}^*(Z) \in \text{int}(\sigma(Z))$, then Z has a unique crossing limit cycle with the same stability of Γ_0;

iii) if $\mathcal{X}^*(Z) \in \partial\sigma(Z)$, then Z has a unique Σ-polycyle containing $m \leq n-1$ regular-tangential singularities of order n_i, with $\sum_{i=1}^{m} n_i \leq n$.

In addition, items (i) and (ii) occur in open regions of the parameter space and item (iii) occurs in a hypersurface of the parameter space.

2.5.3 Σ-Polycycles having a unique regular-cusp singularity

In the previous section, assuming that Z_0 has a Σ-polycycle Γ_0 admitting a unique regular-tangential singularity of order n, $n \geq 2$, we have identified all the possible crossing behavior of nonsmooth vector fields $Z = (X, Y)$ sufficiently near Z_0 in a small annulus $\mathcal{A}_0$ of Γ_0. Nevertheless, the domain $\sigma(Z)$, of the displacement function (2.5.5), has some particularities depending on the order n. In order to illustrate it, we describe the bifurcation diagram of Z_0 around Γ_0 assuming that $n = 3$. Furthermore, we shall see that $\mathcal{S}(\Gamma_0) = 2$ (see Definitions 2.2.12).

As before, the displacement function writes

$$\Delta(Z)(x) = \beta(Z) + \lambda(Z)x + \mathcal{O}_2(x). \tag{2.5.9}$$

For the sake of simplicity, we are omitting the parametrization $h_Z(x)$ in the displacement function (2.5.9).

As we have shown before, $\Delta(Z)$ has a unique zero $x^*(\eta(Z))$ in an interval $(-\varepsilon, \varepsilon)$. Now, we have to study how the domain $\sigma(Z)$ of $\Delta(Z)$ changes with Z. Now, we use the parameter $\lambda(Z)$, defined in (2.5.4), to characterize $\sigma(Z)$. Recall that $\lambda(Z) = \widetilde{d}(Z) - \lambda_1(Z)$ and $\lambda_1(Z)$ is given in the unfolding of $T^u_{Z_0}$. Analogously to the proof of Theorem B, we consider a coordinate system $(\bar{x}, \bar{y})$ which trivializes the flow of X at $(0, 0)$. In this coordinate system, $\Sigma = \{(\bar{x}, \gamma_{\lambda_1(Z)}(\bar{x})); \bar{x} \in (-\varepsilon, \varepsilon)\}$ and the transition map $T^u(Z)$ becomes $T^u_*(Z)(\bar{x}) = \gamma_{\lambda_1(Z)}(\bar{x})$, where $\gamma_{\lambda_1(Z)}(\bar{x}) = \kappa(Z)\bar{x}^3 + \lambda_1(Z)\bar{x} + \mathcal{O}_4(\bar{x})$ and $\kappa(Z_0) = -X_0^3 h(0, 0)$.

There is no loss of generality in assuming that $\kappa(Z_0) < 0$, since the case $\kappa(Z_0) > 0$ is completely analogous. Hence, we have the following situation (see Figure 2.21):

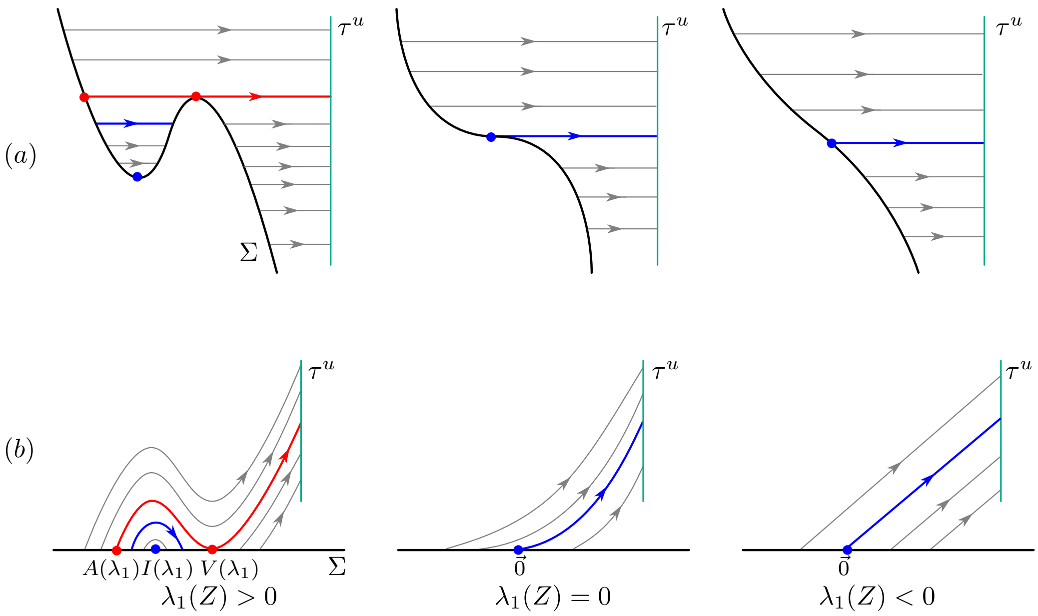


Figure 2.21: Unfolding of the regular-cusp singularity in the coordinate system (a) $(\overline{x}, \overline{y})$ and (b) (x, y).

i) If $\lambda_1(Z) < 0$, all the orbits of X are transversal to Σ, Therefore, $\sigma(Z) = (-\varepsilon, \varepsilon)$;

ii) If $\lambda_1(Z) = 0$, $\sigma(Z) = (-\varepsilon, \varepsilon)$ (see Corollary 2.4.2);

iii) If $\lambda_1(Z) > 0$, then $\gamma_{\lambda_1(Z)}(\bar{x})$ has a minimum at $I(\lambda_1) = -\sqrt{-\dfrac{\lambda_1}{3\kappa}} + \mathcal{O}_1(\lambda_1)$ and a maximum at $V(\lambda_1) = \sqrt{-\dfrac{\lambda_1}{3\kappa}} + \mathcal{O}_1(\lambda_1)$. Therefore, X has a visible regular-fold singularity at $V(\lambda_1)$ and an invisible regular-fold singularity at $I(\lambda_1)$. In addition, the orbit passing through the visible regular-fold singularity intersects Σ backward in time at a point $A(\lambda_1) < I(\lambda_1)$. This means that $\sigma(Z) = (-\varepsilon, A(\lambda_1)] \cup [V(\lambda_1), \varepsilon)$ and $A(\lambda_1), V(\lambda_1) \to 0$ as $\lambda_1 \to 0^+$.

From the discussion above we have the following result.

Theorem C. *Let Z_0 be a nonsmooth vector field having a C-attracting Σ-polycycle Γ_0 containing a unique regular-cusp singularity. Therefore, there exists an annulus $\mathcal{A}_0$ around Γ_0 such that for each annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, there exist neighborhoods $\mathcal{V} \subset \Omega^r$ of Z_0 and $V \subset \mathbb{R}^2$ of $(0,0)$, a surjective function $(\beta, \lambda_1) : \mathcal{V} \to V$, with $(\beta, \lambda_1)(Z_0) = (0,0)$, and three smooth functions $\overline{A}, \overline{V}, \overline{I} : \mathcal{V} \to R$ with $\overline{A}(Z_0) = \overline{V}(Z_0) = \overline{I}(Z_0) = 0$, for which the following statements hold inside $\mathcal{A}$.*

1. *If $\lambda_1(Z) < 0$, then Z has a unique crossing limit cycle of Z, which is hyperbolic attracting.*
2. *If $\lambda_1(Z) = 0$ and $\beta(Z) \neq 0$, then Z has a unique crossing limit cycle of Z, which is hyperbolic attracting.*
3. *If $\lambda_1(Z) = \beta(Z) = 0$, then Z has a unique Σ-polycycle, containing a unique regular-cusp singularity of Z, which is C-attracting.*
4. *If $\lambda_1(Z) > 0$ and $\beta(Z) > \overline{V}(Z)$, then Z has a unique crossing limit cycle of Z, which is hyperbolic attracting.*
5. *If $\lambda_1(Z) > 0$ and $\beta(Z) = \overline{V}(Z)$, then Z has a unique Σ-polycycle, containing a visible unique regular-fold singularity, which is C-attracting.*
6. *If $\lambda_1(Z) > 0$ and $\overline{V}(Z) < \beta(Z) < \overline{I}(Z)$, then Z has a sliding cycle containing a visible regular-fold singularity.*
7. *If $\lambda_1(Z) > 0$ and $\overline{I}(Z) = \beta(Z)$, then Z has a sliding cycle containing a visible regular-fold singularity and an invisible regular-fold singularity.*
8. *If $\lambda_1(Z) > 0$ and $\overline{A}(Z) < \beta(Z) < \overline{I}(Z)$, then Z has a sliding cycle containing a unique visible regular-fold singularity.*
9. *If $\lambda_1(Z) > 0$ and $\beta(Z) = \overline{A}(Z)$, then Z has a unique Σ-polycycle, containing a unique regular-fold singularity, which is C-attracting.*
10. *If $\lambda_1(Z) > 0$ and $\overline{A}(Z) < \beta(Z)$, then Z has a unique crossing limit cycle of Z, which is hyperbolic attracting.*

In addition,

$$\begin{aligned} \overline{A}(Z) = &\ \tilde{d}(Z)A(\lambda_1(Z)) + \mathcal{O}_2(\lambda_1(Z), A(\lambda_1(Z))), \\ \overline{V}(Z) = &\ \tilde{d}(Z)\sqrt{-\frac{\lambda_1(Z)}{3\kappa(Z)}} + \mathcal{O}_1(\lambda_1(Z)), \\ \overline{I}(Z) = &\ -\tilde{d}(Z)\sqrt{-\frac{\lambda_1(Z)}{3\kappa(Z)}} + \mathcal{O}_1(\lambda_1(Z)), \end{aligned} \tag{2.5.10}$$

where $A(\lambda_1(Z))$ *and* $V(\lambda_1(Z))$ *are defined as the extrema of* $\sigma(Z)$ *as follows* $\sigma(Z) = (-\varepsilon, A(\lambda_1(Z))] \cup [V(\lambda_1(Z)), \varepsilon)$.

The theorem above provides the bifurcation diagram of Z_0 in the (β, λ_1)-parameter space (see Figure 2.3).

Proof. From the construction of the auxiliary crossing system (2.5.3), performed in Section 2.5.1, we get the existence of an annulus $\mathcal{A}_0$ around Γ_0 and neighborhoods $\mathcal{V}_0 \subset \Omega^r$ of Z_0 and $V_0 \subset \mathbb{R}^2$ of $(0,0)$, for which the equation (2.5.9) is well defined.

Now, given an annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, let $\varepsilon > 0$ satisfy $(-\varepsilon, \varepsilon) \times \{0\} \subset \mathcal{A}$. Consider the function $\mathcal{X} : \mathcal{V}_0 \times V_0 \to (-\varepsilon, \varepsilon)$ given by (2.5.6), and for a sufficiently small neighborhood $U \subset \mathbb{R}^2$ of the origin, define $\mathcal{B} : \mathcal{V} \times U \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$\mathcal{B}(Z, \beta, \lambda_1, v) = \mathcal{X}(Z, \beta, \lambda_1 - \widetilde{d}(Z)) - v = -\frac{\beta}{\lambda_1 - \widetilde{d}(Z)} - v + \mathcal{O}_2(\beta).$$

Notice that

$$\mathcal{B}(Z_0, 0, 0, 0) = 0, \quad \text{and} \quad \partial_\beta \mathcal{B}(Z_0, 0, 0, 0) = \frac{1}{\widetilde{d}(Z_0)} \neq 0.$$

From the Implicit Function Theorem for Banach Spaces, there exist $\delta > 0$, an open interval J containing 0, and a unique $\mathcal{C}^r$ function $\beta^* : \mathcal{V} \times J \times (-\varepsilon, \varepsilon) \to (-\delta, \delta)$ such that $\mathcal{B}(Z, \beta, \lambda, v) = 0$ if, and only if $\beta = \beta^*(Z, \lambda_1, v)$. Also, we can see that

$$\beta^*(Z, \lambda_1, v) = \widetilde{d}(Z)v - \lambda_1 v + \mathcal{O}_2(v).$$

Notice that, if $\overline{A}(Z) = \beta^*(Z, \lambda_1(Z), A(\lambda_1(Z)))$ and $\overline{V}(Z) = \beta^*(Z, \lambda_1(Z), V(\lambda_1(Z)))$, then $\mathcal{X}^*(Z, \overline{A}(Z), \lambda_1(Z)) = A(\lambda_1(Z))$ and $\mathcal{X}^*(Z, \overline{V}(Z), \lambda_1(Z)) = V(\lambda_1(Z))$. Since

$$V(\lambda_1(Z)) = \sqrt{-\frac{\lambda_1(Z)}{3\kappa(Z)}} + \mathcal{O}_1(\lambda_1(Z)),$$

we get $\overline{V}(Z)$ from (2.5.10).

From construction of the maps $T^u(Z)$, $T^s(Z)$ and $D(Z)$ given in (2.5.1) and (2.5.2), it follows that the points $I(\lambda_1(Z))$ and $V(\lambda_1(Z))$ are connected by an orbit of $Z = (X, Y)$ if, and only if,

$$G(Z) =: T^u(Z)(V(\lambda_1(Z))) - [D(Z)]^{-1} \circ T^s(Z)(I(\lambda_1(Z))) = 0.$$

Notice that

$$G(Z) = \beta(Z) + \widetilde{d}(Z)\sqrt{-\frac{\lambda_1(Z)}{3\kappa(Z)}} + \mathcal{O}_1(\lambda_1(Z)).$$

Thus, applying the Implicit Function Theorem to the function $\mathcal{G} : \mathcal{V} \times (-\delta, \delta) \to \mathbb{R}$, given by $\mathcal{G}(Z, \beta) := G(Z) - \beta(Z) + \beta$, at the point $(Z_0, 0)$, we get a unique $\mathcal{C}^r$ function $\overline{I} : \mathcal{V} \to (-\delta, \delta)$ such that $\mathcal{G}(Z, \beta) = 0$ if, and only if, $\beta = \overline{I}(Z)$. Hence, the points $V(\lambda_1(Z))$ and $A(\lambda_1(Z))$ are connected by an orbit of Z if, and only if $\beta(Z) = \overline{I}(Z)$. In this case,

$$\overline{I}(Z) = -\widetilde{d}(Z)\sqrt{-\frac{\lambda_1(Z)}{3\kappa(Z)}} + \mathcal{O}_1(\lambda_1(Z)).$$

From here, the proof follows directly from the definitions of the curves $\overline{A}, \overline{V}$ and $\overline{I}$, and Propositions 2.5.1 and 2.5.2. □

2.5.4 Σ-Polycycles having several regular-tangential singularities

Now we perform an analysis of a class of Σ-polycyles having several regular-tangential singularities and we obtain similar results for those in Section 2.5.2. Consider the class of nonsmooth vector fields $Z_0 = (X_0, Y_0)$ which admit a Σ-polycycle having k regular-tangential singularities, $p_i \in \Sigma$, of order n_i, $i = 1, \dots, k$ satisfying the following property:

(A) for each $i = 1, \dots, k$, there exists a curve γ_i connecting p_i and p_{i+1}, oriented from p_i to p_{i+1}, such that $\gamma_i \setminus \{p_i, p_{i+1}\}$ is a regular orbit of Z_0, γ_i is tangent to Σ at p_i and transversal to Σ at p_{i+1}, where $p_{k+1} = p_1$ (see Figure 2.22).

In what follows, without loss of generality, we assume that $h(x, y) = y$, $p_1 = (0, 0)$, and $p_i = (a_i, 0)$, $i = 2, \cdots, k$.

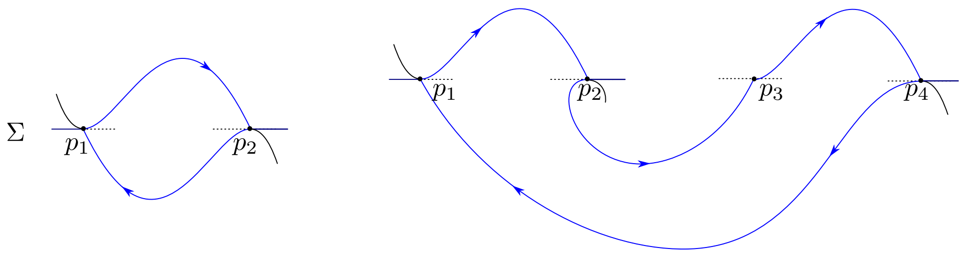


Figure 2.22: Σ-polycycles satisfying hypothesis (A).

Following the constructions presented in Sections 2.5.1 and 2.5.2 the displacement functions $\Delta_i(Z) : \sigma_i(Z) \to \mathbb{R}$ are given by

$$\Delta_i(Z)(h_Z^i(x_i), h_Z^{i+1}(x_{i+1})) = \beta_i(Z) + \widetilde{\lambda}_i(Z)x_{i+1}^s + \mathcal{O}_2(x_i^u) + \mathcal{O}_2(x_{i+1}^s),$$

where $x_i^{s,u} = x_i - a_i$, $\beta_i(Z) = \Delta_i(Z)(h_Z^i(a_i), h_Z^{i+1}(a_{i+1}))$ satisfies $\beta_i(Z_0) = 0$, and $\widetilde{\lambda}_i(Z) = \lambda_1^i(Z) - \widetilde{d}_i(Z)$ satisfies $\widetilde{\lambda}_i(Z_0) = -\widetilde{d}_i(Z_0) \neq 0$. Thus, there exists a neighborhood $\mathcal{V}$ of Z_0 such that for each $Z \in \mathcal{V}$ and $i = 1, \dots, k$, $\widetilde{\lambda}_i(Z) = -\widetilde{d}_i(Z) \neq 0$ and the crossing system (2.5.3) is given by

$$\begin{cases} \Delta_1(Z)(h_Z^1(x_1), h_Z^2(x_2)) = \beta_1(Z) + \widetilde{\lambda}_1(Z)x_2^s + \mathcal{O}_2(x_1^u) + \mathcal{O}_2(x_2^s) = 0, \\ \Delta_2(Z)(h_Z^2(x_2), h_Z^3(x_3)) = \beta_2(Z) + \widetilde{\lambda}_2(Z)x_3^s + \mathcal{O}_2(x_2^u) + \mathcal{O}_2(x_3^s) = 0, \\ \vdots \\ \Delta_{k-1}(Z)(h_Z^{k-1}(x_{k-1}), h_Z^k(x_k)) = \beta_{k-1}(Z) + \widetilde{\lambda}_{k-1}(Z)x_k^s + \mathcal{O}_2(x_{k-1}^u) + \mathcal{O}_2(x_k^s) = 0, \\ \Delta_k(Z)(h_Z^k(x_k), h_Z^1(x_1)) = \beta_k(Z) + \widetilde{\lambda}_k(Z)x_1^s + \mathcal{O}_2(x_k^u) + \mathcal{O}_2(x_1^s) = 0, \\ x_i^{s,u} = x_i - a_i, \ i = 1, \cdots, k, \\ h_Z^i(x_i) \in \sigma_i(Z), \ i = 1, \cdots, k. \end{cases} \tag{2.5.11}$$

So for the Σ-polycycle Γ_0 we have the following proposition.

Proposition 2.5.4. *Let Γ_0 be a Σ-polycycle having k regular-tangential singularities $p_i \in \Sigma$, of order n_i, $i = 1, \dots, k$, satisfying the property (A). Then, Γ_0 attracts the orbits passing through the section $\sigma_1(Z_0)$ (domain of $T_1^u(Z_0)$). In this case, we say that Γ_0 is C-attracting.*

Proof. Notice that the first return map associated with the Σ-polycycle Γ_0 of Z_0 is given by

$$\mathcal{P}_0(x) = ([D_k]^{-1} \circ T_1^s)^{-1} \circ T_k^u \circ ([D_{k-1}]^{-1} \circ T_k^s)^{-1} \circ T_{-1}^u \circ \dots \circ ([D_1]^{-1} \circ T_2^s)^{-1} \circ T_1^u(x)$$

where, from (2.5.1) and (2.5.2) (recall that $h_{Z_0}^1 = Id$),

$$T_i^u(x) = \kappa_i(Z_0)x^{n_i} + \mathcal{O}_{n_i+1}(x) \quad \text{and} \quad [D_{i-1}]^{-1} \circ T_i^s(x) = \widetilde{d}_i(Z_0)\, x + \mathcal{O}_2(x).$$

Hence,

$$\mathcal{P}_0(x) = \prod_{i=1}^{k} \frac{\kappa_i(Z_0)}{\widetilde{d}_i(Z_0)} x^{n_i} + \mathcal{O}_N(x), \quad N = n_1 + n_2 + \dots + n_k + 1.$$

Therefore, for $|x|$ small enough, $|\mathcal{P}_0(x)| < |x|$, which means that Γ_0 attracts the orbits passing through the section $\sigma_1(Z_0)$ (domain of $T_1^u(Z_0)$). □

Set $\eta(Z) = (\beta(Z), \widetilde{\lambda}(Z))$ with $\beta(Z) = (\beta_1(Z), \dots, \beta_k(Z))$ and $\widetilde{\lambda}(Z) = (\widetilde{\lambda}_1(Z), \dots, \widetilde{\lambda}_k(Z))$, and denote $\widetilde{d}(Z) = (\widetilde{d}_1(Z), \dots, \widetilde{d}_k(Z))$. Notice that $\eta : \mathcal{V} \to V$ is surjective onto a neighborhood of $(0, -\widetilde{d}(Z_0)) \in V$. Now, we present the main result of this section which is an extension of the Proposition 2.5.2.

Proposition 2.5.5. *Let Γ_0 be a Σ-polycycle of $Z_0 = (X_0, Y_0) \in \Omega^r$ having k regular-tangential singularities $p_i \in \Sigma$, of order n_i, $i = 1, \dots, k$, satisfying property (A). Then, the following statements hold.*

i) There exists an annulus $\mathcal{A}_0$ at Γ_0 and a neighborhood $\mathcal{V}$ of Z_0 such that each $Z \in \mathcal{V}$ has at most one crossing limit cycle bifurcating from Γ_0 in $\mathcal{A}_0$, which is hyperbolic attracting.

ii) Let $Z_{\beta,\widetilde{\lambda}}$ be a continuous $2k$-parameter family in $\mathcal{V}$ such that $Z_{0,-\widetilde{d}(Z_0)} = Z_0$ and satisfying $Z_{\beta,\widetilde{\lambda}} \in \eta^{-1}(\beta, \widetilde{\lambda})$, for every $(\beta, \widetilde{\lambda}) \in V$. Then, for each $\widetilde{\lambda}$ near $-\widetilde{d}(Z_0)$ and for each connected component C_i of $\sigma_i(Z_{\beta,\widetilde{\lambda}}) \cap \mathcal{A}_0$, there exist non-empty open intervals $I_{\widetilde{\lambda},C_i}$, satisfying $I_{\widetilde{\lambda},C_1} \times \dots \times I_{\widetilde{\lambda},C_k} \times \{\widetilde{\lambda}\} \subset V$, such that $Z_{\beta,\lambda}$ has a hyperbolic attracting crossing limit cycle passing through $C_1 \times \dots \times C_k$, for each $\beta \in I_{\widetilde{\lambda},C_1} \times \dots \times I_{\widetilde{\lambda},C_k}$.

Proof. As seen before, there exists a neighborhood $\mathcal{V}$ of Z_0 in Ω^r such that, for each $Z = (X, Y) \in \mathcal{V}$, we associate the displacement functions $\Delta_i(Z)$, $i = 1, \dots, k$, given in (2.5.11), which can be extended to $\widetilde{\Delta}_i(Z) : (-\varepsilon, \varepsilon) \to \mathbb{R}$ (see Section 2.3.3).

Define the $\mathcal{C}^r$ function $\mathcal{F} : \mathcal{V} \times V \times (-\varepsilon, \varepsilon)^k \to \mathbb{R}^k$ as

$$\mathcal{F}(Z, \beta, \widetilde{\lambda}, x) = (\mathcal{F}_1(Z, \beta, \widetilde{\lambda}, x), \dots, \mathcal{F}_k(Z, \beta, \widetilde{\lambda}, x)),$$

where V is an open neighborhood of $(0, -\widetilde{d}(Z_0)) \in \mathbb{R}^k \times \mathbb{R}^k$ and, for $i = 1, \dots, k$,

$$\mathcal{F}_i(Z, \beta, \widetilde{\lambda}, x) = \widetilde{\Delta}_i(Z)(h_Z^i(x_i), h_Z^{i+1}(x_{i+1})) - \beta_i(Z) - \widetilde{\lambda}_i(Z)x_{i+1} + \beta_i + \widetilde{\lambda}_i x_{i+1},$$

with $x_{k+1} = x_1$ and $h_Z^{k+1} = h_Z^1$.

Notice that $\mathcal{F}(Z_0, 0, -\widetilde{d}(Z_0), 0) = (0, \dots, 0)$ and

$$D_x\mathcal{F}(Z_0, 0, -\widetilde{d}(Z_0), 0) = \begin{pmatrix} 0 & -\widetilde{d}_2(Z_0) & 0 & \cdots & 0 \\ 0 & 0 & -\widetilde{d}_3(Z_0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\widetilde{d}_k(Z_0) \\ -\widetilde{d}_1(Z_0) & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From the Implicit Function Theorem for Banach Spaces and reducing $\mathcal{V}$ and V if necessary, there exists a unique $\mathcal{C}^r$ function $\mathcal{X} : \mathcal{V}\times V \to (-\varepsilon, \varepsilon)^k$ such that $\mathcal{F}(Z, \beta, \widetilde{\lambda}, x) = 0$ if, and only if, $x = \mathcal{X}(Z, \beta, \widetilde{\lambda})$. Since

$$\mathcal{F}(Z, \beta, \widetilde{\lambda}, x) = (\beta_1 + \widetilde{\lambda}_1 x_2, \dots, \beta_{k-1} + \widetilde{\lambda}_{k-1} x_k, \beta_k + \widetilde{\lambda}_k x_1) + \mathcal{O}_2(x),$$

it follows that $\mathcal{X}(Z, 0, \widetilde{\lambda}) = 0$, for any $(Z, 0, \widetilde{\lambda}) \in \mathcal{V} \times V$. Consequently,

$$\mathcal{X}(Z, \beta, \widetilde{\lambda}) = -\left(\frac{\beta_k}{\widetilde{\lambda}_k}, \frac{\beta_1}{\widetilde{\lambda}_1}, \dots, \frac{\beta_{k-1}}{\widetilde{\lambda}_{k-1}}\right) + \mathcal{O}_2(\beta).$$

From the definition of the function $\mathcal{F}$, the unique zero of $\widetilde{\Delta}(Z) = (\widetilde{\Delta}_1(Z), \dots, \widetilde{\Delta}_k(Z))$ in $(-\varepsilon, \varepsilon)^k$ is given by

$$\mathcal{X}^*(Z) = \mathcal{X}(Z, \beta(Z), \widetilde{\lambda}(Z)).$$

Hence, system (2.5.11) has at most one zero in $\sigma_1(Z) \times \cdots \times \sigma_k(Z)$. Moreover, since

$$\frac{\partial \widetilde{\Delta}_i(Z_0)}{\partial x_{i+1}}(\mathcal{X}^*(Z_0)) = -\widetilde{d}_i(Z_0) > 0,$$

it follows that

$$\frac{\partial \widetilde{\Delta}_i(Z)}{\partial x_{i+1}}(\mathcal{X}^*(Z)) = \widetilde{\lambda}_i(Z) + \mathcal{O}_1(\mathcal{X}^*(Z)) > 0, \quad i = 1, \dots, k, \tag{2.5.12}$$

for Z sufficiently near Z_0.

Now, for $Z \in \mathcal{V}$ suppose that the solution $\mathcal{X}^*(Z) = (x_1^*(Z), \dots, x_k^*(Z)) \in \text{int}(\sigma_1(Z) \times \cdots \times \sigma_k(Z))$ of system (2.5.11) is associated with a crossing limit cycle of Z. From the Implicit Function Theorem, for each x_1 sufficiently close to $x_1^*(Z)$ the orbit of Z, starting at $(x_1, 0) \in \sigma_1(Z) \times \{0\}$, intersects each $\text{int}(\sigma_i(Z)) \times \{0\}$ at $(\xi_i(x_1), 0)$ with $\xi_i(x_1)$ near $x_i^*(Z)$. Notice that $\widetilde{\Delta}_i(Z)(h_Z^i(\xi_i(x_1)), h_Z^{i+1}(\xi_{i+1}(x_1))) = 0$, for $i = 1, \dots, k-1$. Consequently,

$$x_1 \mapsto \widetilde{\Delta}_k(Z)(h_Z^k(\xi_k(x_1)), h_Z^1(x_1))$$

is the displacement function associated with the crossing limit cycle defined in neighborhood of $x_1^*(Z)$ in $\text{int}(\sigma_1(Z)) \times \{0\}$. Clearly, the above displacement function vanishes at $x_1^*(Z)$. Moreover, from (2.5.12), the derivative of displacement function at x_1^* is positive. Therefore, when the crossing limit cycle exists, it is hyperbolic and attracting.

The proof of item (i) follows by taking $\mathcal{A}_0 = \{p \in M; d(p, \Gamma_0) < \varepsilon\}$, where d denotes the Hausdorff distance.

Now, consider the family $Z_{\beta,\widetilde{\lambda}}$ given in item (ii). The unique zero of $\widetilde{\Delta}(Z_{\beta,\widetilde{\lambda}})$ is given by

$$\mathcal{X}^*(Z_{\beta,\widetilde{\lambda}}) = -\left(\frac{\beta_k}{\widetilde{\lambda}_k}, \frac{\beta_1}{\widetilde{\lambda}_1}, \dots, \frac{\beta_{k-1}}{\widetilde{\lambda}_{k-1}}\right) + \mathcal{O}_2(\beta). \tag{2.5.13}$$

Recall that each isolated solution x^* of system (2.5.11) represents either a crossing limit cycle (if $x^* \in \text{int}(\sigma_1(Z_{\beta,\widetilde{\lambda}}) \times \cdots \times \sigma_k(Z_{\beta,\widetilde{\lambda}})))$ or a Σ-polycycle (if $x^* \in \partial(\sigma_1(Z_{\beta,\widetilde{\lambda}}) \times \cdots \times \sigma_k(Z_{\beta,\widetilde{\lambda}})))$. So, for $i = 1, \ldots, k$, let C_i be a connected component of $\sigma_i(Z_{\beta,\widetilde{\lambda}}) \subset (-\varepsilon, \varepsilon)$ for some fixed parameter $\widetilde{\lambda} \in \pi_2(V)$. Hence, from (2.5.13), there exists a non-empty open interval $I_{\widetilde{\lambda},C_i}$ such that $I_{\widetilde{\lambda},C_1} \times \ldots \times I_{\widetilde{\lambda},C_i} \times \{\widetilde{\lambda}\} \subset V$ and $\mathcal{X}^*(Z_{\beta,\widetilde{\lambda}}) \in \text{int}(C_1 \times \ldots \times C_k)$ whenever $\beta_i \in I_{\widetilde{\lambda},C_i}$. $\square$

Remark 2.5.6. *Regarding Propositions 2.5.4 and 2.5.5, if we change the orientation in property (A) in order to reverse the orientation of the Σ-polycycle, all the results remain the same reversing the stability the Σ-polycycle and the crossing limit cycle.*

These results are illustrated in the next section for the case where the Σ-polycycle has two fold-regular singularities.

2.5.5 Σ-Polycycles having two regular-fold singularities

Firstly, without loss of generality, we assume some conditions in order to characterize the nonsmooth vector fields $Z_0 = (X_0, Y_0) \in \Omega^r$ which admit a Σ-polycycle Γ_0 satisfying (A) and having only two regular-fold singularities (see Figure 2.23). So, consider a coordinate system (x, y) such that $x(p_1) = a_1$, $y(p_1) = 0$, $x(p_2) = a_2 > 0$, $y(p_2) = 0$ and $h(x, y) = y$ in neighborhoods of p_1 and p_2. Consider the following sets of hypotheses:

(DRF-A):
- p_1 is a visible regular-fold singularity of X_0 and $\pi_1 \circ X_0(p_1) > 0$;
- p_2 is a visible regular-fold singularity of Y_0 and $\pi_1 \circ Y_0(p_2) < 0$;
- $W^u_+(p_1)$ reaches Σ transversally at p_2;
- $W^u_-(p_2)$ reaches Σ transversally at p_1

(DRF-B):
- p_1 is a visible regular-fold singularity of X_0 and $\pi_1 \circ X_0(p_1) < 0$;
- p_2 is a visible regular-fold singularity of Y_0 and $\pi_1 \circ Y_0(p_2) > 0$;
- $W^u_+(p_1)$ reaches Σ transversally at p_2;
- $W^u_-(p_2)$ reaches Σ transversally at p_1

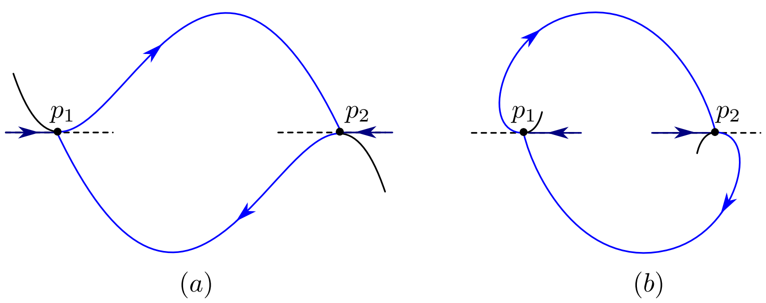


Figure 2.23: Σ-polycycle Γ_0 of Z_0 under the set of hypotheses (a) (DRF-A) and (b) (DRF-B), respectively.

Hypotheses (DRF-A) and (DRF-B) fix the orientation and the stability of the Σ-polycycle Γ_0. Indeed, in this case Γ_0 is C-attracting. According to Remark 2.5.6, the stability of Γ_0 is reversed if we change the orientation.

Remark 2.5.7. *Notice that there are other closed connections containing two regular-fold singularities which are not Σ-polycycles since they violate condition (iv) of Definition 2.2.7 (see Example 2.2.9).*

Here we shall assume that Z_0 satisfies (DRF-A), the case (DRF-B) will follow analogously. In this case, Z_0 admits a Σ-polycycle Γ_0 given by the union $W^u_+(a_1,0)\cup W^u_-(a_2,0)\cup\{(a_1,0),(a_2,0)\}$. We shall see that $\mathcal{S}(\Gamma_0)=2$.

Since regular-fold singularities are locally structurally stable, they persist under small perturbations. Consequently, without loss of generality, we may assume that the diffeomorphisms h^i_Z, $i=1,2$, provenient from Theorem B may be taken as the identity. Accordingly, the displacement functions write

$$\begin{aligned}\Delta_1(Z)(x_1,x_2) =\ & T^u_1(Z)(x_1)-[D_1(Z)]^{-1}\circ T^s_2(Z)(x_2)\\ =\ & \lambda^1_0(Z)+\kappa_1(Z)(x_1-a_1)^2+\mathcal{O}_3(x_1-a_1)\\ & -\widetilde{c}_1(Z)-\widetilde{d}_1(Z)(x_2-a_2)+\mathcal{O}_2(x_2-a_2),\\ \Delta_2(Z)(x_2,x_1) =\ & T^u_2(Z)(x_2)-[D_2(Z)]^{-1}\circ T^s_1(Z)(x_1)\\ =\ & \lambda^2_0(Z)+\kappa_2(Z)(x_2-a_2)^2+\mathcal{O}_3(x_2-a_2)\\ & -\widetilde{c}_2(Z)-\widetilde{d}_2(Z)(x_1-a_1)+\mathcal{O}_2(x_1-a_1),\end{aligned}$$

where $\kappa_1(Z)<0$, $\kappa_2(Z)>0$, and $\widetilde{d}_i(Z)>0$, for $i=1,2$. Therefore, denoting $\beta_i(Z)=\lambda^i_0(Z)-\widetilde{c}_i(Z)$, $i=1,2$, (see Figure 2.24) the auxiliary crossing system (2.5.3) becomes

$$\begin{cases}\beta_1(Z)-\widetilde{d}_1(Z)\xi_2+\kappa_1(Z)\xi_1^2+\mathcal{O}_2(\xi_2)+\mathcal{O}_3(\xi_1)=0,\\ \beta_2(Z)-\widetilde{d}_2(Z)\xi_1+\kappa_2(Z)\xi_2^2+\mathcal{O}_2(\xi_2)+\mathcal{O}_3(\xi_1)=0,\\ \xi_i=x_i-a_i,\ i=1,2,\\ (x_1,x_2)\in\sigma_1(Z)\times\sigma_2(Z)=[a_1,a_1+\varepsilon)\times(a_2-\varepsilon,a_2].\end{cases}\tag{2.5.14}$$

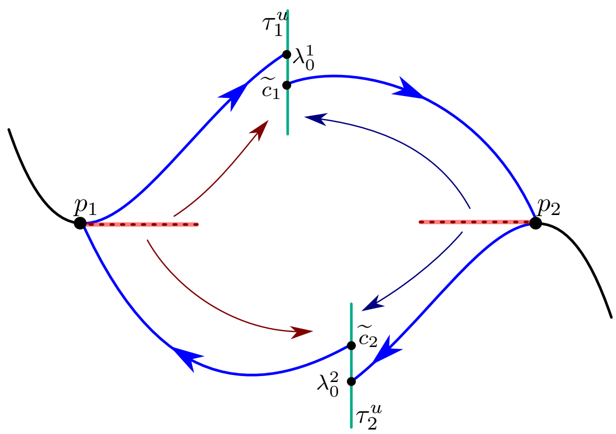


Figure 2.24: Splitting of the separatrices for a perturbed system $Z\in\mathcal{V}$.

In what follows we use the auxiliary crossing system (2.5.14) to describe the bifurcation diagram of Z_0 at Γ_0 assuming the set of hypotheses (DRF-A) (see Figure 2.4).

Theorem D. *Let Z_0 be a nonsmooth vector field having a Σ-polycycle satisfying the set of hypotheses (DRF-A). Therefore, there exists an annulus $\mathcal{A}_0$ around Γ_0 such that for* *each annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, there exist neighborhoods $\mathcal{V} \subset \Omega^r$ of Z_0 and $V \subset \mathbb{R}^2$ of $(0,0)$, a surjective function $(\beta_1, \beta_2) : \mathcal{V} \to V$ with $(\beta_1, \beta_2)(Z_0) = (0,0)$, and two smooth functions $\gamma_1, \gamma_2 : \mathcal{V} \to \mathbb{R}$ with $\gamma_1(Z_0) = \gamma_2(Z_0) = 0$, for which the following statements hold inside $\mathcal{A}$.*

1. *If $\beta_2(Z) > 0$ and $\beta_1(Z) > \gamma_1(Z)$, then Z has a sliding cycle containing the regular-fold singularity p_2 and a unique sliding segment.*

2. *If $\beta_2(Z) > 0$ and $\beta_1(Z) = \gamma_1(Z)$, then Z has a C-attracting Σ-polycycle containing the regular-fold singularity p_2.*

3. *If $\beta_2(Z) > 0$ and $0 < \beta_1(Z) < \gamma_1(Z)$, then Z has a hyperbolic attracting crossing limit cycle.*

4. *If $\beta_2(Z) > 0$ and $\beta_1(Z) = 0$, then Z has a hyperbolic attracting crossing limit cycle and a heteroclinic connection between p_1 and p_2.*

5. *If $\beta_2(Z) > 0$ and $\beta_1(Z) < 0$, then Z has a hyperbolic attracting crossing limit cycle.*

6. *If $\beta_2(Z) = 0$ and $\beta_1(Z) < 0$, then Z has a hyperbolic attracting crossing limit cycle and a heteroclinic connection between p_1 and p_2.*

7. *If $\beta_2(Z) = \beta_1(Z) = 0$, then Z has a C-attracting Σ-polycycle containing two regular-fold singularities.*

8. *If $\beta_1(Z) < 0$ and $\gamma_2(Z) < \beta_2(Z) < 0$, then Z has a hyperbolic attracting crossing limit cycle.*

9. *If $\beta_1(Z) < 0$ and $\beta_2(Z) = \gamma_2(Z)$, then Z has a C-attracting Σ-polycycle containing the regular-fold singularity p_1.*

10. *If $\beta_1(Z) < 0$ and $\beta_2(Z) < \gamma_2(Z)$, then Z has a sliding cycle containing the regular-fold singularity p_1 and a unique sliding segment.*

11. *If $\beta_2(Z) < 0$ and $\beta_1(Z) = 0$, then Z has a sliding cycle containing two regular-fold singularities and one sliding segment.*

12. *If $\beta_2(Z) < 0$ and $\beta_1(Z) > 0$, then Z has a sliding cycle containing two regular-fold singularities and two sliding segments.*

13. *If $\beta_2(Z) = 0$ and $\beta_1(Z) > 0$, then Z has a sliding cycle containing two regular-fold singularities and one sliding segment.*

Here,

$$\gamma_1(Z) = -\frac{\kappa_1(Z)}{\tilde{d}_2(Z)^2}\beta_2(Z)^2 + \mathcal{O}_3(\beta_2(Z)) \text{ and } \gamma_2(Z) = -\frac{\kappa_2(Z)}{\tilde{d}_1(Z)^2}\beta_1(Z)^2 + \mathcal{O}_3(\beta_1(Z)).$$

In addition, in the cases (1), *and* $(10) - (13)$, *Z does not admit limit cycles.*

Proof. From the construction of the auxiliary crossing system (2.5.3), performed in Section 2.5.1, we get the existence of an annulus $\mathcal{A}_0$ around Γ_0 and neighborhoods $\mathcal{V}_0 \subset \Omega^r$ of Z_0 and $V_0 \subset \mathbb{R}^2$ of $(0,0)$, for which the auxiliary crossing system (2.5.14) is well defined.

Now, given an annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, let $\varepsilon > 0$ satisfy $[a_1, a_1+\varepsilon) \times \{0\} \subset \mathcal{A}$ and $(a_2-\varepsilon, a_2] \times \{0\} \subset \mathcal{A}$. Consider the function $F : \mathcal{V}_0 \times (-\varepsilon, \varepsilon)^2 \times V_0 \to \mathbb{R}^2$ given by

$$F(Z, \xi_1, \xi_2, \beta_1, \beta_2) = (F_1(Z, \xi_1, \xi_2, \beta_1), F_2(Z, \xi_1, \xi_2, \beta_2)),$$

where

$$F_1(Z, \xi_1, \xi_2, \beta_1) = \widetilde{F_1}(Z, \xi_1, \xi_2) - \beta_1(Z) + \beta_1,$$
$$F_2(Z, \xi_1, \xi_2, \beta_2) = \widetilde{F_2}(Z, \xi_1, \xi_2) - \beta_2(Z) + \beta_2,$$

and $\widetilde{F_1}$ and $\widetilde{F_2}$ are given by the left-hand side of the first two equations of (2.5.14).

Notice that $F(Z_0, 0, 0, 0, 0) = (0, 0)$ and $\det[D_{(\xi_1,\xi_2)} F(Z_0, 0, 0, 0, 0)] = -\widetilde{d}_1(Z_0)\widetilde{d}_2(Z_0) \neq 0$. From the Implicit Function Theorem for Banach Spaces, there exist neighborhoods $\mathcal{V} \subset \mathcal{V}$ and $V \subset V_0$ and unique $\mathcal{C}^r$ functions $\Xi_1, \Xi_2 : \mathcal{V} \times V \to (-\varepsilon, \varepsilon)$ such that

$$F(Z, \Xi_1(Z, \beta_1, \beta_2), \Xi_2(Z, \beta_1, \beta_2), \beta_1, \beta_2) = (0, 0).$$

Consequently, for each $Z \in \mathcal{V}$, the auxiliary crossing system (2.5.14) has at most one solution. In fact, (2.5.14) is satisfied if, and only if,

$$(\xi_1, \xi_2) = (\Xi_1(Z, \beta_1(Z), \beta_2(Z)), \Xi_2(Z, \beta_1(Z), \beta_2(Z))) \in [0, \varepsilon) \times (-\varepsilon, 0]. \tag{2.5.15}$$

Therefore, each $Z \in \mathcal{V}$ has either a Σ-polycycle having a unique regular-fold singularity (which occurs when $\xi_1 = 0$ or $\xi_2 = 0$) or at most one crossing limit cycle.

In what follows, we find parameters $(\beta_1(Z), \beta_2(Z))$ satisfying (2.5.15).

First, $\Xi_2(Z, \beta_1(Z), \beta_2(Z)) = 0$ implies the existence of a Σ-polycycle of Z passing through the regular-fold singularity p_2. Applying the Implicit Function Theorem to $g(Z, \xi_1, \beta_2) = F_2(Z, \xi_1, 0, \beta_2)$ at $(Z_0, 0, 0)$, we obtain the existence of a unique $\mathcal{C}^r$ function $\widetilde{\Xi}_1(Z, \beta_2)$ such that $g(Z, \widetilde{\Xi}_1(Z, \beta_2), \beta_2) = 0$. In addition,

$$\widetilde{\Xi}_1(Z, \beta_2) = \frac{\beta_2}{\widetilde{d}_2(Z)} + \mathcal{O}_2(\beta_2) = \mathcal{O}_1(\beta_2).$$

Now, applying the Implicit Function Theorem to $h(Z, \beta_1, \beta_2) = F_1(Z, \widetilde{\Xi}_1(Z, \beta_2), 0, \beta_1)$ at the point $(Z_0, 0, 0)$ we obtain a function $\overline{\beta_1}(Z, \beta_2)$ such that $h(Z, \overline{\beta_1}(Z, \beta_2), \beta_2) = 0$. It follows directly from the expression of h that

$$\overline{\beta_1}(Z, \beta_2) = -\frac{\kappa_1(Z)}{\widetilde{d}_2(Z)^2}\beta_2^2 + \mathcal{O}_3(\beta_2).$$

Hence, it shows that $F(Z, \widetilde{\Xi}_1(Z, \beta_2(Z)), 0, \overline{\beta_1}(Z, \beta_2(Z)), \beta_2(Z)) = (0, 0)$. From uniqueness of the solution,

$$\Xi_1(Z, \overline{\beta_1}(Z, \beta_2(Z)), \beta_2(Z)) = \widetilde{\Xi}_1(Z, \beta_2(Z)) \quad \text{and} \quad \Xi_2(Z, \overline{\beta_1}(Z, \beta_2(Z)), \beta_2(Z)) = 0.$$

Thus, $\Xi_2(Z, \beta_1(Z), \beta_2(Z)) = 0$ if, and only if, $\beta_1(Z) = \overline{\beta_1}(Z, \beta_2(Z))$. Moreover, since $\widetilde{d}_2(Z) > 0$, it follows that $\widetilde{\Xi}_1(Z, \beta_2(Z)) \in [0, \varepsilon)$ if, and only if, $\beta_2(Z) \geq 0$. Finally, defining $\gamma_1(Z) = \overline{\beta_1}(Z, \beta_2(Z))$, we have that each $Z \in \mathcal{V}$, satisfying $\beta_1(Z) = \gamma_1(Z)$ and $\beta_2(Z) \geq 0$, has a Σ-polycycle containing a unique regular-fold singularity, namely $p_2 = (a_2, 0)$.

Analogously, $\Xi_1(Z, \beta_1(Z), \beta_2(Z)) = 0$ implies the existence of a Σ-polycycle of Z passing through the regular-fold singularity p_1. Following the same ideas above, we obtain a unique $\mathcal{C}^r$ function $\widetilde{\Xi}_2(Z, \beta_1)$ such that $F_1(Z, 0, \widetilde{\Xi}_2(Z, \beta_1), \beta_1) = 0$. Furthermore

$$\widetilde{\Xi}_2(Z, \beta_1) = \frac{\beta_1}{\widetilde{d}_1(Z)} + \mathcal{O}_2(\beta_1).$$

Also, we obtain a unique $\mathcal{C}^r$ function $\overline{\beta_2}(Z,\beta_1)$ such that $F_2(Z,0,\tilde{\Xi}_2(Z,\beta_1),\overline{\beta_2}(Z,\beta_1))=0$ and

$$\overline{\beta_2}(Z,\beta_1)=-\frac{\kappa_2(Z)}{\tilde{d}_1(Z)^2}\beta_1^2+\mathcal{O}_3(\beta_1).$$

Therefore, $F(Z,0,\tilde{\Xi}_2(Z,\beta_1(Z)),\beta_1(Z),\overline{\beta_2}(Z,\beta_1(Z)))=(0,0)$. Again, from uniqueness of the solution, it follows that

$$\Xi_1(Z,\beta_1(Z),\overline{\beta_2}(Z,\beta_1(Z)))=0 \quad \text{and} \quad \Xi_2(Z,\beta_1(Z),\overline{\beta_2}(Z,\beta_1(Z)))=\tilde{\Xi}_2(Z,\beta_1(Z)).$$

Hence, $\Xi_1(Z,\beta_1(Z),\beta_2(Z))=0$ if, and only if, $\beta_2(Z)=\overline{\beta_2}(Z,\beta_1(Z))$. Also, since $\tilde{d}_1(Z)>0$, it follows that $\tilde{\Xi}_2(Z,\beta_1(Z))\in(-\varepsilon,0]$ if, and only if, $\beta_1(Z)\leq 0$. Defining $\gamma_2(Z)=\overline{\beta_2}(Z,\beta_1(Z))$, we have that each $Z\in\mathcal{V}$ satisfying $\beta_2(Z)=\gamma_2(Z)$ and $\beta_1(Z)\leq 0$ has a Σ-polycycle containing a unique regular-fold singularity given by $p_1=(a_1,0)$.

The C-attractiveness of the Σ-polycycle detected above is given by Proposition 2.5.4. Hence, items (2), (7) and (9) are proved.

In what follows we shall identify when the solution $(\Xi_1(Z,\beta_1(Z),\beta_2(Z)),\Xi_2(Z,\beta_1(Z),\beta_2(Z)))$ of the auxiliary crossing system (2.5.14) corresponds to a crossing limit cycle.

Note that

$$\begin{aligned}\Xi_1(Z,\beta_1(Z),\beta_2(Z))&=\frac{1}{\tilde{d}_2(Z)}\beta_2(Z)+\mathcal{O}_2(\beta_1(Z),\beta_2(Z)),\\ \Xi_2(Z,\beta_1(Z),\beta_2(Z))&=\frac{1}{\tilde{d}_1(Z)}\beta_1(Z)+\mathcal{O}_2(\beta_1(Z),\beta_2(Z)).\end{aligned}\tag{2.5.16}$$

Recall that $\Xi_2(Z,\gamma_1(Z),\beta_2(Z))=0$. Using (2.5.16), we expand $\Xi_2(Z,\beta_1(Z),\beta_2(Z))$ around $\beta_1(Z)=\gamma_1(Z)$ as

$$\Xi_2(Z,\beta_1(Z),\beta_2(Z))=\left(\frac{1}{\tilde{d}_1(Z)}+\mathcal{O}_1(\beta_2(Z))\right)(\beta_1(Z)-\gamma_1(Z))+\mathcal{O}_2(\beta_1(Z)-\gamma_1(Z)).$$

Since $\tilde{d}_1(Z)>0$, it follows that $\Xi_2(Z,\beta_1(Z),\beta_2(Z))\in(-\varepsilon,0)$ if, and only if, $\beta_1(Z)<\gamma_1(Z)$. Also, $\Xi_1(Z,\gamma_1(Z),\beta_2(Z))\in(0,\varepsilon)$ for $\beta_2(Z)>0$ and, thus, $\Xi_1(Z,\beta_1(Z),\beta_2(Z))\in(0,\varepsilon)$ for $\beta_2(Z)>0$ and $\beta_1(Z)$ sufficiently close to $\gamma_1(Z)$. Finally, we conclude that $(\Xi_1(Z,\beta_1(Z),\beta_2(Z)),\ \Xi_2(Z,\beta_1(Z),\beta_2(Z)))\in(0,\varepsilon)\times(-\varepsilon,0)$ with $\beta_2(Z)>0$ if, and only if, $\beta_1(Z)<\gamma_1(Z)$. Hence, we get the existence or not of crossing limit cycles in items (1), (3), (4), and (5).

Analogously, since $\Xi_1(Z,\beta_1(Z),\gamma_2(Z))=0$, the expansion of $\Xi_1(Z,\beta_1(Z),\beta_2(Z))$ around $\beta_2(Z)=\gamma_2(Z)$ writes

$$\Xi_1(Z,\beta_1(Z),\beta_2(Z))=\left(\frac{1}{\tilde{d}_2(Z)}+\mathcal{O}_1(\beta_1(Z))\right)(\beta_2(Z)-\gamma_2(Z))+\mathcal{O}_2(\beta_2(Z)-\gamma_2(Z)).$$

Recalling that $\tilde{d}_2(Z)>0$, we obtain $\Xi_1(Z,\beta_1(Z),\beta_2(Z))\in(0,\varepsilon)$ if, and only if, $\beta_2(Z)>\gamma_2(Z)$. Also, $\Xi_2(Z,\beta_1(Z),\gamma_2(Z))\in(-\varepsilon,0)$ for $\beta_1(Z)<0$. Therefore, $\Xi_2(Z,\beta_1(Z),\beta_2(Z))\in(-\varepsilon,0)$, for $\beta_1(Z)<0$ and $\beta_2(Z)$ sufficiently close to $\gamma_2(Z)$. Finally, we conclude that $(\Xi_1(Z,\beta_1(Z),\beta_2(Z)),\ \Xi_2(Z,\beta_1(Z),\beta_2(Z)))\in(0,\varepsilon)\times(-\varepsilon,0)$ with $\beta_1(Z)<0$ if, and only if, $\beta_2(Z)>\gamma_2(Z)$. Hence, we get the existence or not of crossing limit cycles in items (6), (8) and (10).

Now, notice that

$$\Xi_1(Z,0,\beta_2(Z)) = \frac{1}{\widetilde{d}_2(Z)}\beta_2(Z) + \mathcal{O}_2(\beta_2(Z)),$$
$$\Xi_2(Z,\beta_1(Z),0) = \frac{1}{\widetilde{d}_1(Z)}\beta_1(Z) + \mathcal{O}_2(\beta_1(Z)).$$

Therefore, $\Xi_1(Z,0,\beta_2(Z)) < 0$ and $\Xi_2(Z,0,\beta_2(Z)) > 0$, provided that $\beta_2(Z) < 0$ and $\beta_1(Z) > 0$. This means that (2.5.14) has no solutions when $\beta_1(Z) = 0$ and $\beta_2(Z) < 0$ or $\beta_2(Z) = 0$ and $\beta_1(Z) > 0$. From continuity, if follows that $\Xi_1(Z,\beta_1(Z),\beta_2(Z)) \in (-\varepsilon,0)\times(0,\varepsilon)$ for $\beta_1(Z) > 0$ and $\beta_2(Z) < 0$. Hence, we conclude the non-existence of crossing limit cycles in items (11), (12) and (13).

Notice that $\beta_1(Z) = T_1^u(Z)(a_1) - [D_1(Z)]^{-1}\circ T_2^s(Z)(a_2)$ and $\beta_2(Z) = T_2^u(Z)(a_2) - [D_2(Z)]^{-1}\circ T_1^s(Z)(a_1)$. Heteroclinc connections exist when $\beta_1(Z) = 0$ or $\beta_1(Z) = 0$. If either $\beta_1(Z) = 0$ and $\beta_2(Z) > 0$ or $\beta_1(Z) < 0$ and $\beta_2(Z) = 0$, the heteroclinic connection is not contained in a sliding cycle. This correspond to items (4) and (6).

Finally, the sliding region corresponding to Z is given by $\Sigma^s = (a_1-\varepsilon,a_1)\times\{0\}\cup(a_2,a_2-\varepsilon)\times\{0\}$, for every $Z\in\mathcal{V}$, the sliding vector field F_Z is regular in Σ^s, $\pi_1\circ F_Z(a_1,0) > 0$, and $\pi_1\circ F_Z(a_2,0) < 0$. Therefore, the sliding phenomena detected in items (1) and $(10)-(13)$ follows straightforwardly. Hence, the proof is concluded. □

Remark 2.5.8. *We notice that the set of displacement functions associated with a non-smooth vector field Z_0 at a Σ-polycycle satisfying the hypotheses (DRF-B) generates the same system of equations* (2.5.14) *obtained for the case (DRF-A). Nevertheless, the domain $\sigma_1\times\sigma_2$ will be given by $\sigma_1\times\sigma_2 = (a_1-\varepsilon,a_1]\times[a_2,a_2+\varepsilon)$. The bifurcation diagram of Z_0 can be obtained analogously and has the same structure and objects of the case (DRF-A). Therefore, we shall omit it here.*

2.6 Fold-Fold Σ-Polycycle

This section is devoted to study Σ-polycycles having fold-fold singularities by means of the displacement functions method described in Section 2.3. More specifically, in Section 2.6.1, we describe the displacement functions appearing in the crossing system (2.3.1) for such Σ-polycycles. In Section 2.6.2, the bifurcation diagram of a Σ-polycycle having a unique Σ-singularity of visible-invisible fold-fold type is completely described.

2.6.1 Description of the Crossing System

Assume that $Z_0 = (X_0,Y_0)\in\Omega^r$ has a Σ-polycycle Γ_0 containing k Σ-singularities p_i, where p_i is either a regular-tangential singularity of order n_i, for some $n_i\in\mathbb{N}$, or a fold-fold singularity. Consider a coordinate system (x,y) satisfying that, for each $i\in\{1,2,\dots,k\}$, $x(p_i) = a_i$, $y(p_i) = 0$, and $h(x,y) = y$ near p_i.

Assume that, for some $i\in\{1,\dots,k\}$, p_i is a fold-fold singularity and consider a small neighborhood U_i of p_i. Notice that p_i is not an invisible-invisible fold-fold point, since there are no Σ-polycycles containing this type of singularity. Accordingly, one of the following properties hold for p_i:

(F_1) Either $\Gamma_0\cap W_+^{u,s}(p_i)\neq\emptyset$ or $\Gamma_0\cap W_-^{u,s}(p_i)\neq\emptyset$ (see Figure 2.25 (a) and (b));

(F_2) Either $\Gamma_0 \cap W^u_+(p_i) \neq \emptyset$ and $\Gamma_0 \cap W^s_-(p_i) \neq \emptyset$ or $\Gamma_0 \cap W^s_+(p_i) \neq \emptyset$ and $\Gamma_0 \cap W^u_-(p_i) \neq \emptyset$ (see Figure 2.25 (c)).

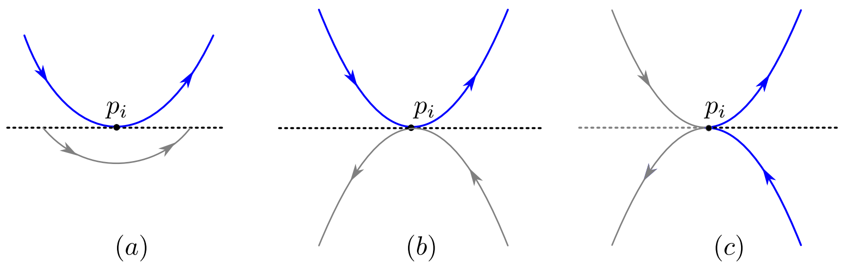


Figure 2.25: Examples of fold-fold singularity of type F_1 ((a) and (b)) and F_2 (c).

In what follows the description of the crossing system will be distinguished in two cases, namely visible-visible and visible-invisible.

Visible-Visible Fold-Fold Singularity

Let p_i be a visible-visible fold-fold singularity. If p_i satisfies (F_1), then the transfer functions $T_i^{u,s} : \sigma_i(Z_0) \to \tau_i^{u,s}$ can be obtained analogously to the case (R_1) in Section 2.5.1. In this case, these functions are restrictions of germs of diffeomorphisms. If p_i satisfies (F_2), then the maps $T_i^u : \sigma_i(Z_0) \to \tau_i^u$ and $T_i^s : \sigma_i(Z_0) \to \tau_i^s$ can be obtained following the case **(O)** in Section 2.3.3. Without loss of generality, we assume that $\Gamma_0 \cap W^u_+(p_i) \neq \emptyset$, $\Gamma_0 \cap W^s_-(p_i) \neq \emptyset$, $\pi_1(X_0(p_i)) > 0$ and $\pi_1(Y_0(p_i)) < 0$ (see Figure 2.25 (c)). In this case, the tangential section $\sigma_i(Z_0)$ is given by $\sigma_i(Z_0) = [a_i, a_i + \varepsilon_i) \times \{0\}$, where $\varepsilon_i > 0$ is sufficiently small.

From Theorem B there exists a neighborhood $\mathcal{V}$ of Z_0 such that for each $Z = (X, Y) \in \mathcal{V}$ the transfer functions corresponding to p_i are given by

$$T_i^\star(Z)(h_Z^{\star,i}(x)) = \lambda_0^{\star,i}(Z) + \kappa_{\star,i}(X)(x - a_i)^2 + \mathcal{O}_3(x - a_i), \quad \star \in \{u, s\},$$

where $h_Z^{\star,i} : (a_i - \varepsilon_i, a_i + \varepsilon_i) \to (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \{0\}$ is a diffeomorphism and $\mathrm{sgn}(\kappa_{\star,i}(Z)) = \mathrm{sgn}(\kappa_{\star,i}(Z_0))$.

Without loss of generality, we can assume for each $Z \in \mathcal{V}$ the fold point of X is fixed at $(a_i, 0)$ and also $h_Z^{u,i}(x) = (a_i + (x - a_i) + \mathcal{O}_2(x - a_i), 0)$. In this case, the fold point of Y is given by $(\alpha_i(Z), 0)$, where $\alpha_i(Z) = (h_Z^{u,i})^{-1} \circ h_Z^{s,i}(a_i)$. Moreover, the domain $\sigma_i(Z)$ of the transfer functions $T_i^{s,u}(Z)$ is given by $\sigma_i(Z) = [\max\{a_i, \alpha_i(Z)\}, a_i + \varepsilon_i)$ and

$$\begin{aligned} T_i^s(Z)(h_Z^{u,i}(x)) &= T_i^s(Z) \circ h_Z^{s,i}((h_Z^{s,i})^{-1} \circ h_Z^{u,i}(x)) \\ &= \lambda_0^{s,i}(Z) + \overline{\kappa}_{s,i}(X)(x - \alpha_i(Z))^2 + \mathcal{O}_3(x - \alpha_i(Z)), \end{aligned}$$

where $\mathrm{sgn}(\overline{\kappa}_{s,i}(Z)) = \mathrm{sgn}(\kappa_{s,i}(Z_0))$ and $\alpha_i(Z_0) = a_i$.

Thus, for each $Z \in \mathcal{V}$, we have characterized the maps $T_i^{s,u}(Z)$ in the domain $\sigma_i(Z)$ under the parameterization $h_Z^{u,i}$. Since the transversal section τ_{i-1}^u is connected to τ_i^s via a diffeomorphism $D_i(Z)$, we obtain

$$[D_{i-1}(Z)]^{-1} \circ T_i^s(Z)(h_Z^{u,i}(x)) = \widetilde{c}_{i-1}(Z) + \widetilde{d}_{i-1}(Z)(x - \alpha_i(Z))^2 + \mathcal{O}_3(x - \alpha_i(Z)),$$

where $\widetilde{c}_{i-1}(Z_0) = q^u_{i-1}$, and $\text{sgn}(\widetilde{d}_{i-1}(Z)) = \text{sgn}(\widetilde{d}_{i-1}(Z_0))$.

It is worthwhile to say that the parameter $\alpha_i(Z)$ locally unfolds the fold-fold singularity p_i (see [55]).

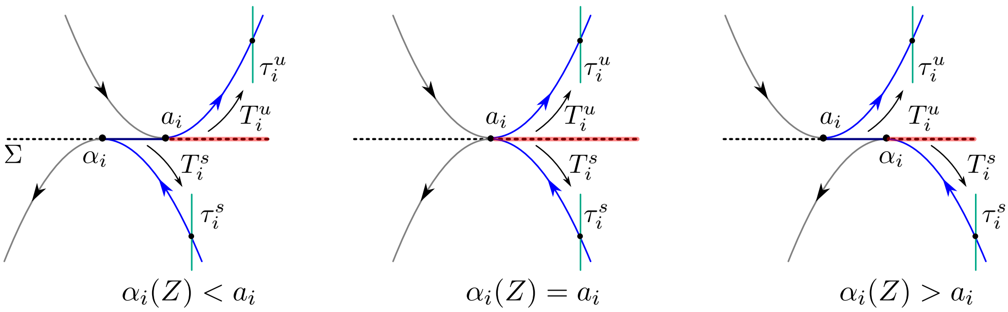


Figure 2.26: Unfolding of a visible-visible fold-fold singularity.

Visible-Invisible Fold-Fold Singularity

Let p_i be a visible-invisible fold-fold singularity. In this case, p_i satisfies (F_1). As we have seen in Section 2.3.3, the transfer functions associated with p_i are defined in the domain $\sigma_i(Z_0)$ which has two components $\sigma_i^{\pitchfork}(X_0)$ and $\sigma_i^t(Z_0) \cap \sigma_i^-(X_0)$. The first one is a restriction to M^+ of a transversal section of X_0 at p_i and the second one is contained in Σ.

For Z sufficiently near Z_0 the transfer functions $T_i^{u,s}(Z) : \sigma_i(Z) \to \tau_i^{u,s}$ restricted to $\sigma_i^{\pitchfork}(X)$ can be obtained analogously to the case (R_1) in Section 2.5.1. In this case, these functions are restrictions of germs of diffeomorphisms (see Figure (2.27)).

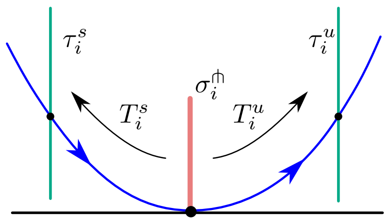


Figure 2.27: Transfer functions $T_i^{u,s}$ restricted to the transversal section $\sigma_i^{\pitchfork}(X_0)$.

Now, the transfer functions $T_i^{u,s}$ restricted to $\sigma_i^t(Z_0) \cup \sigma_i^-(X_0)$ are obtained following the case **(E-II)** in Section 2.3.3. Without loss of generality, we assume that $\Gamma_0 \cap W_+^{u,s}(p_i) \neq \emptyset$, $\pi_1(X_0(p_i)) > 0$ and $\pi_1(Y_0(p_i)) > 0$ (Figure 2.25 (a)). In this case, the tangential section $\sigma_i^t(Z_0) \cap \sigma_i^-(X_0)$ is given by $\sigma_i^t(Z_0) \cap \sigma_i^-(X_0) = (a_i - \varepsilon_i, a_i] \times \{0\}$, where ε_i is sufficiently small.

From Theorem B there exists a neighborhood $\mathcal{V}$ of Z_0 such that for each $Z = (X, Y) \in \mathcal{V}$ the trasition maps corresponding to p_i are given by

$$T_{\pm}^X(h_X(x)) = \lambda_0^{\pm}(Z) + \kappa_{\pm}(X)(x - a_i)^2 + \mathcal{O}_3(x - a_i), \tag{2.6.1}$$

where $h_X : (a_i - \varepsilon_i, a_i + \varepsilon_i) \to (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \{0\}$ is a diffeomorphism and $\text{sgn}(\kappa_{\pm}(X)) = \text{sgn}(\kappa_{\pm}(X_0))$. As before, we can assume that $h_X(x) = (a_i + (x - a_i) + \mathcal{O}_2(x - a_i), 0)$.

Since p_i is an invisible fold point of Y_0, we have that Y has a unique fold point $p_i^Y = (a_i^Y, 0)$ in a neighborhood of p_i, for every $Z = (X, Y) \in \mathcal{V}$. Hence, from (2.3.6) the involution ρ_i^Y associated to Y is given by

$$h_Y^{-1}(\rho_i^Y(h_Y(x))) = a_i^Y - (x - a_i^Y) + \mathcal{O}_2(x - a_i^Y),$$

where $h_Y : (a_i - \varepsilon_i, a_i + \varepsilon_i) \to (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \{0\}$ is a diffeomorphism such that $h_Y(x) = (a_i^Y + l(Y)(x - a_i^Y) + \mathcal{O}_2(x - a_i^Y), 0)$ and $l(Y) > 0$.

Now, from (2.3.7) the transfer functions $T_i^{u,s} : \sigma_i(Z) \to \tau_i^{u,s}$, restricted to $\sigma_i^t(Z) \cup \sigma_i^-(X)$, are given by $T_i^s = T_-^X$ and $T_i^u = T_+^X \circ \rho_i^Y$. Therefore, taking $\alpha_i(Z) = h_X^{-1}(p_i^Y)$, we get

$$T_i^s(Z)(h_X(x)) = \lambda_0^{s,i}(Z) + \kappa_{s,i}(X)(x - a_i)^2 + \mathcal{O}_3(x - a_i),$$

and

$$\begin{aligned} T_i^u(Z)(h_X(x)) = \;& \lambda_0^{u,i}(Z) + \kappa_{u,i}(X)(x - 2\alpha_i(Z) + a_i + \mathcal{O}_2(x - \alpha_i(Z)))^2 \\ & + \mathcal{O}_3(x - 2\alpha_i(Z) + a_i + \mathcal{O}_2(x - \alpha_i(Z))), \end{aligned}$$

where $\lambda_0^{u,i}(Z) = \lambda_0^+(Z)$, $\lambda_0^{s,i}(Z) = \lambda_0^-(Z)$, $\kappa_{u,i}(X) = \kappa_+(X)$, $\kappa_{s,i}(X) = \kappa_-(X)$, and $\alpha_i(Z_0) = a_i$.

Notice that, if $Z = (X, Y) \in \mathcal{V}$, then X has a visible fold point at $(a_i, 0)$ and Y has an invisible fold point at $h_X(\alpha_i(Z)) = p_i^Y$. In this case, the domain $\sigma_i^t(Z) \cap \sigma_i^-(X)$ of the transfer functions $T_i^{s,u}(Z)$ is given by

$$\sigma_i^t(Z) \cap \sigma_i^-(X) = \begin{cases} (a_i - \varepsilon_i, \rho_i^Y(a_i, 0)], & \alpha_i(Z) \le a_i, \\ (a_i - \varepsilon_i, a_i], & \alpha_i(Z) > a_i, \end{cases}$$

see Figure 2.28.

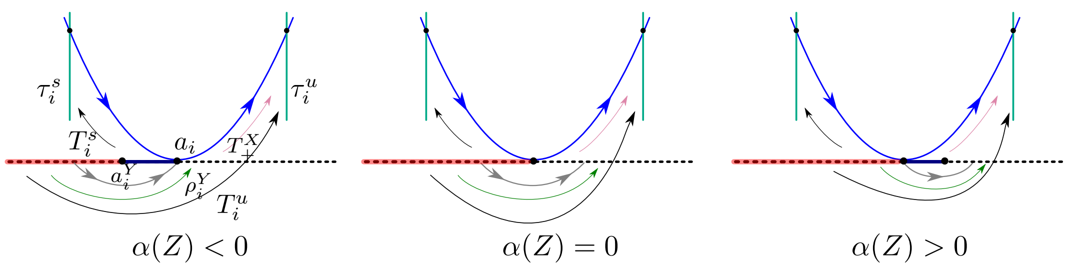


Figure 2.28: Unfolding of a visible-invisible fold-fold singularity.

Thus, for each $Z \in \mathcal{V}$, we have characterized the maps $T_i^{s,u}(Z)$ in the domain $\sigma_i(Z)$ under the parameterization h_X. Since the transversal section τ_{i-1}^u is connected to τ_i^s via a diffeomorphism $D_i(Z)$, we obtain

$$[D_{i-1}(Z)]^{-1} \circ T_i^s(Z)(h_X(x)) = \widetilde{c}_{i-1}(Z) + \widetilde{d}_{i-1}(Z)(x - a_i)^2 + \mathcal{O}_3(x - a_i), \qquad (2.6.2)$$

where $\widetilde{c}_{i-1}(Z_0) = q_{i-1}^u$, and $\operatorname{sgn}(\widetilde{d}_{i-1}(Z)) = \operatorname{sgn}(\widetilde{d}_{i-1}(Z_0))$.

As in the visible-visible case, the parameter $\alpha_i(Z)$ locally unfolds the fold-fold singularity p_i.

2.6.2 Σ-Polycycles having a unique fold-fold singularity

We characterize the nonsmooth vector fields $Z_0 = (X_0, Y_0) \in \Omega^r$ which admit a Σ-polycycle Γ_0 having a unique singularity p_1 of fold-fold type. So, consider a coordinate system (x, y) such that $x(p_1) = 0$, $y(p_1) = 0$, and $h(x, y) = y$ in a neighborhoods of p_1. Consider the following sets of hypotheses:

(VV-A):
- $(0,0)$ is a visible-visible fold-fold singularity of Z_0;
- $\pi_1 \circ X_0(0,0) > 0$ and $\pi_1 \circ Y_0(0,0) < 0$;
- $W^u_+(0,0)$ reaches Σ transversally at p;
- $W^u_-(0,0)$ reaches Σ transversally at q;
- p and q are connected by regular orbit fo Z_0;

(VV-B):
- $(0,0)$ is a visible-visible fold-fold singularity of Z_0;
- the trajectory of Z_0 through $(0,0)$ crosses Σ transversally $n-$times at $q_1, \cdots, q_n$, satisfying:
  - if $n = 0$, then Γ_0 is a hyperbolic limit cycle of X_0;
  - if $n \neq 0$, then, for each $i = 1, \ldots, n$, there exists $t_i > 0$ such that $\varphi_{Z_0}(t_i; q_i) = q_{i+1}$, where $q_{n+1} = (0,0)$. Moreover, $\Gamma_0 \cap \Sigma = \{q_1, \ldots, q_n, (0,0)\}$.

(VI):
- $(0,0)$ is a visible fold point of X_0 and an invisible fold point of Y_0;
- $\pi_1 \circ X_0(0,0) > 0$, and $\pi_1 \circ Y_0(0,0) > 0$;
- the trajectory of Z_0 through $(0,0)$ crosses Σ transversally $n-$times at $q_1, \cdots, q_n$, satisfying:
  - if $n = 0$, then Γ_0 is a hyperbolic limit cycle of X_0;
  - if $n \neq 0$, then, for each $i = 1, \ldots, n$, there exists $t_i > 0$ such that $\varphi_{Z_0}(t_i; q_i) = q_{i+1}$, where $q_{n+1} = (0,0)$. Moreover, $\Gamma_0 \cap \Sigma = \{q_1, \ldots, q_n, (0,0)\}$.

Notice that the analysis remains similar if we change the roles of X_0 and Y_0 and the orientation of the orbits. Some examples of this kind of Σ-polycycle are presented in Figure 2.29.

It is worthwhile to emphasize that the case (VV-A) has been mentioned only for completeness, since the complete description of the bifurcation diagram in this case has been provided in [79]. The case (VV-B) will be avoided since it can be easily obtained by combining grazing bifurcation and visible-visible fold-fold bifurcation [65].

In what follows, we consider Filippov systems satisfying (VI). For simplicity, we assume that $n = 0$, nevertheless, we stress that similar results can be obtained for Σ-polycycles satisfying (VI) with $n > 0$.

From (2.3.6), the involution ρ associated with the smooth vector field Y_0 at $(0,0)$ is given by

$$\rho(x) = -x + \mathcal{O}_2(x),$$

for x small enough.

Remark 2.6.1. *Observe that, fixing a system of coordinates (x, y), the first derivative of ρ does not depend on the vector field Y_0. It is an intrinsic property of an invisible $2n$-order contact point (see Section 2.3.2).*

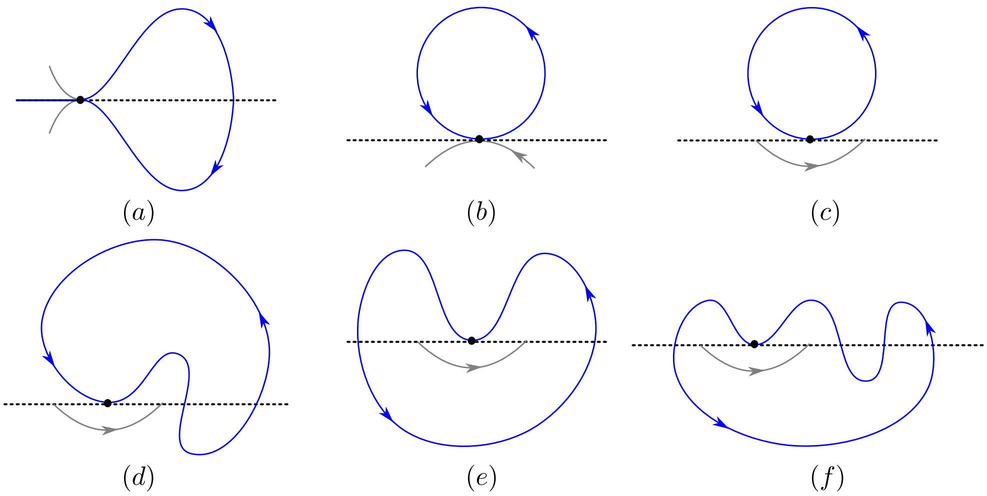


Figure 2.29: Examples of Σ-polycycles satisfying $(VV-A)$ (a), $(VV-B)$ (b), and (VI) with $n=0$ (c), $n=2$ (d,e) and $n=4$ (f).

Consider two local transversal sections τ^s and τ^u to X_0 at $q^s \in W^s_+(0,0)$ and $q^u \in W^u_+(0,0)$, which are sufficiently near to $(0,0)$. Since Γ_0 does not have other singularities and is transversal to Σ up to $(0,0)$, the first return map $\mathcal{P}_0 : \tau^u \to \tau^u$, given by the flow of Z_0, is a piecewise diffeomorphism for which $\mathcal{P}_0(p_0) = p_0$. Now, we show that $\mathcal{P}_0$ is, in fact, a diffeomorphism in a neighborhood of p_0.

Lemma 2.6.2. *Let $Z_0 = (X_0, Y_0)$ be a nonsmooth vector field which admits a Σ-polycycle Γ_0 satisfying (VI) for $n=0$. Then, the first return map defined around Γ_0 is a local diffeomorphism.*

Proof. Let τ^s and τ^u be the two local transversal sections defined above and let $\mathcal{P}_0$ be the first return map of Z_0, defined in a neighborhood of q^u in τ^u. Since $(0,0)$ is the unique Σ-singularity of Z_0 in Γ_0, $\mathcal{P}_0$ is written as $\mathcal{P}_0(x) = E \circ D(Z_0)(x)$, where $D(Z_0) : \tau^u \to \tau^s$ is the diffeomorphism induced by the flow of X_0 and E is the piecewise $\mathcal{C}^r$ function defined by

$$E(q) = \begin{cases} \varphi_{X_0}(t(q); q), & \text{if } q > q^s, \\ q^u, & \text{if } q = q^s, \\ T^u(Z_0) \circ (T^{X_0}_-)^{-1}(q), & \text{if } q < q^s, \end{cases}$$

where $t(q) > 0$ is the flying time from τ^s to τ^u, $T^u(Z_0)$ is the transfer function associated with Z_0 corresponding to the unstable invariant manifold of $(0,0)$ and $T^{X_0}_-$ is the transition map of X_0 with respect to the stable invariant manifold of $(0,0)$ (see Figure 2.30). It is sufficient to prove that E is a local diffeomorphism around q^s.

Now, recall that $T^u(Z_0) = \rho^{Y_0} \circ T^{X_0}_+$, where $T^{X_0}_+$ is the transition map of X_0 with respect to the unstable invariant manifold of $(0,0)$ and ρ^{Y_0} is the involution associated with Y_0 at the invisible fold point $(0,0)$.

As we have seen, the derivative of ρ^{Y_0} at $(0,0)$ does not depend on Y_0, it only depends on the fact that $(0,0)$ is an invisible fold point. In particular, if $Y_0 = X_0$, we have that $T^u(Z_0) \circ (T^{X_0}_-)^{-1}(q) = \varphi_{X_0}(t(q); q)$, which is a local diffeomorphism from τ^s to τ^u. Therefore, it follows that the left lateral derivative of E at q^s is equal to the right lateral derivative of E, and they coincide with the derivative $\varphi_{X_0}(t(q); q)$ at q^s. Hence E is derivative at q^s and $E'(q_s) = \dfrac{d}{dq}\,\varphi_{X_0}(t(q);q)|_{q=q^s}$. Since $\varphi_{X_0}(t(q);q)$ is a diffeomorphism at q^s, we conclude that E is a local diffeomorphism around q^s. □

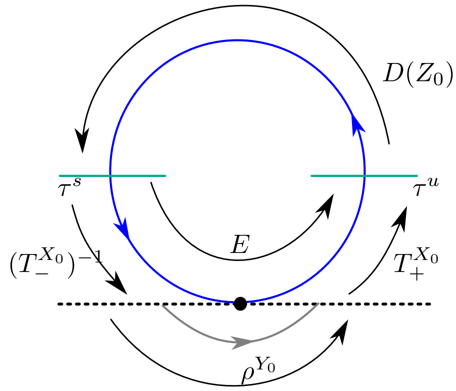


Figure 2.30: First return map $\mathcal{P}_0$.

Remark 2.6.3. *Notice that, if Γ_0 is a hyperbolic limit cycle of X_0 then a hyperbolic cycle persists under small perturbations of Z_0 that do not break the fold-fold singularity. The persistent hyperbolic cycle can be either a crossing limit cycle or a limit cycle of X_0 contained in M^+. These cases will be distinguished latter.*

Now, following Section 2.6.1, we have that the displacement function associated with $Z = (X, Y) \in \mathcal{V}$ near Z_0 in the domain $\sigma^t(Z) \cap \sigma^-(X)$ is written as

$$\begin{aligned}\Delta(Z)(h_X(x)) = &\; T^u(Z)(h_X(x)) - [D(Z)]^{-1} \circ T^s(Z)(h_X(x))\\ = &\; \lambda_0(Z) + \kappa(Z)y^2 - \tilde{c}(Z) - \tilde{d}(Z)x^2 + \mathcal{O}_3(y) + \mathcal{O}_3(x),\end{aligned}$$

where $h_X : (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon) \times \{0\}$ is a diffeomorphism such that $h_X(x) = (x + \mathcal{O}_2(x), 0)$, with $\operatorname{sgn}(\kappa(Z)) = \operatorname{sgn}(\kappa(Z_0))$, $\operatorname{sgn}(\tilde{d}(Z)) = \operatorname{sgn}(\tilde{d}(Z_0))$. The new variable y is given by

$$y(x) = x - 2\alpha(Z) + \mathcal{O}_2(x, \alpha(Z)),$$

and $\alpha(Z_0) = 0$. Also, we notice that X has a visible fold point at $(0, 0)$ and Y has an invisible fold point at $h_X(\alpha(Z))$. From assumption (VI), $\kappa(Z_0) > 0$ and $\tilde{d}(Z_0) > 0$.

Taking

$$\beta(Z) = \lambda_0(Z) - \tilde{c}(Z), \quad \text{and} \quad \eta(Z) = (\alpha(Z), \beta(Z)), \tag{2.6.3}$$

the displacement function $\Delta(Z)(h_X(x))$ writes

$$\Delta(Z)(h_X(x)) = \beta(Z) + \kappa(Z)y^2 - \tilde{d}(Z)x^2 + \mathcal{O}_3(y) + \mathcal{O}_3(x), \tag{2.6.4}$$

and

$$y = x - 2\alpha(Z) + \mathcal{O}_2(x, \alpha(Z)). \tag{2.6.5}$$

Notice that $\eta : \mathcal{V} \to V$ is a surjective function onto a small neighborhood V of $(0, 0)$ satisfying $\alpha(Z_0) = \beta(Z_0) = 0$. In this case, the auxiliary crossing system (2.5.3) is reduced to the system

$$\begin{cases} \beta(Z) + \kappa(Z)y^2 - \tilde{d}(Z)x^2 + \mathcal{O}_3(y) + \mathcal{O}_3(x) = 0,\\ y = x - 2\alpha(Z) + \mathcal{O}_2(x, \alpha(Z)),\\ h_X(x) \in (-\varepsilon, \xi(\alpha(Z))], \end{cases} \tag{2.6.6}$$

where ξ satisfies $\xi(\alpha(Z)) = \alpha(Z) - |\alpha(Z)| + \mathcal{O}_2(\alpha(Z))$ and $\xi(\alpha(Z)) = 0$, for every $\alpha(Z) > 0$. The parameter β controls the existence of connections and Σ-polycycles while α unfolds the fold-fold singularity (see Figure 2.31).

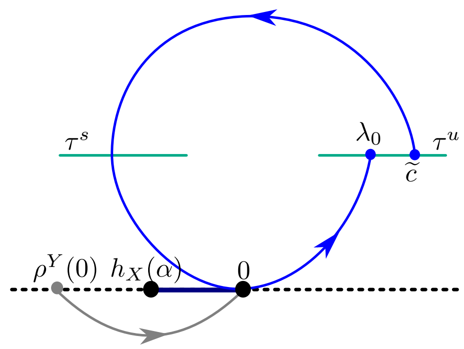


Figure 2.31: Illustration of the parameters α and β.

From the hyperbolicity of the Σ-polycycle, we deduce the following property:

Lemma 2.6.4. *Consider the notation above. If $Z_0 = (X_0, Y_0)$ has a Σ-polycycle Γ_0 satisfying (VI) which is a hyperbolic limit cycle of X_0, then $\tilde{d}(Z_0) \neq \kappa(Z_0)$. In addition:*

(i) If Γ_0 is attracting, then $\kappa(Z_0) - \tilde{d}(Z_0) < 0$;

(ii) If Γ_0 is repelling, then $\kappa(Z_0) - \tilde{d}(Z_0) > 0$.

Proof. Let $\mathcal{P}_0 : \tau^u \to \tau^u$ be the first return map of Z_0 at Γ_0 and notice that, if $x \in \overline{\tau^u_+}$, then $\mathcal{P}_0(x) = T^u(Z_0) \circ ([D(Z_0)]^{-1} \circ T^s(Z_0))^{-1}$, with $\lambda_0(Z_0) = \tilde{c}(Z_0)$ and $\tilde{\alpha}(Z_0) = 0$. From [79], we have that $([D(Z_0)]^{-1} \circ T^s(Z_0))^{-1}$ is given by

$$([D(Z_0)]^{-1} \circ T^s(Z_0))^{-1}(x) = -\sqrt{\frac{x - \tilde{c}(Z_0)}{\tilde{d}(Z_0)}} + \mathcal{O}_1(x - \tilde{c}(Z_0)),$$

for each $x \in [\tilde{c}(Z_0), \tilde{c}(Z_0) + \delta)$ and some $\delta > 0$ sufficiently small.

Using the expansions of $T^u(Z_0)$, we have that

$$\mathcal{P}_0(x) = \tilde{c}(Z_0) + \frac{\kappa(Z_0)}{\tilde{d}(Z_0)}(x - \tilde{c}(Z_0)) + \mathcal{O}_2(x - \tilde{c}(Z_0)),$$

for each $x \in [\tilde{c}(Z_0), \tilde{c}(Z_0) + \delta)$. Hence,

$$\lim_{x \to \tilde{c}(Z_0)^+} \frac{\mathcal{P}_0(x) - \mathcal{P}_0(\tilde{c}(Z_0))}{x - \tilde{c}(Z_0)} = \frac{\kappa(Z_0)}{\tilde{d}(Z_0)}.$$

The result follows directly from this expression. □

Now, we reduce $\mathcal{V}$ in such a way that, for each $Z \in \mathcal{V}$, either $\kappa(Z) - \tilde{d}(Z) < 0$ if Γ_0 is attracting or $\kappa(Z) - \tilde{d}(Z) > 0$ if Γ_0 is repelling.

Using the expansion of h_X, we have that there exists a $\mathcal{C}^r$ function ζ such that $h_X(x) \in (-\varepsilon, \xi(\alpha(Z))]$ if, and only if, $x \in (-\varepsilon, \zeta(\alpha(Z))]$, where

$$\zeta(\alpha(Z)) = \alpha(Z) - |\alpha(Z)| + \mathcal{O}_2(\alpha(Z)). \tag{2.6.7}$$

Following similar technicalities used in [79], we obtain the next theorem.

Theorem E. *Let Z_0 be a nonsmooth vector field having a Σ-polycycle Γ_0 satisfying the hypothesis (VI) and assume that Γ_0 is an attracting hyperbolic limit cycle of X_0. Therefore, there exists an annulus $\mathcal{A}_0$ such that for each annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, there exist neighborhoods $\mathcal{V} \subset \Omega^r$ of Z_0 and $V \subset \mathbb{R}^2$ of $(0,0)$, a surjective function $(\alpha,\beta) : \mathcal{V} \to V$, with $(\alpha,\beta)(Z_0) = (0,0)$, and three smooth functions $\beta_1, \beta_2, \beta_3 : \mathcal{V} \to (-\delta,\delta)$, for which the following statements hold inside $\mathcal{A}$.*

1. *If either $\beta(Z) < \beta_1(Z)$ and $\alpha(Z) > 0$ or $\beta(Z) < \beta_3(Z)$ and $\alpha(Z) < 0$, then Z admits no crossing limit cycles.*
2. *If $\beta(Z) = \beta_1(Z)$ and $\alpha(Z) > 0$, then Z has a semi-stable crossing limit cycle, which is repelling from inside and attracting from outside;*
3. *If $\beta_1(Z) < \beta(Z) < \beta_2(Z)$ and $\alpha(Z) > 0$, then Z has two nested hyperbolic crossing limit cycles such that the outer one is attracting and the inner one is repelling.*
4. *If $\beta(Z) = \beta_2(Z)$ and $\alpha(Z) > 0$, then Z has a hyperbolic repelling crossing limit cycle and a Σ-polycycle passing through a unique regular-fold singularity $(0,0)$ (with $Xh(0,0) = 0$).*
5. *If $\beta_2(Z) < \beta(Z)$ and $\alpha(Z) \geq 0$ or if $\beta(Z) > \beta_3(Z)$ and $\alpha(Z) < 0$, then Z has a unique crossing limit cycle in $\mathcal{V}_0$, which is hyperbolic attracting.*
6. *If $\beta(Z) = \beta_3(Z)$ and $\alpha(Z) < 0$, then Z has Σ-polycycle passing through a unique regular-fold singularity $(0,0)$ (with $Xh(0,0) = 0$) and admits no crossing limit cycles.*

In addition,

$$\beta_1(Z) = \frac{4\kappa(Z)\widetilde{d}(Z)}{\kappa(Z) - \widetilde{d}(Z)}\alpha(Z)^2 + \mathcal{O}_3(\alpha(Z)), \ \beta_2(Z) = -4\kappa(Z)\alpha(Z)^2 + \mathcal{O}_3(\alpha(Z)),$$

and

$$\beta_3(Z) = 4\widetilde{d}(Z)\alpha(Z)^2 + \mathcal{O}_3(\alpha(Z)).$$

Proof. From the construction of the auxiliary crossing system (2.5.3), performed in Section 2.5.1, we get the existence of an annulus $\mathcal{A}_0$ around Γ_0 and neighborhoods $\mathcal{V}_0 \subset \Omega^r$ of Z_0 and $V_0 \subset \mathbb{R}^2$ of $(0,0)$, for which the auxiliary crossing system (2.6.6) is well defined.

Now, given an annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, let $\varepsilon > 0$ satisfy $(-\varepsilon,\varepsilon) \times \{0\} \subset \mathcal{A}$. Considering (2.6.4) and (2.6.5), the displacement function $\Delta(Z)$ satisfies

$$\Delta(Z)(h_X(x)) = \beta(Z) + \kappa(Z)(x - 2\alpha(Z))^2 - \widetilde{d}(Z)x^2 + \Delta_E(Z,\alpha(Z),x),$$

for $x \in (-\varepsilon,\varepsilon)$, where $\Delta_E(Z,\alpha(Z),x) = \mathcal{O}_3(x,\alpha(Z))$. Define the auxiliary function $\mathcal{F} : \mathcal{V}_0 \times (-\delta,\delta)^2 \times (-\varepsilon,\varepsilon) \to \mathbb{R}$, given by

$$\mathcal{F}(Z,\alpha,\beta,x) = \beta + \kappa(Z)(x - 2\alpha)^2 - \widetilde{d}(Z)x^2 + \Delta_E(Z,\alpha,x),$$

and notice that $\mathcal{F}(Z,\alpha(Z),\beta(Z),x) = \Delta(Z)(h_X(x))$.

Throughout this proof, in order to simplify the notation, the parameters $\delta > 0$ and $\varepsilon > 0$ will be taken smaller (if necessary) with no distinction.

Claim: There exist a neighborhood $\mathcal{V} \subset \mathcal{V}_0$ of Z_0 and functions $x_{\mathrm{SN}} : \mathcal{V} \times (-\delta, \delta) \to (-\varepsilon, \varepsilon)$ and $\beta_{\mathrm{SN}} : \mathcal{V} \times (-\delta, \delta) \to (-\delta, \delta)$ such that $\partial_x \mathcal{F}(Z, \alpha, \beta, x) = \mathcal{F}(Z, \alpha, \beta, x) = 0$ if, and only if, $\beta = \beta_{\mathrm{SN}}(Z, \alpha)$ and $x = x_{\mathrm{SN}}(Z, \alpha)$. In addition

$$x_{\mathrm{SN}}(Z, \alpha) = \frac{2\kappa(Z)}{\kappa(Z) - \widetilde{d}(Z)} \alpha + \mathcal{O}_2(\alpha),$$

and

$$\beta_{\mathrm{SN}}(Z, \alpha) = 4 \frac{\kappa(Z) \widetilde{d}(Z)}{\kappa(Z) - \widetilde{d}(Z)} \alpha^2 + \mathcal{O}_3(\alpha).$$

In fact, notice that

$$\partial_x \mathcal{F}(Z, \alpha, \beta, x) = 2(\kappa(Z) + \widetilde{d}(Z))x - 4\kappa(Z)\alpha + \mathcal{O}_2(x, \alpha).$$

Therefore, $\partial_x \mathcal{F}(Z, 0, \beta, 0) = 0$, for every $Z, \beta \in \mathcal{V}_0 \times (-\delta, \delta)$, and $\partial_x^2 \mathcal{F}(Z_0, 0, 0, 0) = 2(\kappa(Z_0) - \widetilde{d}(Z_0)) \neq 0$. It follows from the Implicit Function Theorem for Banach Spaces that there exist a neighborhood $\mathcal{V} \subset \mathcal{V}_0$ of Z_0 and a function $\widehat{x_{\mathrm{SN}}} : \mathcal{V} \times (-\delta, \delta)^2 \to \mathbb{R}$ such that $\partial_x \mathcal{F}(Z, \alpha, \beta, x) = 0$ if , and only if, $x = \widehat{x_{\mathrm{SN}}}(Z, \alpha, \beta)$. In addition,

$$\widehat{x_{\mathrm{SN}}}(Z, \alpha, \beta) = \frac{2\kappa(Z)}{\kappa(Z) - \widetilde{d}(Z)} \alpha + \mathcal{O}_2(\alpha).$$

Now, consider the function $\widehat{\mathcal{F}}(Z, \alpha, \beta) = \mathcal{F}(Z, \alpha, \beta, \widehat{x_{\mathrm{SN}}}(Z, \alpha, \beta))$. Notice that

$$\widehat{\mathcal{F}}(Z, \alpha, \beta) = \beta - 4 \frac{\kappa(Z) \widetilde{d}(Z)}{\kappa(Z) - \widetilde{d}(Z)} \alpha^2 + \mathcal{O}_3(\alpha).$$

Again, reducing $\mathcal{V}$ if necessary, it follows from the Implicit Function Theorem that there exists a function $\beta_{\mathrm{SN}} : \mathcal{V} \times (-\delta, \delta) \to (-\delta, \delta)$ such that $\widehat{\mathcal{F}}(Z, \alpha, \beta) = 0$ if, and only if, $\beta = \beta_{\mathrm{SN}}(Z, \alpha)$. Hence, the proof of Claim 1 follows by taking $x_{\mathrm{SN}}(Z, \alpha) = \widehat{x_{\mathrm{SN}}}(Z, \alpha, \beta_{\mathrm{SN}}(Z, \alpha))$.

Now, in order to find all the zeroes of $\mathcal{F}$, we use the curve β_{SN} provided in Claim 1 . Define

$$P = \{(Z, \alpha, \beta) \in \mathcal{V} \times (-\delta, \delta)^2; \ \beta \geq \beta_{\mathrm{SN}}(Z, \alpha)\}.$$

Claim: There exist functions $x_{\pm} : P \to (-\varepsilon, \varepsilon)$ such that $\mathcal{F}(Z, \alpha, \beta, x) = 0$ if, and only if, $x = x_+(Z, \alpha, \beta)$ or $x = x_-(Z, \alpha, \beta)$. In addition

$$x_+(Z, \alpha, \beta_{\mathrm{SN}}(Z, \alpha)) = x_-(Z, \alpha, \beta_{\mathrm{SN}}(Z, \alpha)) = x_{\mathrm{SN}}(Z, \alpha),$$

and

$$x_{\pm}(Z, \alpha, \beta) = \frac{2\kappa(Z)}{\kappa(Z) - \widetilde{d}(Z)} \alpha \pm \sqrt{-\frac{\beta - \beta_{\mathrm{SN}}}{\kappa(Z) - \widetilde{d}(Z)}} + \mathcal{O}_2\left(\alpha, \sqrt{\beta - \beta_{\mathrm{SN}}}\right),$$

where $\beta_{\mathrm{SN}} = \beta_{\mathrm{SN}}(Z, \alpha)$.

Recall that the remainder term Δ_E in the function $\mathcal{F}$ does not depend on β. Also, denoting $\beta_{\mathrm{SN}} = \beta_{\mathrm{SN}}(Z, \alpha)$ and $x_{\mathrm{SN}} = x_{\mathrm{SN}}(Z, \alpha)$, we have $\mathcal{F}(Z, \beta_{\mathrm{SN}}, \alpha, x_{\mathrm{SN}}) = \partial_x \mathcal{F}(Z, \beta_{\mathrm{SN}}, \alpha, x_{\mathrm{SN}}) = 0$ and $\partial_x^2 \mathcal{F}(Z, \beta_{\mathrm{SN}}, \alpha, x_{\mathrm{SN}}) = 2(\kappa(Z) - \widetilde{d}(Z)) + \mathcal{O}(\alpha) \neq 0$. Thus, $\mathcal{F}$ writes

$$\mathcal{F}(Z, \alpha, \beta, x) = \beta - \beta_{\mathrm{SN}} + \left(\kappa(Z) - \widetilde{d}(Z) + \mathcal{O}(\alpha)\right)(x - x_{\mathrm{SN}})^2 + \mathcal{O}_3(x - x_{\mathrm{SN}}).$$

Now, define

$$\mathcal{G}(Z,\alpha,\beta,u)=\beta-\beta_{\rm SN}+\Big(\kappa(Z)-\widetilde{d}(Z)+\mathcal{O}(\alpha)\Big)\,u+\mathcal{O}_{3/2}(u)$$

in such way that $\mathcal{F}(Z,\alpha,\beta,x)=0$ if, and only if, $\mathcal{G}(Z,\alpha,\beta,u)=0$ and $u=(x-x_{\rm SN})^2$. Since $\mathcal{G}(Z,\alpha,\beta_{\rm SN},0)=0$ and $\partial_u\mathcal{G}(Z_0,0,0,0)=\kappa(Z_0)-\widetilde{d}(Z_0)\neq 0$, it follows from the Implicit Function Theorem that there exists a function $u_0:\mathcal{V}\times(-\delta,\delta)^2\times(-\varepsilon,\varepsilon)$ such that $\mathcal{G}(Z,\alpha,\beta,u)=0$ if, and only if, $u=u_0(Z,\alpha,\beta)$. In addition

$$u_0(Z,\alpha,\beta)=-\frac{\beta-\beta_{\rm SN}}{\kappa(Z)-\widetilde{d}(Z)}+\mathcal{O}_1(\alpha(\beta-\beta_{\rm SN}),(\beta-\beta_{\rm SN})^2).$$

Since $u_0(Z,\alpha,\beta)\geq 0$ if, and only if, $\beta\geq\beta_{\rm SN}$, the proof Claim 2 follows by taking $x_{\pm}(Z,\alpha,\beta)=x_{\rm SN}(Z,\alpha)\pm\sqrt{u_0(Z,\alpha,\beta)}$.

From Claim 2, we have found all the zeroes of $\mathcal{F}$ in a neighborhood of $(Z_0,0,0,0)$. Now, we must analyze whether $x_{\pm}(Z,\alpha,\beta)\in(-\varepsilon,\zeta(\alpha)]$, where ζ is given by (2.6.7).

First, assume that $\alpha\geq 0$, hence $\zeta(\alpha)=\mathcal{O}_2(\alpha)$. In this case, since $\kappa(Z)-\widetilde{d}(Z)>0$, $\kappa(Z),\widetilde{d}(Z)>0$, it follows that

$$x_-(Z,\alpha,\beta)-\zeta(\alpha)=\frac{2\kappa(Z)}{\kappa(Z)-\widetilde{d}(Z)}\alpha-\sqrt{-\frac{\beta-\beta_{\rm SN}}{\kappa(Z)-\widetilde{d}(Z)}}+\mathcal{O}_2\left(\alpha,\sqrt{\beta-\beta_{\rm SN}}\right)<0,$$

for every $\beta>\beta_{\rm SN}$. Thus, $x_-(Z,\alpha,\beta)\in int(\sigma(Z))$ corresponds to a crossing limit cycle. Also, since $\partial_x\mathcal{F}(Z,\alpha,\beta,x_-(Z,\alpha,\beta))<0$ for $\beta>\beta_0$, it follows that the crossing limit cycle corresponding to x_- is hyperbolic attracting.

Notice that

$$x_+(Z,\alpha,\beta)-\zeta(\alpha)=\mu+\frac{2\kappa(Z)}{\kappa(Z)-\widetilde{d}(Z)}\alpha+\mathcal{O}_2(\alpha,\mu),$$

where

$$\mu=\sqrt{-\frac{\beta-\beta_{\rm SN}}{\kappa(Z)-\widetilde{d}(Z)}}. \tag{2.6.8}$$

Define $\mathcal{H}(Z,\alpha,\mu):=x_+(Z,\alpha,\beta)-\zeta(\alpha)$. Applying the Implicit Function Theorem to $\mathcal{H}$, we obtain the existence of a function $\mu_0^+:\mathcal{V}\times(-\delta,\delta)\to\mathbb{R}$ such that $\mathcal{H}(Z,\alpha,\mu)=0$ if, and only if, $\mu=\mu_0^+(Z,\alpha)$. Also,

$$\mu_0^+(Z,\alpha)=-\frac{2\kappa(Z)}{\kappa(Z)-\widetilde{d}(Z)}\alpha+\mathcal{O}_2(\alpha)\geq 0.$$

Thus, taking $\beta^+(Z,\alpha)=\beta_{\rm SN}(Z,\alpha)=-(\kappa(Z)-\widetilde{d}(Z))(\mu_0^+(Z,\alpha))^2$, we obtain that

$$\beta^+(Z,\alpha)=-4\kappa(Z)\alpha^2+\mathcal{O}_3(\alpha).$$

Moreover, $x_+(Z,\alpha,\beta)\in\partial\sigma(Z)$ if, and only if, $\beta=\beta^+(Z,\alpha)$. In this case, $x_+(Z,\alpha,\beta)$ corresponds to a Σ-polycycle passing through the visible point $(0,0)$ of X. On the other hand, since $\mathcal{H}$ is increasing in the variable μ, $x_+(Z,\alpha,\beta)\in\mathrm{int}(\sigma(Z))$ if, and only if, $\beta_{\rm SN}(Z,\alpha)<\beta<\beta^+(Z,\alpha)$. In this case, $x_+(Z,\alpha,\beta)$ corresponds to a crossing limit cycle of Z. Also, $\partial_x\mathcal{F}(Z,\alpha,\beta,x_+(Z,\alpha,\beta))>0$ for $\beta>\beta_{\mathrm{SN}}$, then the crossing limit cycle (resp. Σ-polycycle) corresponding to x_+ is hyperbolic repelling (resp. C-unstable).

If $\beta=\beta_{\mathrm{SN}}(Z,\alpha)$, then $x_{\mathrm{SN}}(Z,\alpha)-\zeta(\alpha)\geq 0$ (with equality if, and only if, $\alpha=0$), and thus $x_{\mathrm{SN}}(Z,\alpha)$ corresponds to a crossing limit cycle. Since $x_-(Z,\alpha,\beta_{\mathrm{SN}})=x_+(Z,\alpha,\beta_{\mathrm{SN}})=x_{\mathrm{SN}}(Z,\alpha)$, and $\beta>\beta_{\mathrm{SN}}$ $x_-(Z,\alpha,\beta)<x_+(Z,\alpha,\beta)$ corresponds, respectively, to a repelling and an attracting crossing limit cycle, it follows that x_{SN} corresponds to a semi-stable crossing limit cycle which is repelling from inside and attracting from outside.

Now, assume that $\alpha<0$. In this case $\zeta(\alpha)=2\alpha+\mathcal{O}_2(\alpha)$ and

$$x_+(Z,\alpha,\beta)-\zeta(\alpha)=\frac{2\widetilde{d}(Z)}{\kappa(Z)-\widetilde{d}(Z)}\alpha+\sqrt{-\frac{\beta-\beta_{\mathrm{SN}}}{\kappa(Z)-\widetilde{d}(Z)}}+\mathcal{O}_2\left(\alpha,\sqrt{\beta-\beta_{\mathrm{SN}}}\right)>0,$$

for every $\beta\geq\beta_{\mathrm{SN}}(Z,\alpha)$. It means that $x_+(Z,\alpha,\beta)\notin\sigma(Z)$ for every $\alpha<0$ and $\beta\geq\beta_{\mathrm{SN}}$. For the other zero, we have that

$$x_-(Z,\alpha,\beta)-\zeta(\alpha)=-\mu+\frac{2\widetilde{d}(Z)}{\kappa(Z)-\widetilde{d}(Z)}\alpha+\mathcal{O}_2\left(\alpha,\mu\right),$$

where μ is given by (2.6.8). Similarly, we obtain that there exists a function $\beta^-(Z,\alpha)$ satisfying

$$\beta^-(Z,\alpha)=4\widetilde{d}(Z)\alpha^2+\mathcal{O}_3(\alpha).$$

In this case, $x_-(Z,\alpha,\beta)\in\partial\sigma(Z)$ if, and only if, $\beta=\beta^-(Z,\alpha)$ which corresponds to a Σ-polycycle of Z passing through the origin. Also, $x_-(Z,\alpha,\beta)\in\mathrm{int}(\sigma(Z))$ if, and only if, $\beta_{\mathrm{SN}}(Z,\alpha)<\beta<\beta^-(Z,\alpha)$, which corresponds to a hyperbolic attracting crossing limit cycle of Z.

The proof follows by taking $\beta_1(Z)=\beta_{\mathrm{SN}}(Z,\alpha(Z))$, $\beta_2(Z)=\beta^+(Z,\alpha(Z))$ and $\beta_3(Z)=\beta^-(Z,\alpha(Z))$. □

In the remainder of this section, in order to complete the bifurcation diagram of Z_0 satisfying the hypotheses of Theorem E, we study the existence of limit cycles of $Z\in\mathcal{V}$ passing through the section $\sigma^{\pitchfork}(Z)$ (see Proposition 2.6.5) as well as the sliding phenomena (see Propositions 2.6.6 and 2.6.7).

Proposition 2.6.5. *Let $Z_0=(X_0,Y_0)$ be a nonsmooth vector field in the setting of Theorem E. Therefore, for an annulus $\mathcal{A}$, with $\Gamma_0\subset\mathcal{A}\subset\mathcal{A}_0$, the following statements hold inside $\mathcal{A}$.*

1. *If $\beta(Z)<0$, then Z admits a unique limit cycle, which is hyperbolic attracting limit cycle of X in M^+.*

2. *If $\beta(Z)=0$, then Z admits a unique Σ-polycycle passing through $(0,0)$, which is a hyperbolic attracting limit cycle of X in $\overline{M^+}$.*

3. *If $\beta(Z)>0$, then Z has no limit cycles contained in $\overline{M^+}$ or $\overline{M^-}$.*

Proof. From the study on the tangential section already done, we have that $T^u(Z)(0,0)=\lambda_0(Z)$ and $[D(Z)]^{-1}\circ T^s(Z)(0,0)=\widetilde{c}(Z)$, which means that

$$\beta(Z)=T^u(Z)(0,0)-[D(Z)]^{-1}\circ T^s(Z)(0,0).$$

Now, since Γ_0 is a hyperbolic attracting limit cycle of X_0, we have that its associated first return map $\mathcal{P}_0^X$ defined in the section $\{0\} \times (-\varepsilon, \varepsilon)$ has a unique attractor hyperbolic fixed point $(0, p_X)$. So, $(0, p_X)$ corresponds to a hyperbolic attracting limit cycle of X in M^+ if, and only if, $p_X > 0$. Also, $p_X = 0$ if, and only if, X has a hyperbolic attracting limit cycle tangent to Σ at the origin. Finally, the result follows by noticing that $p_X \beta(Z) < 0$, and $p_X = 0$ if, and only if, $\beta(Z) = 0$. $\square$

Now we proceed with the analysis of the sliding dynamics. In this present setting, $(0,0)$ is a visible-invisible fold-fold singularity of Z_0 and for each $Z = (X, Y) \in \mathcal{V}$ we have that X has a visible fold point at $(0,0)$ and Y has an invisible fold point at $h_X(\alpha(Z))$ of Y. Recall that the parameter $\alpha(Z)$ locally unfolds the visible-invisible fold-fold singularity. For $\alpha \neq 0$, there exists either a stable sliding region or an unstable sliding region between the two regular-fold singularities, $(0,0)$ and $h_X(\alpha(Z))$. In both cases, the sliding vector field F_Z has no pseudo-equilibria. Moreover, $\pi_1 \circ F_Z(x, 0) > 0$ for all $(x, 0)$ between $(0,0)$ and $h_X(\alpha(Z))$ (see Figure 2.32).

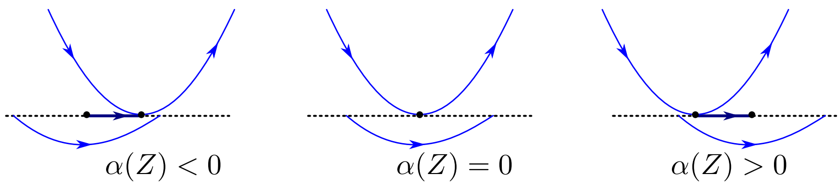


Figure 2.32: Bifurcation diagram of the visible-invisible fold-fold singularity.

Proposition 2.6.6. *Let $Z_0 = (X_0, Y_0)$ be a nonsmooth vector field in the setting of Theorem E. Therefore, for an annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, there exists a $\mathcal{C}^r$ function $\beta_4 : \mathcal{V} \to \mathbb{R}$ such that, for $\alpha(Z) > 0$ and $\beta_2(Z) < \beta(Z) < 0$, the following statements hold inside $\mathcal{A}$.*

1. *If $\beta(Z) < \beta_4(Z)$, then Z has a sliding cycle through $(0,0)$ for which the trajectory through $(0,0)$ crosses Σ^c once before it reaches Σ^s from M^-;*

2. *If $\beta(Z) = \beta_4(Z)$, then Z has a sliding cycle that contains $p_0 = (0,0)$ and $p_Y = h_X(\alpha(Z))$ for which the arc-orbit $\widehat{p_0 p_Y}|_Z$ is contained in M^+;*

3. *If $\beta(Z) > \beta_4(Z)$, then Z has a sliding cycle through $(0,0)$ for which the trajectory through $(0,0)$ reaches Σ^s from M^+ without crossing Σ^c.*

In addition,

$$\beta_4(Z) = -\kappa(Z)\alpha(Z)^2 + \mathcal{O}_3(\alpha(Z)).$$

Proof. For $\alpha > 0$, a connection of $Z = (X, Y)$ between p_0 and p_Y is characterized as the zero of the function

$$S_+(Z, \alpha, \beta) = T_+^X(h_X(\alpha)) - [D(Z)]^{-1} \circ T^s(Z)(0,0),$$

where T_+^X is the transition map of X given by (2.6.1) (with $a_i = 0$) and $[D(Z)]^{-1} \circ T^s(Z)$ is given by (2.6.2). Thus, it follows from (2.6.3) that

$$\begin{aligned} S_+(Z, \alpha, \beta) &= \lambda_0(Z) + \kappa(Z)\alpha^2 + \mathcal{O}_2(\alpha) - \widetilde{c}(Z) \\ &= \beta + \kappa(Z)\alpha^2 + \mathcal{O}_2(\alpha). \end{aligned}$$

From the Implicit Function Theorem, there exists a function $\beta_s^+(Z,\alpha)$ such that $S_+(Z,\alpha,\beta) = 0$ if, and only if, $\beta = \beta_s^+(Z,\alpha)$.

Notice that $S_+(Z,\alpha,\beta) < 0$ is equivalent to $\beta < \beta_s^+(Z,\alpha)$. In this case, since $\beta > \beta_2(\alpha)$, the trajectory through $(0,0)$ crosses Σ^c once and reaches Σ^s from M^-. Then it slides to $(0,0)$.

Finally, $S_+(Z,\alpha,\beta) > 0$ is equivalent to $0 > \beta > \beta_s^+(Z,\alpha)$. In this case, the trajectory through $(0,0)$ reaches Σ^s directly from M^+. The proof follows by taking $\beta_4(Z) = \beta_s^+(Z,\alpha(Z))$. □

Proposition 2.6.7. *Let $Z_0 = (X_0, Y_0)$ be a nonsmooth vector field in the setting of Theorem E. Therefore, for an annulus $\mathcal{A}$, with $\Gamma_0 \subset \mathcal{A} \subset \mathcal{A}_0$, there exists a $\mathcal{C}^r$ function $\beta_5 : \mathcal{V} \to \mathbb{R}$ such that, for $\alpha(Z) < 0$ and $0 < \beta(Z) < \beta_3(Z)$, the following statements hold inside $\mathcal{A}$.*

1. *If $0 < \beta(Z) < \beta_5(Z)$, then Z has a sliding cycle through $(0,0)$ for which the negative trajectory through $(0,0)$ reaches Σ^s from M^+ without crossing Σ^c;*
2. *If $\beta(Z) = \beta_5(Z)$, then Z has a sliding cycle containing $p_0 = (0,0)$ and $p_Y = h_X(\alpha(Z))$ for which the arc-orbit $\widehat{p_Y p_0}|_Z$ is contained in M^+;*
3. *If $\beta_5(Z) < \beta(Z)$, then Z has a sliding cycle through $(0,0)$ for which the negative trajectory through $(0,0)$ reaches Σ^s from M^- after it crosses Σ^c once.*

In addition,

$$\beta_5(Z) = \widetilde{d}(Z)\alpha(Z)^2 + \mathcal{O}_3(\alpha(Z)).$$

Proof. For $\alpha < 0$, a connection of $Z = (X,Y)$ between p_Y and p_0 is characterized as the zero of the function

$$S_-(Z,\alpha,\beta) = T_+^X(0,0) - [D(Z)]^{-1} \circ T^s(Z)(h_X(\alpha)),$$

where T_+^X is the transition map of X given by (2.6.1) (with $a_i = 0$) and $[D(Z)]^{-1} \circ T^s(Z)$ is given by (2.6.2). Thus, it follows from (2.6.3) that

$$\begin{aligned} S_-(Z,\alpha,\beta) &= \lambda_0(Z) - \widetilde{c}(Z) - \widetilde{d}(Z)\alpha^2 + \mathcal{O}_2(\alpha) \\ &= \beta - \widetilde{d}(Z)\alpha^2 + \mathcal{O}_2(\alpha). \end{aligned}$$

Following the same steps of the proof of Proposition 2.6.6, we obtain the result. □

A complete description of the bifurcation diagram of a nonsmooth vector field Z_0 satisfying the hypotheses of Theorem E is achieved by combining Theorem E, Propositions 2.6.5, 2.6.6 and 2.6.7, and noticing that Z has a visible-invisible fold-fold singularity at the origin if, and only if, $\alpha(Z) = 0$. This bifurcation diagram is illustrated in Figure 2.5.

Remark 2.6.8. *Suppose that Z_0 has a Σ-polycycle Γ_0 satisfying conditions (VI) with $n \geq 1$. If the first return map $\mathcal{P}_0$ defined around Γ_0 has a hyperbolic fixed point, then a similar analysis can be performed in order to describe the bifurcation diagram of the unfolding of Γ_0.*

2.7 Conclusion and Further Directions

In this work, we provided a method to study the unfolding of Σ-polycycles in planar Filippov systems under certain hypotheses, and we have used such a mechanism to completely describe the bifurcation diagrams of three different types of Σ-polycycles.

Despite the generality of Σ-polycycles covered by the mentioned methodology, there are some classes of Σ-polycycles for which the Method of Displacement Functions does not detect all bifurcating phenomena in their unfoldings. In fact, such Σ-polycycles seems to exhibit a behavior much more complicated than the ones considered herein. Roughly speaking, if global connections appear in the local unfolding of a Σ-singularity, which is contained in a Σ-polycycle Γ, then such a mechanism does not detect all the crossing dynamics in the unfolding of Γ. Nevertheless, it is worth mentioning that, even in these cases, our method detects all the bifurcating crossing limit cycles with the same topological type of the Σ-polycycle.

In light of this, an accurate description of local bifurcations of Σ-singularities of tangential-tangential type is needed. In particular, a detailed analysis of the unfolding of a cusp-cusp singularity is very welcome, since Σ-polycycles through a visible-visible fold-fold singularity bifurcate from such a singularity. The knowledge of the local structure of degenerated Σ-singularities might lead us to the comprehension of Σ-polycycles in a most general scenario.

In addition, Σ-polycycles passing through Σ-singularities of $Z = (X, Y) \in \Omega^r$ involving equilibria of X or Y should be considered. As an example, we mention the homoclinic-like connection through a boundary-saddle singularity studied in [4]. We emphasize that the analysis of the problem becomes harder in this case due to the lack of normal forms (via change of coordinates) for such a type of Σ-singularity.

Chapter 3

Generic Singularities of $3D$ Filippov Systems

The aim of this chapter is to provide a discussion on current directions of research involving typical singularities of $3D$ nonsmooth vector fields. A brief survey of known results is also presented.

We describe the dynamical features of a fold-fold singularity in its most basic form and we give a complete and detailed proof of its local structural stability (or instability). In addition, classes of all topological types of a fold-fold singularity are intrinsically characterized. Such proof essentially follows from some lines laid out by Colombo, García, Jeffrey, Teixeira and others and it offers a rigorous mathematical treatment under clear and crisp assumptions and solid arguments.

One should to highlight that the geometric-topological methods employed lead us to the mathematical understanding of the dynamics around a T-singularity. This approach lends itself to applications in generic bifurcation theory. It is worth saying that such subject is still poorly understood in higher dimension.

3.1 Introduction

Certain aspects of the theory of nonsmooth vector fields (piecewise smooth vector fields) has been mainly motivated by the study of vector fields near the boundary of a manifold. Concerning this topic, many authors provided results and techniques which have been very useful in piecewise smooth systems. It is worthwhile to cite in the 2-dimensional case works from Andronov et al., Peixoto, Teixeira (see [5, 83, 97]) and in higher dimensions the works from Sotomayor and Teixeira, Vishik and Percell (see [95, 84, 104]). In particular, in [104] (1972), Vishik provided a classification of generic points lying in the boundary of a manifold, using techniques from Theory of Singularities.

Many papers have contributed to the analysis and generic classification of singularities of 2D Filippov systems (Kuznetsov et al., Guardia et al., Kozlova among others, see [55, 63, 65]). Specifically with respect to the fold-fold singularity we point Ekeland (see [34]) and Teixeira (see [98]). Regarding the n-dimensional problem, we point out the work from Colombo and Jeffrey (see [26]) which analyzes an n-dimensional family having a two-fold singularity, nevertheless the generic classification for $n > 2$ is much more complicated and still poorly understood.

As far as we know, the first approach where a generic 3D fold-fold singularity was studied was offered by Teixeira in [99] (1981) where one finds a discussion on some features of the first return mapping defined around this singularity. Maybe due to this fact, the invisible fold-fold singularity is known as T-singularity.

In [39] (1988), Filippov provided a mathematical formalization of the theory of piecewise smooth vector fields. In the last chapter of [39], Filippov studied generic singularities in $3D$ piecewise smooth systems, and a systematic mathematical analysis of the behavior around a fold-fold singularity was officially arisen. However, most of proofs were only roughly sketched and would require a better explanation and interpretation. In particular, the proofs of the results concerning the fold-fold singularity were obscure and unfinished. Many works appeared lately trying to explain it (see [24, 25, 38, 101]).

In [101], Teixeira established necessary conditions for the structural stability of the fold-fold singularity and he proved that it is not a generic property. Nevertheless, the case of the invisible fold-fold point having a hyperbolic first return map was not understood. He also provided results concerning asymptotic stability.

In [24, 25, 38], Jeffrey et al. also studied the problem of the classification of the structural stability around a fold-fold singularity. More specifically, in [38], the authors studied the behavior of a 2-parameter semi-linear model $Z_{\alpha,\beta}$ having a T-singularity at $Z_{0,0}$. By studying the first return map explicitly, they have found countably many curves γ_k in a region of the parameter space, where the topological type β_k of a system in γ_k satisfies $\beta_k \neq \beta_l$ provided $k \neq l$.

Guided by these results, we show that in the region of the parameter space considered in [38], a general Filippov system Z having a T-singularity at p always has a first return map with complex eigenvalues. It brings several consequences to the behavior of Z around p, in particular, it produces a foliation of this region in the parameter space depending on the argument of the eigenvalues of Z such that, two systems in different leaves are not topologically equivalent near the T-singularity, which means that there is no class of stability in this region of parameters.

A $3D$ fold-fold singularity is an intriguing phenomenon that has no counterparts in smooth systems, and the complete characterization of the local structural stability of a $3D$ nonsmooth system around an elliptic fold-fold singularity has been an open problem over the last 30 years. In this work, we believe that all mathematical existing gaps were filled up and the precise statement of results and proofs were well established.

It is worth mentioning that the methods and techniques used in this chapter provide a solution from a geometric-topological point of view. In addition, we present a generic and qualitative characterization of a fold-fold singularity, in order to clarify any fact concerning the generality of the results.

3.2 Setting the Problem

In what follows we summarize a rough overall description of the basic concepts and results in order to set the problem.

Let M be a connected bounded region of $\mathbb{R}^3$, let $f : M \to \mathbb{R}$ be a smooth function having 0 as a regular value and assume that $\Sigma = f^{-1}(0)$ is compact. Throughout this chapter, we consider germs of piecewise smooth vector fields at Σ

Remark 3.2.1. *Notice that, in this chapter, Ω^r stands for the set of germs of tridimensional piecewise smooth vector fields at Σ.*

3.2.1 Σ-Equivalence

An orbital equivalence relation is defined in Ω^r as follows.

Definition 3.2.2. *Let $Z_0, Z \in \Omega^r$ be two germs of nonsmooth vector fields. We say that Z_0 is* ***topologically equivalent*** *to Z at p if there exist neighborhoods U and V of p in M and an order-preserving homeomorphism $h : U \to V$ such that it carries orbits of Z_0 onto orbits of Z, and it preserves Σ, i.e. $h(\Sigma \cap U) = \Sigma \cap V$.*

The concept of local structural stability at a point $p \in \Sigma$ is defined in the natural way.

Definition 3.2.3. *$Z_0 \in \Omega^r$ is said to be* ***Σ-locally structurally stable*** *if Z_0 is locally structurally stable at p, for each $p \in \Sigma$.*

Denote the space of germs of nonsmooth vector fields $Z \in \Omega^r$ which are Σ-locally structurally stable by Σ_0.

3.2.2 Reversible mappings

We introduce concepts which will be useful throughout this chapter. More details can be found in [73, 100].

Definition 3.2.4. *A germ of an* ***involution*** *at 0 is a $\mathcal{C}^r$ germ of a diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(0) = 0$, $\varphi^2(x, y) = (x, y)$ and $\det[\varphi'(0, 0)] = -1$.*

The set of all germs of involutions at 0 is denoted by I^r and it is endowed with the $\mathcal{C}^r$ topology. Consider $W^r = I^r \times I^r$ endowed with the product topology.

Definition 3.2.5. *Let $\varphi = (\varphi_0, \varphi_1)$, $\psi = (\psi_0, \psi_1) \in W^r$ be two pairs of involutions at 0. Then φ and ψ are said to be* ***topologically equivalent*** *at 0 if there exists a germ of a homeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ which satisfies $h\varphi_0 = \psi_0 h$ and $h\varphi_1 = \psi_1 h$, simultaneously.*

The local structural stability of a pair of involutions in W^r is defined in the natural way. The proof of the next theorem can be found in [99] as well as more details about involutions.

Theorem 3.2.6. *A pair of involutions (φ, ψ) is locally and simultaneous structurally stable at 0 if and only if 0 is a hyperbolic fixed point of the composition $\varphi \circ \psi$. Moreover, the structural stability in the space of pairs of involutions is not a generic property.*

3.3 Generic Singularities

Recall that, in a PSVF, if only one component of $Z = (X, Y)$ is considered, say X, then it is a germ of $\mathcal{C}^r$ vector field defined on a manifold with boundary $\overline{M^+}$. Therefore, the theory of vector fields on manifolds with boundary (see [83, 95, 97, 104]) is used to distinguish some points of Σ.

Denote the space of germs of $\mathcal{C}^r$ vector fields defined on the manifold with boundary N by $\chi^r(N)$ $(r > 1)$. If N is not specified, then consider $N = \overline{M^+}$ or $N = \overline{M^-}$.

Definition 3.3.1. *A point $p \in \Sigma$ is said to be a* ***fold*** *point of $X \in \chi^r(\overline{M^+})$ if $Xf(p) = 0$ and $X^2 f(p) \neq 0$. If $X^2 f(p) > 0$ (resp. $X^2 f(p) < 0$), then p is a* ***visible fold*** *(resp.* ***invisible fold****).*

Remark 3.3.2. *If $X \in \chi^r(\overline{M^-})$, the visibility condition is switched.*

Definition 3.3.3. *A point $p \in \Sigma$ is said to be a **cusp** of $X \in \chi^r(N)$ if $Xf(p) = X^2f(p) = 0$, $X^3f(p) \neq 0$ and $\{df(p),\ dXf(p),\ dX^2f(p)\}$ is a linearly independent set.*

Generically, a fold point of X belongs to a local curve of fold points of X with the same visibility, and cusp points occur as isolated points located at the extreme of curves of fold points.

Definition 3.3.4. *$X \in \chi^r(N)$ is said to be **simple** if either $S_X = \emptyset$ or S_X is just composed by fold and cusp points of X. The set of all simple germs of $\chi^r(N)$ will be denoted by χ^r_S.*

In [104], S. M. Vishik used tools from Theory of Singularities to obtain sharpen results on vector fields near the boundary of an n-manifold. In particular, when $n = 3$, the following result is stated.

Theorem 3.3.5 (Vishik's Normal Form)**.** *Let $X \in \chi^r_S$. If $p \in S_X$ then there exist a neighborhood $V(p)$ of p in M, a system of coordinates (x_1, x_2, x_3) at p defined in $V(p)$ ($x_i(p) = 0$, $i = 1, 2, 3$) and an integer $k = k(p)$, $k = 1, 2$, such that:*

1. *If p is a fold point, then $k = 1$ and $X|_{V(p)}$ is a germ at $V(p) \cap \Sigma$ of the vector field given by*

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = 1, \\ \dot{x_3} = 0. \end{cases} \tag{3.3.1}$$

2. *If p is a cusp point, then $k = 2$ and $X|_{V(p)}$ is a germ at $V(p) \cap \Sigma$ of the vector field given by*

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = x_3, \\ \dot{x_3} = 1. \end{cases} \tag{3.3.2}$$

3. *Σ is given by the equation $x_1 = 0$ in $V(p)$.*

The set χ^r_S is open and dense in $\chi^r(N)$.

Remark 3.3.6. *If we perform the change of coordinates $y_1 = x_3$, $y_2 = x_3^2 - 2x_2$, and $y_3 = 2x_1$, then system* (3.3.2) *is carried to the system $\dot{y_1} = 1$, $\dot{y_2} = 0$, $\dot{y_3} = y_1^2 - y_2$. Analogously, if we consider the change $y_1 = x_2$, $y_2 = x_3$, and $y_3 = x_1$, then* (3.3.1) *is carried to $\dot{y_1} = 1$, $\dot{y_2} = 0$, $\dot{y_3} = y_1$. In both cases, Σ is given by the equation $y_3 = 0$.*

In the piecewise smooth context, we consider the following tangential singularities.

Definition 3.3.7. *Let $Z = (X, Y) \in \Omega^r$. A tangential singularity $p \in \Sigma$ is said to be **elementary** if it satisfies one of the following conditions:*

*(FR) - $Xf(p) = 0$, $X^2f(p) \neq 0$ and $Yf(p) \neq 0$ (resp. $Xf(p) \neq 0$, $Yf(p) = 0$ and $Y^2f(p) \neq 0$). In this case, p is said to be a **fold-regular** (resp. regular-fold) point of Σ.*

*(CR) - $Xf(p) = 0$, $X^2f(p) = 0$, $X^3f(p) \neq 0$ and $Yf(p) \neq 0$ (resp. $Xf(p) \neq 0$, $Yf(p) = 0$, $Y^2f(p) = 0$ and $Y^3f(p) \neq 0$), and $\{df(p), dXf(p), dX^2f(p)\}$ (resp. $\{df(p), dYf(p), dY^2f(p)\}$) is a linearly independent set. In this case, p is said to be a **cusp-regular** (resp. regular-cusp) point of Σ.*

*(FF) - If $Xf(p) = 0$, $X^2f(p) \neq 0, Yf(p) = 0$, $Y^2f(p) \neq 0$ and $S_X \pitchfork S_Y$ at p. In this case, p is said to be a **fold-fold** point of Σ.*

See Figure 3.1.

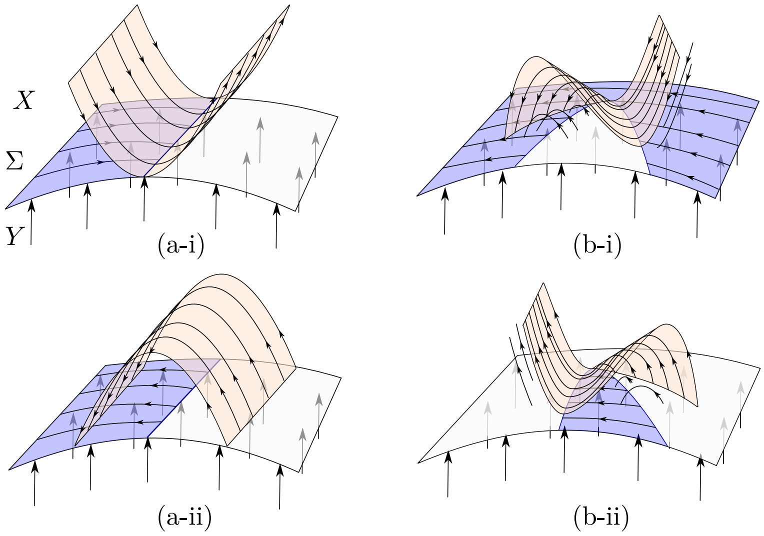


Figure 3.1: (a) Fold-Regular singularities ((i) visible and (ii) invisible) and (b) Cusp-Regular singularities ((i) $X^3f(p) < 0$ and (ii) $X^3f(p) > 0$).

Definition 3.3.8. *Define Ξ_0 as the set of all $Z \in \Omega^r$ such that each $p \in \Sigma$ is either a regular-regular point of Z or an elementary tangential singularity of Z.*

Remark 3.3.9. *An element $Z \in \Xi_0$ is referred as an **elementary piecewise smooth vector field**.*

From Theorem 3.3.5, we derive the following proposition.

Proposition 3.3.10. *Ξ_0 is an open dense set of Ω^r.*

The elementary tangential singularities of type (FR) and (CR) determine certain local behavior of the sliding solutions lying on Σ^s, as we can see in the following result proved in [101].

Lemma 3.3.11. *Let $Z = (X, Y) \in \Omega^r$ and assume that R is a connected component of Σ^s. Then:*

1. *The sliding vector field F_Z is of class $\mathcal{C}^r$ and it can be smoothly extended beyond the boundary of R through the normalized sliding vector field F_Z^N.*

2. *If $p \in \partial R$ is a fold point of X and a regular point of Y, then F_Z is transverse to ∂R at p.*

3. *If $p \in \partial R$ is a cusp point of X and a regular point of Y, then F_Z has a quadratic contact with ∂R at p.*

Theorem 3.3.12. *Let $Z = (X, Y) \in \Omega^r$, then:*

1. *Z is locally structurally stable at a regular-regular point $p \in \Sigma$ if and only if $p \in \Sigma^c$ or $p \in \Sigma^s$ and, in the second case, p is either a regular point or a hyperbolic singularity of F_Z.*
2. *Z is locally structurally stable at any fold-regular singularity $p \in \Sigma$.*
3. *Z is locally structurally stable at any cusp-regular singularity $p \in \Sigma$.*

The proof of this result can be found in [39, 55].

3.4 Statement of the main results

Define the following subsets of Ω^r:

- $\Sigma(G)$: $Z \in \Omega^r$ such that each point $p \in \Sigma$ is either a tangential singularity or a regular-regular point.
- $\Sigma(R)$: $Z \in \Omega^r$ such that for each regular-regular point $p \in \Sigma$ of Z we have either $p \in \Sigma^c$ or $p \in \Sigma^s$ and, in the second case, p is either a regular point or a hyperbolic singularity of F_Z;
- $\Sigma(H)$: $Z \in \Omega^r$ such that for each visible fold-fold point $p \in \Sigma$, the normalized sliding vector field F_Z^N has no center manifold in Σ^s.
- $\Sigma(P)$: $Z \in \Omega^r$ such that for each invisible-visible point $p \in \Sigma$, the normalized sliding vector field F_Z^N is either transient in Σ^s or it has a hyperbolic singularity at p. Moreover, if ϕ_X is the involution associated to Z then it satisfies:
    1. $\phi_X(S_Y) \pitchfork S_Y$ at p;
    2. F_Z^N and $\phi_X^* F_Z^N$ are transversal at each point of $\Sigma^{ss} \cap \phi_X(\Sigma^{us})$;
    3. $\phi_X(S_Y) \pitchfork F_Z^N$ in a neighborhood of p.
- $\Sigma(E)$: $Z \in \Omega^r$ such that for each T-singularity $p \in \Sigma$, the first return map ϕ_Z associated to Z has a fixed point at p of type saddle with both local invariant manifolds $W_{loc}^{u,s}$ contained in Σ^c.

Remark 3.4.1. *If Z has a visible-invisible fold-fold singularity at p, then the roles of X and Y in the condition $\Sigma(P)$ are interchanged.*

The main result of this chapter s the following theorem.

Theorem F. *$Z \in \Omega^r$ is locally structurally stable at a T-singularity p if and only if it satisfies condition $\Sigma(E)$ at p.*

The following theorem is proved in [24, 39] and a detailed proof clarifying some obscure points is presented.

Theorem G. *i) $Z \in \Omega^r$ is locally structurally stable at a hyperbolic fold-fold singularity p if and only if it satisfies condition $\Sigma(H)$ at p.*

ii) $Z \in \Omega^r$ is locally structurally stable at a parabolic fold-fold singularity p if and only if it satisfies condition $\Sigma(P)$ at p.

Theorem H. $\Sigma_0 = \Sigma(G) \cap \Sigma(R) \cap \Sigma(H) \cap \Sigma(P) \cap \Sigma(E)$.

Theorem I. *Σ_0 is not residual in Ω^r.*

As a corollary of the characterization Theorem H, we obtain:

Corollary 3.4.2. *i) Σ_0 is an open dense set in $\Sigma(E)$. Moreover, $\Sigma(E)$ is maximal with respect to this property.*

ii) If $Z \notin \Sigma(E)$ then Z has ∞-moduli of stability.

In addition, if Z has a T-singularity at p and ϕ_Z has complex eigenvalues, then a neighborhood $\mathcal{V}$ of Z in Ω^r is foliated by codimension one submanifolds of Ω^r corresponding to the value of the argument of the eigenvalues of the first return map. Moreover, the topological type along the corresponding leaf is locally constant.

We conclude that the local behavior around a T-singularity implies in the non-genericity of Σ_0 in Ω^r.

3.5 Fold-Fold Singularity

3.5.1 A Normal Form

In this section we derive a normal form to study the fold-fold singularity and we present some consequences. This section is mainly motivated by the normal form of a fold point obtained by S. M. Vishik in [104] and some variants such as [24, 38, 39].

Proposition 3.5.1. *If $Z = (X, Y) \in \Omega^r$ is a nonsmooth vector field having a fold-fold point at p such that $S_X \pitchfork S_Y$ at p, then there exist coordinates (x, y, z) around p such that $f(x, y, z) = z$ and Z is given by:*

$$X(x,y,z) = \begin{pmatrix} \alpha \\ 1 \\ \delta y \end{pmatrix} \text{ and } Y(x,y,z) = \begin{pmatrix} \gamma + \mathcal{O}(|(x,y,z)|) \\ \beta + \mathcal{O}(|(x,y,z)|) \\ x + \mathcal{O}(|(x,y,z)|^2) \end{pmatrix}, \tag{3.5.1}$$

where $\delta = sgn(X^2 f(p))$, $sgn(\gamma) = sgn(Y^2 f(p))$, $\alpha, \beta, \gamma \in \mathbb{R}$.

Outline. Use the coordinates (x, y, z) of Theorem 2 from [104] to put X in the form $X(x, y, z) = (0, 1, \delta y)$ and $f(x, y, z) = z$. Now, consider the Taylor expansion of Y in this coordinate system and perform changes to put $Y f(x, y, z) = x + \mathcal{O}(|(x, y, z)|^2)$. □

Definition 3.5.2. *If $Z \in \Omega^r$ has a fold-fold singularity at p, then the coordinate system of Proposition 3.5.1 will be called **normal coordinates** of Z at p and the parameters of Z in the normal coordinates will be referred as **normal parameters** of Z at p. Denote $Z = Z(\alpha, \beta, \gamma)$.*

Remark 3.5.3. *If $\gamma = \pm 1$, $\alpha = V^+$ and $\beta = V^-$, then this normal form and the model used in [24, 25, 38], have the same semi-linear part. Geometrically, V^+ (V^-) measures the cotangent of the angle θ^+ (θ^-) between $X(0)$ ($Y(0)$) and the fold line S_X (S_Y). See [25] for more details.*

Corollary 3.5.4. *If $Z = (X, Y) \in \Omega^r$ is a nonsmooth vector field having a fold-fold point at p such that $S_X \pitchfork S_Y$ at p, then there exist coordinates (x, y, z) around p defined in a neighborhood U of p in M, such that:*

1. $f(x, y, z) = z$;

2. $S_X \cap U = \{(x, 0, 0);\ x \in (-\varepsilon, \varepsilon)\}$, *for* $\varepsilon > 0$ *sufficiently small;*

3. $S_Y \cap U = \{(g(y), y, 0);\ y \in (-\varepsilon, \varepsilon)\}$, *for* $\varepsilon > 0$ *sufficiently small, where* g *is a* $\mathcal{C}^r$ *function such that* $g(y) = \mathcal{O}(y^2)$*, i.e.,* S_{Y_0} *is locally a smooth curve tangent to the* y*-axis.*

Outline. It follows directly from Proposition 3.5.1 and the Implicit Function Theorem. □

Proposition 3.5.5. *Let* $Z = (X, Y) \in \Omega^r$ *be a nonsmooth vector field having a fold-fold point at* p *such that* $S_X \pitchfork S_Y$ *at* p*. Then, the normalized sliding vector field of* Z *has a singularity at* p *and it is given by*

$$F_Z^N(x, y) = \begin{pmatrix} \alpha & -\delta\gamma \\ 1 & -\delta\beta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}(|(x, y)|^2),$$

in the normal coordinates of Z *at* p*, where* $\delta = sgn(X^2 f(p))$*,* $sgn(\gamma) = sgn(Y^2 f(p))$*,* $\alpha, \beta, \gamma \in \mathbb{R}$*.*

Outline. It follows directly from the expression of Z in this coordinate system. □

Finally, we can classify a fold-fold singularity in four topologically distinct classes:

Definition 3.5.6. *A fold-fold point* p *of* $Z = (X, Y) \in \Omega^r$ *is said to be:*

- *a* ***visible fold-fold*** *if* $X^2 f(p) > 0$ *and* $Y^2 f(p) < 0$*;*
- *an* ***invisible-visible fold-fold*** *if* $X^2 f(p) < 0$ *and* $Y^2 f(p) < 0$*;*
- *a* ***visible-invisible fold-fold*** *if* $X^2 f(p) > 0$ *and* $Y^2 f(p) > 0$*;*
- *an* ***invisible fold-fold*** *if* $X^2 f(p) < 0$ *and* $Y^2 f(p) > 0$*, in this case,* p *is also called a* ***T-singularity****.*

Remark 3.5.7. *Notice that the visible-invisible case can be obtained from the invisible-visible one by performing an orientation reversing change of coordinates. Also, we refer to a visible, invisible-visible/visible-invisible, invisible fold-fold point as a hyperbolic, parabolic, elliptic fold-fold point, respectively. See Figure 3.2.*

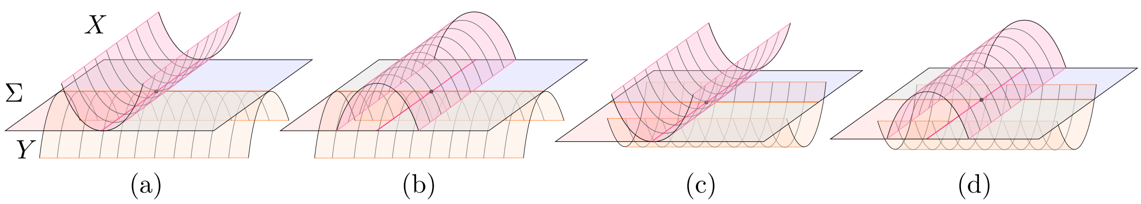


Figure 3.2: Fold-Fold Singularity: (a) Hyperbolic, (b,c) Parabolic and (d) Elliptic.

3.5.2 Sliding Dynamics

In this subsection we discuss the sliding dynamics around a fold-fold singularity. This is a matured topic which has been well developed in [25, 39, 102].

From Proposition 3.5.1 and Lemma 3.3.11, we already know the behavior of the sliding vector field near a fold-fold singularity in a generic scenario (not only for the truncated system).

Let $Z = Z(\alpha, \beta, \gamma) \in \Omega^r$ having a fold-fold singularity at p, and consider its normalized sliding vector field F_Z^N in normal coordinates.

Consider:

$$
\begin{aligned}
&R_E^1 = \{(\alpha,\beta,\gamma) \in \mathbb{R}^2 \times \mathbb{R}^+;\ \alpha\beta > \gamma \text{ and } \alpha < 0,\ \beta < 0\},\\
&R_E^2 = \mathbb{R}^2 \times \mathbb{R}^+ \setminus \overline{R_E^1},\\
&R_H^1 = \{(\alpha,\beta,\gamma) \in \mathbb{R}^2 \times \mathbb{R}^-;\ \alpha\beta < \gamma \text{ and } \alpha > 0,\ \beta < 0\},\\
&R_H^2 = \mathbb{R}^2 \times \mathbb{R}^- \setminus \overline{R_H^1},\\
&R_P^1 = \{(\alpha,\beta,\gamma) \in \mathbb{R}^2 \times \mathbb{R}^-;\ \alpha\beta < \gamma \text{ and } \beta - \alpha > -2\sqrt{-\gamma}\},\\
&R_P^2 = \{(\alpha,\beta,\gamma) \in \mathbb{R}^2 \times \mathbb{R}^-;\ \alpha\beta < \gamma \text{ and } \alpha > 0\},\\
&R_P^3 = \{(\alpha,\beta,\gamma) \in \mathbb{R}^2 \times \mathbb{R}^-;\ \alpha\beta > \gamma,\ \beta + \alpha > 0 \text{ and } \beta - \alpha < -2\sqrt{\gamma}\},\\
&R_P^4 = \{(\alpha,\beta,\gamma) \in \mathbb{R}^2 \times \mathbb{R}^-;\ \alpha\beta > \gamma,\ \beta + \alpha < 0 \text{ and } \beta - \alpha < -2\sqrt{-\gamma}\}.
\end{aligned}
$$

We claim that:

Claim 1: If p is an elliptic fold-fold singularity and $(\alpha, \beta, \gamma) \in R_E^1$ then F_Z has an invariant manifold W in Σ^s passing through p and each orbit of F_Z is transverse to S_Z and reaches p asymptotically to W (for a finite positive time in Σ^{ss} and negative time in Σ^{us}).

Claim 2: If p is an elliptic fold-fold singularity and $(\alpha, \beta, \gamma) \in R_E^2$ then F_Z has an invariant manifold W in Σ^s passing through p and each orbit is transverse to S_Z and does not reach p, with exception of W.

Claim 3: If p is a hyperbolic fold-fold singularity and $(\alpha, \beta, \gamma) \in R_H^1$ (resp. $(\alpha, \beta, \gamma) \in R_H^2$) then F_Z is of the same type of claim 1 (resp. claim 2) for reverse time.

Claim 4: If p is a parabolic fold-fold singularity and $(\alpha, \beta, \gamma) \in R_P^1$ then each orbit in Σ^{ss} (resp. Σ^{us}) is transverse to S_X (resp. S_Y) and reaches S_Y (resp. S_X) transversally for a positive finite time. In this case we say that F_Z has transient behavior in Σ^s.

Claim 5: If p is a parabolic fold-fold singularity and $(\alpha, \beta, \gamma) \in R_P^2$ then there exist two invariant manifolds W_1 and W_2 in Σ^s passing through p which divide Σ^{ss} (and Σ^{us}) in three sectors. The intermediate sector is of hyperbolic type and in the other sectors the orbits are transversal to S_Z and go away from p (the orientation of the orbits is given in Figure 3.3).

Claim 6: If p is a parabolic fold-fold singularity and $(\alpha, \beta, \gamma) \in R_P^3$ then there exist two invariant manifolds W_1 and W_2 in Σ^s passing through p which divide Σ^{ss} in three sectors. In the intermediate sector each orbit reaches p for a finite positive time asymptotically to W_1. In the left one each orbit is transverse to S_Y and reaches p for a finite positive time asymptotically to W_1. In the right one, each orbit is transverse to S_X and goes away from p. The behavior in Σ^{us} is similar and can be seen in Figure 3.3.

Claim 7: If p is a parabolic fold-fold singularity and $(\alpha, \beta, \gamma) \in R_P^4$ then F_Z has the same behavior as in claim 6 for reverse time and changing the role of W_1 and W_2, S_X and S_Y, right and left.

Claim 8: If (α, β, γ) is not in any of these regions then F_Z presents bifurcations in Σ^s.

All these claims can be straightforward verified by analyzing the linear part of the normalized sliding vector field F_Z^N. We omitted the proofs due to its simplicity.

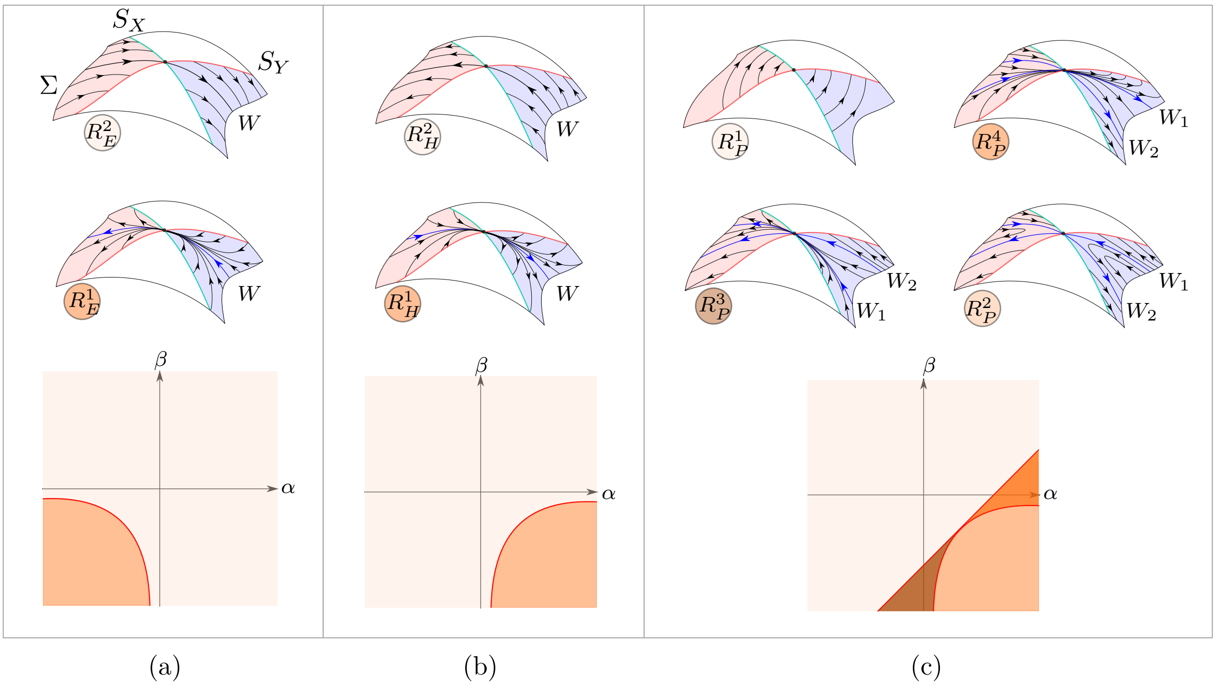


Figure 3.3: Sliding dynamics near a fold-fold singularity of type elliptic (a), hyperbolic (b) and parabolic (c). In each case, the regions above are outlined in the (α, β)-parameter space for a fixed value of γ.

3.6 Proofs of Theorems F and I

This section is devoted to prove Theorems F and I. In the sequel we develop some Lemmas and Propositions which will lead us to the proof of the Theorems.

Assume that $Z \in \Omega^r$ has a T-singularity at p. Therefore, we have a first return map ϕ of Z defined around p. In order to study the local structural stability of Z, it will be crucial to study the dynamics of ϕ. Now, we derive the existence and some properties of ϕ.

Lemma 3.6.1. *Let $Z = (X, Y) \in \Omega^r$ be a nonsmooth vector field having a T-singularity at p such that $S_X \pitchfork S_Y$ at p. There exist two involutions $\phi_X : (\Sigma, p) \to (\Sigma, p)$ and $\phi_Y : (\Sigma, p) \to (\Sigma, p)$ associated to the folds X and Y such that:*

- *$Fix(\phi_X) = S_X$;*
- *$Fix(\phi_Y) = S_Y$;*
- *$\phi = \phi_X \circ \phi_Y$ is a first return map of Z such that $\phi(p) = p$.*

The proof of Lemma 3.6.1 can be found in [23] (Lemma 1). A straightforward verification shows the following results.

Lemma 3.6.2. *If $\phi = \varphi \circ \psi$, where φ and ψ are involutions of $\mathbb{R}^2$ at 0, then $\phi^n \circ \varphi = \varphi \circ \phi^{-n}$ and $\psi \circ \phi^n = \phi^{-n} \circ \psi$, for each $n \in \mathbb{Z}$.*

Proposition 3.6.3. *If $\phi = \varphi \circ \psi$, where φ and ψ are involutions of Σ at p, then the invariant manifolds W^s and W^u of ϕ at p are interchanged by φ and ψ in the following way:*

$$\psi(W^s) \subset W^u \text{ and } \varphi(W^u) \subset W^s.$$

Now, using the normal coordinates of $Z = (X, Y)$ at an elliptic fold-fold singularity, we obtain the following expressions for the associated involutions.

Lemma 3.6.4. *Let $Z = (X, Y) \in \Omega^r$ be a nonsmooth vector field having a T-singularity at p such that $S_X \pitchfork S_Y$ at p. Consider the normal coordinates (x, y, z) of Z at p. Then the involutions ϕ_X and ϕ_Y are given by*

$$\phi_X(x, y) = (x - 2\alpha y, -y) \text{ and } \phi_Y(x, y) = \left(-x, -\frac{2\beta}{\gamma}x + y\right) + h.o.t.,$$

in these coordinates, where α, β, γ are the normal parameters of Z at p.

Finally, we associate the local structural stability of Z at an elliptic fold-fold singularity with the local structural stability of the pair of involutions associated to Z.

Lemma 3.6.5. *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ such that p is a T-singularity for Z_0. If Z_0 is locally structurally stable at p in Ω^r then the pair of involutions (ϕ_{X_0}, ϕ_{Y_0}) associated to Z_0 is locally and simultaneously structurally stable at 0 in W^r.*

Proof. In fact, since p is a T-singularity of Z_0, there exist neighborhoods $\mathcal{V}$ of Z_0 in Ω^r and V of p in M such that, each $Z \in \mathcal{V}$ has a unique Teixeira singularity at $q(Z) \in V \cap \Sigma$.

Consider the map $F : \mathcal{V} \to W^r$ given by:

$$F(X, Y) = (\phi_X, \phi_Y),$$

where ϕ_X and ϕ_Y are the involutions at $(0, 0)$ of $\mathbb{R}^2$ associated to X and Y, respectively.

From the continuous dependence of solutions with respect to initial conditions and parameters, it follows that F is a continuous map.

Moreover, there exists a neighborhood $\mathcal{U}$ of (ϕ_{X_0}, ϕ_{Y_0}) in W^r, such that, for each $(\tau, \psi) \in \mathcal{U}$, there exists a vector field $Z = (X, Y) \in \mathcal{V}$ such that $\tau = \phi_X$ and $\psi = \phi_Y$, and it can be done in a continuous fashion.

Then, reducing $\mathcal{V}$ if necessary, it follows that $F : \mathcal{V} \to W^r$ is an open continuous map.

Since Z_0 is locally structurally stable at p in Ω^r, $\mathcal{V}$ can be reduced such that every $Z \in \mathcal{V}$ is topologically equivalent to Z_0.

Thus, if $Z \in \mathcal{V}$, there exist a fold-fold singularity $q(Z) \in \Sigma$ of Z (with the same type of p) and a topological equivalence $h : (V_1, p) \to (V_2, q(Z))$ between Z_0 and Z, where V_1 and V_2 are neighborhoods of p in M, such that $q(Z) \in V_2$.

In particular, it induces a homeomorphism $h : \Sigma \cap V_1 \to \Sigma \cap V_2$ such that $h(p) = q(Z)$. Using coordinates, (x, y, z) around p and (u, v, w) around $q(Z)$ such that $f(x, y, z) = z$ and $f(u, v, w) = w$, the induced homeomorphism h can be seen as $h : U_1 \to U_2$, where U_1 and U_2 are neighborhoods of $(0, 0)$ in $\mathbb{R}^2$ and $h(0, 0) = (0, 0)$.

Now, given $(x, y) \in \Sigma - S_{X_0}$ (sufficiently near from $(0, 0)$), it follows from the definition of the involution ϕ_{X_0} that, the points (x, y) and $\phi_{X_0}(x, y)$ are connected by an orbit γ_0 of

X_0 contained in M^+. Analogously, the points $h(x,y)$ and $\phi_X(h(x,y))$ are connected by an orbit γ of X contained in M^+.

Since h is a topological equivalence such that $h(\Sigma) \subset \Sigma$, it follows that $h(\gamma_0) = \gamma$ and

$$h(\phi_{X_0}(x,y)) = \phi_X(h(x,y)). \tag{3.6.1}$$

It is trivial to see that 3.6.1 is also true when $(x,y) \in S_{X_0}$, by observing that $h(S_{X_0}) = S_X$. Hence h is an equivalence between the germs of involution ϕ_{X_0} and ϕ_X.

Analogously, by changing the roles of X and Y, it can be shown that h is also an equivalence between the involutions ϕ_{Y_0} and ϕ_Y.

We conclude that h is a (simultaneous) topological equivalence between the pairs of involutions (ϕ_{X_0}, ϕ_{Y_0}) and (ϕ_X, ϕ_Y).

Since Z is arbitrary in $\mathcal{V}$, it follows that every pair of involutions in $\mathcal{U}$ is topologically equivalent to (ϕ_{X_0}, ϕ_{Y_0}), and since $\mathcal{U}$ is open in W^r, it follows that (ϕ_{X_0}, ϕ_{Y_0}) is local and simultaneous structurally stable in W^r. $\square$

The following result is obtained by combining Theorem 3.2.6 and Lemma 3.6.5.

Proposition 3.6.6. *Let $Z_0 \in \Omega^r$ having a T-singularity at p, and let (ϕ_{X_0}, ϕ_{Y_0}) be the pair of involutions of $\mathbb{R}^2$ at $(0,0)$ associated to Z_0. If 0 is not a hyperbolic fixed point of $\phi_{Y_0} \circ \phi_{X_0}$, then Z_0 is locally structurally unstable at p.*

A simple computation of eigenvalues and eigenvectors allows us to study the fixed point p of the first return map ϕ (see Figure 3.4):

Lemma 3.6.7. *Let $Z = (X,Y) \in \Omega^r$ be a nonsmooth vector field having a T-singularity at p such that $S_X \pitchfork S_Y$ at p. Let (α, β, γ) be the normal parameters of Z at p.*

1. *If $\alpha\beta(\alpha\beta - \gamma) \leq 0$, then 0 is not a hyperbolic fixed point of ϕ. In addition, if $\alpha\beta(\alpha\beta - \gamma) < 0$, then ϕ has complex eigenvalues.*
2. *If $\alpha\beta(\alpha\beta - \gamma) > 0$, then 0 is a saddle point of ϕ. In addition, if λ, μ are the eigenvalues of ϕ such that $|\mu| < 1 < |\lambda|$, and v_μ, v_λ are the correspondent eigenvectors, then:*
   - *(a) If $\alpha > 0$ and $\beta > 0$, then $v_\mu, v_\lambda \in \Sigma^s$.*
   - *(b) If $\alpha > 0$ and $\beta < 0$, then $v_\mu \in \Sigma^c$ and $v_\lambda \in \Sigma^s$.*
   - *(c) If $\alpha < 0$ and $\beta > 0$, then $v_\mu \in \Sigma^s$, and $v_\lambda \in \Sigma^c$.*
   - *(d) If $\alpha < 0$ and $\beta < 0$ then $v_\mu, v_\lambda \in \Sigma^c$.*

Proposition 3.6.8. *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a germ of nonsmooth vector field having a T-singularity at p. Let (α, β, γ) be the normal parameters of Z_0 at p. If $\alpha\beta(\alpha\beta - \gamma) \leq 0$, then Z_0 is locally structurally unstable at p.*

Proof. It follows directly from Proposition 3.6.6 and the fact that p is not a hyperbolic fixed point of the first return map $\phi_0 = \phi_{X_0} \circ \phi_{Y_0}$ associated to Z_0. In the sequel we present an explicit argument for the local structural instability of Z_0. It is mainly based on [14] and the Blow-up procedure (see [6]).

Let $\phi_0 : (\Sigma, p) \to (\Sigma, p)$ be the (germ of) first return map associated to Z_0 at p. From the conditions assumed in the Theorem, it follows that ϕ_0 has eigenvalues $\lambda_\pm = a \pm ib$,

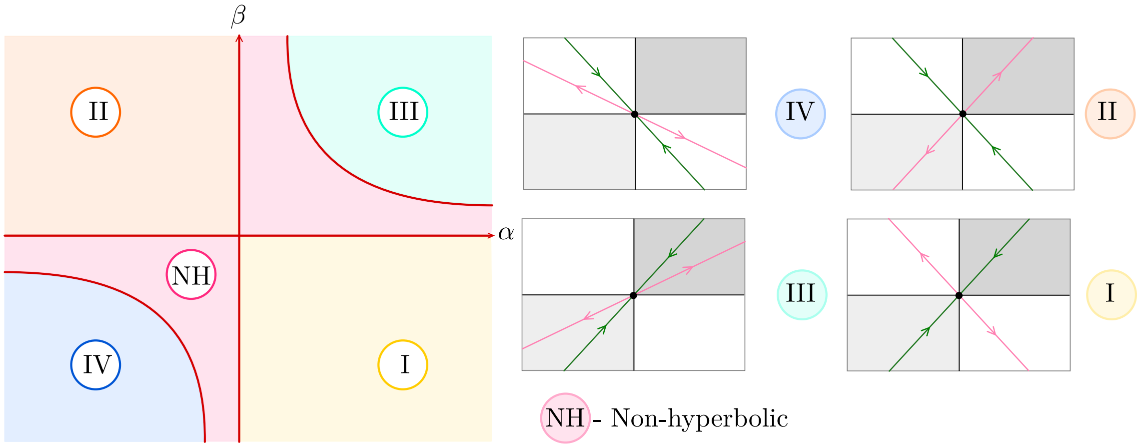


Figure 3.4: Regions of the (α, β)-parameter space with the corresponding behavior of the first return map φ, for a fixed value of $\gamma > 0$.

where $a^2 + b^2 = 1$. Using the normal form of Z_0 and basic linear algebra, it is easy to find coordinates (x, y) of Σ at p, such that:

$$\phi_0(x, y) = (ax - by, bx + ay) + \mathcal{O}(|(x, y)|^2).$$

Consider the germs of functions $h_1, h_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, given by:

$$h_1(x, y) = (x, y) \text{ and } h_2(x, y) = \sqrt{x^2 + y^2}(x, y).$$

Notice that h_1, h_2 are germs of homeomorphisms if we exclude the origin in their domains.

If $(x, y) \neq (0, 0)$, a straightforward computation shows that:

$$\psi_0(x, y) = h_2^{-1} \circ \phi_0 \circ h_1(x, y) = \frac{1}{\sqrt{x^2 + y^2}}\phi_0(x, y).$$

Therefore, ϕ_0 and ψ_0 are topologically equivalent. Using the polar change of coordinates $\zeta(r, \theta) = (r\cos(\theta), r\sin(\theta))$, where $r > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we write $\psi_0 \circ \zeta$ as

$$\psi_0 \circ \zeta(r, \theta) = \begin{pmatrix} \cos(\theta + \tau) \\ \sin(\theta + \tau) \end{pmatrix} + \mathcal{O}(r),$$

where $a + ib = e^{i\tau}$.

If $r \to 0$, ζ blows up the singularity $r = 0$ into the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and the map $\zeta^{-1} \circ \psi_0 \circ \zeta$ induces a dynamics in S^1 (see Figure 3.5) given by

$$\overline{\psi_0}([\theta]) = [\theta + \tau].$$

Let Z be a small perturbation of Z_0, take it small enough such that the normal parameters $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of Z are close enough to (α, β, γ).

If ϕ is the first return map associated to Z at the fold-fold point $q(Z) \approx p$, then it has eigenvalues $\tilde{\lambda}_\pm = \tilde{a} \pm i\tilde{b}$.

Applying the same procedure to ϕ, we can blow-up its singularity $q(Z)$ into S^1, and the dynamics in S^1 is induced by $\overline{\psi} : S^1 \to S^1$, given by $\overline{\psi}(\theta) = \theta + \tilde{\tau}$, where $\tilde{a} + i\tilde{b} = e^{i\tilde{\tau}}$.

Now, if $h : V(p) \to V(q(Z))$ is an equivalence between Z_0 and Z, then $h(S_{X_0}) = S_X$. In adequate coordinates, it means that $h(x, 0) = (f(x), 0)$, where f is a homeomorphism of the real line such that $f(0) = 0$.

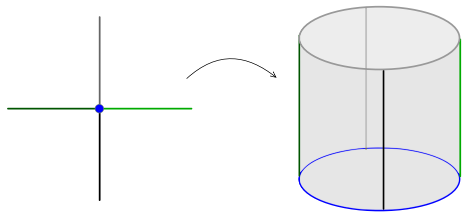


Figure 3.5: Blow-up of p into S^1.

Notice that the motion of $S_{X_0} \cap \{x \geq 0\}$ (resp. $S_X \cap \{x \geq 0\}$) around the origin through ϕ_0 (resp. ϕ) is given by the orbit $\gamma_0 = \{\overline{\psi_0}^n(0), n \in \mathbb{Z}\}$ (resp. $\gamma = \{\overline{\psi}^n(0), n \in \mathbb{Z}\}$).

Since h is an equivalence, it follows that the orbits γ_0 and γ have the same topology. Nevertheless, if $\tau \in \mathbb{Q}$ (resp. $\tau \notin \mathbb{Q}$) we can take Z (sufficiently near of Z_0) such that $\tilde{\tau} \notin \mathbb{Q}$ (resp. $\tilde{\tau} \in \mathbb{Q}$). Therefore, γ_0 is a periodic orbit and γ is dense in S^1 (resp. γ_0 is dense in S^1 and γ is a periodic orbit).

It means that, when $\tau \in \mathbb{Q}$ (and γ_0 is periodic), the curves $\phi^n(S_X)$ are tangent to a finite number of directions at p, i.e., there exist m vectors $v_1, \cdots, v_m$ in $T_p\Sigma$ such that $T_p\phi^n(S_X) = \text{span}\{v_{i(n)}\}$, for some $i(n) \in \{1, \cdots, m\}$, for each $n \in \mathbb{N}$. Hence, we conclude that $\bigcup \phi^n(S_X)$ has zero measure in Σ.

On the other hand, if $\tau \notin \mathbb{Q}$ (and γ_0 is dense), we have that for each $v \in T_p\Sigma$, there exist a sequence $\phi^{n_k}(S_X)$, such that $T_p\phi^{n_k}(S_X) = \text{span}\{v_k\}$, and $v_k \to v$ when $k \to \infty$. We conclude that $\bigcup \phi^n(S_X)$ has full measure in Σ.

From these facts, we can see that the orbits $\phi_0^n(S_{X_0})$ and $\phi^n(S_X)$ do not have the same topology (Figure 3.6).

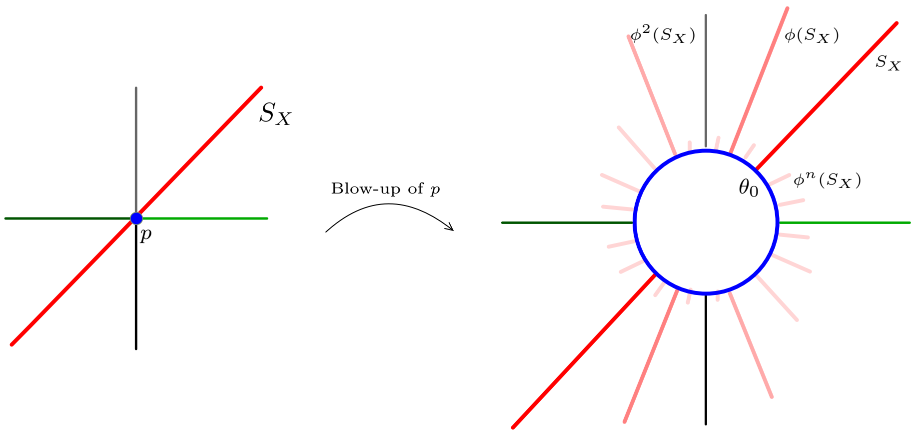


Figure 3.6: Behavior of S_X when $\theta \notin \mathbb{Q}$.

Now, a Σ-equivalence between Z_0 and Z has to satisfy $h(S_{X_0}) = S_X$ and $h \circ \phi_0 = \phi \circ h$. Since $\phi_0^n(S_{X_0})$ and $\phi^n(S_X)$ have different topological type, it follows that there is no Σ-equivalence between Z_0 and Z.

We conclude that, in any neighborhood of Z_0 in Ω^r we can find a nonsmooth vector field Z such that Z_0 is not topologically equivalent to Z at p. Therefore, Z_0 is locally structurally unstable at p. $\square$

Remark 3.6.9. *Let τ_Z be the argument of the eigenvalues $a \pm ib$ of the first return map*

ϕ associated to Z.

If Z_0 is a nonsmooth vector field satisfying the hypotheses of Proposition 3.6.8, then a neighborhood $\mathcal{V}_0$ of Z_0 in Ω^r is foliated by codimension one submanifolds of Ω^r corresponding to the value of τ_Z, i.e., $Z_1 \in \mathcal{V}_0$ and $Z_2 \in \mathcal{V}_0$ lies on the same leaf if and only if $\tau_{Z_1} = \tau_{Z_2}$.

The topological type of the first return map is locally constant along each leaf. Moreover, if Z_1 and Z_2 are elements of $\mathcal{V}_0$ lying on different leaves of the foliation then they are not topologically equivalent.

We conclude that Z_0 has ∞-moduli of stability. (See [14, 28, 81] for more details.)

Now we can prove Theorem D.

Theorem 3.6.10. *Σ_0 is not residual in Ω^r.*

Proof of Theorem D. It follows directly from Theorem 3.6.8. In fact, let $Z_0 \in \Omega^r$ and let $(\alpha_0, \beta_0, \gamma_0)$ be the normal parameters of Z_0 at p, they satisfy $\alpha_0\beta_0(\alpha_0\beta_0 - \gamma_0) < 0$.

From continuity (and Implicit Function Theorem), there exist neighborhoods $\mathcal{V}$ of Z_0 in Ω^r and V of p in M such that, each Z has a T-singularity at $q(Z) \in V$.

Moreover, if we apply Proposition 3.5.1 to Z at $q(Z)$, the normal parameters (α, β, γ) of Z at $q(Z)$ also satisfy $\alpha\beta(\alpha\beta - \gamma) < 0$.

From Theorem 3.6.8, each $Z \in \mathcal{V}$ is locally structurally unstable at the fold-fold singularity $q(Z) \in V \cap \Sigma$. It means that each $Z \in \mathcal{V}$ is locally structurally unstable at a point $q(Z) \in \Sigma$, hence each $Z \in \mathcal{V}$ is Σ-locally structurally unstable. Thus, $\mathcal{V} \subset \Omega^r \setminus \Sigma_0$ and Σ_0 is not residual in Ω^r. □

Notice that the results obtained until this point are mainly concerned with the foliation $\mathcal{F}$ generated by a nonsmooth vector field near a T-singularity. The sliding dynamics does not have influence on these results. Nevertheless, the existence of sliding vector fields will be extremely important in the classification of the structural stability of a T-singularity having a first return map with hyperbolic fixed point.

Proposition 3.6.11. *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a germ of nonsmooth vector field having a T-singularity at p. Let (α, β, γ) be the normal parameters of Z_0 at p. If either $\alpha\beta \geq \gamma$ and $\alpha, \beta > 0$ or $\alpha\beta < 0$, then Z_0 is locally structurally unstable at p.*

Proof. In the conditions of the theorem, we can use Lemma 3.6.7 to conclude that the first return map ϕ_0 of Z_0 has a local invariant manifold of the saddle contained in Σ^s.

Without loss of generality, assume that $W^s \subset \Sigma^s$. Notice that the map ϕ_0^2 has the same invariant manifolds of ϕ_0, but it has both positive eigenvalues $0 < \lambda < 1 < \mu$.

Generically (i.e. $W^s \pitchfork W$ at p, where W is the invariant manifold of claim 2 in Section 3.5.2), we have that the sliding vector field F_0 of Z_0 is transverse to $W^s \cap \Sigma^{ss}$ for a small neighborhood of p. Let $V = U \cap \Sigma^s$, where U is a neighborhood of p such that F_0 is transverse to $W^s \cap V$. See Figure 3.7.

Since $\lambda > 0$, we have that $\phi_0^2(W^s) \subset \phi_0^2(V) \cap V$. Moreover,

$$\phi_0^{2n}(W^s) \subset \phi_0^{2n}(V) \cap \phi_0^{2(n-1)}(V) \cap \cdots \cap \phi_0^2(V) \cap V,$$

for each $n \in \mathbb{N}$.

Let R_n be the open set $\phi_0^{2n}(V) \cap \phi_0^{2(n-1)}(V) \cap \cdots \cap \phi_0^2(V) \cap V$. Notice that, in each region $\phi_0^{2i}(V)$, we have a (push-forwarded) vector field

$$F_i = (\phi_0^{2i})^*(F_0),$$

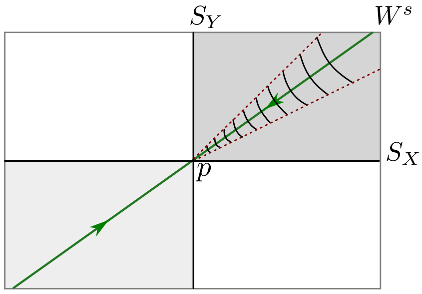


Figure 3.7: Vector field F_0 near W^s.

defined on it. Therefore, there are $n+1$ vector fields defined on R_n. Moreover, we can reduce R_n such that F_i and F_j are transversal at each point of R_n, for $i \neq j$, generically. In fact, consider the expressions of ϕ_X, ϕ_Y and F_Z^N in the normal coordinates. Consider the curves $\gamma_\pm(t) = tv_\pm$, where $v_\pm$ are the eigenvectors associated to the eigenvalues $\lambda_\pm$ of $d\phi_0^2$. A simple computation shows that:

$$F_{ij}^{\pm}(t) = \det(F_i(\gamma_\pm(t)), F_j(\gamma_\pm(t))) = A_{ij}^{\pm}(\alpha, \beta, \gamma)t^2 + \mathcal{O}(t^3),$$

where $A_{ij}^{\pm}$ is a rational function depending on α, β and γ.

Clearly, if $A_{ij}^{\pm} \neq 0$, then F_i and F_j are transversal in a neighborhood of $\gamma_\pm$. In particular, they are transversal in a neighborhood of W^s.

Since $A_{ij}^{\pm} = 0$, for each $i, j = 0, 1, 2$, defines a zero measure set in the parameter space (α, β, γ), we achieved our goal.

Notice that, each vector field F_i in R_n defines a codimension one foliation $\mathcal{F}_i$ of R_n (R_n is foliated by the integral curves of the vector field F_i). Moreover, $(\mathcal{F}_0, \cdots, \mathcal{F}_n)$ is in general position (by the reduction of R_n). In particular, for $n = 2$, we obtain 3 foliations $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ of R_2 (see Figure 3.8). This is called a **3-web** in R_2 (see [13] and [85]).

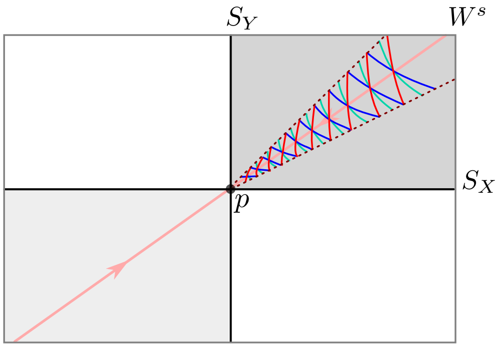


Figure 3.8: Foliations $\mathcal{F}_0, \mathcal{F}_1$, and $\mathcal{F}_2$ originated from the vector fields F_0, F_1 and F_2, respectively, near W^s.

Since R_2 is a 2-dimensional manifold, it follows that these foliations are structurally unstable in the following sense. If $(\widetilde{\mathcal{F}_0}, \widetilde{\mathcal{F}_1}, \widetilde{\mathcal{F}_2})$ are the foliations correspondent to a non-smooth vector field $\widetilde{Z} \approx Z_0$, then there exists at least one $\widetilde{Z}$ such that there is no homeomorphism $h : R_2 \to \widetilde{R_2}$ satisfying $h(\mathcal{F}_i) = \widetilde{\mathcal{F}_i}$, for every $i = 0, 1, 2$, preserving the leaves of each foliation.

Clearly the property above has to be preserved by a Σ-equivalence, hence there exists a Z sufficiently near of Z_0 which is topologically different from Z_0 near p.

The instability of Z_0 at p follows directly from these facts. □

Remark 3.6.12. *In general, the Theory of Webs used in the last Theorem is developed for foliations on $\mathbb{C}^n$. Nevertheless, we can identify Σ with $\mathbb{C}$ at p (since Σ is 2-dimensional) and apply the results of this theory for this case.*

Now, let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a germ of nonsmooth vector field having a Teixeira singularity at p. Let (α, β, γ) be the normal parameters of Z_0 at p and assume that $\alpha\beta \geq \gamma$ and $\alpha, \beta < 0$.

Let $Z \in \Omega^r$ be any small perturbation of Z_0 and denote their first return maps by ϕ and ϕ_0, respectively. Our goal is to construct a topological equivalence between Z and Z_0.

Using the Implicit Function Theorem and the continuous dependence between Z_0 and its normal parameters, we can deduce the following result.

Lemma 3.6.13. *There exists a neighborhood $\mathcal{V}$ of Z_0 such that, for each $Z \in \mathcal{V}$, F_Z^N and $F_{Z_0}^N$ have the same topological type and the first return map ϕ of Z has a saddle at the origin with both local invariant manifolds in Σ^c.*

Remark 3.6.14. *In what follows, $\mathcal{V}$ will denote the neighborhood of Lemma 3.6.13.*

Now we prove the existence of an invariant nonsmooth diabolo in an analytic way, this result was achieved by M. Jeffrey and A. Colombo for the semi-linear case (see [24]).

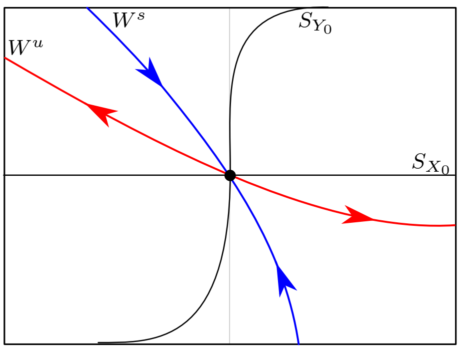
Proposition 3.6.15. *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a nonsmooth vector field having a T-singularity at p such that the normal parameters (α, β, γ) of Z_0 at p satisfy $\alpha\beta \geq \gamma$ and $\alpha, \beta < 0$. Then Z_0 has an invariant nonsmooth diabolo D_0 which prevents connections between points of Σ^{us} and Σ^{ss} through orbits of Z.*

Proof. From Lemma 3.6.13, it follows that the first return map $\phi_0 = \phi_{X_0} \circ \phi_{Y_0}$ associated to Z_0 has a hyperbolic saddle at p with both eigenvectors in Σ^c.

Notice that the local stable manifold of the saddle W^s is tangent to the eigenvector v_- correspondent to the eigenvalue λ and the local unstable manifold of the saddle W^u is tangent to the eigenvector v_- correspondent to the eigenvalue μ, where $|\lambda| < 1 < |\mu|$.

Moreover, W^s and W^u are curves on Σ passing through p transverse to $S_X \cup S_Y$ at p and $W^s \pitchfork W^u$ at p (p is hyperbolic). Using coordinates (x, y) at p (which put Z_0 in the normal form 3.5.1), we can see that, $S_{X_0} = \text{Fix}(\phi_{X_0})$ is the x-axis, $S_{Y_0} = \text{Fix}(\phi_{Y_0})$ is a curve tangent to the y-axis at 0, and W^s and W^u are curves passing through 0 contained in the second and the fourth quadrants which are transverse to $S_{X_0} \cup S_{Y_0}$ at 0.

Therefore we have the following situation:



From Proposition 3.6.3, it follows that $\phi_{X_0}(W^u) \subset W^s$.Now, the image of a point in the semi-plane $\{y > 0\}$ through ϕ_{X_0} is a point in the semi-plane $\{y < 0\}$ by the construction of ϕ_{X_0}. It means that the branch of W^u in the second quadrant has to be taken into the branch of W^s in the fourth quadrant.

Also, $\phi_{Y_0}(W^s) \subset W^u$. Notice that, S_{Y_0} splits $\mathbb{R}^2$ in two connected components, C_- and C_+. From the construction of ϕ_{Y_0}, the image of a point in C_- through ϕ_{Y_0} is a point in C_+. It means that the branch of W^s in the fourth quadrant is taken into the branch of W^u in the second quadrant.

These connections produce an invariant (nonsmooth) cone with vertex at the fold-fold point which contains Σ^{us} in its interior. Analogously, we prove that there exists an invariant (nonsmooth) cone with vertex at the fold-fold point which contains Σ^{ss} in its interior. These two cones produce the required nonsmooth diabolo (see Figure 3.9). $\square$

Remark 3.6.16. *In other words, there is no communication between Σ^{us} and Σ^{ss} in this case.*

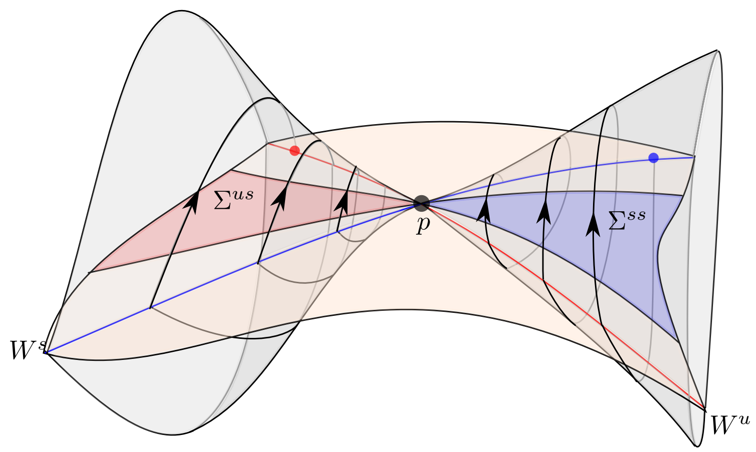


Figure 3.9: A nonsmooth diabolo D_0 of Z_0.

Remark 3.6.17. *Notice that, the existence of the invariant diabolo D_0 implies that the T-singularity p_0 has stable and unstable invariant manifolds of dimension 2, and this is a phenomena which has no counterpart in smooth vector fields of dimension 3.*

Now we proceed by constructing a homeomorphism between $Z \in \mathcal{V}$ and Z_0.

Lemma 3.6.18. *If $Z \in \mathcal{V}$, there exists an order-preserving homeomorphism $h : \Sigma^s(Z_0) \to \Sigma^s(Z)$ which carries orbits of F_{Z_0} onto orbits of F_Z.*

The proof of this lemma follows straightforward from Lemmas 3.3.11 and 3.6.13.

Definition 3.6.19. *If $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a germ of diffeomorphism at 0 having a saddle at 0, then the* ***deMelo-Palis invariant*** *of ϕ is defined as:*

$$P(\phi) = \frac{\log(|\lambda|)}{\log(|\mu|)},$$

where λ, μ are the eigenvalues of $d\phi(0)$ such that $|\lambda| < 1 < |\mu|$.

Remark 3.6.20. *In fact, the deMelo-Palis invariant P is a moduli of stability for ϕ. (See [28, 81].)*

Proposition 3.6.21. *If $Z \in \mathcal{V}$, there exists a homeomorphism $h : \Sigma \to \Sigma$ which is a continuous extension of the homeomorphism $h : \Sigma^s(Z_0) \to \Sigma^s(Z)$ given by Lemma 3.6.18, such that $\phi \circ h = h \circ \phi_0$, i.e. it is a topological equivalence between ϕ and ϕ_0.*

Proof. The proof of this proposition is divided into steps.

Let $h : \Sigma^s(Z_0) \to \Sigma^s(Z)$ be the homeomorphism obtained in Lemma 3.6.18.

Notice that Z has a T-singularity at $q(Z) \approx p$. Since $F^N_{Z_0}$ and F^N_Z are transversal to $S_{Z_0} \setminus \{p\}$ and $S_Z \setminus \{q(Z)\}$, respectively, we can easily continuously extend h on $\overline{\Sigma^s(Z_0)}$ via limit to obtain

$$h : \overline{\Sigma^s(Z_0)} \to \overline{\Sigma^s(Z)}.$$

Step 1: The first task is to define a **fundamental domain** for the first return maps, ϕ and ϕ_0.

We will detail it for ϕ_0. The process to construct the fundamental domain of ϕ is completely analogous.

By the Linearization Theorem (see [57]), we may assume that ϕ_0 is linear. Moreover, we can consider coordinates (x, y) of Σ at p such that:

$$\phi_0(x, y) = (\lambda_0 x, \mu_0 y),$$

where λ_0, μ_0 are the eigenvalues of ϕ_0 such that $|\mu_0| < 1 < |\lambda_0|$.

By the position of S_{X_0}, S_{Y_0} and the invariant manifolds of the saddle, obtained in Proposition 3.6.15, it follows that:

- S_{X_0} is a curve passing through 0, with one branch in the first quadrant and another in the fourth;
- S_{Y_0} is a curve passing through 0, with one branch in the first quadrant and another in the fourth;
- S_{X_0} is tangent to the line $y = k_0 x$;
- S_{Y_0} is tangent to the line $y = K_0 x$;
- $0 < k_0 < K_0$.

We have the situation illustrated in Figure 3.10.

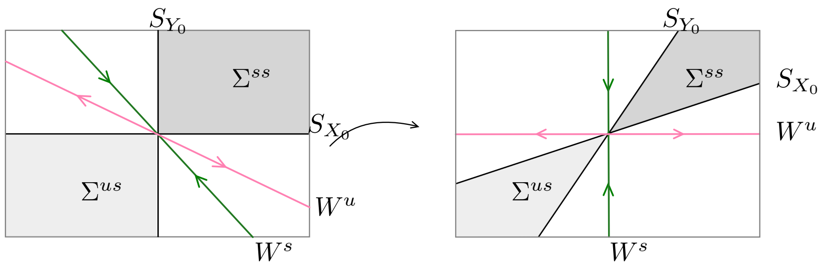


Figure 3.10: Change of coordinates.

Without loss of generality, consider that $S_{X_0} = \{y = k_0 x\}$ and $S_{Y_0} = \{y = K_0 x\}$ and assume that these lines are the fixed points of ϕ_{X_0} and ϕ_{Y_0}, respectively. It will reduce our work, nevertheless it generates no loss of generality, since the same can be done with the original sets.

From the existence of the invariant diabolo in Proposition 3.6.15, it follows that, $\phi_0^{-1}(S_{X_0})$ is a line in the same quadrants containing S_{X_0}, moreover, its inclination is greater than K_0.

Define:

$$\omega_0 = \{(x,y);\ k_0 x \leq y \leq K_0 x\} \text{ and } \widetilde{\omega}_0 = \phi_{Y_0}(\omega_0).$$

Notice that $R_0 = \omega_0 \cup \widetilde{\omega}_0$ is the region delimited by the lines S_{X_0} and $\phi_0^{-1}(S_{X_0})$.

Now it is immediate that $\phi_0^n(S_{X_0}) \to W^u$ when $n \to \infty$ and $\phi_0^n(S_{X_0}) \to W^s$ when $n \to -\infty$. Therefore, the first and the third quadrants are partitioned by $\phi_0^n(R_0)$, $n \in \mathbb{Z}$.

In another words, if $Q = \{(x,y);\ xy > 0\}$, then

$$Q = \bigcup_{n\in\mathbb{Z}} \phi_0^n(R_0).$$

Therefore, we say that R_0 is the fundamental domain of ϕ_0. See Figure 3.11

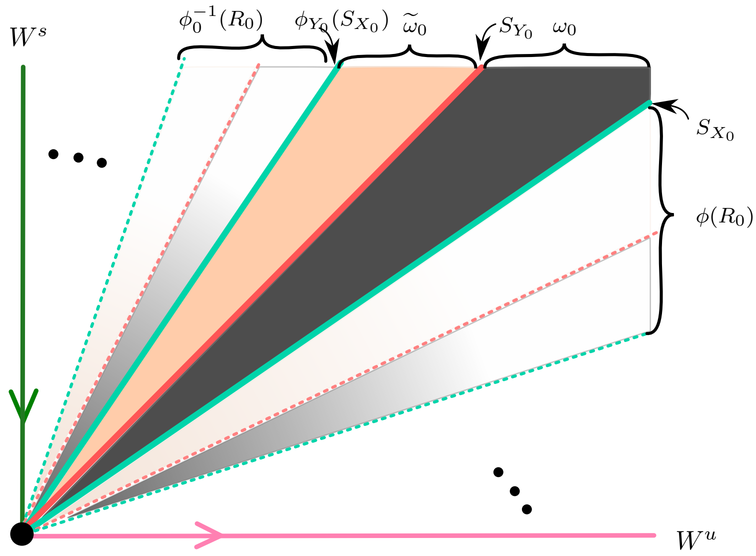


Figure 3.11: Fundamental domain $R_0 = \omega_0 \cup \widetilde{\omega}_0$ in the first quadrant.

Similarly, we can consider coordinates (x,y) of Σ at p such that:

$$\phi(x,y) = (\lambda x, \mu y),$$

where λ, μ are the eigenvalues of ϕ such that $|\mu| < 1 < |\lambda|$. Therefore, there exists $R = \omega \cup \widetilde{\omega}$, where ω is the region delimited by S_X and S_Y and $\widetilde{\omega} = \phi_Y(\omega)$.

Also $Q = \bigcup_{n\in\mathbb{Z}} \phi^n(R)$, and R is the region delimited by S_X and $\phi^{-1}(S_X)$.

In both cases, each orbit of ϕ_0 (and ϕ) passes a unique time in each sector of the partition of Q.

Step 2: Extending the domain of h into $h : Q \to Q$.

Notice that $h : \omega_0 \to \omega$ is already defined (it is the homeomorphism $h : \overline{\Sigma^s(Z_0)} \to \overline{\Sigma^s(Z)}$ in these coordinates).

If $q \in \widetilde{\omega}_0$, then $q = \phi_{Y_0}(\tilde{q})$, for some $\tilde{q} \in \omega_0$, therefore, define:

$$h(q) = \phi_Y(h(\tilde{q})).$$

Clearly, it is a continuous extension of h from ω_0 into R_0. Now, we have defined a homeomorphism $h : R_0 \to R$.

The extension to Q follows in a natural way (since it is defined in a fundamental domain).

In fact, if $q \in Q$, there exists a unique $\tilde{q} \in R_0$ and a unique $n \in \mathbb{Z}$, such that $q = \phi_0^n(\tilde{q})$. Define:

$$h(q) = \phi^n(h(\tilde{q})).$$

Clearly, $h : Q \to Q$ is a homeomorphism satisfying:

$$h(\phi_0(q)) = \phi(h(q)),$$

for each $q \in Q$.

Step 3: Extending h on both W^u and W^s in a continuous fashion.

This is the most delicate part of the proof. Consider an arbitrary continuous extension of h on W^s.

Now, the difficult task is to continuously extend it to W^u, and it will be only possible because

$$P(\phi_0) = -1 = P(\phi),$$

where P is the deMelo-Palis invariant.

Only the extension in the first quadrant will be detailed. The extensions in the other quadrants are similar.

We extend ϕ in the following way.

Fix $w = (d, 0) \in W^u$, then, there exists a sequence $w_i = \phi_0^{N_i}(y_i)$ such that $N_i \to \infty$ when $i \to \infty$ and y_i is a sequence contained in $S_{X_0} \cap \{x, y > 0\}$ such that $y_i \to 0$ when $i \to \infty$, which satisfies:

$$\lim_{i \to \infty} \phi_0^{N_i}(y_i) = w.$$

Notice that, the homeomorphism h is already defined for the sequence w_i. Since we want a continuous extension and an equivalence, we must define:

$$h(w) = \lim_{i \to \infty} h(\phi_0^{N_i}(y_i)) = \lim_{i \to \infty} \phi^{N_i}(h(y_i)).$$

Our work is to prove that the limit above exists. In this case, h will be extended on W^u by doing this process for every $q \in [w, \phi_0(w)]$ and then extend it through the images of this fundamental domain by ϕ_0.

Now, we prove the existence of the limit.

Since $h(S_{X_0}) = S_X$ and $\phi^n(S_X) \to W^u$ as $n \to \infty$, it follows directly that:

$$\lim_{i \to \infty} \pi_2(\phi^{N_i}(h(y_i))) = 0.$$

Therefore, $\pi_2(h(w)) = 0$ and it is well-defined. The problem happens for the first coordinate. Consider:

1. $w = (d, 0)$;
2. $y_i \to 0$, $y_i \in S_{X_0}$, for every i;
3. $N_i \to$ such that $\phi_0^{N_i}(y_i) = w_i \to w$;

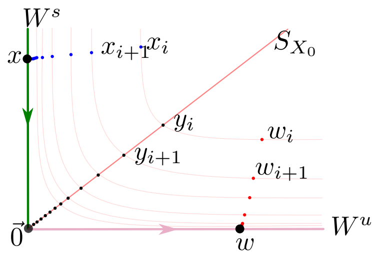


Figure 3.12: Sequences (x_i), (y_i) and (w_i).

4. $t_i \to \infty$, $x_i \to x \in W^s$ such that $y_i = \phi_0^{t_i}(x_i)$.

See Figure 3.12.

Now, denote $\widetilde{y_i} = h(y_i)$, $\widetilde{x_i} = h(x_i)$, $\widetilde{w_i} = \phi^{N_i}(\widetilde{y_i})$, $d_i = \pi_1(w_i)$, $\widetilde{d_i} = \pi_1(\widetilde{w_i})$, $a_i = \pi_2(x_i)$ and $\widetilde{a_i} = \pi_2(h(x_i))$. Hence, we must prove that $\widetilde{d_i}$ converges.

Notice that, since h is continuously extended for W^s and the sequence x_i converges to $x \in W^s$, it follows that $\widetilde{a_i}$ is a convergent sequence. Denote $\widetilde{a} = \lim \widetilde{a_i}$, and notice that

$$\widetilde{d_i} \quad = \quad \pi_1(\phi^{N_i}(h(y_i))) \quad = \quad \lambda^{N_i}\pi_1(\widetilde{y_i}).$$

Now, observe that:

$$\widetilde{y_i} = h(y_i) = h(\phi_0^{t_i}(x_i)) = \phi^{t_i}(\widetilde{x_i}) = (\lambda^{t_i}\pi_1(\widetilde{x_i}), \mu^{t_i}\pi_2(\widetilde{x_i})).$$

Since $\widetilde{y_i} \in S_X = \{y = kx\}$, it follows that:

$$\pi_1(\widetilde{y_i}) = \frac{1}{k}\pi_2(\widetilde{y_i}) = \frac{1}{k}\mu^{t_i}\pi_2(\widetilde{x_i}).$$

Hence:

$$\widetilde{d_i} = \frac{1}{k}\lambda^{N_i}\mu^{t_i}\pi_2(\widetilde{x_i}) = \frac{1}{k}\lambda^{N_i}\mu^{t_i}\widetilde{a_i},$$

and applying the logarithm, we obtain:

$$\log(\widetilde{d_i}k) = N_i\log(\lambda) + t_i\log(\mu) + \log(\widetilde{a_i}).$$

With the same process, we also obtain:

$$\log(d_ik_0) = N_i\log(\lambda_0) + t_i\log(\mu_0) + \log(a_i).$$

Since $\log(d_ik_0)$ and $\log(a_i)$ converge, it follows that $N_i\log(\lambda_0) + t_i\log(\mu_0)$ converges. Now, using that $P(\phi_0) = P(\phi)$, it is immediate that $N_i\log(\lambda) + t_i\log(\mu)$ converges. Since $\widetilde{a_i} \to \widetilde{a}$, it follows that $\widetilde{d_i}$ converges and the proof is complete. □

Remark 3.6.22. *Notice that, both ϕ and ϕ_0 are composition of elements of W^r, therefore a perturbation of the first return map ϕ_0 still is a composition of two involutions. Hence the diffeomorphism ϕ_0 is perturbed only over the codimension one submanifold $P^{-1}(-1)$ of Diff$(\mathbb{R}^2, 0)$ (space of germs of diffeomorphisms at 0.).*

It follows straightforward from the previous results:

Proposition 3.6.23. *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a germ of nonsmooth vector field having a Teixeira singularity at p. Let (α, β, γ) be the normal parameters of Z_0 at p. If $\alpha\beta \geq \gamma$ and $\alpha, \beta < 0$, then Z_0 is locally structurally stable at p.*

Finally, we conclude the proof of Theorem F:

Proof of Theorem F. Notice that Z satisfies condition $\Sigma(E)$ at p if, and only if, the normal parameters (α,β,γ) of Z at p satisfy $\alpha\beta \geq \gamma$ and $\alpha, \beta < 0$.

The result follows directly from Propositions 3.6.8, 3.6.11 and 3.6.23, □

3.7 Proofs of Theorems G, H and Corollary 3.4.2

In this section we intend to discuss the hyperbolic and the parabolic case of the fold-fold singularity in order to complete the characterization of Σ_0.

3.7.1 Hyperbolic Fold-Fold

Let $Z = (X,Y) \in \Omega^r$ be a nonsmooth vector field having a hyperbolic fold-fold point at p such that $S_X \pitchfork S_Y$ at p. Consider the normal coordinates (x,y,z) of Z at p and let (α,β,γ) be the normal parameters of Z at p. In this case we do not have any orbit of X or Y connecting points of Σ, therefore the local structural stability of Z at p depends only on the sliding dynamics which is generically characterized in section 3.5.2.

Proposition 3.7.1. *Let $Z_0 = (X_0,Y_0) \in \Omega^r$ be a nonsmooth vector field having a visible fold-fold point at p such that $S_{X_0} \pitchfork S_{Y_0}$ at p. Let $(\alpha_0,\beta_0,\gamma_0)$ be the normal parameters of Z_0 at p. Then, Z_0 is locally structurally stable at p if and only if $(\alpha_0,\beta_0,\gamma_0) \in R^1_H \cup R^2_H$.*

Outline. The first implication is obvious since F_{Z_0} presents bifurcations in Σ^s. To prove the converse, let $(\alpha_0,\beta_0,\gamma_0)$ be the normal parameters of Z_0 at p. Using Implicit Function Theorem we can find a neighborhood $\mathcal{V}$ of Z_0 in Ω^r such that every $Z \in \mathcal{V}$ has a hyperbolic fold-fold point $q(Z)$ near p and the normal parameters of Z at $q(Z)$ are close to $(\alpha_0,\beta_0,\gamma_0)$.

Now, it is easy to construct a homeomorphism $h : \Sigma \to \Sigma$ carrying sliding orbits of F_{Z_0} onto sliding orbits of F_Z. Extend it to a germ of homeomorphism $h : (M,p) \to (M,q(Z))$ using the flows in the same way of [39] (Lemma 3, page 271). □

3.7.2 Parabolic Fold-Fold

Let $Z = (X,Y) \in \Omega^r$ be a nonsmooth vector field having an invisible-visible fold-fold point at p such that $S_X \pitchfork S_Y$ at p. Consider the normal coordinates (x,y,z) of Z at p, and let (α,β,γ) be the normal parameters of Z at p.

Proceeding as in the elliptic case, Z has an involution ϕ_X associated to the invisible fold of X, and recall that it is given by

$$\phi_X(x,y) = (x - 2\alpha y, -y),$$

in normal coordinates. Now we use it to study the connections between sliding orbits, when they exist.

Lemma 3.7.2. *Let $Z = (X,Y) \in \Omega^r$ be a nonsmooth vector field having an invisible-visible fold-fold point at p such that $S_X \pitchfork S_Y$ at p. Let (α,β,γ) be the normal parameters of Z at p. Then, $\phi_X(S_Y) \pitchfork S_Y$ at p if and only if $\alpha \neq 0$.*

Proof. From Corollary 3.5.4, we have that $S_Y = \{(g(y),y);\ y \in (-\varepsilon,\varepsilon)\}$, for some $\varepsilon > 0$, where g is a smooth function with $g(y) = \mathcal{O}(y^2)$. Therefore $T_0 S_Y = \text{span}\{(0,1)\}$.

On the other hand, $\phi_X(S_Y) = \{(g(y) - 2\alpha y, -y);\ y \in (-\varepsilon,\varepsilon)\}$. Then $T_0\phi_X(S_Y) = \text{span}\{(-2\alpha,-1)\}$. The result follows from these expressions. □

Lemma 3.7.3. *Let $Z = (X, Y) \in \Omega^r$ be a nonsmooth vector field having an invisible-visible fold-fold point at p such that $S_X \pitchfork S_Y$ at p. Let (α, β, γ) be the normal parameters of Z at p. Then, $\phi_X(\Sigma^{us}) \cap \Sigma^{ss} = \emptyset$ if and only if $\alpha > 0$.*

Proof. In fact, in these coordinates, $S_Y = \{(g(y), y);\ y \in (-\varepsilon, \varepsilon)\}$, and $\phi_X(S_Y) = \{(g(y) - 2\alpha y, -y);\ y \in (-\varepsilon, \varepsilon)\}$, for some $\varepsilon > 0$, where g is a smooth function with $g(y) = \mathcal{O}(y^2)$.

Therefore, $T_0\phi_X(S_Y) = \text{span}\{(-2\alpha, -1)\}$. The sliding region Σ^s is the region delimited by S_X and S_Y.

Since $T_0S_Y = \text{span}\{(0, 1)\}$ and $T_0S_X = \text{span}\{(1, 0)\}$, it follows that $\phi_X(S_Y) \subset \Sigma^s$ if and only if $\alpha > 0$.

We conclude the proof by noticing that, if $\phi_X(S_Y) \subset \Sigma^c$, then $\phi_X(\Sigma^{us}) \subset \Sigma^c$. Nevertheless, if $\phi_X(S_Y) \subset \Sigma^s$, then the region delimited by S_Y and $\phi_X(S_Y)$ in Σ^{us} is carried into the region delimited by S_Y and $\phi_X(S_Y)$ in Σ^{ss}. □

Remark 3.7.4. *In another words, there exist orbits of X in M^+ connecting distinct points in the sliding region Σ^s if and only if $\alpha > 0$.*

Definition 3.7.5. *If $\phi : \Sigma \to \Sigma$ is a diffeomorphism and F is a vector field in Σ, then define the **reflected vector field of** F **by** ϕ as $\phi^* F$.*

Remark 3.7.6. *The reflected vector field of F by ϕ can also be referred as **transport of** F **by** ϕ.*

Lemma 3.7.7. *Let $Z = (X, Y) \in \Omega^r$ be a nonsmooth vector field having an invisible-visible fold-fold point at p such that $S_X \pitchfork S_Y$ at p. Let (α, β, γ) be the normal parameters of Z at p.*

Assume that there exist a region $S \subset \Sigma^{us}$ such that $\widetilde{S} = \phi_X(S) \subset \Sigma^{ss}$, and suppose that S is maximal with respect to this property. If $2(\alpha + \beta)(\alpha\beta - \gamma) \neq 0$, then F_Z^N and the transport of F_Z^N by ϕ_X are transversal vector fields defined in $\widetilde{S}$.

Proof. Consider $F_0 = F_Z^N$ and $F_1 = \phi^* F_Z^N$, where ϕ_X is the involution associated to X.

Clearly, F_0 and F_1 are transversal at $q \in \Sigma$ if and only if $F_0(q)$ and $F_1(q)$ are linearly independent vectors.

Considering the normal coordinates (x, y, z) at p. Define

$$D(x, y) = \det \begin{pmatrix} F_0(x, y) \\ F_1(x, y) \end{pmatrix}.$$

Notice that $D(x, y) \neq 0$ if and only if F_0 and F_1 are transversal at (x, y). Now, we use the expressions of the vector field in these coordinates to derive an approximation for the function D.

Since ϕ_X is a linear involution, it follows that $\phi_X^{-1} = \phi_X$ and $d\phi_X = \phi_X$, therefore:

$$\begin{aligned} F_1(x, y) &= d\phi_X(F_Z^N(\phi_X^{-1}(x, y))) \\ &= \phi_X(F_Z^N(\phi_X(x, y))) \end{aligned}$$

In order to compute D, we must analyze the influence of the higher order terms in the computation of F_Z^N. From Proposition 3.5.1, we have that:

$$X(x, y, z) = \begin{pmatrix} \alpha \\ 1 \\ -y \end{pmatrix} \text{ and } Y(x, y, z) = \begin{pmatrix} \gamma + \widetilde{F}(x, y, z) \\ \beta + \widetilde{G}(x, y, z) \\ x + \widetilde{H}(x, y, z) \end{pmatrix},$$

where $\widetilde{F}(x,y,z) = \mathcal{O}(|(x,y,z)|)$, $\widetilde{G}(x,y,z) = \mathcal{O}(|(x,y,z)|)$ and $\widetilde{H}(x,y,z) = \mathcal{O}(|(x,y,z)|^2)$.

Hence, the sliding vector field is given by:

$$F_Z^N(x,y) = \begin{pmatrix} \alpha & \gamma \\ 1 & \beta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha H(x,y) + yF(x,y) \\ H(x,y) + yG(x,y) \end{pmatrix},$$

where $F(x,y) = \widetilde{F}(x,y,0) = \mathcal{O}(|(x,y)|)$, $G(x,y) = \widetilde{G}(x,y,0) = \mathcal{O}(|(x,y)|)$ and $H(x,y) = \widetilde{H}(x,y,0) = \mathcal{O}(|(x,y)|^2)$.

Using the expressions of F_Z^N and $\phi_X(x,y) = (x - 2\alpha y, -y)$, we obtain:

$$D(x,y) = y^2[-2(\alpha+\beta)(\alpha\beta-\gamma) + P_1(x,y)].$$

where $P_1(x,y) = \mathcal{O}(|(x,y)|)$.

Now, if $(\alpha+\beta)(\alpha\beta-\gamma) \neq 0$, then the x-axis is the only solution of $D(x,y) = 0$, near the origin. Therefore the vector fields F_0 and F_1 are transversal in the region $S \cup \widetilde{S}$, since it does not contain points of the x-axis.

□

Remark 3.7.8. *Notice that, in the curves $\alpha+\beta = 0$ and $\alpha\beta = \gamma$, the higher order terms may produce curves in $S \cup \widetilde{S}$ where the vector fields are not transversal, and they can be broken by small perturbations (making $\alpha+\beta \neq 0$ or $\alpha\beta \neq \gamma$). Clearly, this situation implies the instability of the system.*

Lemma 3.7.9. *Let $Z = (X,Y) \in \Omega^r$ be a nonsmooth vector field having an invisible-visible fold-fold point at p such that $S_X \pitchfork S_Y$ at p. Let (α,β,γ) be the parameters given by Proposition 3.5.1 associated to Z at p. If $2\alpha(\alpha+\beta) - \gamma \neq 0$, then F_Z^N is transversal to $\phi_X(S_Y)$ in Σ^s.*

Proof. In the coordinates of Proposition 3.5.1, we have that $S_Y = \{(g(y), y, 0);\ y \in (-\varepsilon, \varepsilon)\}$, for $\varepsilon > 0$ sufficiently small, where g is a $\mathcal{C}^r$ function such that $g(y) = \mathcal{O}(y^2)$.

Therefore $\phi_X(S_Y) = \{(g(y) - 2\alpha y, -y);\ y \in (-\varepsilon, \varepsilon)\}$. Since $\phi_X(S_Y)$ is tangent to the curve $\gamma(y) = (-2\alpha y, -y)$ at the origin, it is sufficient to prove that F_Z^N is transversal to γ.

Clearly, F_Z^N is transversal to γ at $\gamma(y)$ if and only if

$$T(y) = F_Z^N(\gamma(y)) \cdot (\gamma'(y))^{\perp} \neq 0. \tag{3.7.1}$$

Now, we use the expression of F_Z^N in these coordinates to obtain an approximation of T. In fact,

$$F_Z^N(\gamma(y)) = F_Z^N(-2\alpha y, -y) = (-2\alpha^2 y - \gamma y, -2\alpha y - \beta y) + \mathcal{O}(y^2)$$

and

$$(\gamma'(y))^{\perp} = (-2\alpha, -1)^{\perp} = (1, -2\alpha).$$

Substituting these expressions in 3.7.1, we obtain:

$$T(y) = [2\alpha(\alpha+\beta) - \gamma]y + \mathcal{O}(y^2)$$

Therefore, if the condition $2\alpha(\alpha+\beta) - \gamma \neq 0$ is assumed and $y \neq 0$ then F_Z^N is transversal to $\phi_X(S_Y)$. Since Σ^s does not contain points where $y \neq 0$ (because they belong to S_X), the result follows. □

Remark 3.7.10. *In the curve $2\alpha(\alpha+\beta)-\gamma=0$, the higher order terms can be used to produce a curve such that F_Z^N is tangent to $\varphi_X(S_Y)$ in every point. Such structurally unstable phenomena is avoided.*

Proposition 3.7.11. *Let $Z_0=(X_0,Y_0)\in\Omega^r$ be a nonsmooth vector field having an invisible-visible fold-fold point at p such that $S_{X_0}\pitchfork S_{Y_0}$ at p. Let $(\alpha_0,\beta_0,\gamma_0)$ be the normal parameters of Z_0 at p. Then, Z_0 is locally structurally stable at p if and only if the following statements hold*

1. $(\alpha_0,\beta_0,\gamma_0)\in\cup_{i=1}^4 R_P^i$;
2. $\alpha_0\neq 0$;
3. $2\alpha_0(\alpha_0+\beta_0)-\gamma_0\neq 0$;
4. $\alpha_0+\beta_0\neq 0$, *if* $\alpha_0>0$.

Moreover, there exist only eleven topologically distinct classes of local structural stable systems at invisible-visible fold-fold points.

Outline. Proceeding as is the proof of Theorem 3.7.1. Consider the neighborhood $\mathcal{V}$ of Z_0 such that the correspondent parameters (α,β,γ) of any $Z\in\mathcal{V}$ are in the same region of $(\alpha_0,\beta_0,\gamma_0)$.

Let $Z=(X,Y)\in\mathcal{V}$. If there is no orbits of X connecting points of Σ^{ss} and Σ^{us}, then the proof can be done in the following way. We omit some details in this case, since it is very similar to the visible case.

- Construct $h:\Sigma^s(Z_0)\to\Sigma^s(Z)$ carrying orbits of F_0 onto orbits of F_Z. In addition extend it to $S_{X_0}\cup S_{Y_0}$ via limit. Hence $h(S_{X_0})=S_X$ and $h(S_{Y_0})=S_Y$;
- For each $p\in\Sigma\setminus S_{X_0}$, there exists $t_0(p)\neq 0$ such that $\varphi_{X_0}(t_0(p),p)\in\Sigma$. Similarly, there exists an analogous time $t(p)\neq 0$ for the vector field X;
- If $p\in\Sigma^s$, then $h(p)$ is already defined. Assume that $p\in\Sigma^c$. If $\varphi_{X_0}(t_0(p),p)\in\Sigma^s$, then define:
$$h(p)=\varphi_X(-t(\varphi_{X_0}(t_0(p),p)),h(\varphi_{X_0}(t_0(p),p))).$$
- Using Tietze Extension Theorem, we can extend h over Σ^c;
- Now, using the same idea of the third item, we can extend it to the whole Σ;
- Extend it to M^+ using the flow of X_0, X and $h:\Sigma\to\Sigma$;
- Following the same idea of the hyperbolic case, extend it to M^-;
- Hence we construct a germ of homeomorphism $h:M\to M$ at p, with $h(p)=q(Z)$, which is an equivalence between Z_0 and Z. Then Z_0 is locally structurally stable at p.

Suppose that there exists a connection between Σ^{ss} and Σ^{us} for Z_0 and Z. Denote by S_0 and S, the regions of Σ^s presenting connections.

From the previous Lemmas of this subsection, it is possible to say that F_0 and $\phi_{X_0}^*F_0$ are transversal in each point of S_0, and the same holds for F_Z and $\phi_X^*F_Z$ in S.

Therefore, the orbits of F_0 and $\phi^*_{X_0}F_0$ define a coordinate system in S_0, such as the orbits of F_Z and $\phi^*_X F_Z$ in S.

Hence, let h be a function carrying S_{Y_0} onto S_Y, and $h(0) = 0$. Now we can use these coordinate systems to construct $h : S_0 \to S$ satisfying

$$h \circ \phi_{X_0} = \phi_X \circ h.$$

By the transversality of F_0 to $\phi_{X_0}(S_{Y_0})$ (resp. F_Z to $\phi_X(S_Y)$), it is possible to extend h on $\Sigma^s(Z_0)$ using the sliding orbits. Then we have a homeomorphism $h : \Sigma^s(Z_0) \to \Sigma^s(Z)$ carrying sliding orbits onto sliding orbits.

By construction, if $x \in S$, then $\phi_X(h(x)) = h(\phi_{X_0}(x))$. With this, we can use the same idea from the previous case without connections to extend such map to a germ of homeomorphism $h : M \to M$ at p, with $h(p) = q(Z)$, which is a topological equivalence between Z_0 and Z at p. □

3.7.3 Proof of Theorem G

Notice that Z satisfies condition $\Sigma(H)$ at p if, and only if, the normal parameters (α, β, γ) of Z at p satisfy the hypotheses of Proposition 3.7.1.

Moreover, Z satisfies condition $\Sigma(P)$ at p if, and only if, the normal parameters (α, β, γ) of Z at p satisfy the hypotheses of Proposition 3.7.11.

The result follows directly from Propositions 3.7.1, 3.7.11.

3.7.4 Proof of Theorem H

From Proposition 3.3.5 it follows that $\Sigma_0 \subset \Sigma(G)$.

The result follows from Theorem 3.3.12, F and G.

3.7.5 Proof of Corollary 3.4.2

From the characterization of Σ_0, we can see that $\Sigma(G)$, $\Sigma(R)$, $\Sigma(H)$, $\Sigma(P)$ are open dense sets in Ω^r.

Nevertheless, we also prove that $\Sigma(E)$ is not residual in Ω^r. Therefore, it follows that $\Sigma_0 \cap \Sigma(E)$ is open dense in $\Sigma(E)$ and $\Sigma(E)$ is the biggest set with this property.

3.8 Conclusion and Further Directions

In this chapter we have obtained a complete characterization of the locally structurally stable $3D$ Filippov systems. As a consequence, we have proved that it is not a generic property in Ω^r. Also, it is worthwhile to mention that the geometrical comprehension of the problem was imperative to the characterization of the local structural stability in dimension 3.

In light of this, we believe that the characterization of local structural stability in higher dimensions is a challenging problem which deserves attention. In addition, the characterization of Σ-singularities generic in k-parameter families, $k \geq 1$ for n-dimensional Filippov systems $n \geq 3$ is an arduous task which might reveal interesting behavior.

Chapter 4

Semi-Local Structural Stability

In this chapter, our main purpose is to provide a non-local approach to study aspects of structural stability of $3D$ Filippov systems. We introduce a notion of semi-local structural stability which detects when a piecewise smooth vector field is robust around the entire switching manifold, as well as, provides a complete characterization of such systems. In particular, we present some methods in the qualitative theory of piecewise smooth vector fields, which make use of geometrical analysis of the foliations generated by their orbits. Such approach displays surprisingly rich dynamical behavior which is studied in detail in this work.

It is worth mentioning that this subject has not been treated in dimensions higher than two from a non-local point of view, and we hope that the approach adopted herein contributes to the understanding of structural stability for piecewise smooth vector fields in its most global sense.

4.1 Introduction

In the classical theory of smooth vector fields, the structural stability concept determines the robustness of a model with respect to the initial conditions and parameters as well as its efficiency. From our point of view, it is of the most importance to establish this concept to PSVF in a systematic way.

In attempt to reach this goal, many papers have emerged with the purpose of the characterization of the structural stability for PSVF. In dimension 2, the concept of local structural stability was extensively studied in [55, 63, 65]. In [19], Broucke et al. have studied the problem in dimension 2 from a global point of view. In dimension 3, the local approach has been completely characterized from papers [24, 25, 44, 95, 101]. In higher dimension, some models were treated in [26], but it remains poorly understood, even locally.

To the best of the authors' knowledge, in dimension 3, non-local aspects of structural stability of PSVF have not yet been studied, maybe due to its high complexity. In light of this, we introduce in this work a concept of semi-local structural stability, in order to understand what happens around the whole switching manifold (not only point-wise) of a robust PSVF. We attempt to provide all results in the most rigorous way by considering the problem from a geometric-topological point of view.

We consider piecewise smooth vector fields Z defined in $\mathbb{R}^3$ having a compact switching manifold Σ, and we denote this set by Ω^r. Roughly speaking, $Z_0 \in \Omega^r$ is semi-local structurally stable at Σ if all systems $Z \in \Omega^r$ sufficiently near Z_0 present the same behavior as Z_0 in a neighborhood $V \subset \mathbb{R}^3$ of Σ. In this work, we completely characterize all the semi-local structurally stable systems at Σ, and conclude that it is not a generic property in Ω^r. Also, a version of Peixoto's Theorem for sliding vector fields is obtained.

It is worth mentioning that the characterization of structural stability of $3D$ PSVF from a global point of view is one of the most complex and intriguing topic in the theory of PSVF. The semi-local approach studied herein, allows us to find constraints in the characterizing problem, and it is our hope that it can be used to study global connections between points of Σ (see [66], for example) and serves as a guideline to solve more general problems.

This chapter is structured as follows. An overview of basic concepts and $3D$ generic tangential singularities is given in Section 4.2. The topological orbital equivalences used throughout this work are described in Section 4.3. Section 4.4 presents a formal language to deal with this problem. In Section 4.5 the main results are presented. Sections 4.6, 4.7 and 4.8 are devoted to proving the main results. In Section 4.9, we discuss future directions this work can take.

4.2 Preliminaries

In what follows we present an overall description of some useful basic concepts and results.

Throughout this chapter, let $M = \mathbb{R}^3$ and let $f : M \to \mathbb{R}$ be a smooth function having 0 as a regular value. Suppose that $\Sigma = f^{-1}(0)$ is an embedded codimension 1 submanifold of M. Assume that Σ is compact, connected and simply connected (i.e. Σ is homeomorphic to $\mathbb{S}^2$). We consider germs of piecewise smooth vector fields at Σ.

Remark 4.2.1. *As in Chapter 3, Ω^r stands for the set of tridimensional piecewise smooth vector fields at Σ.*

Also, in this chapter, Σ can be denoted by $\Sigma(Z)$, in order to distinguish the regions of Σ corresponding to Z, when necessary.

As we are interested in studying structural stability in Ω^r it is imperative to take into account all the leaves of the foliation in M generated by the orbits of $Z = (X, Y)$ (orbits of X, Y and F_Z). For more details see Section 4.8.1.

In this chapter, we consider all the notations, definitions and results introduced in Section 3.3. We also consider the following concept.

Definition 4.2.2. *Let $Z_0 \in \Omega^r$, we say that $\Gamma_0 \subset \Sigma^s$ is a **Σ-separatrix of a fold-fold point** p_0 of Z_0, if it satisfies one of the following conditions:*

1. *p_0 is a singularity of saddle type of $F^N_{Z_0}$ and Γ_0 is a saddle separatrix of $F^N_{Z_0}$ at p_0;*
2. *p_0 is a singularity of nodal type of $F^N_{Z_0}$ and Γ_0 is a strong manifold of $F^N_{Z_0}$ at p_0.*

If Γ_0 is both a Σ-separatrix of two distinct fold-fold points, we say that Γ_0 is a connection of Σ-separatrices of fold-fold points.

4.3 Topological Equivalences in Ω^r

We are concerned with the persistence of the foliation of the state space generated by a vector field $Z = (X, Y)$ in Ω^r. In light of this, we consider orbital equivalences throughout this work.

4.3.1 Sliding Topological Equivalence

Firstly, we consider a topological equivalence to relate piecewise smooth vector fields having similar behavior in the sliding region.

Definition 4.3.1. *Let $Z_0, Z \in \Omega^r$ be two germs of piecewise smooth vector fields at Σ. We say that Z_0 is* ***sliding equivalent*** *to Z if there exists a homeomorphism $h : \Sigma \to \Sigma$, which carries S_{Z_0} onto S_Z preserving the topological type of the singularity and sliding orbits of Z_0 onto sliding orbits of Z.*

The concept of sliding structural stability is defined in a natural way. We stress that such kind of stability only concerns with the sliding features (contained in Σ) of $Z_0 \in \Omega^r$.

The set of all sliding structurally stable piecewise smooth vector fields is denoted by Ω^r_{SLR}.

4.3.2 Semi-Local Topological Equivalence

In the literature, the local topological equivalence is commonly used to relate piecewise smooth vector fields presenting similar behavior around a point. In this work, we shall consider an extension of this type of equivalence with the purpose of understanding the behavior of a piecewise smooth vector field around a compact set.

Definition 4.3.2. *Let $N \neq \emptyset$ be a compact subset of Σ and let $Z_0, Z \in \Omega^r$. We say that Z_0 is* ***semi-locally equivalent*** *to Z at N if there exist a neighborhood U of N in M and a Σ-invariant homeomorphism $h : U \to U$ which carries orbits of Z_0 onto orbits of Z.*

The concept of semi-local structural stability at a compact subset N of Σ is defined in a natural way.

We remark that the local term is frequently used with respect to phenomena occurring around a point, and for this reason we use the semi-local term to refer to a phenomenon occurring in a neighborhood (in M) of a compact set.

In particular, if N is a point of Σ, say it p, then Definition 4.3.2 turns out to be the classical local topological equivalence at a point $p \in \Sigma$, which is extensively studied in [19, 24, 25, 26, 44, 101, 104]. It follows from [39, 55] that each $Z_0 \in \Omega^r$ is locally structurally stable at regular-regular, fold-regular and cusp-regular points.

Notice that, if $N = \Sigma$, then the semi-local equivalence is quite different from the sliding equivalence. Indeed, the sliding equivalence is concerned only with the elements lying in Σ (dimension 2), whereas the semi-local equivalence at Σ regards all orbits lying in an open set of M (dimension 3) containing Σ.

4.3.3 A Review on the Fold-Fold Singularity

In Chapter 3, one can find a complete intrinsic characterization of piecewise smooth vector fields which are locally structurally stable at fold-fold points. For the sake of clarity, we outline the results of this chapter which will be used throughout this work.

The local structural stability of $Z_0 = (X_0, Y_0) \in \Omega^r$ at a fold-fold singularity p_0 is strongly related with the existence of local connections between sliding regions (Σ^{ss} and Σ^{us}) through orbits of X_0 and Y_0.

Indeed, if p_0 is a hyperbolic fold-fold singularity, then there is no local connections between Σ^{ss} and Σ^{us}. In this case, the local structural stability of Z_0 at p_0 depends only on the local sliding dynamics at p_0.

If p_0 is a parabolic fold-fold point, say it is of invisible-visible type, then the flow of X_0 induces an involution $\phi_{X_0} : (\Sigma, p_0) \to (\Sigma, p_0)$. A deep analysis of the dynamics of such mapping allows us to conclude the existence of local connections between the regions Σ^{us} and Σ^{ss} via the orbits of X_0. In the presence of some transversality hypotheses on these connections, it can be proven that Z_0 is locally structurally stable at p_0. The visible-invisible case is analogous.

Finally, if p_0 is an elliptic fold-fold singularity, then we have a first return map defined around p_0, and it was that Z_0 is locally structurally stable at p_0 if, and only if, there is no local connection between the regions Σ^{us} and Σ^{ss} through orbits of X_0 and Y_0. In this case, Z_0 presents piecewise smooth invariant cones (with vertices at p_0) isolating such regions.

All formal conditions to characterize the local structural stability at fold-fold singularities are stated in Section 4.5 of this article.

4.4 Σ-Blocks' Mechanism

The main purpose of this work is to classify all $Z \in \Omega^r$ which are semi-locally structurally stable at Σ. Now, we introduce a formal language to deal with this problem. We highlight that it is useful to prove the results obtained in the present paper for piecewise smooth vector fields having a non-simply connected switching manifold (e.g. 2-dim torus). Also, the present mechanism can be easily adapted to attack the problem in higher dimension.

The following definition is motivated by the isolating blocks theory considered in [27].

Definition 4.4.1. *A subset $U \neq \emptyset$ of Σ is said to be a Σ-**block** of $Z \in \Omega^r$ if U is a compact connected set such that:*

1. *$\mu(S_Z) = 0$, where μ is the volume measure on Σ (with respect to the euclidean metric defined on Σ);*
2. *int(U) is a 2-dimensional manifold;*
3. *int(U) is Z-invariant;*
4. *int(U) is maximal, i.e., every neighborhood of int(U) in Σ is not Z-invariant.*

*In addition, if $U = \Sigma$, then U is said to be a **trivial** Σ-**block** of Z.*

Remark 4.4.2. *Notice that, if the condition* 1 *in Definition 4.4.1 is dropped, then we may face degenerate situations. As an example, we point out the system $X(x, y, z) = (-y, x, 0)$, $Y(x, y, z) = (x, y, z)$ and $f(x, y, x) = x^2 + y^2 + z^2 - 1$. In this case, $S_Z = \Sigma$, and X induces a dynamics on Σ which is Z-invariant. It is easy to see that it is a structurally unstable situation (consider the perturbation $X_\varepsilon(x, y, z) = (-y, x, \varepsilon z)$). Also, condition* 1 *of Definition 4.4.1 is satisfied for every $Z \in \Xi_0$.*

Notice that, a Σ-block of $Z \in \Xi_0$ is a connected component of $\overline{\Sigma^s(Z)}$. Also, if Z has a trivial Σ-block, then $S_Z = \emptyset$. In this case, either $\Sigma = \Sigma^{ss}$ or $\Sigma = \Sigma^{us}$.

Proposition 4.4.3. *If $Z_0 = (X_0, Y_0) \in \Xi_0$ has no Σ-blocks, then $S_{Z_0} = \emptyset$, $\Sigma = \Sigma^c$ and Z_0 is semi-locally structurally stable at Σ.*

Proof. In fact, if $S_{Z_0} \neq \emptyset$, then from Theorem 3.3.5, it follows that Σ^s has non-empty interior in Σ, which means that Z_0 would have at least one Σ-block. It follows that $S_{Z_0} = \emptyset$ and consequently $\Sigma = \Sigma^c$.

From continuity of the maps $F, G : \chi^r \times \Sigma \to \mathbb{R}$, given by $F(X,p) = Xf(p)$ and $G(Y,p) = Yf(p)$, and compactness of Σ, there exist neighborhoods $\mathcal{U}$ of X_0 and $\mathcal{V}$ of Y_0, such that $Xh(p)Yh(p) > 0$, for each $X \in \mathcal{U}$, $Y \in \mathcal{V}$ and $p \in \Sigma$.

Therefore, $\Sigma^c(Z) = \Sigma$, and thus Z_0 and Z are semi-locally equivalents at Σ, for each $Z = (X,Y) \in \mathcal{U} \times \mathcal{V}$. □

Definition 4.4.4. *A vector field $Z_0 \in \Omega^r$ is said to be **Σ-block structurally stable** if either Z_0 has no Σ-blocks or Z_0 is semi-locally structurally stable at each Σ-block of Z_0. Denote the set of all $Z \in \Omega^r$ which are Σ-block structurally stable by Ω^r_Σ.*

Proposition 4.4.5. *Let $Z_0 \in \Xi_0$. Then, Z_0 is Σ-block structurally stable if and only if Z_0 is semi-locally structurally stable at Σ.*

Proof. To prove the non-trivial implication, assume that Z_0 is Σ-block structurally stable. If Z_0 has no Σ-blocks then, from Proposition 4.4.3, Z_0 is semi-locally structurally stable at Σ.

Let $U_1, \cdots, U_k$ be all the Σ-blocks of Z_0. From hypothesis, for each $i = 1, \cdots, k$, there exists a neighborhood $\mathcal{U}_i$ of Z_0 in Ω^r such that Z_0 and Z are semi-locally equivalent at U_i, for each $Z \in \mathcal{U}_i$.

Take $\mathcal{U} = \mathcal{U}_1 \cap \cdots \mathcal{U}_k$, and let $Z \in \mathcal{U}$. Hence, there exist disjoint compact neighborhoods V_i of U_i in M, and homeomorphisms $h_i : V_i \to V_i$ which carry orbits of Z_0 onto orbits of Z, for each $i = 1, \cdots, k$. Notice that Z_0 and Z are transverse to $\partial V_i \cap \Sigma$ and we can construct h_i such that $h_i|_{\partial V_i} = id$.

Now, let $V = V_1 \cup \cdots \cup V_k$. Since $Z_0 \in \Xi_0$, $\Sigma \setminus V \subset \Sigma^c(Z_0)$ and $\Sigma \setminus V \subset \Sigma^c(Z)$. Setting $h|_{V_i} = h_i$ and $h|_{\Sigma \setminus V} = id$, we can use the flows of Z_0 and Z to construct a homeomorphism $h : U \to U$ (see [55]), carrying orbits of Z_0 onto orbits of Z, where U is a neighborhood of Σ. Hence Z_0 is semi-locally structurally stable at Σ. □

From Proposition 4.4.5, to classify the vector fields in Ω^r which are robust around Σ, it is enough to understand the Σ-block structurally stable systems. Notice that all results in this section remain valid if we drop the simply connectedness condition on the switching manifold.

4.5 Main Goal and Statement of the Main Results

Our strategy is to use informations of $Z \in \Omega^r$ around points of Σ to understand its behavior around the switching manifold. In order to do this, we use the concepts of sliding and Σ-block structural stability introduced in Section 4.3 to formalize the problem and we give a complete characterization of the sets Ω^r_{SLR} and Ω^r_Σ.

Let $\Sigma_0(SLR)$ be the set of $Z = (X,Y) \in \Omega^r$ such that:

$G)$ $Z \in \Xi_0$;

$F_1)$ If $p \in \Sigma$ is either a hyperbolic or an elliptic fold-fold singularity of Z then F_Z^N has no center manifold in $V_p \cap \overline{\Sigma^s}$, where V_p is a neighborhood of p in Σ;

$F_2)$ If $p \in \Sigma$ is a parabolic fold-fold singularity of Z then F_Z^N is transient in $V_p \cap \overline{\Sigma^s}$ or it has a hyperbolic singularity at p, where V_p is a neighborhood of p in Σ;

$F_3)$ There is no connection between Σ-separatrices of fold-fold points of Z in Σ^s;

$F_4)$ There is no connection between a Σ-separatrix of fold-fold of Z and a saddle separatrix of F_Z^N contained in Σ^s;

$I_1)$ $F_Z^N|_{\overline{\Sigma^s}}$ has a finite number of pseudo-equilibria. All of them are hyperbolic and contained in $\text{int}(\Sigma^s)$;

$I_2)$ $F_Z^N|_{\overline{\Sigma^s}}$ has a finite number of periodic orbits. All of them are hyperbolic and contained in $\text{int}(\Sigma^s)$;

$I_3)$ F_Z^N does not present any saddle connection in $\overline{\Sigma^s}$;

$B_1)$ There is no orbit of F_Z^N contained in Σ^s connecting two tangency points of F_Z^N with $\partial\Sigma^s$;

$B_2)$ Each saddle separatrix of F_Z^N is transversal to $\partial\Sigma^s$ (except at fold-fold points).

$R)$ F_Z^N has no recurrent orbit.

Theorem J (Peixoto's Theorem - Sliding Version)**.** *The set Ω^r_{SLR} is residual in Ω^r and it coincides with $\Sigma_0(SLR)$.*

Now, consider the following properties (see Chapter 3 for more details):

$\Xi(P)$: If $p \in \Sigma$ is an invisible-visible fold-fold point of $Z \in \Omega^r$, then the germ of the involution ϕ_X at p associated to Z satisfies:

1. $\phi_X(S_Y) \pitchfork S_Y$ at p;
2. F_Z^N and $\phi_X^* F_Z^N$ are transversal at each point of $\Sigma^{ss} \cap \phi_X(\Sigma^{us})$;
3. $\phi_X(S_Y) \pitchfork F_Z^N$ at p.

$\Xi(E)$: If $p \in \Sigma$ is an elliptic fold-fold point of $Z \in \Omega^r$, then the germ of the first return map ϕ_Z at p associated to Z has a fixed point at p of saddle type with both local invariant manifolds $W^{u,s}_{loc}$ contained in Σ^c.

Remark 4.5.1. *If Z has a visible-invisible fold-fold point at p, then the roles of X and Y are interchanged in the property $\Xi(P)$.*

Let Σ_0 be the set of $Z \in \Omega^r$ satisfying the following properties:

$S)$ $Z \in \Omega^r_{SLR}$;

F) If $p \in \Sigma$ is a fold-fold point of Z then $\Xi(P)$ and $\Xi(E)$ are satisfied at p.

Theorem K (Classification of Ω^r_Σ)**.** *The following statements hold:*

(i) $\Omega^r_\Sigma = \Sigma_0$*;*

(ii) Ω^r_Σ *is not residual in* Ω^r*;*

(iii) Ω^r_Σ *is residual in* $\Sigma(E)$*, where* $\Sigma(E)$ *is the set of* $Z \in \Omega^r$ *satisfying* $\Xi(E)$*. Moreover,* $\Sigma(E)$ *is maximal with respect to this property.*

4.6 Robustness of Tangency Sets

In this section we discuss about the structure of the tangency set of $Z \in \Xi_0$. Firstly, we analyze the local behavior of an elementary tangential singularity and secondly, some global features of the tangency set of Z are also discussed.

4.6.1 Local Analysis

Let $X \in \chi^r$ be a $\mathcal{C}^r$ vector field defined around Σ (which is the common boundary of M^+ and M^-). The local behavior of X at a point $p \in \Sigma$ is a very matured topic and the results of this section can be found in [104, 95, 97] from a different point of view.

The following propositions provide a geometric interpretation of fold and cusp points in Σ.

Proposition 4.6.1. *Let* $X_0 \in \chi^r$ *having a fold point at* $p_0 \in \Sigma$*, then there exist neighborhoods* $\mathcal{V}$ *of* X_0 *in* χ^r *and* V *of* p_0 *in* Σ *such that*

(a) for each $X \in \mathcal{V}$*, there exists a unique* $\mathcal{C}^r$ *curve of fold points* $\gamma_X \subset V$ *of* X *in* Σ *which intersects* ∂V *transversally at only two points;*

(b) $p_0 \in \gamma_{X_0}$ *and* $sgn(X^2f(p)) = sgn(X_0^2f(p_0))$*, for each* $p \in \gamma_X$ *and* $X \in \mathcal{V}$*.*

Proof. Consider the map $F : \chi^r \times \Sigma \to \mathbb{R}$ given by $F(X,p) = X^2f(p)$. It satisfies $F(X_0,p_0) \neq 0$. From continuity, there exist neighborhoods $\mathcal{V}_1$ of X_0 in χ^r and V_1 of p_0 in Σ such that $X^2f(p) \neq 0$ for each $X \in \mathcal{V}_1$ and $p \in V_1$, and the sign of $X_0^2f(p_0)$ is preserved.

Let $\phi : (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \to V_2$ be a local chart of Σ around p_0 such that $V_2 \subset V_1$, and notice that ϕ is a $\mathcal{C}^r$ diffeomorphism. Consider $G : \chi^r \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \to \mathbb{R}$ given by $G(X, x_1, x_2) = Xf(\phi(x_1, x_2))$, and notice that G is a $\mathcal{C}^r$ function such that $G(X_0, 0, 0) = 0$.

Since $X_0^2f(p) \neq 0$, we have that $dX_0f(p_0)$ is a nonzero linear transformation, and since ϕ is a diffeomorphism, it follows that $\dfrac{\partial G}{\partial(x_1,x_2)}(X_0,0,0) = dX_0f(p_0) \circ d\phi(0,0)$ is nonzero. We conclude that $\dfrac{\partial G}{\partial x_1}(X_0,0,0) \neq 0$ or $\dfrac{\partial G}{\partial x_2}(X_0,0,0) \neq 0$.

Without loss of generality, assume that $\dfrac{\partial G}{\partial x_2}(X_0,0,0) \neq 0$, now we use the Implicit Function Theorem (for Banach Spaces) to find a neighborhood $\mathcal{V}$ of X_0 in χ^r contained in $\mathcal{V}_1$, real numbers a, b such that $-\varepsilon_1 < a < 0 < b < \varepsilon_1$, and a $\mathcal{C}^r$ function $\alpha : \mathcal{V} \times (a,b) \to (-\varepsilon_2, \varepsilon_2)$ such that $G(X, x_1, x_2) = 0$ with $X \in \mathcal{V}$ and $x_1 \in (a,b)$ if and only if $x_2 = \alpha(X, x_1)$.

Notice that, for each $X \in \mathcal{V}$, the curve $c_X : (a,b) \to (a,b) \times (-\varepsilon_2, \varepsilon_2)$ given by $c_X(t) = (t, \alpha(X,t))$, is transverse to each horizontal line $x_1 = x_0$, with $x_0 \in (a,b)$.

Setting $V = \phi([a_0,b_0] \times (-\varepsilon_2,\varepsilon_2))$, for some $a < a_0 < 0 < b_0 < b$, it follows that, for each $X \in \mathcal{V}$, the curve $\gamma_X = \phi \circ c_X$ intersects ∂V (transversally) only at the points $\gamma_X(a_0)$ and $\gamma_X(b_0)$, therefore it satisfies part (a) of the statement. In addition, $\gamma_{X_0}(0) = \phi(0,0) = p_0$, which proves part (b).

□

Proposition 4.6.2. *Let $X_0 \in \chi^r$ having a cusp point at $p_0 \in \Sigma$. Then, there exist a neighborhood $\mathcal{V}$ of X_0 in χ^r, real numbers $a_0 < 0 < b_0$, and a neighborhood V of p_0 in Σ such that, for each $X \in \mathcal{V}$:*

(a) there exists a unique $\mathcal{C}^r$ curve $\gamma_X : [a_0,b_0] \to V$ of tangential singularities of X in V which intersects ∂V transversally at only two points;

(b) there exists $a_0 < t(X) < b_0$ such that $p(X) = \gamma_X(t(X))$ is a cusp point of X. In addition, $sgn(X^3 f(p(X))) = sgn(X_0^3 f(p_0))$.

(c) $\gamma_X(t)$ is a fold point of X for every $t \in [a_0,b_0]$ such that $t \neq t(X)$. In addition, $X^2 f(\gamma_X(t)) X^2 f(\gamma_X(s)) < 0$, for each $a_0 \leq t < t(X)$ and $t(X) < s \leq b_0$.

Proof. From the linearly independence of $\{df(p_0), dX_0 f(p_0), dX_0^2 f(p_0)\}$, it follows that $dX_0 f(p_0) \neq 0$. Using the same notation and ideas as in the proof of Proposition 4.6.1, we can find neighborhoods $\mathcal{V}$ of X_0 in χ^r, V of p_0 in Σ, real number $a_0 < 0 < b_0$ and curves $\gamma_X : [a_0,b_0] \to V$, for each $X \in \mathcal{V}$, such that

$$Xf(p) = 0, \text{ with } X \in \mathcal{V} \text{ and } p \in V \Leftrightarrow p = \gamma_X(t), \text{ for some } t \in [a_0,b_0],$$

and $\mathrm{sgn}(X^3 f(p)) = \mathrm{sgn}(X_0^3 f(p_0))$, for each $X \in \mathcal{V}$ and $p \in V$.

In addition, γ_X intersects ∂V transversally at $\gamma_X(a_0)$ and $\gamma_X(b_0)$, and $\gamma_X(t) \in \mathrm{int}(V)$ for each $t \in (a_0,b_0)$. Therefore, item (a) is proved.

To prove (b), consider the $\mathcal{C}^r$ function $H : \chi^r \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $H(X,p) = (f(p), Xf(p), X^2 f(p))$. Since p_0 is a cusp point of X_0, it follows that $H(X_0,p_0) = (0,0,0)$ and $\dfrac{\partial H}{\partial p}(X_0,p_0)$ is invertible. Now, we use the Implicit Function Theorem for Banach spaces to obtain a $\mathcal{C}^r$ function $\beta : \mathcal{V} \to V$ (reduce the initial neighborhoods $\mathcal{V}$ and V, if necessary) such that $H(X,p) = (0,0,0)$, with $X \in \mathcal{V}$ and $p \in V$, if and only if $p = \beta(X)$.

Reducing $\mathcal{V}$ to $\mathcal{V} \cap \beta^{-1}(\mathrm{int}(V))$, we conclude that each $X \in \mathcal{V}$ has a unique cusp point $p(X) = \beta(X)$ in V which is contained in $\mathrm{int}(V)$. Since $Xf(p(X)) = 0$, it follows that, there exists $t(X) \in (a_0,b_0)$ such that $\gamma_X(t(X)) = p(X)$.

To prove (c), notice that, for $t \neq t(X)$, $H(X,\gamma_X(t)) \neq (0,0,0)$, and $f(\gamma_X(t)) = Xf(\gamma_X(t)) = 0$. Thus, $X^2 f(\gamma_X(t)) \neq 0$ and $\gamma_X(t)$ is a fold point.

Let $X \in \mathcal{V}$, then $h(t) = X^2 f(\gamma_X(t))$ is a real smooth function such that $h(t(X)) = 0$ and $h(t) \neq 0$ otherwise. Notice that $h'(t) = dX^2 f(\gamma_X(t)) \cdot \gamma_X'(t)$.

If $h'(t(X)) = 0$, then $dX^2 f(p(X))$ is orthogonal to $\gamma_X'(t(X))$, and since $Xf(\gamma_X(t)) = f(\gamma_X(t)) = 0$ for every $t \in [a_0,b_0]$, it follows that $dXf(p(X))$ and $df(p(X))$ are orthogonal to $\gamma_X'(t(X))$. Since $\mathrm{span}\{\gamma_X'(t(X))\}^{\perp}$ has dimension 2, we have that $\{df(p(X)),\ dXf(p(X)),\ dX^2 f(p(X))\}$ is linearly dependent, which is a contradiction because $p(X)$ is a cusp point of X.

Therefore, $h'(t(X)) \neq 0$ and $h(t(X)) = 0$. If follows that $h(t)h(s) < 0$ for each $-\varepsilon < t < t(X) < s < \varepsilon$, for some $\varepsilon > 0$ sufficiently small. □

4.6.2 Global Analysis

Now, the tangency set S_X of $X \in \chi^r$ is analyzed. We shall prove that a Σ-block of a piecewise smooth vector field $Z \in \Xi_0$ is robust under small perturbations in Ω^r.

Proposition 4.6.3. *If $X \in \chi^r$ is simple and $S_X \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that $S_X = \sqcup_{i=1}^n S_X^i$, where each S_X^i is diffeomorphic to the unit circle $\mathbb{S}^1$. Moreover, S_X has at most a finite number of cusp points.*

Proof. From continuity of Xf in Σ, it follows that $S_X = Xf^{-1}(0)$ is a compact subset of Σ. In addition, from Propositions 4.6.1 and 4.6.2, it follows that S_X is locally connected. Therefore, the connected components of S_X are open in the induced topology of S_X and by compactness, we can conclude that S_X has only a finite number of connected components.

Let $\phi : \mathbb{S}^2 \to \Sigma$ be a diffeomorphism and consider the $\mathcal{C}^r$ function $F : \mathbb{S}^2 \to \mathbb{R}$ given by $F(p) = Xf(\phi(p))$.

Notice that, $F^{-1}(0) = \phi^{-1}(S_X)$ and $dF(x) = dXf(\phi(x)) \circ d\phi(x)$, for every $x \in \mathbb{S}^2$. Since S_X is composed by fold and cusp points, and $\phi(p) \in S_X$, for each $p \in F^{-1}(0)$, it follows that $dXf(\phi(p)) \neq 0$. As $d\phi(p)$ is an isomorphism, we conclude that $dF(p) \neq 0$, and thus 0 is a regular value of F.

So, $F^{-1}(0)$ is a 1-dimensional embedded submanifold of $\mathbb{S}^2$. Also, S_X has a finite number of connected components, and thus $F^{-1}(0)$ has a finite number of connected components.

Since every connected component is a closed set in Σ, it follows that each connected component of $F^{-1}(0)$ is a compact connected 1-dimensional embedded submanifold of $\mathbb{S}^2$, and thus, diffeomorphic to $\mathbb{S}^1$.

Finally, we use Propositions 4.6.1 and 4.6.2 to construct an open cover of S_X such that each element of this cover has only fold points and at most one cusp point. By compactness, we conclude that S_X has just a finite number of cusp points. □

See Figure 4.1.

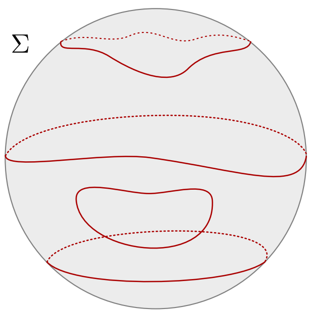


Figure 4.1: Structure of the tangency set S_X of a simple vector field $X \in \chi^r$.

Remark 4.6.4. *Since Σ is a compact manifold, the tangency set S_X of $X \in \chi_S^r$ with Σ is diffeomorphic to a union of circles. From Proposition 4.6.2, the cusp points occur as isolated points in a circle of fold points. In addition, if p is a cusp of X, then, there exists a smooth curve of fold points of X in Σ passing through p, which has their visibility changed at p. Therefore, the number k of cusp points in a fold circle is always even and it has $k/2$ arcs of visible fold points and $k/2$ arcs of invisible fold points.*

Now, we prove the persistence of the connected components of the tangency set S_X of $X \in \chi^r_S$.

Proposition 4.6.5. *Let $X_0 \in \chi^r$ be a simple vector field such that $S_{X_0} \neq \emptyset$, and let C_0 be a connected component of S_{X_0} containing k_0 cusp points. Then there exist neighborhoods $\mathcal{V}$ of X_0 and V of C_0 in Σ such that, for each $X \in \mathcal{V}$:*

(a) S_X has a unique connected component in V containing exactly k_0 cusp points.

(b) The number of connected components of S_X and S_{X_0} coincide for any $X \in \mathcal{V}$.

Proof. Given $p \in C_0$, from Propositions 4.6.1 and 4.6.2, there exist neighborhoods $V_p \subset \Sigma$ of p and $\mathcal{V}_p \subset \chi^r$ of X_0 such that, for each $X \in \mathcal{V}_p$, there exists a smooth curve $\gamma^p_X : [a_p, b_p] \to V_p$ satisfying the following properties:

(i) $\gamma^p_X(t(X))$ is a point of same nature of p, for some $a_p < t(X) < b_p$;

(ii) $\gamma^p_X(t)$ is a fold point of X for each $t \neq t(X)$.

In addition, this curve contains all tangency points of X inside V_p, and intersects V_p transversally at $\gamma^p_X(a_p)$ and $\gamma^p_X(b_p)$. Notice that $\gamma^p_{X_0}$ is a local parametrization of C_0 at p.

Clearly, $\mathcal{U} = \{V_p; p \in C_0\}$ covers C_0, and from compactness, we extract a finite subcover of $\mathcal{U}$. Then, $C_0 \subset V_1 \cup \cdots \cup V_k$, with $V_i \in \mathcal{U}$, $i = 1, \cdots, k$.

From connectedness of C_0, and the fact that C_0 is a circle, we can order $V_1, \cdots, V_k$ such that, for each $1 \leq i \leq k-1$, there exists $p_i \in C_0$ such that $p_i \in \text{int}(V_i \cap V_{i+1})$ and $p_k \in C \cap V_k \cap V_1$.

From the construction of the neighborhoods, p_i is contained in both curves $\gamma^i_{X_0}$ and $\gamma^{i+1}_{X_0}$, and from continuity of γ on X, γ^i_X and γ^{i+1}_X have at least a point (for each curve) in $V_i \cap V_{i+1}$, for each $X \in \mathcal{V}_i \cap \mathcal{V}_{i+1}$ (reduce $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$ if necessary). From uniqueness, γ^i_X and γ^{i+1}_X must coincide in $V_i \cap V_{i+1}$.

Let $V = V_1 \cup \cdots \cup V_k$ and $\mathcal{V} = \mathcal{V}_1 \cap \cdots \cap \mathcal{V}_k$. Therefore, for each $X \in \mathcal{V}$, we construct a $\mathcal{C}^r$ curve $\gamma_X : [a_1, b_n] \to V$ which is injective in (a_1, b_n) such that $\gamma_X(a_1) = \gamma_X(b_n)$ and $\gamma_X([a_1, b_n]) \cap V_i = \text{Im}(\gamma^i_X)$, $i = 1, \cdots, k$.

It follows that, for each $X \in \mathcal{V}$, $C = \text{Im}(\gamma_X)$ is a connected component of S_X in V and $p \in V$ is a tangency point of X if and only if $p \in C$. The part (a) of the statement is proved.

To prove (b), let $C^0_1, \cdots, C^0_k$ be all connected components of S_{X_0}. From (a), there exist disjoint open neighborhoods W_i of C^0_i and $\mathcal{W}_i$ of X_0 such that, for every $X \in \mathcal{W}_i$, S_X has a unique connected component contained in W_i.

Define $W = W_1 \cup \cdots \cup W_k$ and $\mathcal{W} = \mathcal{W}_1 \cap \cdots \cap \mathcal{W}_k$, and notice that $X_0 f(p) \neq 0$ for each $p \in \Sigma \setminus W$. From continuity, for each $p \in \Sigma \setminus W$, there exist neighborhoods V_p of p in Σ and $\mathcal{V}_p$ of X_0 in χ^r, such that $Xf(q) \neq 0$, for each $X \in \mathcal{V}_p$ and $q \in V_p$.

From compactness of $\Sigma \setminus W$, we find a neighborhood $\mathcal{V} \subset \mathcal{W}$ of X_0 such that $Xf(p) \neq 0$, for each $X \in \mathcal{V}$ and $p \in \Sigma \setminus W$. Therefore, $S_X \subset W$, for each $X \in \mathcal{V}$, and we are done. □

Notice that Proposition 4.6.5 concerns with smooth vector fields defined in manifolds with boundary. Now, we extend this analysis to elementary piecewise smooth vector fields.

Let $Z \in \Xi_0$. If C is a connected component of S_Z composed only by fold-regular and cusp-regular points, then the normalized sliding vector field F^N_Z of Z is transversal to C, except at cusp points. Indeed, at each cusp-regular point p, F_Z^N has a quadratic contact with C.

At each quadratic contact of F_Z^N with C, say it p, the orientation of the orbits of F_Z^N reaching C is changed in a neighborhood V of p. More specifically, F_Z^N reaches $C \cap V$ in either negative or positive time, depending on the side of $C \setminus \{p\}$. Also, if we compute F_Z^N along C, it gives a complete turn between two cusp-regular points.

Hence, the classical index $I(F_Z^N, S_X)$ of F_Z^N along C provides the number of complete turns that it gives along C, and from Remark 4.6.4, we conclude that it coincides with half of the number of cusp-regular points of C.

Based on the discussion above, we have the following result.

Proposition 4.6.6. *Let $Z = (X, Y) \in \Xi_0$. On each connected component C of S_Z, composed by fold-regular and cusp-regular points, the number of cusp-regular points of Z in C is given by*

$$N_{cusp} = 2I(F_Z^N, C),$$

where I is the index of F_Z^N along C.

Recalling that the fold-fold points of $Z \in \Xi_0$ are isolated in Σ, we obtain the next result directly from Proposition 4.6.3 and compactness of Σ.

Proposition 4.6.7. *Let $Z = (X, Y) \in \Xi_0$ such that $S_Z \neq \emptyset$. Then:*

(a) there exists $n \in \mathbb{N}$ such that $S_Z = \cup_{i=1}^n S_Z^i$, where each S_Z^i is diffeomorphic to the unit circle $\mathbb{S}^1$ and is contained in either S_X or S_Y. In addition, S_Z has at most a finite number of cusp-regular and fold-fold points.

(b) $p \in S_Z$ is a fold-fold point of Z if and only if p is contained in the intersection of two circles S_Z^i and S_Z^j, for some $1 \leq i, j \leq n$. In this case, $S_Z^i \subset S_X$ and $S_Z^j \subset S_Y$.

See Figure 4.2.

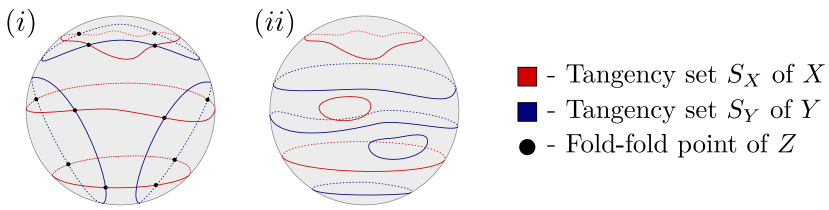


Figure 4.2: Structure of the tangency set S_Z of an elementary piecewise smooth vector field $Z = (X, Y)$ with (i) and without (ii) fold-fold points.

Combining Proposition 4.6.5 with the transversality of S_X and S_Y at fold-fold points of $Z = (X, Y)$, we obtain the following result.

Proposition 4.6.8. *Let $Z_0 = (X_0, Y_0) \in \Xi_0$ be an elementary piecewise smooth vector field such that $S_{Z_0} \neq \emptyset$, and let C_0 be a connected component of S_{Z_0} containing n_0 cusp-regular points and m_0 fold-fold points, then there exist neighborhoods $\mathcal{V}$ of Z_0 and V of C_0 in Σ such that:*

(a) for each $Z = (X, Y) \in \mathcal{V}$, S_Z has a unique connected component C in V containing exactly n_0 cusp-regular points and m_0 fold-fold points.

(b) *For each circle S_0 of S_{X_0} (resp. S_{Y_0}) contained in C_0, there exists a neighborhood $V_0 \subset V$ of S_0 such that each $Z = (X, Y) \in \mathcal{V}$ has a unique circle S of S_X (resp. S_Y) contained in V_0, with the same number of cusp-regular and fold-fold points of S_0.*

(c) *If two circles S_0^1 and S_0^2 of S_{X_0} contained in C_0 intersect themselves at fold-fold points $p_1, p_2, \cdots, p_{2k}$, then, for each $Z \in \mathcal{V}$, the correspondent circles of item (b) intersect themselves at fold-fold points $p_1(Z), \cdots, p_{2k}(Z)$. In addition, $p_i(Z)$ is sufficiently close to p_i and they have the same visibility.*

(d) *for each $Z = (X, Y) \in \mathcal{V}$, S_Z has the same number of connected components of S_{Z_0}, for each $Z \in V$.*

We see that, for a small neighborhood $\mathcal{V}$ of $Z_0 \in \Xi_0$, each $Z \in \mathcal{V}$ has a tangency set S_Z with exactly the same characteristics of S_{Z_0}, i.e., each circle of S_Z is near to a circle of S_{Z_0} and they present the same configuration of intersections. It allows us to conclude that, if Z_0 has a Σ-block U_0, then there exists a neighborhood V_0 of U_0 in Σ, such that each $Z \in \mathcal{V}$ has a unique Σ-block U contained in V_0 and it has the same structure as U_0.

We complete the characterization of the tangency sets by exhibiting another property concerning the number of fold-fold points of $Z \in \Xi_0$.

Proposition 4.6.9. *If $Z \in \Xi_0$, then the number of fold-fold points of Z is even.*

Proof. In fact, if p is a fold-fold points of Z, then p is contained in the transversal intersection of two circles of S_Z, say it C_1 and C_2.

From Jordan Curve Theorem, C_1 divides $\Sigma \setminus C_1$ into two connected components, and since C_2 is a closed curve, it follows that γ must intersect C_1 again in another point different from p, say it q. Since $Z \in \Xi_0$, q is a fold-fold point.

To complete the proof, it is enough to notice that if $\tilde{p}$ is another fold-fold point different from p and q, then by the same argument as above we can find another fold-fold point $\tilde{q}$ different from the others, and the result follows by induction and the fact that Z has a finite number of fold-fold points (due to compactness of Σ). $\square$

Remark 4.6.10. *Due to the similarity on the behavior between a fold-fold point and the vertex defined in [83], it will be referred as a **vertex** of S_Z. In addition, if S_Z has no vertices, then each Σ-block of $Z \in \Xi_0$ has a smooth boundary.*

Finally, we notice that, if $Z_0 \in \Xi_0$, the transversality of S_{X_0} and S_{Y_0} at vertices can not be dropped. Indeed, if S_{X_0} is tangent to S_{Y_0} at p then $T_p S_{X_0} = T_p S_{Y_0}$ and the intersection between S_X and S_Y can be easily broken by translations. In this case, the structure of the tangency set is quite different for small perturbations of Z_0.

4.7 Sliding Structural Stability

In this section we discuss the concept of sliding structural stability and we prove Theorem J.

Let $Z_0 \in \Xi_0$ having a Σ-block U_0. As we can see in [44, 55], the first element to construct a semi-local equivalence at U_0 between Z_0 and some $Z \in \Omega^r$ is the existence of a sliding equivalence between Z_0 and Z. Based on this, we propose a sliding version of Peixoto's Theorem for typical piecewise smooth vector fields. It is worth mentioning that in [83], the authors have mentioned that the case where the boundary is piecewise smooth can be considered with the addition of some conditions on the boundary.

The most relevant difference between the classical case and the one to be considered in Theorem J is the existence of persistent singularities of PSVF at vertices in the boundary (fold-fold points of Z_0). Notice that it is a typical object of the nonsmooth universe. Actually, the boundary changes with the sliding vector field in such a way that the singularity remains in the vertex.

Recall that F_{Z_0} is defined on $\text{int}(U_0)$, but it is not defined on ∂U_0. Since U_0 is connected, Lemma 3.3.11 allows us to extend F_{Z_0} to ∂U_0 through the normalized sliding vector field $F^N_{Z_0}$, and then the stability of F_{Z_0} on U_0 is determined by the stability of $F^N_{Z_0}$ on $\overline{U_0}$.

Notice that a vertex is a singularity of $F^N_{Z_0}$, but it is not a critical point of F_{Z_0} (the vector field is not even defined on these points). Thus, if a trajectory of F_{Z_0} reaches a vertex, it does in a finite time, differently from a trajectory of $F^N_{Z_0}$.

4.7.1 Proof of Theorem J

Let $Z_0 = (X_0, Y_0) \in \Sigma_0(SLR)$. If $\Sigma = \Sigma^s(Z_0)$, then F_{Z_0} is defined in the entire Σ and for a small neighborhood $\mathcal{V}$ of Z_0 in Ω^r, the sliding vector field F_Z of $Z \in \mathcal{V}$ is also defined in $\Sigma = \Sigma^s(Z)$. Therefore, the result follows from Lemma 3.3.11 and the classical version of Peixoto's Theorem.

Assume for instance that $\Sigma^s(Z_0) \neq \Sigma$. Since Z_0 has a typical tangency set, it follows that $\overline{\Sigma^s(Z_0)}$ is a compact set which is properly contained in $\Sigma \setminus \{p_0\}$, for some $p_0 \in \Sigma$. Thus, we perform a change of coordinates in $\Sigma \setminus \{p_0\}$ (stereographic projection) which brings $\overline{\Sigma^s(Z_0)}$ into a compact subset of $\mathbb{R}^2$. Denote this identification by $\overline{\Sigma^s(Z_0)} \simeq_\rho M_0$. See Figure 4.3

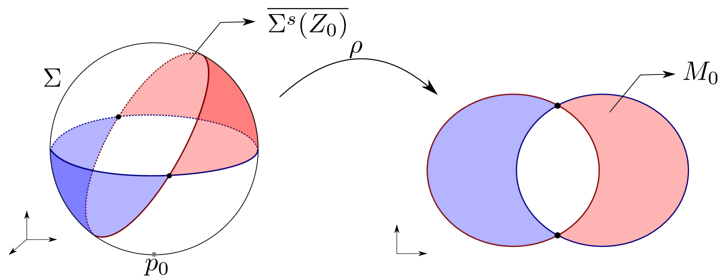


Figure 4.3: Interpretation of the identification $\overline{\Sigma^s(Z_0)} \simeq_\rho M_0$.

Since $Z_0 \in \Xi_0$, the boundary ∂M_0 is composed by fold-regular points, a finite (even) number n_c of cusp-regular points, $p_i(Z_0)$, $i = 1, \cdots, n_c$, and a finite (even) number n_f of fold-fold points, $q_j(Z_0)$, $j = 1, \cdots, n_f$.

Notice that each Σ-block $U_0 \subset M_0$ of Z_0 is a path-connected region, and the boundary ∂U_0 of U_0 is composed by a union of circles which are pairwise transversal. Also, the circles of ∂U_0 intersect themselves only at a finite number of points. In addition, each circle is composed by fold-regular, cusp-regular and fold-fold points of Z_0, and two circles intersect themselves at p if, and only if, p is a fold-fold point of Z_0. See Figure 4.4.

Now each Σ-block U_0 of Z_0 is path-connected, but $\widetilde{U_0} = U_0 \setminus \{q_1(Z_0), \ \cdots, \ q_{n_f}(Z_0)\}$ is not connected. Call the closure of the connected components of $\widetilde{U_0}$ of all the Σ-blocks U_0 of Z_0 by $R_i(Z_0)$, $i = 1, \cdots, l$, where l is the number of connected components of $\widetilde{U_0}$.

From the characterization of U_0, it follows that, each $R_i(Z_0)$ is a simple polygon with $a_i \leq n_f$ vertices and a finite number of holes in its interior. Notice that, if $a_i = 0$, then

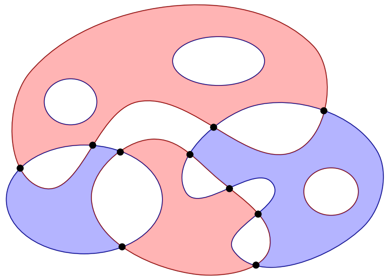


Figure 4.4: Example of a Σ-block M_0 of $Z_0 = (X_0, Y_0)$. The stable and unstable sliding regions are represented by the colors blue and red, respectively.

$\partial R_i(Z_0)$ is smooth. See Figure 4.5.

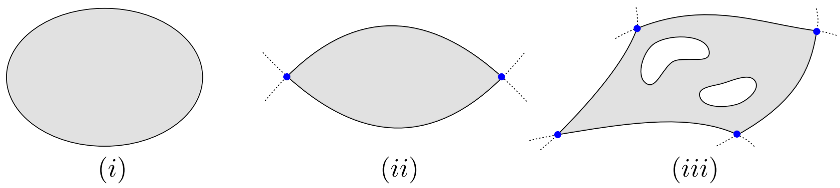


Figure 4.5: Example of regions $R_i(Z_0)$: (i) $a_i = 0$, (ii) $a_i = 2$ and (iii) $a_i = 4$.

From Proposition 4.6.8, it follows that, for a small neighborhood $\mathcal{V}$ of Z_0 in Ω^r, the region $M_Z = \overline{\Sigma^s(Z)} \subset \mathbb{R}^2$ (use the same previous change of coordinates) has exactly the same configuration as M_0, for each $Z \in \mathcal{V}$. Indeed, $M_Z = R_1(Z) \cup \cdots \cup R_l(Z)$, where $R_i(Z)$ is a region with the same characteristics as $R_i(Z_0)$, i.e., $R_i(Z)$ is a simple polygon with a_i vertices and the same number of holes as $R_i(Z_0)$ in its interior.

Also, for each fold-fold (vertex) $q(Z_0)$ of $R_i(Z_0)$, there exists a unique fold-fold of $q(Z)$ of $R_i(Z)$ with the same visibility of $q(Z_0)$, which is sufficiently close to $q(Z_0)$.

Lema 4.7.1. There exists a homeomorphism $h_Z^i : R_i(Z_0) \to R_i(Z)$ which preserves the type of the singularity of the boundary and carries sliding orbits of Z_0 onto sliding orbits of Z, for $i = 1, \cdots, l$. In addition, $h_Z^i(q_j(Z_0)) = q_j(Z)$ for each $q_j(Z_0) \in R_i(Z_0)$, $j = 1, \cdots, n_f$, $i = 1, \cdots, l$.

Proof. Let R be one of the regions $R_i(Z_0)$ and let $Z \in \mathcal{V}$.

If R has no vertices, then ∂R is smooth and so there exists a diffeomorphism $\Psi : R \to \widetilde{R}$, where $\widetilde{R} = R_i(Z)$. Thus, we construct a homeomorphism $\widetilde{h} : R \to R$ between Z_0 and $\Psi_* Z$, via the classical Peixoto's Theorem, and $h = \Psi \circ \widetilde{h}$ satisfies the properties of the Lemma.

Now, for simplicity, assume that R is a simple polygon with only two vertices q_1 and q_2, which has no holes. We stress that there is no loss of generality in this assumption, since the following construction can be easily extended to any configuration of R.

Therefore, $\widetilde{R}$ has two vertices $\widetilde{q_1}$ and $\widetilde{q_2}$ which are sufficiently near to q_1 and q_2, respectively. See Figure 4.6.

Notice that $\partial R = A_1 \cup A_2$, where A_i, $i = 1, 2$, is an open arc with extrema q_1 and q_2 just composed by fold-regular and cusp-regular points of Z_0. Recall that, from Lemma 3.3.11, $F_{Z_0}^N$ is transverse to A_i at fold-regular points and it has a quadratic contact with A_i

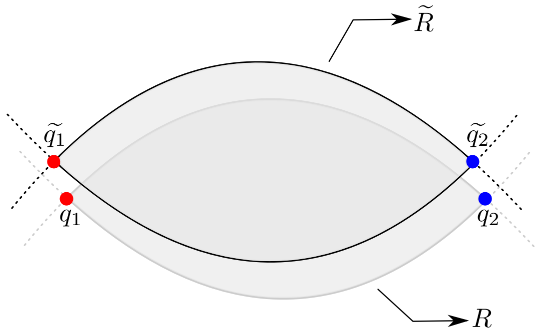


Figure 4.6: Persistence of the region R.

at cusp-regular points, $i = 1, 2$. Hence, $F^N_{Z_0}$ satisfies all hypotheses of Peixoto's Theorem, with exception of the points q_1 and q_2.

Now, we must clarify what happens around points q_1 and q_2, i.e., fold-fold points of Z_0. Let q be either q_1 or q_2. From the description of the sliding dynamics around q given in Section 3.5.2, we obtain that under hypotheses F_1 and F_2, there exists a neighborhood V of q in $\mathbb{R}^2$, such that A_i is transversal to ∂V and $F^N_{Z_0}$ satisfies either one of the following:

I $-$ (i) $F^N_{Z_0}$ is transversal to $\partial V \cap R$, (ii) there exists a unique orbit Γ_0 of $F^N_{Z_0}$ departing from q and reaching $\partial V \cap R$, and (iii) each orbit passing through another point of $V \cap R$ departs from an arc A_i, $i = 1, 2$, and reaches $\partial V \cap R$ at finite time.

II $-$ (i) $F^N_{Z_0}$ is transversal to $\partial V \cap R$, (ii) each orbit passing through a point of $V \cap R$ departs from $(\partial V \cap R) \cup (A_1 \cap V) \cup (A_2 \cap V)$ and reaches q at infinite positive time.

III $-$ (i) $F^N_{Z_0}$ is transversal to $\partial V \cap R$, with exception of a point $x_0 \in \partial V \cap \text{int}(R)$, where $F^N_{Z_0}$ has a quadratic contact with $\partial V \cap R$. (ii) Each orbit of $V \cap R$ either departs from A_1 and reaches A_2 or it departs from ∂V and reaches ∂V.

IV $-$ (i) $F^N_{Z_0}$ is transversal to $\partial V \cap R$, (ii) there exists a Σ-separatrix (nodal type) Γ_0 of q which reaches $\partial V \cap R$ at x_0, and (ii) the orbit passing through another point of $V \cap R \setminus \Gamma_0$ either departs from A_{i_1} and reaches ∂V or it departs from q and reaches $A_{i_2} \cap \partial V$, $\{i_1, i_2\} = \{1, 2\}$.

V $-$ (i) $F^N_{Z_0}$ is transversal to $\partial V \cap R$, with exception of a point $x_0 \in \partial V \cap \text{int}(R)$, where $F^N_{Z_0}$ has a quadratic contact with $\partial V \cap R$. (ii) There exist two separatrices (type saddle) of q which reaches ∂V at x_1 and x_2, respectively, and x_0 is between x_1 and x_2. (iii) Each orbit through a point of $V \cap R$ either departs from ∂V and reaches A_{i_1} or it departs from ∂V and reaches ∂V or it departs from A_{i_2} and reaches ∂V, $\{i_1, i_2\} = \{1, 2\}$.

See Figure 4.7. We remark that all situations $I - V$ can happen also for negative time.

Let $\widetilde{q}$ be the point $\widetilde{q}_i$ which is near to q. Since Z is sufficiently near to Z_0, it follows that $\widetilde{q} \in V$ and F^N_Z also satisfies the same property of $F^N_{Z_0}$ $(I - V)$ in V. Hence it is straightforward to construct a homeomorphism $h_q : \overline{V} \cap R \to \overline{V} \cap \widetilde{R}$, such that $h_q(q) = \widetilde{q}$, which carries sliding orbits of Z_0 onto sliding orbits of Z, see [24, 44, 97, 102].

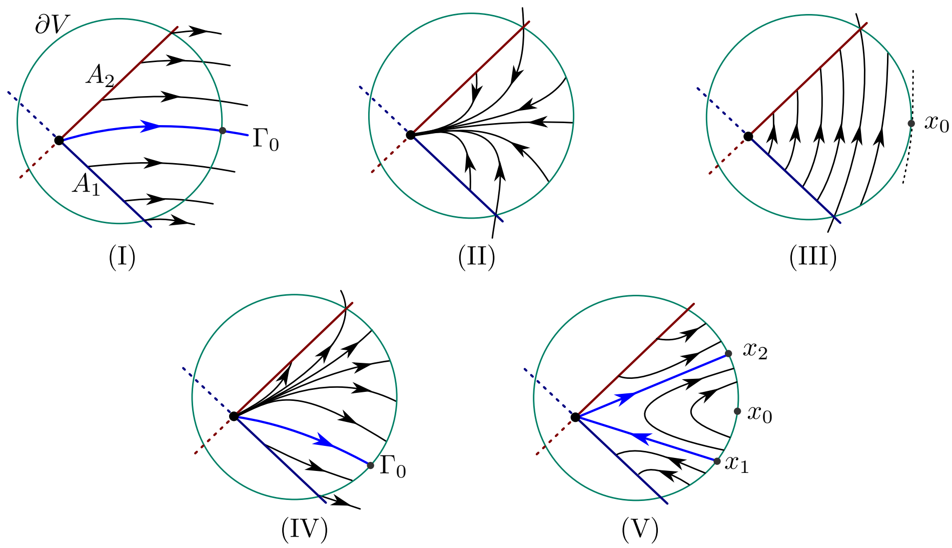


Figure 4.7: Description of the sliding dynamics near a fold-fold point: I-V.

Notice that $v \in \partial V \cap A_i$ is a vertex point in the sense of [83] (it is not a fold-fold point of Z_0), and it satisfies that $F^N_{Z_0}$ is transversal to both ∂V and A_i at v. Therefore $F^N_{Z_0}$ satisfies condition **B5** of [83].

In addition, we reduce the neighborhoods V of each fold-fold point of R in order that there is no trajectory of $F^N_{Z_0}$ connecting two vertices. Hence it also satisfies condition **B6** of [83].

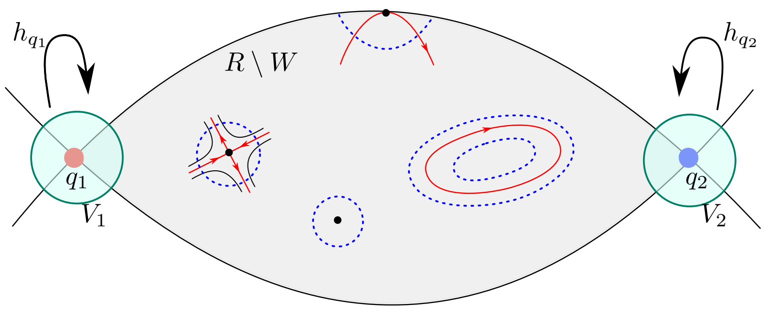


Figure 4.8: Construction of the homeomorphism h^i_Z using distinguished neighborhoods of hyperbolic equilibrium points, hyperbolic periodic orbits, and quadratic tangency points of $F^N_{Z_0}$ in $R \setminus W$ obtained via Peixoto's Theorem.

After doing this process for the points q_1 and q_2, we obtain two neighborhoods V_1 and V_2 as above, let $W = V_1 \cup V_2$. Since Z_0 satisfies the hypotheses from Peixoto's Theorem inside $R \setminus W$, we use the same methods as used in [83] to extend the homeomorphisms h_{q_1} and h_{q_2} into a homeomorphism $h_Z : R \to \widetilde{R}$ satisfying the properties of the claim. See Figure 4.8. $\square$

Thus, we use all the homeomorphisms h^i_Z to construct a homeomorphism $h^s_Z : M_0 \to M_Z$. Now, returning to the initial coordinates of $\Sigma - \{p_0\}$, we obtain a homeomorphism $\widetilde{h^s_Z} : \overline{\Sigma^s(Z_0)} \to \overline{\Sigma^s(Z)}$. Hence, considering any extension $h_Z : \Sigma \to \Sigma$ of $\widetilde{h^s_Z}$, we conclude that $Z_0 \in \Omega^r_{SLR}$.

Therefore, $\Sigma_0(SLR) \subset \Omega^r_{SLR}$. Now, if Z_0 does not satisfy some condition of $\Sigma_0(SLR)$, then we conclude that $Z_0 \notin \Omega^r_{SLR}$ (see [44, 82, 83]). It follows that $\Omega^r_{SLR} = \Sigma_0(SLR)$.

Since $\Sigma_0(SLR)$ is a residual set in Ω^r, the proof is complete.

Remark 4.7.1. *From the proof of Theorem J, it follows that the construction of the equivalence $h : \Sigma \to \Sigma$ around fold-fold points can be made in several ways. In particular, any local equivalence h_q at a fold-fold q can extend itself into a sliding equivalence $h : \Sigma \to \Sigma$ of Definition 4.3.1.*

4.8 Proof of Theorem K

Let $Z_0 \in \Sigma_0$, we shall prove that Z_0 is Σ-block structurally stable. If Z_0 has no Σ-blocks, then $Z_0 \in \Omega^r_\Sigma$ (see Proposition 4.4.3). Let U_0 be a Σ-block of Z_0.

If $U_0 = \Sigma$, then condition **S** of Σ_0 allows us to find a neighborhood $\mathcal{V}$ of Z_0 such that, for each $Z \in \mathcal{V}$, there exists a homeomorphism $h_s : \Sigma \to \Sigma$ carrying sliding orbits of Z_0 onto sliding orbits of Z preserving the tangential singularities. We follow the same idea as the proof of Proposition 4.4.3 to construct a neighborhood U of Σ and a homeomorphism $h : U \to U$ satisfying Definition 4.3.2 ($N = \Sigma$), such that $h|_\Sigma = h_s$, and conclude that $Z_0 \in \Omega^r_\Sigma$.

Finally, assume that $U_0 \neq \Sigma$. It implies that $S_{Z_0} \neq \emptyset$. Hence, ∂U_0 is a reunion of circles of S_{X_0} and S_{Y_0} intersecting themselves transversally at fold-fold points of Z_0 (Proposition 4.6.7).

From Proposition 4.6.8, there exist neighborhoods $\mathcal{V}_0$ of Z_0 in Ω^r and V_0 (compact) of U_0 in Σ, such that $\partial V_0 \subset \Sigma^c(Z_0)$, and each $Z \in \mathcal{V}_0$ has a unique Σ-block U in V_0 with the same characteristics of U_0, i.e., $Z \in \mathcal{V}$ satisfies the following properties:

(i) U_0 and U have the same number of cusp-regular and fold-fold points of the same type;

(ii) There exists $\varepsilon_0 > 0$, such that, if $p \in \partial U$ is either a cusp-regular or a fold-fold point, then there exists a unique point $p_0 \in \partial U_0$ of the same type of p such that $|p - p_0| < \varepsilon_0$;

(iii) If p_0^1 and p_0^2 are points of ∂U_0 connected by a curve Γ_0 of either visible or invisible fold-regular points contained in ∂U_0, then there exist points p^1 and p^2 of ∂U of the same type of p_0^1 and p_0^2, respectively, and a unique curve $\Gamma \subset \partial U$ of fold-regular points of the same type of Γ_0, such that $d(\Gamma, \Gamma_0) < \varepsilon_0$ (d denotes the Hausdorff distance).

Notice that, it implies that both U_0 and U have the same circles configuration, for each $Z \in \mathcal{V}_0$. Given $Z \in \mathcal{V}$, we shall construct a semi-local equivalence between Z_0 and Z at U_0.

4.8.1 Local Description of the Invariant Manifolds of Elementary Tangential Singularities

Firstly, we use Vishik's Normal Form Theorem 3.3.5 to distinguish the local invariant manifolds of elementary tangential singularities.

Fold-Regular

Let p_0 be a fold-regular point of Z_0 in ∂U_0, and without loss of generality, assume that $p_0 \in S_{X_0}$.

From Theorem 3.3.5, there exists a diffeomorphism $\Psi : V_{p_0} \to R_{p_0}$ (denote the coordinate functions of Ψ by (x_Ψ, y_Ψ, z_Ψ)) such that $\Psi(p_0) = \vec{0}$, V_{p_0} is a compact neighborhood of p_0 in M, $R_{p_0} = [-l(p_0), l(p_0)] \times [-H(p_0), H(p_0)]^2 \subset \mathbb{R}^3$, for some $H(p_0), l(p_0) > 0$, $f(x_\Psi, y_\Psi, z_\Psi) = z_\Psi$ and the orbit through a point $p \in V_{p_0}$ of X_0 is carried into the orbit of $\widetilde{X_0}(x, y, z) = (0, 1, \xi y)$ passing through $\Psi(p)$, where $\xi = \operatorname{sgn}(X_0^2 f(p_0))$.

Notice that $l(p_0)$ can be taken small enough such that $Y_0 f(q) \neq 0$, for each $q \in V_{p_0}$. The flow of $\widetilde{X_0}$ is given by

$$\varphi_{\widetilde{X_0}}(t; x_0, y_0, z_0) = \left(x_0,\ t + y_0,\ \xi\frac{(t + y_0)^2}{2} + z_0 - \xi\frac{y_0^2}{2}\right),$$

Firstly, consider $\xi > 0$ and notice that $\widetilde{X_0}$ is not transverse to the sides of $R_{p_0} \cap M^+$ only at the fold-regular lines $L(\alpha) = \{(x, y, z); |x| \leq l(p_0), y = 0, z = \alpha\}$, $\alpha = 0, H(p_0)$, and $\operatorname{sgn}(\widetilde{X_0}^2 f(p)) = \operatorname{sgn}(\widetilde{X_0}^2 f(p_0))$, for each $p \in L(0) \cup L(H(p_0))$.

Now, the trajectory of $\widetilde{X_0}$ through $(x, 0, 0) \in L(0)$ reaches $z = H(p_0)$ at the points $x^\pm = \left(x, \pm\sqrt{2H(p_0)}, H(p_0)\right)$, when $t = \pm\sqrt{2H(p_0)}$, respectively. Choosing $H(p_0)$ sufficiently small, it follows that $x^\pm \in R_{p_0}$, for every $x \in L(0)$.

In addition, using the Flow Box Theorem, we reduce $H(p_0)$ and find a diffeomorphism $\Phi : R_{p_0} \cap M^- \to R_{p_0} \cap M^-$ such that $\Phi|_{z=0} = \mathrm{Id}$ and the orbit of Y_0 through $p \in R_{p_0} \cap M^-$ is carried onto the orbit of $\widetilde{Y_0}(x, y, z) = (0, 0, 1)$ passing through $\Phi(p)$. Considering the homeomorphism $\Theta : V_{p_0} \to R_{p_0}$ (which is a piecewise diffeomorphism) given by

$$\Theta(p) = \begin{cases} \Psi(p) & \text{if} \quad p \in M^+, \\ \Psi \circ \Phi(p) & \text{if} \quad p \in M^-, \end{cases}$$

we define the **local 2-dimensional invariant manifold of X_0 at p_0** as:

$$W_{p_0} = \Theta^{-1}\left(W_{p_0}^+ \cup W_{p_0}^-\right),$$

where $W_{p_0}^+ = \left\{(x, t, t^2/2); |x| \leq l(p_0),\ |t| \leq \sqrt{2H(p_0))}\right\}$, and $W_{p_0}^- = \{(x, 0, t); |x| \leq l(p_0),\ -H(p_0) \leq t \leq 0\}$.

Remark 4.8.1. *Notice that Θ preserves the foliation generated by Z_0 and $\widetilde{Z_0} = (\widetilde{X_0}, \widetilde{Y_0})$, however it does not preserve the orientation of the orbits.*

It is worth mentioning that, since W_{p_0} depends exclusively on the flow of X_0 and Y_0 and on the tangential curve of X_0 with Σ, it follows that the existence of W_{p_0} does not depend on Θ. Although, Θ provides a complete characterization of W_{p_0}.

In this case, the foliation $\mathcal{F}$ generated by Z_0 in $\operatorname{int}(V_{p_0})$ is characterized in the following way. Let N_p be the normal vector of Σ at p and consider the 2-dimensional manifold $\Lambda = \{p + \lambda N_p;\ \lambda > 0, p \in S_{Z_0}\} \cap V_{p_0}$. Thus, each leaf of $\mathcal{F} \backslash W_{p_0}$ is either a piecewise smooth curve which passes transversally through a point of Σ and intersects ∂V_{p_0} transversally in two points (one in M^+ and another in M^-) or it is a smooth curve which passes transversally through a point in Λ and intersects ∂V_{p_0} in two points of M^+. See Figure 4.9.

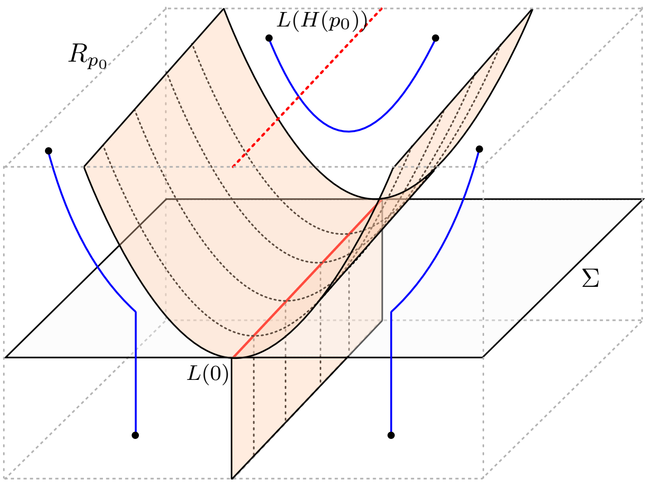


Figure 4.9: The local invariant manifold W_{p_0} for a visible fold-regular point.

Now, if $\xi < 0$, we define the same objects by changing the roles of $L(0)$ and $L(H(p_0))$, and we consider $W_{p_0}^{-} = \{(x, y, t); |x| \leq l(p_0),\ -H(p_0) \leq t \leq 0,\ y = 0 \text{ or } y = \pm\sqrt{2H(p_0)}\}$. Also, in this case, the leaves of the foliation passing through Λ are also piecewise smooth, intersect ∂V_{p_0} at two points of M^{-} and intersect Σ at two points, which are in opposite sides in relation to S_{Z_0}). See Figure 4.10.

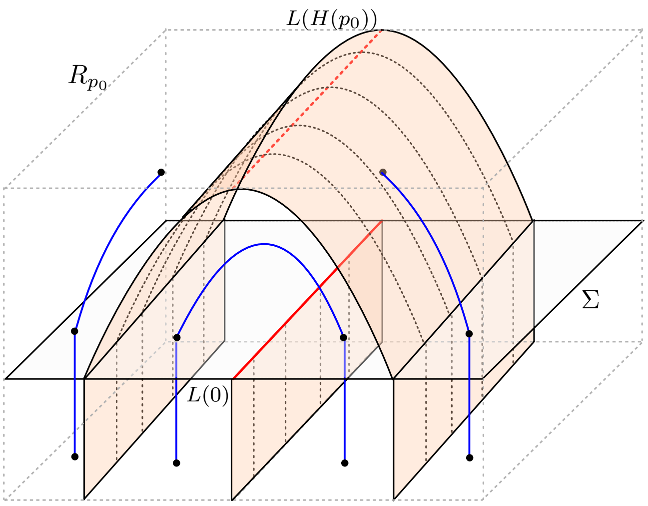


Figure 4.10: The local invariant manifold W_{p_0} for an invisible fold-regular point.

Remark 4.8.2. *Notice that, if $X_0 \in \chi^r$ is a smooth vector field defined in $\overline{\Sigma^+}$ having a fold point at p_0, then we construct the local invariant manifold of X_0 at p_0 in the same way.*

Cusp-Regular

Let p_0 be a cusp-regular point of Z_0 in ∂U_0, and assume that $p_0 \in S_{X_0}$.

Following the same arguments and notation as used in Section 4.8.1, there exists a diffeomorphism $\Psi(p_0) = \vec{0}$, V_{p_0} such that $f(x_\Psi, y_\Psi, z_\Psi) = z_\Psi$ and the orbit of X_0 through a point $p \in V_{p_0}$ is carried into the orbit of $\widetilde{X_0}(x, y, z) = (1, 0, \xi x^2 + y)$ passing through $\Psi(p)$, where $\xi=$ sgn$(X_0^3 f(p_0))$ in this case.

We only consider $\xi = 1$, since the invariant manifolds obtained when $\xi = -1$ are completely analogous. The flow of $\widetilde{X_0}$ is given by

$$\varphi_{\widetilde{X_0}}(t; x_0, y_0, z_0) = \left(t + x_0,\ y_0,\ \frac{(t + x_0)^3}{3} + y_0 t + z_0 - \frac{x_0^3}{3} \right).$$

Now, $\widetilde{X_0}$ is not transverse to the sides of $R_{p_0} \cap M^+$ only at the points of the curves $L(\alpha) = \{(x, y, z); |x| \leq \sqrt{l(p_0)}, y = -x^2, z = \alpha\}$, $\alpha = 0, H(p_0)$. In addition,

$$\text{sgn}(\widetilde{X_0}^2 f((x, y, z))) = \text{sgn}(x),$$

for each $(x, y, z) \in L(0) \cup L(H(p_0))$ with $x \neq 0$, and $(0, 0, \alpha)$ are cusp points of $\widetilde{X_0}$ such that $\text{sgn}(\widetilde{X_0}^3 f((0, 0, \alpha))) = 1$.

Let, $p_x = (x, -x^2, 0)$ be a visible fold-regular point of $L(0)$ $(x > 0)$, then the orbit through p_x is given by $\gamma_x(t) = (t + x, -x^2, (t + x)^3/3 - x^2 t - x^3/3)$. In particular, if $h_0 = 3/4H(p_0) > 0$, then γ_{h_0} intersects $z = H(p_0)$ at $(-h_0, -h_0^2, H(p_0)) \in L(H(p_0))$ (take $H(p_0)$ such that $h_0 < \sqrt{l(p_0)}$). Also, notice that γ_{h_0} is the orbit through the visible fold-regular point $(-h_0, -h_0^2, H(p_0))$.

In addition, γ_h is tangent to the planes $z = 0$ and $z = 4/3h^3 = \delta_h$ and it is contained in the plane $y = -h^2$. Also, γ_h intersects $z = 0$ at the points $(h, -h^2, 0)$ and $P_h = (-2h, -h^2, 0)$, and it intersects $z = \delta_h$ at $Q_h = (-h, -h^2, \delta_h)$ and $(2h, -h^2, \delta_h)$. Notice that, $P_h, Q_h \to \vec{0}$ and $\delta_h \to 0$ as $h \to 0$. Therefore, it presents the behavior illustrated in Figure 4.11. Consider $W_{p_0}^+(0) = \{\gamma_x(t);\ 0 \leq x \leq \sqrt{l(p_0)},\ t \in I_x = [T_-(x), T_+(x)]\}$, where I_x is the maximal interval such that $\gamma_x(I_x) \subset R_{p_0}$.

Changing the roles of $z = 0$ and $z = H(p_0)$, we define an analogous 2-manifold $W_{p_0}^+(H(p_0)) = \{\varphi_{\widetilde{X_0}}(t; x, -x^2, H(p_0));\ -\sqrt{l(p_0)} \leq x \leq 0,\ t \in I_x = [T_-(x), T_+(x)]\}$, where I_x is the maximal interval such that $\varphi_{\widetilde{X_0}}(I_x; x, -x^2, H(p_0)) \subset R_{p_0}$. See Figure 4.12.

Notice that, $W_{p_0}^+(0)$ and $W_{p_0}^+(H(p_0))$ intersect themselves transversally at the curve γ_{h_0}. Let $W_{p_0}^+ = W_{p_0}^+(0) \cup W_{p_0}^+(H(p_0))$ and consider $S = (W_{p_0} \cup S_{\widetilde{X_0}}) \cap \{z = 0\}$, hence the invariant manifold $W_{p_0}^-$ is constructed as in the fold-regular case, but herein we take it as the image of the flow of $\widetilde{Y_0}$ through S, for $-H(p_0) \leq t \leq 0$.

In this case, the local 2-dimensional invariant manifold of Z_0 at p_0 is given by $W_{p_0} = \Theta^{-1}\left(W_{p_0}^+ \cup W_{p_0}^-\right)$.

In Figure 4.13, the foliation of Z_0 in R_{p_0} is described. For simplicity, we characterize it on each plane $y = k$, where $-l(p_0) \leq k \leq l(p_0)$. Notice that R_{p_0} is partitioned in regions where the behavior is of type either transversal or visible fold-regular or invisible fold-regular, and the formal description of this regions can be found in Section 4.8.1.

Fold-Fold

If p_0 is a fold-fold point of Z_0, then we construct the invariant manifolds of X_0 and Y_0 for the fold-lines of X_0 and Y_0, respectively, by following Section 4.8.1 (see Remark 4.8.2). The resultant manifolds can be seen in Figure 3.2 for each type of fold-fold singularity.

In addition, if p_0 satisfies the conditions of local structural stability at p_0, then Theorems 3 and 4 in [44] allows us to construct a homeomorphism $h_{p_0} : V_{p_0} \to V_{p_0}$ which carries orbits of Z_0 onto orbits of Z.

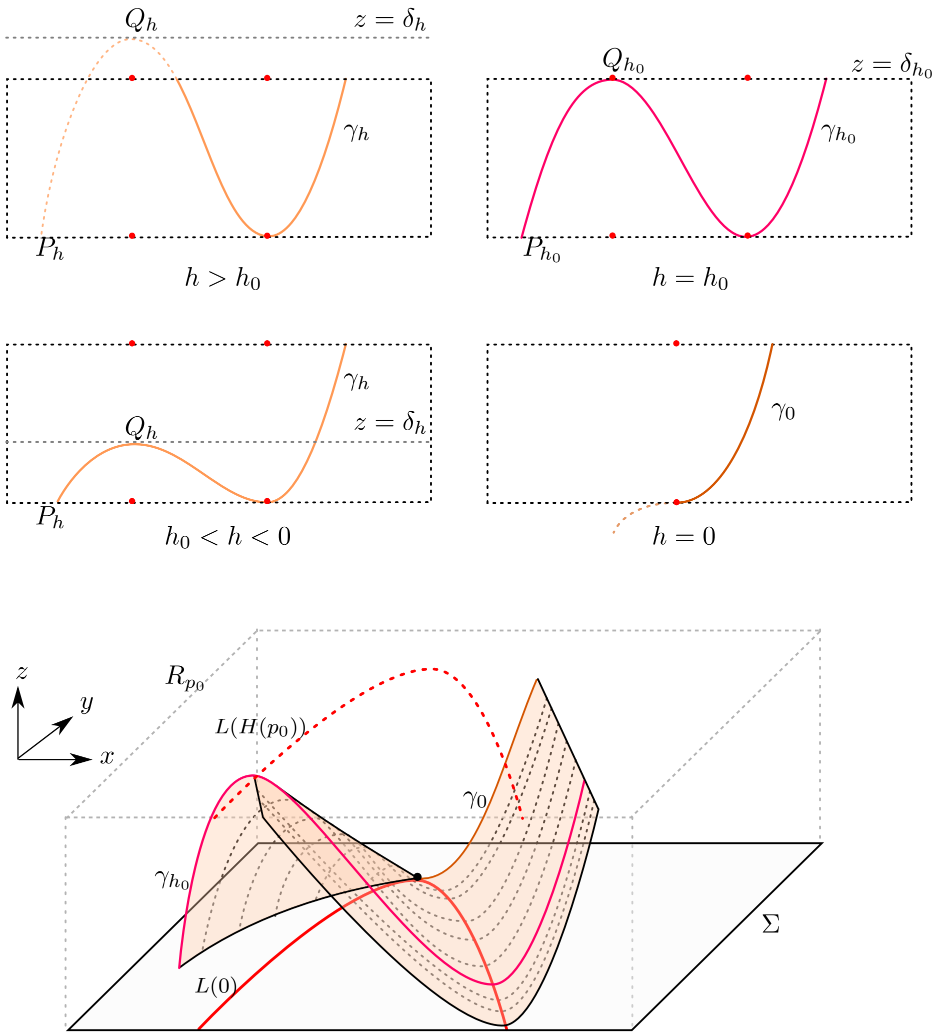


Figure 4.11: The local invariant manifold $W^+_{p_0}(0)$ for a cusp point and its description in the slice $y = -h^2$ of R_{p_0}.

Also, notice that, all trajectories outside the local invariant manifolds intersect ∂V_p transversally, and if we consider a neighborhood V of Σ in M sufficiently small, an orbit contained in V can intersect Σ more than one time only inside neighborhoods V_p of elliptic fold-fold points.

Therefore, there exist only local first return maps in V, and since h_{p_0} is a local equivalence between Z_0 and Z at p_0, we extend it into a semi-local equivalence between Z_0 and Z at Σ. Hence, local first returns are not an obstruction to have semi-local structural stability at Σ.

4.8.2 Existence of the Invariant Manifold of a Tangency Set

Now, we must show that the local invariant manifolds of the elementary tangential singularities give rise to global invariant manifolds of S_{Z_0} defined in a neighborhood Λ of the entire Σ in M. The process mainly consists in the use of the compactness of Σ to concatenate the local manifolds in a smooth way.

Remark 4.8.3. *The term global invariant manifold is used to emphasize that it is defined*

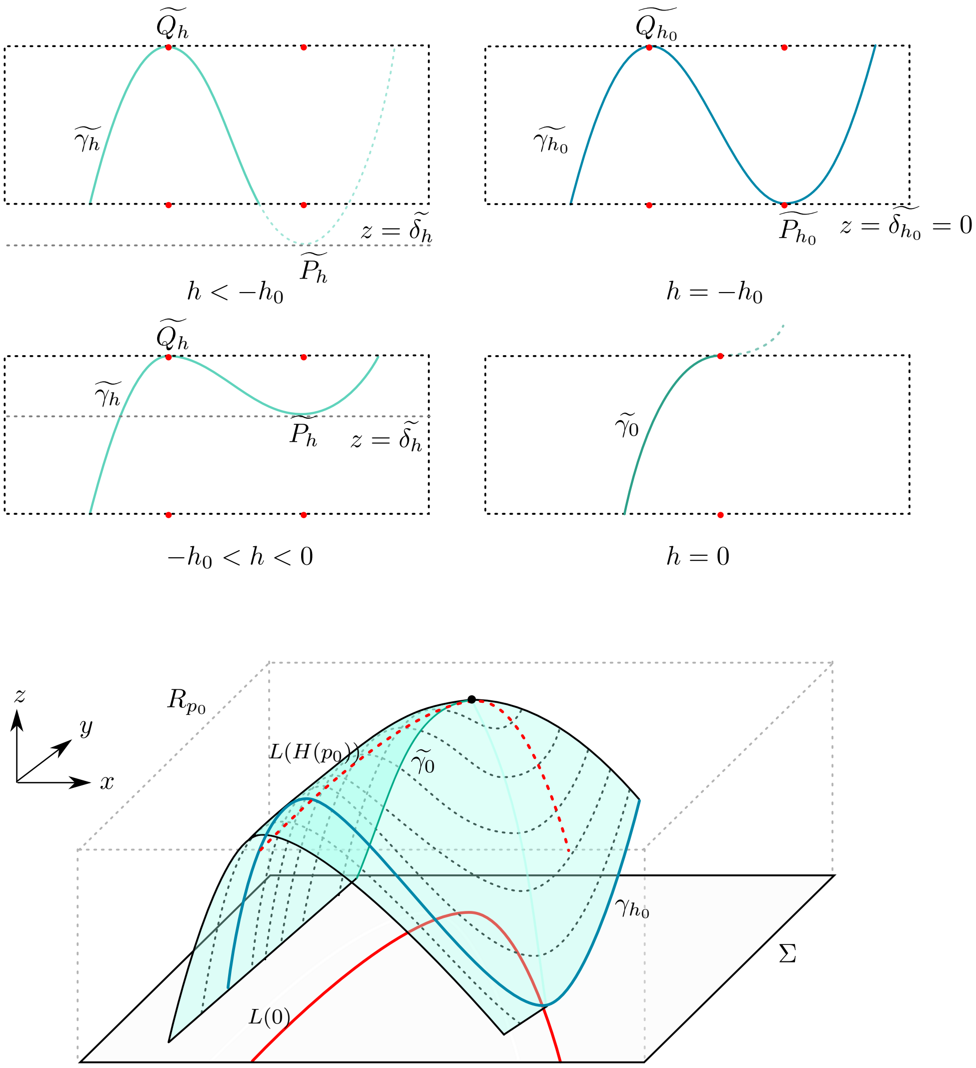


Figure 4.12: The local invariant manifold $W^+_{p_0}(H(p_0))$ for a cusp point and its description in the slice $y = -h^2$ of R_{p_0}. We denote by $\widetilde{P_h}$, $\widetilde{Q_h}$ and $\widetilde{\gamma_h}$ the elements analogous to P_h, Q_h and γ_h in Figure 4.11, respectively.

in a neighborhood of the entire Σ. In fact, it is a collection of local invariant manifolds.

Let N_p be the normal vector of Σ at p pointing toward Σ. Consider the following **λ-lamination of Σ**

$$\Sigma_\lambda = \{p + \lambda N_p;\ p \in \Sigma\},$$

where $\lambda \in \mathbb{R}$.

Let $p \in \Sigma$, since $Z_0 \in \Xi_0$, p is either a regular-regular or a fold-regular or a cusp-regular or a fold-fold point of Z_0. Hence, let V_p be a compact neighborhood of p in M such that:

(i) If p is regular-regular, then each $q \in \Sigma \cap V_p$ is a regular-regular point of Z_0 and X_0, Y_0 are transverse to ∂V_p;

(ii) If p is an elementary tangential singularity, consider the neighborhood V_p given in Section 4.8.1.

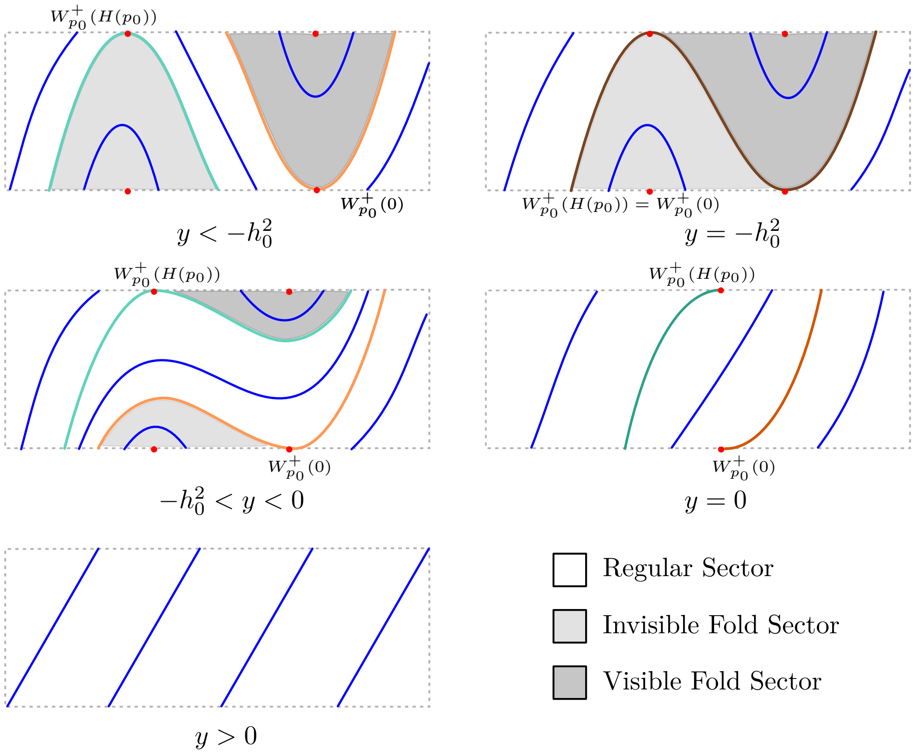


Figure 4.13: Foliation generated by Z_0 in the slices $y = k$ (k is a constant) of the neighborhood $R_{p_0} \cap \{z \geq 0\}$.

From compactness of Σ, we find a finite subcover $V = V_{p_1} \cup \cdots V_{p_n}$ of Σ. Thus, there exists $\lambda^* > 0$ such that $\Lambda = \cup_{\lambda \in [-\lambda^*, \lambda^*]} \Sigma_\lambda$ is contained in V.

Notice that, for each $p \in S_{Z_0}$, the laminations $\Sigma_{\pm\lambda^*} \cap V_p$ correspond to the planes $z = \pm k$ in the neighborhood R_p, for some $k > 0$. For simplicity, we assume $H(p) = k$.

Since Λ is constructed by laminations of Σ in the direction of the normal vectors of Σ, it follows that the same tangency set S_{Z_0} persists on $\partial\Lambda$. See Figure 4.14.

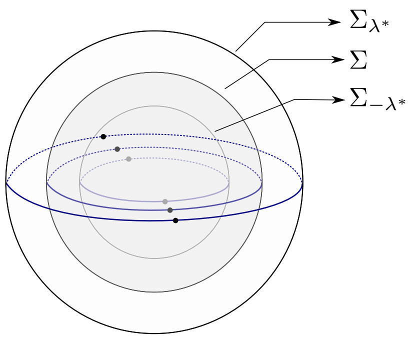


Figure 4.14: Example of neighborhood Λ, where $S_{Z_0} = \{(x, y, 0);\ x^2 + y^2 = 1\}$. The distinguished points represent cusp points of X_0.

Recall that the local invariant manifolds of an elementary tangential singularity p_0 depends only on the tangency set of $\widetilde{Z}_0$ with $z = 0$, $z = H(p)$ and $z = -H(p)$. Therefore, they depend intrinsically on the tangency set of Z_0 with Σ and $\Sigma_{\pm\lambda^*}$.

It is enough to prove that all the local invariant manifolds characterized in Section 4.8.1 extend themselves to global invariant manifolds of S_{Z_0}.

In order to clarify these ideas, we explain how the local invariant manifolds originate a global invariant manifold when $S_{Z_0} = S_{X_0}$, and S_{X_0} is a connected set composed by fold-regular points and two cusp-regular points.

Let p, q be the cusp-regular points of Z_0, therefore S_{X_0} is composed by two arcs A_1 and A_2 with extrema p, q, such that the fold-regular points of $A_1 \setminus \{p, q\}$ (resp. $A_2 \setminus \{p, q\}$) are visible (resp. invisible).

At each point $p \in A_1$, consider the neighborhoods V_p found in Section 4.8.1. From compactness of A_1, a finite number of them covers A_1, say it $V_1, \cdots, V_n$. By connectedness, they intersect each other at least in one point. See Figure 4.15.

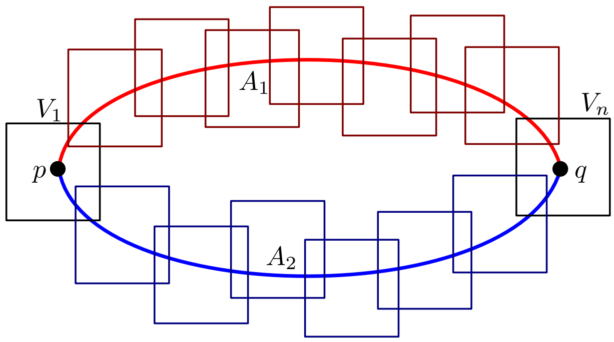


Figure 4.15: Arcs A_1 and A_2 of S_{Z_0}.

Now, in each V_i, the local manifold is given by the image of the flow $\varphi_{X_0}(t; p)$, with $T_-^i(p) \leq t \leq T_+^i(p)$, where $T_-^i(p) < 0 < T_+^i(p)$ and $p \in A_1 \cap V_i$. Let $q \in A_1 \cap \text{int}(V_i \cap V_j)$, and restrict the values of t to the interval with extrema $T_+(q) = \min\{T_+^{i,j}(q)\}$ and $T_-(q) = \min\{T_-^{i,j}(q)\}$. It is enough to reduce the heights of the neighborhoods R_i to concatenate the local manifolds. Repeating this process, we extend the manifolds to the arc A_1 obtaining W_1^+.

Notice that, in the neighborhoods V_1 and V_n, we have cusp-regular points, therefore, the invariant manifold in $V_2 \cup \cdots \cup V_{n-1}$ concatenates with the local invariant manifolds of the cusp-regular points having visible fold-regular points.

The construction of the global manifold of the arc A_2 is done in an analogous way. Notice that, in this case, the concatenation has to be done in the visible fold-regular points at the lamination. Following this process, we obtain the global invariant manifold W_2^+ (see Figure 4.16).

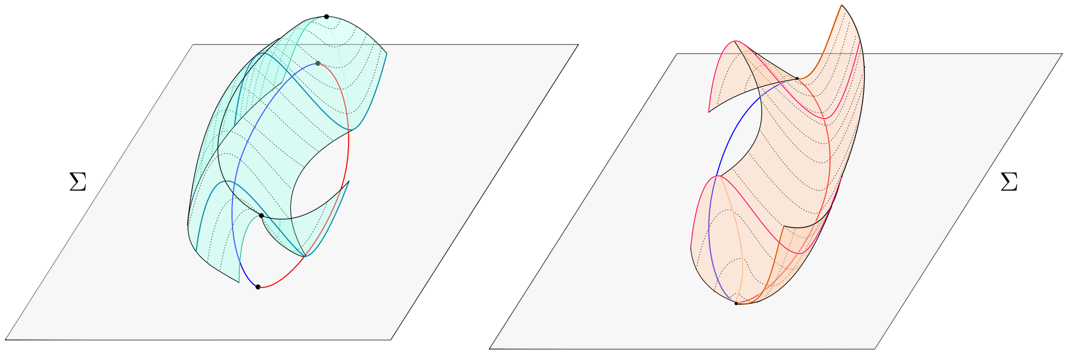
4.8.3 Construction of the Homeomorphism

Finally, we construct the semi-local equivalence between Z_0 and Z at U_0. Firstly, define $h : U_0 \to U$ by using Theorem J.

Consider the neighborhood Λ of the previous section. From Remark 4.7.1, if $p_0 \in \partial U_0$ is a fold-fold singularity, then we extend h into a neighborhood $W_{p_0} = V_{p_0} \cap \Lambda$, i.e. $h : W_{p_0} \to W_{p_0}$ through the remarks in Section 4.8.1.

Since h carries the tangential singularities of Z_0 onto the tangential singularities of Z of the same type, then h carries ∂U_0 onto ∂U. Therefore, we use the flows of Z_0 and Z to carry the global invariant manifolds of Z_0 onto the global invariant manifolds of Z.

Recall that, outside the global manifolds, the flows of Z_0 and Z are transversal to $\partial\Lambda$. Consider any extension of h into a small compact neighborhood W of $U_0 \cup U$ in Σ.



(a) (b)

Figure 4.16: Global manifolds W_2^+ (a) and W_1^+ (b).

Let $V = \{p + \lambda N_p;\ p \in W,\ \lambda \in [-\lambda^*, \lambda^*]\}$, then, we extend h into V through the flow of Z_0 and Z. In fact, the behavior of both piecewise smooth vector fields are trivial outside global manifolds, and we use the local foliations characterized in Section 4.8.1 and transversality arguments to do this extension (see [39, 44, 95, 97] for more details).

It follows from the construction that h carries orbits of Z_0 onto orbits of Z. Hence Z_0 is semi-local equivalent to Z at U_0.

Conclusion of the Proof

We have shown that $\Sigma_0 \subset \Omega^r_\Sigma$. From Local Theory, it is easy to see that, if $Z_0 \notin \Sigma_0$ then Z_0 is not semi-locally structurally stable at Σ. Therefore, we have proven item (i). Items (ii) and (iii) of Theorem K follows directly from Corollary 4.1 in [44].

4.9 Conclusion and Further Directions

In this chapter, we have found necessary conditions for the structural stability in Ω^r. First of all, remark that all the results stated in Section 4.5 hold for vector fields having a compact oriented switching manifold Σ (without the simply connectedness assumption). In fact, the proofs of Theorems J and K for Filippov systems having compact oriented non-simply connected switching manifold Σ follow in the same way as those ones exhibited here. For simplicity, we have considered Σ diffeomorphic to $\mathbb{S}^2$ just for technical reasons.

We highlight that the problem remains open for non-orientable switching manifolds. In this case, even the definition of piecewise smooth vector fields is still not established. It certainly presents lots of mathematical challenges.

The behavior of continuous piecewise smooth vector fields is trivial around the switching manifold, nevertheless they may present a completely non-trivial dynamics from the global point of view. In light of this, the characterization of structural stability is a rather challenging problem.

Finally, the most natural extension of this work is to study global continuations of the invariant manifolds defined in Section 4.8.2. It originates applications in generic bifurcation theory.

Chapter 5

Quasi-Generic Loops in $3D$ Filippov Systems

Aiming to contribute to the characterization of structural stability of $3D$ Filippov systems from a global point of view, we analyze a homoclinic-like loop at a fold-regular singularity. We provide conditions on a piecewise smooth vector field to have a loop which is robust in one-parameter families. Moreover, the basin of attraction at the loop is computed as well as its bifurcation diagram.

5.1 Introduction

The study of global connections in smooth systems is a challenging problem which has been extensively studied throughout the last decades due to its importance in the understanding of the dynamics of a smooth vector field. In fact, once the singular elements (singularities, limit cycles, etc.) of the system are detected, the dynamics of the system inside a region is determined by the existence or not of global connections between them.

In the nonsmooth context, one finds new types of singular elements, such as the so-called Σ-singularities (see Definition 1.1.5), and thus, it gives rise to an extensive class of global connections which has no counterparts in the smooth context.

Therefore, in order to follow the Peixoto's program to characterize the structurally stable $3D$ Filippov systems from a global point of view, it is imperative to understand non-local connections between generic Σ-singularities.

5.1.1 Historical Facts

In dimension 2, there are plenty of works dealing with global connections to Σ-singularities of Filippov systems. In fact, homoclinic-like loops at a fold-regular singularity have been studied in [67, 87], and in [40], the authors have described the bifurcation diagram of such connection. In [2], a review on such results is provided. Also, in [17], the authors regularized the bifurcation diagram of this kind of connection. It is worth mentioning that, such loops appear in the unfolding of more degenerate phenomena, such as the fold-saddle singularity studied in [21] and the homoclinic-like loop at a visible-visible fold-fold singularity approached in [79].

As we have seen in Chapter 2, we provide a method to deal with Σ-polycycles in planar system in a general scenario, and we also study the generic bifurcation of some codimension 2 global connections to Σ-singularities. As far as we know, this topic has not been treated for $3D$ Filippov systems in the literature. So, with the recent development of planar phenomena, it is natural to extend these studies to dimension 3.

5.1.2 Description of the Results

Now, we provide a roughly description of the results of this chapter. We consider global connections involving fold-regular singularities in $3D$ Filippov systems. More specifically, we present a class Λ_1 of Filippov systems Z_0 having a homoclinic-like loop Γ_0 at a fold-regular singularity.

We prove that $Z_0 \in \Lambda_1$ is generic in one-parameter families. It means that, given a $\mathcal{C}^r$ family of Filippov systems $\mathcal{Z}(\lambda)$, $\lambda \in [-\varepsilon_0, \varepsilon_0]$, such that $\mathcal{Z}(0) = Z_0 \in \Lambda_1$, then any one-parameter family $\widetilde{\mathcal{Z}}$ sufficiently near to $\mathcal{Z}$ (in the $\mathcal{C}^r$ topology) has a point $\lambda_0 \in [-\varepsilon_0, \varepsilon_0]$ such that $\widetilde{\mathcal{Z}}(\lambda_0) \in \Lambda_1$.

We provide the bifurcation diagram of $Z_0 \in \Lambda_1$ around Γ_0 under certain generic conditions. Also, we compute the basin of attraction of Γ_0. It is worth mentioning that, the use of sliding features of Z_0 is crucial to obtain these results.

Finally, we introduce a notion of weak equivalence in Λ_1 and we obtain a modulus of stability which allows us to conclude that there are infinitely many different elements in Λ_1 under this equivalence relation.

This chapter is organized as follows. In Section 5.2 we discuss some scenarios where a $3D$ Filippov system admits a global connection involving a fold-regular singularity. In Section 5.3 we present the Filippov systems approached in this chapter and we state our main results. Section 5.4 is devoted to present the necessary tools to prove our results. In Section 5.5 we present some models realizing the bifurcation diagram presented in Section 5.3 in order to illustrate the result. Finally, in Section 5.6 we prove the main results stated in Section 5.3.

5.2 A discussion on some global connections

Throughout this chapter, we consider Filippov systems defined in $\mathbb{R}^3$ with switching manifold $\Sigma = f^{-1}(0)$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is a $\mathcal{C}^r$ function having 0 as a regular value (see Chapter 1). Denote the set of such systems by $\Omega^r = \chi^r \times \chi^r$, where χ^r is the space of $\mathcal{C}^r$ vector fields. Endow Ω^r with the product topology.

Let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a Filippov system having a visible fold-regular singularity at $p_0 \in \Sigma$ (see Definition 3.3.7). Denote the flows of X_0 and Y_0 by $\varphi_{X_0}(t; p)$ and $\varphi_{Y_0}(t; p)$, respectively. Assume that Z_0 satisfies the following set of global hypotheses **(G)**:

(G_1) There exists $T_+ > 0$ such that $p_0^+ = \varphi_{X_0}(T_+; p_0) \in \Sigma$;

(G_2) $\Gamma_0^+ = \{\varphi_{X_0}(t; p_0);\ t \in (0, T_+)\} \subset M^+$ and X_0 is transverse to Σ at p_0^+;

(G_3) There exist a point $q_0 \in \Sigma$ and a regular orbit Γ_0^R of Z_0 connecting p_0^+ and q_0.

Without loss of generality, assume that Γ_0^R in (G_3) is a regular orbit of Y_0 contained in M^-. Using properties of a fold-regular singularity (see [104]) and the transversality condition (G_2), we define the germs $\mathcal{P}_0^+ : (\Sigma, p_0) \to (\Sigma, p_0^+)$ and $\mathcal{P}_0^- : (\Sigma, p_0^+) \to (\Sigma, q_0)$ induced by the flows of X_0 and Y_0, respectively (see Figure 5.1). Thus, consider

$$\mathcal{P}_0 = \mathcal{P}_0^- \circ \mathcal{P}_0^+, \tag{5.2.1}$$

and notice that the restriction of $\mathcal{P}_0$ to $\overline{\Sigma^c}$ is a first return map of Z_0 in Σ.

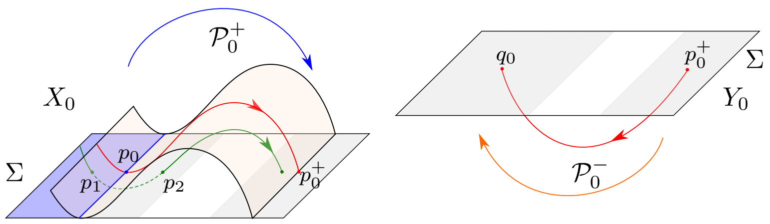


Figure 5.1: Action of the maps $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$.

Remark 5.2.1. *Notice that, in Figure 5.1, the points $p_1 \neq p_2$ have the same image through $\mathcal{P}_0^+$. We will see that $\mathcal{P}_0$ is a non-invertible $\mathcal{C}^r$ map and its restriction to $\overline{\Sigma^c}$ is a $\mathcal{C}^r$ homeomorphism.*

Since p_0 is a visible fold-regular singularity, it follows that Z_0 has a (compact) $\mathcal{C}^r$ curve $\gamma_0 \subset \Sigma$ of visible fold-regular singularities containing p_0 (see [104]). It follows that, γ_0 is brought to a (compact) $\mathcal{C}^r$ curve $\zeta_0 \subset \Sigma$ by $\mathcal{P}_0$ such that $q_0 \in \zeta_0$.

Also, still from Local Theory, the sliding vector field F_{Z_0} of Z_0 is transverse to the curve γ_0 anywhere, and there exists a neighborhood V_0 of $\text{int}(\gamma_0)$ in Σ (with compact closure) such that:

i) Y_0 is transverse to Σ at any point of V_0;

ii) γ_0 divides $\overline{V_0}$ into two connected components, one contained in Σ^s and the other one contained in Σ^c;

iii) F_{Z_0} is $\mathcal{C}^r$-extended onto $\overline{V_0}$ (see Lemma 3.3.11).

See Figure 5.2.

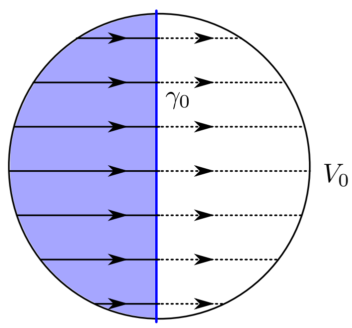


Figure 5.2: Neighborhood $V_0 \subset \Sigma$.

For our purposes, we assume that $\zeta_0 \subset V_0$. Accordingly, we consider $\mathcal{P}_0 : V_0 \to V_0$. In this case, we distinguish the following situations: (a) $\zeta_0 \subset \Sigma^s$, (b) $\zeta_0 \subset \Sigma^c$, (c) ζ_0 is transverse to γ_0 at q_0, and (d) ζ_0 is tangent to γ_0 at q_0 (see Figure 5.3).

Notice that configurations (a), (b) and (c) are robust in Ω^r. However, configuration (d) is easily broken by small perturbations. In fact, the degree of degeneracy in case (d) depends on the degree of contact between γ_0 and ζ_0 at q_0. The most degenerate situation occurs when $\gamma_0 = \zeta_0$ (as shown in Figure 5.3).

Now, we discuss the possible dynamics concerning the robust situations (a), (b) and (c).

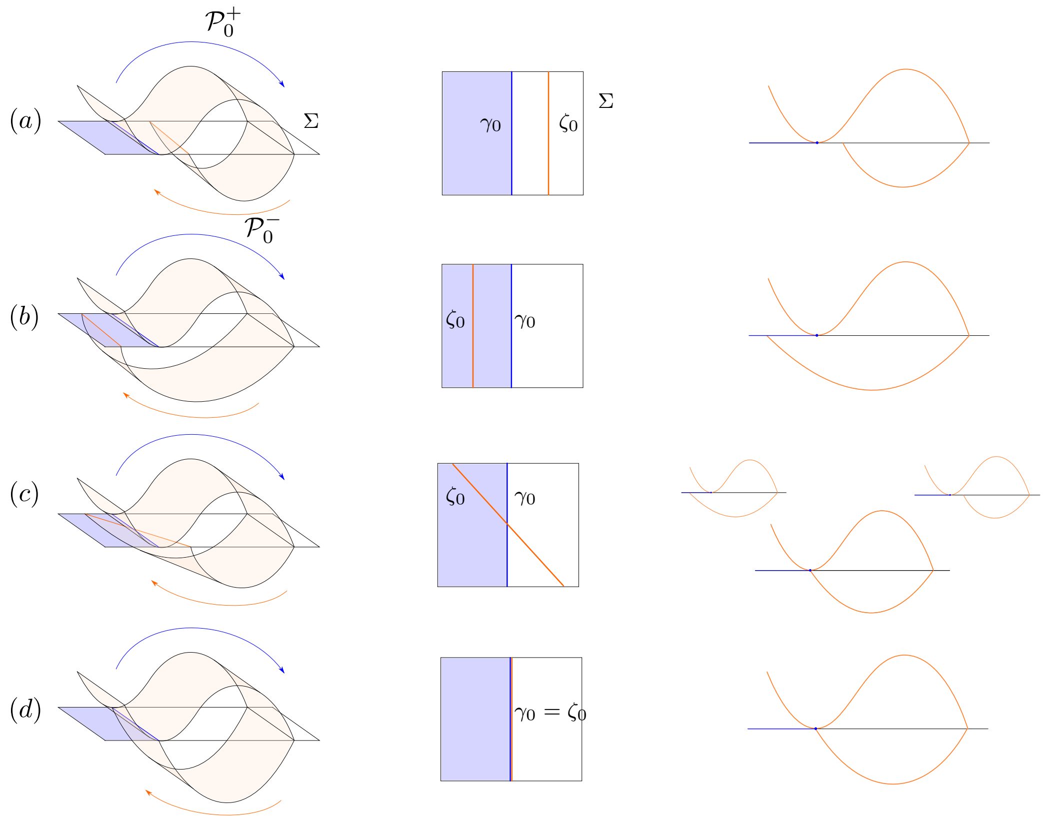


Figure 5.3: Relative positions between the curves γ_0 of fold-regular singularities and its image ζ_0 through the flow of Z_0: (a) $\zeta_0 \subset \Sigma^s$, (b) $\zeta_0 \subset \Sigma^c$, (c)$\gamma_0 \pitchfork \zeta_0$ and (d) $\zeta_0 = \gamma_0$.

5.2.1 Cases (a) $\zeta_0 \subset \Sigma^s$ and (b) $\zeta_0 \subset \Sigma^c$

If $\zeta_0 \subset \Sigma^c$, then the dynamics of Z_0 is trivial around the orbit connecting p_0 and q_0. In fact, consider

i) a section Π^+ at p_0 such that Π^+ is the restriction to M^+ of a local transversal section of X_0 at p_0 which intersects Σ at γ_0;

ii) a section Π^- which consists on a neighborhood of p_0 intersected with $\overline{\Sigma^c}$;

iii) $\Pi = \Pi^+ \cup \Pi^-$.

Thus, using the local structure of a fold-regular singularity (see [104]), we obtain that all orbits of Z_0 in a neighborhood of p_0 intersect Π. Also, for a neighborhood N_0 of q_0 contained in Σ^c, we construct a tubular flow box between Π and N_0 along the orbits of X_0 and Y_0 (see Figure 5.4).

Now, consider that $\zeta_0 \subset \Sigma^s$. As we have seen, each point $p \in \gamma_0$ is brought to a point $\mathcal{P}_0(p) \in \zeta_0$ through the flow of Z_0 (orbits of X_0 and Y_0). Since F_{Z_0} is regular in V_0 and transverse to γ_0, each point $q \in \zeta_0$ reaches γ_0 at a unique point $\psi_0^*(q)$ through a sliding trajectory of Z_0. It defines the $\mathcal{C}^r$ map

$$\psi_0 : p \in \gamma_0 \longmapsto \psi_0^*(\mathcal{P}_0(p)) \in \gamma_0, \tag{5.2.2}$$

which induces a dynamics in the fold curve γ_0. We refer to ψ_0 as the **fold line map** associated to Z_0.

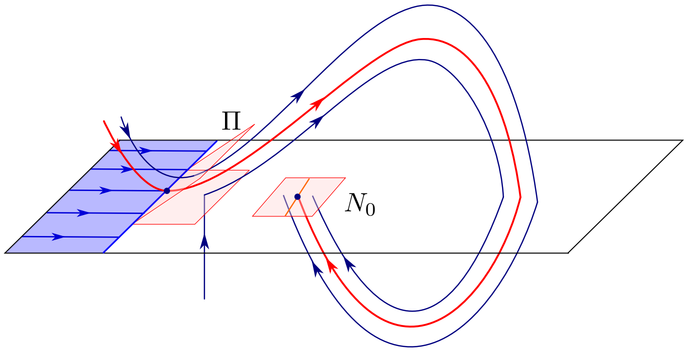


Figure 5.4: Tubular flow of Z_0 between Π and N_0.

In this case, the orbits of X_0, Y_0 and F_{Z_0} connect γ_0 to itself and they give rise to a Z_0-invariant manifold $\mathcal{M}$ which is a piecewise-smooth $2D$-cylinder or a piecewise-smooth Möbius strip, depending on the identification provided by ψ_0. Also, the dynamics of Z_0 in $\mathcal{M}$ is completely characterized by the dynamics of ψ_0 in γ_0. Thus,

(Sl_R) if $p_0 \in \gamma_0$ is a regular point of ψ_0, then the dynamics of Z_0 in $\mathcal{M}$ is trivial. It means that there are no minimal sets contained in $\mathcal{M}$;

(Sl_S) if $p_0 \in \gamma_0$ is a fixed point of ψ_0, then Z_0 has a sliding connection Γ_0 through p_0 contained in $\mathcal{M}$ (see Figure 5.5).

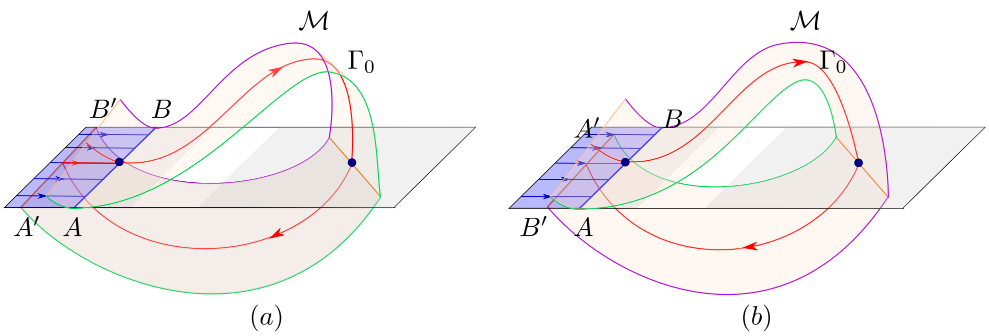


Figure 5.5: A sliding connection Γ_0 of Z_0 in $\mathcal{M}$ where $\mathcal{M}$ is a piecewise-smooth cylinder (a) or a piecewise-smooth Möbius strip (b).

If (Sl_S) is satisfied, then the sliding connection Γ_0 of Z_0 can be persistent, depending on the properties of ψ_0 at p_0. In fact, we mention the following cases.

i) If p_0 is a hyperbolic fixed point of ψ_0, then each $Z \in \Omega^r$ nearby Z_0 presents a sliding connection Γ near Γ_0, in the Hausdorff distance, with the same stability of Γ_0;

ii) If p_0 is a fixed point of ψ_0 of saddle-node type, i.e. $|\psi_0'(p_0)|= 1$ and $|\psi_0''(p_0)|\neq 1$, then Z_0 belongs to a codimension one submanifold of Ω^r. A versal unfolding of Z_0 in Ω^r around Γ_0 is illustrated in Figure 5.6.

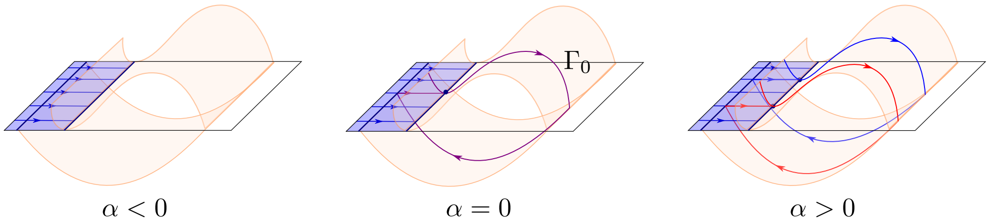


Figure 5.6: A versal unfolding Z_α of Z_0 around Γ_0 when conditions (Sl_S) and (ii) are satisfied (saddle-node bifurcation).

Remark 5.2.2. *Observe that, if p_0 is a fixed point of ψ_0 having a higher degree of degeneracy, then $Z \in \Omega^r$ nearby Z_0 presents complicated sliding features contained in $\widetilde{\mathcal{M}}$ (which is a Z-invariant manifold nearby $\mathcal{M}$ having the same topological type of $\mathcal{M}$) bifurcating from Γ_0.*

5.2.2 Case (c): ζ_0 and γ_0 are transverse at q_0

Assume that $Z_0 = (X_0, Y_0) \in \Omega^r$ satisfies **(G)** and the following assumption

$$\textbf{(T)}{:}\zeta_0 \cap \gamma_0 = \{q_0\} \text{ and } \zeta_0 \pitchfork \gamma_0 \text{ at } q_0.$$

If $q_0 \neq p_0$, then for $Z \approx Z_0$ ($\approx$ stands for nearby), hypothesis (T) implies that there exist curves in Σ, γ and ζ, analogous to γ_0 and ζ_0, satisfying $\zeta \cap \gamma = \{q\}$ for some $q \approx q_0$ and $\zeta \pitchfork \gamma$ at q. Also, there exists $p \approx p_0$ which is mapped to q through the flow of Z, and $q \neq p$. It follows that the connection between p_0 and q_0 of Z_0 is persistent for $Z \in \Omega^r$ nearby Z_0 (see Figure 5.7).

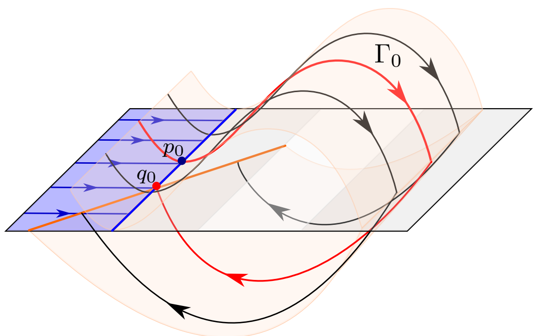


Figure 5.7: A robust connection Γ_0 of Z_0 between p_0 and q_0.

Now, if hypothesis

$$\textbf{(H)}{:}\ p_0 = q_0,$$

is also satisfied, then Z_0 has a homoclinic-like loop Γ_0 at p_0 (see Figure 5.8). In contrast to the previous case, this phenomenon is not persistent in Ω^r.

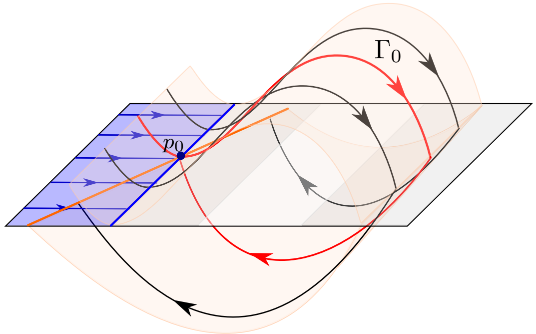


Figure 5.8: A homoclinic-like loop Γ_0 of Z_0 at a fold-regular singularity p_0.

5.3 Quasi-generic loops

Our aim is to describe the bifurcation diagram of vector fields $Z_0 \in \Omega^r$ satisfying hypotheses **(G)**, **(T)**, and **(H)** around its homoclinic-like loop Γ_0 at p_0 (see Figure 5.8), and characterize the dynamical features arising from such connection.

Generally speaking, we prove that, under some constraints, such connection is generic for one-parameter families in Ω^r. In what follows, we consider some classes of vector fields in Ω^r and we state our main results concerning this topic.

Consider $Z_0 \in \Omega^r$ satisfying **(G)**, **(T)**, and **(H)**, and recall that the sliding vector field F_{Z_0} is defined in the entire neighborhood V_0 (via extension) and it foliates V_0 by curves transverse to γ_0. In light of this, the fold line map $\psi_0 : \gamma_0 \to \gamma_0$ given in (5.2.2) is still defined herein in the same way. Nevertheless, in this case, ψ_0 is defined through orbits of X_0, Y_0 and virtual sliding orbits of Z_0 for some points of γ_0.

In fact, remark that p_0 splits the curve γ_0 into two connected components named C^1_γ and C^2_γ. Analogously, p_0 splits ζ_0 into C^1_ζ and C^2_ζ. Without loss of generality, assume that C^1_γ and C^2_γ are mapped onto C^1_ζ and C^2_ζ through the orbits of X_0 and Y_0, respectively. Now, one of the components of ζ_0, say it C^1_ζ, is contained in Σ^s and the other one is contained in Σ^c.

Thus, the points $p \in \gamma_0$ and $\psi_0(p) \in \gamma_0$ are connected by an orbit of Z_0 if, and only if $p \in \overline{C^1_\gamma}$ (which is mapped onto $\overline{C^1_\zeta} \subset \overline{\Sigma^s}$ by orbits of Z_0). It follows that only the restriction of ψ_0 to $\overline{C^1_\gamma}$ describes the dynamics of Z_0.

Definition 5.3.1. *Define Λ_1 as the set of vector fields $Z_0 \in \Omega^r$ such that*

*i) Z_0 satisfies hypotheses **(G)**, **(T)** and **(H)**;*

ii) F_{Z_0} is transverse to ζ_0 at p_0;

iii) The fold line map $\psi_0 : \gamma_0 \to \gamma_0$ induced by Z_0 has a hyperbolic fixed point at p_0.

*If $Z_0 \in \Lambda_1$, then we say that Z_0 has a **quasi-generic loop** Γ_0 **at the fold-regular singularity** p_0.*

Remark 5.3.2. *Throughout the text, we also refer to a quasi-generic loop at a fold-regular singularity simply by quasi-generic loop.*

In the result below, we show the robustness of quasi-generic loops in one-parameter families of Filippov systems.

Theorem L. *Given $Z_0 \in \Lambda_1$. There exist a solid torus $\mathcal{A}_0$ around Γ_0, a neighborhood $\mathcal{V}_0$ of Z_0 in Ω^r and a $\mathcal{C}^r$ function $\zeta : \mathcal{V}_0 \to \mathbb{R}$, such that $\zeta(Z_0) = 0$, and $\zeta(Z) = 0$ if, and only if, Z has a unique quasi-generic loop Γ at a fold-regular singularity p contained in $\mathcal{A}_0$. Furthermore, 0 is a regular value of ζ, and thus Λ_1 is a codimension one $\mathcal{C}^r$-submanifold of Ω^r.*

Now, we distinguish the following situations

(Cyl): ψ_0 preserves the connected components C^1_γ and C^2_γ of γ_0;

(Mob): ψ_0 exchanges the connected components C^1_γ and C^2_γ of γ_0.

Define Λ_1^C and Λ_1^M as the subsets of Λ_1 containing the Filippov systems Z_0 satisfying **(Cyl)** and **(Mob)**, respectively, and consider the cases

(N): The hyperbolic fixed point p_0 of ψ_0 is attractive;

(S): The hyperbolic fixed point p_0 of ψ_0 is repulsive.

Notice that, if $Z_0 \in \Lambda_1^C$, then γ_0 self-connects through orbits of X_0, Y_0, F_{Z_0} and virtual orbits of F_{Z_0} as a topological cylinder. Nevertheless, if $Z_0 \in \Lambda_1^M$, then γ_0 self-connects as a topological Möbius strip. See Figure 5.9.

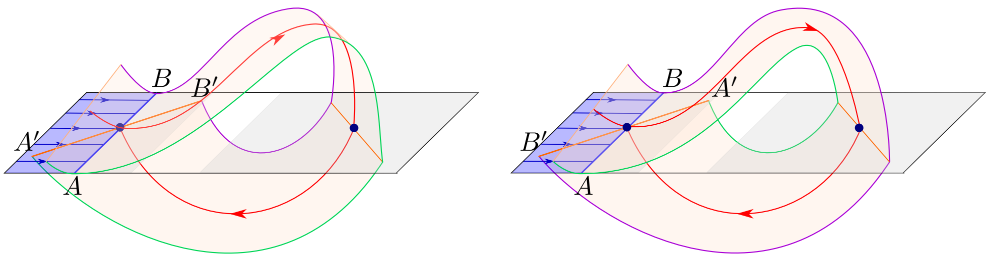


Figure 5.9: Quasi-generic loop Γ_0 of (a) $Z_0 \in \Lambda_1^C$ and (b) $Z_0 \in \Lambda_1^M$.

It is worth mentioning that, if $Z_0 \in \Lambda_1^C$, all the iterations of the fold line map ψ_0 (defined in the fold line γ_0) captures the dynamics of Z_0, since $\psi_0(\overline{C^1_\gamma}) \subset \overline{C^1_\gamma}$ and thus $\psi_0|_{\overline{C^1_\gamma}}$ defines a dynamical system in $\overline{C^1_\gamma}$. Although, it does not hold when $Z_0 \in \Lambda_1^M$, since $\psi_0(\overline{C^1_\gamma}) \subset \overline{C^2_\gamma}$, which means that $\psi_0|_{\overline{C^1_\gamma}}$ can not be iterated. In Section 5.4.4 below, we discuss how to adapt the fold line map ψ_0 to correctly describe the dynamics of $Z_0 \in \Lambda_1^M$ in γ_0.

In the remaining results of this section, we consider only vector fields $Z_0 \in \Lambda_1^C$, in order to provide an amenable analysis, nevertheless we believe that the same conclusions hold for vector fields in Λ_1^M through slight modifications.

The next result is devoted to identify minimal sets bifurcating from a quasi-generic loop of a Filippov system $Z_0 \in \Lambda_1^C$.

Theorem M. *Let $Z_0 \in \Lambda_1^C$ having a quasi-generic loop Γ_0 at a fold-regular singularity $p_0 \in \Sigma$ and consider the torus $\mathcal{A}_0$ given by Theorem L. If $\mathcal{Z} : (-\varepsilon, \varepsilon) \to \Omega^r$ is a one-parameter $\mathcal{C}^1$ family such that $\mathcal{Z}(0) = Z_0$ and $\mathcal{Z}$ is transverse to Λ_1, then the following statements hold.*

1. *If Z_0 satisfies condition (N), then $\mathcal{Z}(\gamma)$ has a unique closed connection Γ_γ in $\mathcal{A}_0$ which is a sliding cycle when $\gamma < 0$ and an attractive hyperbolic crossing limit cycle when $\gamma > 0$, or vice-versa (see Figure 5.10).*

2. *If Z_0 satisfies condition (S), then $\mathcal{Z}(\gamma)$ has a unique hyperbolic crossing limit cycle for either $\gamma < 0$ or $\gamma > 0$, and it has at most a unique sliding cycle in $\mathcal{A}_0$.*

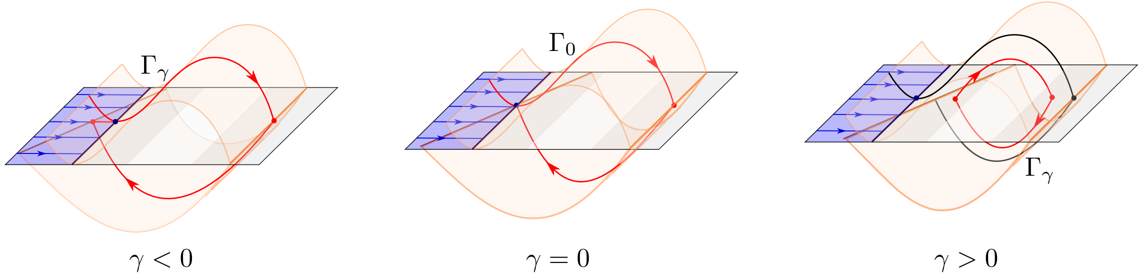


Figure 5.10: A versal unfolding of $Z_0 \in \Lambda_1^C$ satisfying (N) in Ω^r.

Now, we combine the informations encoded by the first return map and the sliding dynamics to analyze the stability of a quasi-generic loop.

Theorem N. *Let $Z_0 \in \Lambda_1^C$ having a quasi-generic loop Γ_0 at a fold-regular singularity $p_0 \in \Sigma$ and consider the torus $\mathcal{A}_0$ given by Theorem L. The following statements hold.*

1. *If Z_0 satisfies condition (N), then Γ_0 is an asymptotically stable minimal set;*

2. *If Z_0 satisfies condition (S), then there exists a piecewise-smooth curve β passing through p_0 such that the basin of attraction of Γ_0 is given by*

$$\mathcal{B} = \{p \in \mathcal{A}_0;\ \textit{there exist a time } t(p) \textit{ such that } \varphi_{Z_0}(t(p);p) \in \beta\}.$$

Furthermore, β has one of the two connected components of $\beta \setminus \{p_0\}$ contained in Σ^s and the other one contained in Σ^c.

We introduce a notion of equivalence in Λ_1 which allows us to obtain a modulus of stability for Z_0.

Definition 5.3.3. *Let $Z, Z_0 \in \Lambda_1$ having quasi-generic loops Γ and Γ_0 at fold-regular singularities $p \in \Sigma$ and $p_0 \in \Sigma$, respectively. We say that Z and Z_0 are* ***weakly topologically equivalent*** *at (Γ, Γ_0) if there exist sufficiently small solid tori $\mathcal{A}_0$ and $\mathcal{A}$ containing Γ_0 and Γ, respectively, and an order-preserving homeomorphism $h : \mathcal{A} \to \mathcal{A}_0$ such that*

i) $\mathcal{A} \cap \Sigma$ and $\mathcal{A}_0 \cap \Sigma$ have connected curves S_Z and S_{Z_0} of fold-regular singularities of Z and Z_0 intersecting $\partial\mathcal{A} \cap \Sigma$ and $\partial\mathcal{A}_0 \cap \Sigma$ transversally, and there are no more Σ-singularities of Z and Z_0 contained in $\mathcal{A} \cap \Sigma$ and $\mathcal{A}_0 \cap \Sigma$, respectively;

ii) $h : S_Z \to S_{Z_0}$ is a diffeomorphism such that $h(p) = p_0$;

iii) $h(\Gamma) = \Gamma_0$ and $h(\mathcal{A} \cap \Sigma) = \mathcal{A}_0 \cap \Sigma$;

iv) h carries orbits of Z onto orbits of Z_0.

Remark 5.3.4. *Notice that, it follows from Section 5.2 that, given $Z \in \Lambda_1$, we can find a sufficiently small torus $\mathcal{A}$, such that item (i) of Definition 5.3.3 is satisfied.*

Finally, given $Z_0 \in \Lambda_1$, we define the **modulus of weak-stability** of Z_0 as

$$\mathcal{W}(Z_0) = \psi_0'(p_0).$$

Theorem O. *Let $Z_0, \widetilde{Z_0} \in \Lambda_1^C$ have quasi-generic loops Γ_0 and $\widetilde{\Gamma_0}$ at fold-regular singularities $p_0, \widetilde{p_0} \in \Sigma$ of type (S), respectively. If Z_0 and $\widetilde{Z}_0$ are weakly topologically equivalent at $(\Gamma_0, \widetilde{\Gamma_0})$, then*

$$\mathcal{W}(Z_0) = \mathcal{W}(\widetilde{Z_0}).$$

A direct consequence of Theorem D is given in the next corollary.

Corollary 5.3.5. *If $Z_0 \in \Lambda_1^C$ satisfies (S), then Z_0 has ∞-moduli of weak-stability in Λ_1^C. It means that there are infinitely many Filippov systems $Z_n \in \Lambda_1^C$, $n \in \mathbb{N}$, such that Z_{n_1} and Z_{n_2} are not weakly topologically equivalent, for every $n_1, n_2 \geq 0$ and $n_1 \neq n_2$.*

5.4 Structure of a homoclinic-like loop

In this section, we characterize the first return map $\mathcal{P}_0$ and the fold line map ψ_0 associated to a homoclinic-like loop Γ_0 of a system $Z_0 \in \Omega^r$. Furthermore, given a small solid torus $\mathcal{A}_0$ around Γ_0 and a vector field Z sufficiently near to Z_0, we associate a first return map $\mathcal{P}_Z$ and a fold line map ψ_Z which describe the dynamics of Z inside $\mathcal{A}_0$.

Let $Z_0 = (X_0, Y_0) \in \Omega^r$ satisfying **(G)**, **(T)** and **(H)**. In order to characterize the first return map $\mathcal{P}_0$ given in (5.2.1), we shall write it as

$$\mathcal{P}_0 = \mathcal{D}_0 \circ \mathcal{T}_0,$$

where $\mathcal{D}_0$ is a diffeomorphism and $\mathcal{T}_0$ is a $\mathcal{C}^r$ map describing the trajectories around a fold-regular singularity. We refer $\mathcal{T}_0$ as the **transition map** of Z_0 at the fold-regular singularity p_0 (see Section 2.3.1 for a planar version of transition maps).

In Section 5.4.1, we construct and characterize the transition map $\mathcal{T}_0$. In Section 5.4.2, we describe the complete first return map $\mathcal{P}_0$. Finally, in Section 5.4.3, we characterize the fold line map ψ_0.

5.4.1 Transition Map

Without loss of generality, assume that p_0 is a fold point of X_0 and a regular point of Y_0. In this case, the transition map will depend only on the smooth vector field X_0.

Since p_0 is a visible fold-regular singularity of Z_0, it follows from Proposition 4.6.1 that there exist $a_0 < 0 < b_0$, and neighborhoods $\mathcal{V}_0$ of Z_0 in Ω^r and V_0 of p_0 in Σ such that:

i) V_0 is compact;

ii) each $Z \in \mathcal{V}_0$ has a curve $\gamma_Z : [a_0, b_0] \to V_0$, composed by visible fold-regular singularities of Z;

iii) $V_0 \setminus \mathrm{Im}(\gamma_Z)$ has only regular-regular points of Z;

iv) γ_Z intersects V_0 transversally at $\gamma_Z(a_0)$ and $\gamma_Z(b_0)$;

v) $\gamma_Z(t) \in \mathrm{int}(V_0)$ for each $t \in (a_0, b_0)$;

vi) $\gamma_{Z_0}(0) = p_0$.

From Vishik's Normal Form Theorem (see Theorem 3.3.5), there exist neighborhoods $U_0 \subset \mathbb{R}^3$ of p_0 and $W_0 \subset \mathbb{R}^3$ of the origin such that $V_0 \subset U_0$, and a local coordinate system $(x, y, z) : (U_0, p_0) \to (W_0, 0)$ such that $f(x, y, z) = z$ and X_0 is given by

$$X_0(x, y, z) = (0, 1, y).$$

We denote the set V_0 in the coordinates (x, y, z) by $\widetilde{V_0}$. Notice that $\mathrm{Im}(\gamma_{Z_0})$ coincides with a segment of the x-axis in the plane $z = 0$ containing the origin, and the flow of X_0 is given by

$$\varphi_{X_0}(t; x, y, z) = \left(x, y + t, z + \frac{(y+t)^2}{2} - \frac{y^2}{2}\right).$$

Given $\varepsilon > 0$ sufficiently small, let τ be a local transversal section of X_0 at $p^*_\varepsilon = (0, \sqrt{2\varepsilon}, \varepsilon)$ contained in the plane $z = \varepsilon$ and notice that the origin is connected to p^*_ε through an orbit of X_0. From the Implicit Function Theorem for Banach Spaces, we reduce $\mathcal{V}_0$ such that, for each $Z = (X, Y) \in \mathcal{V}_0$, a point $(x, y, 0) \in \widetilde{V_0}$ reaches τ through the flow of X for a positive time $t(X; x, y)$.

Therefore, given $Z = (X, Y) \in \mathcal{V}_0$, we define the **full transition map** $\mathcal{T}_Z : \widetilde{V_0} \to \tau$ of Z by

$$\mathcal{T}_Z(x, y, 0) = \varphi_X(t(X; x, y); x, y, 0), \tag{5.4.1}$$

and notice that the dependence of $\mathcal{T}_Z$ on Z is of class $\mathcal{C}^r$. See Figure 5.11.

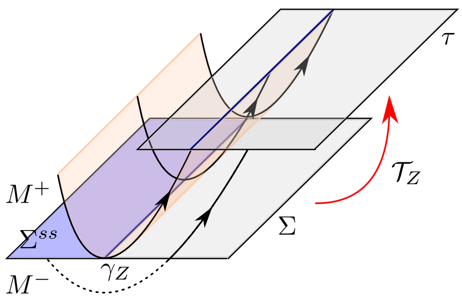


Figure 5.11: Full transition map $\mathcal{T}_Z$ for a vector field Z near Z_0.

Using the expression of the flow of X_0, an easy computation allows us to see that $\mathcal{T}_0 := \mathcal{T}_{Z_0}$ is given by

$$\mathcal{T}_0(x, y, 0) = (x, \sqrt{y^2 + 2\varepsilon}, \varepsilon).$$

Finally, for each $Z = (X, Y) \in \mathcal{V}_0$, we construct a finite cover of $\mathrm{Im}(\gamma_Z)$ by domains of Vishik's coordinate system (due to the compactness of γ_Z), to see that the orbit of X connecting $p \in \widetilde{V_0}$ and a point of τ is contained in M^+ if, and only if, $p \in \widetilde{V_0} \cap \overline{\Sigma^c}$. Therefore, $\mathcal{T}_Z$ describes the real behavior of the trajectories of Z between Σ and τ only in the domain

$$\sigma_Z = \widetilde{V_0} \cap \overline{\Sigma^c}.$$

Accordingly, we define the **transition map** of Z as $T_Z = \mathcal{T}_Z|_{\sigma_Z}$.

Notice that T_Z is a homeomorphism onto its image and $\widetilde{\mathcal{T}_Z}$ is a natural $\mathcal{C}^r$ extension of T_Z to V_0 induced by the setting of the problem. Nevertheless, $\mathcal{T}_Z$ is a non-invertible map.

5.4.2 First Return Map

Consider the coordinate system and the local transversal section τ introduced in Section 5.4.1, and recall that Y_0 is transverse to each point of $\widetilde{V_0}$ and X_0 is transverse to τ. From conditions **(G)**, **(T)**, and **(H)**, it follows from the Implicit Function Theorem that, for each $Z \in \mathcal{V}_0$ (reduce $\mathcal{V}_0$ if necessary), there exists a diffeomorphism $\mathcal{D}_Z : \tau \to \Sigma$ onto its image induced by regular orbits of Z. In particular, denoting $\mathcal{D}_0 = \mathcal{D}_{Z_0}$, we obtain

$$\mathcal{D}_0(0, \sqrt{\varepsilon}, \varepsilon) = (0, 0, 0).$$

We define the **full first return map** $\mathcal{P}_Z : \widetilde{V_0} \to \Sigma$ of $Z \in \mathcal{V}_0$ as

$$\mathcal{P}_Z(x, y, 0) = \mathcal{D}_Z \circ \mathcal{T}_Z(x, y, 0), \tag{5.4.2}$$

where $\mathcal{T}_Z$ is the full transition map of Z given in (5.4.1). Accordingly, the **first return map** of Z is defined by $P_Z = \mathcal{D}_Z \circ T_Z$, where T_Z is the transition map of Z.

If $p \in \sigma_Z$, then p and $\mathcal{P}_Z(p)$ are connected by a trajectory of Z, nevertheless, if $p \in \widetilde{V_0} \setminus \sigma_Z$, then p and $\mathcal{P}_Z(p)$ are related by a virtual trajectory of Z. It follows that P_Z is a $\mathcal{C}^r$ homeomorphism (onto its image) which completely describes the crossing dynamics of Z inside the torus $\mathcal{A}_0$ generated by Γ_0 and $\widetilde{V_0}$.

Notice that both P_Z and $\mathcal{P}_Z$ have a $\mathcal{C}^r$ dependence on Z. Also, $\mathcal{P}_Z$ is a non-invertible map which is a $\mathcal{C}^r$ extension of P_Z to $\widetilde{V_0}$. In particular, the origin is a fixed point of $\mathcal{P}_0 = \mathcal{P}_{Z_0}$, corresponding to the homoclinic-like loop Γ_0 of Z_0.

5.4.3 Fold Line Map

Finally, we characterize the fold line map ψ_0 of Z_0 induced by the sliding dynamics. In addition, we construct this map for every $Z \in \Omega^r$ sufficiently near Z_0. Consider the same notation used above in Section 5.4.

Denote the fold line γ_Z of Z by S_Z. Since $S_{Z_0} \cap \widetilde{V_0}$ is composed by fold-regular singularities of Z_0, it follows from Lemma 3.3.11 that, reducing $\widetilde{V_0}$ if necessary, the sliding vector field F_{Z_0} is extended onto $\widetilde{V_0}$, and it is transverse to S_{Z_0} at p_0. Define the $\mathcal{C}^r$ map $\mathcal{G} : \mathcal{V}_0 \times \widetilde{V_0} \times \mathbb{R} \to \Sigma$ given by

$$\mathcal{G}(Z, (x, y, 0), s) = Xf(\varphi_{F_Z}(s; x, y, 0)),$$

where $Z = (X, Y)$. Since $S_{Z_0} = X_0 f^{-1}(0)$, it follows that

$$\mathcal{G}(Z_0, (0, 0, 0), 0) = X_0 f(0, 0, 0) = 0 \quad \text{and} \quad \partial_s \mathcal{G}(Z_0, (0, 0, 0), 0) = dX_0 f(0, 0, 0) \cdot F_{Z_0}(0, 0, 0) \neq 0.$$

From the Implicit Function Theorem, reducing $\widetilde{V_0}$ and $\mathcal{V}_0$ if necessary, there exists a unique $\mathcal{C}^r$ function $s^* : \mathcal{V}_0 \times \widetilde{V_0} \to \mathbb{R}$ such that $\mathcal{G}(Z, (x, y, 0), s^*(Z, (x, y, 0))) = 0$.

Consider the full first return map $\mathcal{P}_Z : \widetilde{V_0} \to \Sigma$ given by (5.4.2). Now, for a sufficiently small neighborhood $\widetilde{V_1}$ of $(0, 0, 0)$ contained in $\widetilde{V_0}$ and reducing $\mathcal{V}_0$ if necessary, we define the **full fold line map** $\Psi_Z : S_Z \cap \widetilde{V_1} \to S_Z \cap \widetilde{V_0}$ by

$$\Psi_Z(p) = \varphi_{F_Z}(s^*(Z, \mathcal{P}_Z(p)); \mathcal{P}_Z(p)), \tag{5.4.3}$$

for each $Z \in \mathcal{V}_0$.

In order to analyze the dynamics encoded by the full fold line map, it is convenient to restrict it to the following domain

$$\sigma_Z^{FL} = \mathcal{P}_Z^{-1}(\mathcal{P}_Z(S_Z \cap \widetilde{V_1}) \cap \Sigma^s). \tag{5.4.4}$$

Accordingly, we define the **fold line map** as $\psi_Z = \Psi_Z|_{\sigma_Z^{FL}}$. Notice that, $(0, 0, 0)$ is a fixed point of $\psi_0 = \psi_{Z_0}$, and Ψ_Z is a $\mathcal{C}^r$ extension of ψ_Z.

Remark 5.4.1. *Consider a map* $\mathcal{H} : \mathcal{V}_0 \times [a_0, b_0] \to \widetilde{V_0}$ *such that, for each* $Z \in \mathcal{V}_0$, $\mathcal{H}_Z := \mathcal{H}(Z, \cdot) : [a_0, b_0] \to S_Z \cap \widetilde{V_0}$ *is a diffeormorphism onto its image in such a way that, for some* $a_0 < a_1 < 0 < b_1 < b_0$, $\mathcal{H}_Z|_{[a_1,b_1]}$ *parameterizes* $S_Z \cap \widetilde{V_1}$. *Therefore,* $\mathcal{H}_Z^{-1} \circ \Psi_Z \circ \mathcal{H}_Z : [a_1, b_1] \to [a_0, b_0]$ *is a family of real diffeomorphisms (onto their image) which is of class* $\mathcal{C}^r$ *on* Z*. Therefore, if* p_0 *is a hyperbolic fixed point of* ψ_0*, we can use such parameterizations to see that, reducing* $\mathcal{V}_0$ *if necessary, the full fold line map* Ψ_Z *has a unique hyperbolic fixed point (with the same type) in* $S_Z \cap \widetilde{V_0}$*, for each* $Z \in \mathcal{V}_0$*.*

5.4.4 Properties

In what follows, we use the full transition map $\mathcal{P}_0$ and the full fold line map Ψ_0 to characterize $P_0 = P_{Z_0}$ and $\psi_0 = \psi_{Z_0}$. We consider the coordinate system (x, y, z) at p_0 as in Section 5.4.1, and from now on, we identify the points $(x, y, 0) \in \Sigma$ and $(x, y, \varepsilon) \in \tau$ with (x, y). Also, consider the neighborhoods $\widetilde{V_1}$ and $\mathcal{V}_0$ of $(0, 0, 0)$ and Z_0 given in Section 5.4.3, respectively.

Lemma 5.4.2. *Given* $Z_0 \in \Omega^r$ *satisfying conditions **(G)**, **(H)**, and **(T)**, there exist real constants* $\alpha_{i,j}, \beta_{i,j} \in \mathbb{R}$, $i = 0, 1, 2$ *and* $j = 0, 1$ *such that the Taylor expansion of the full first return map* $\mathcal{P}_0$ *of* Z_0 *at the origin is given by*

$$\mathcal{P}_0(x, y) = \begin{pmatrix} \alpha_{1,0}x + \alpha_{0,1}y^2 + \alpha_{2,0}x^2 + \alpha_{1,1}xy^2 + \mathcal{O}(x^3, x^2y^2, y^4) \\ \beta_{1,0}x + \beta_{0,1}y^2 + \beta_{2,0}x^2 + \beta_{1,1}xy^2 + \mathcal{O}(x^3, x^2y^2, y^4) \end{pmatrix}. \tag{5.4.5}$$

Furthermore, the following statements hold.

i) $d = \alpha_{1,0}\beta_{0,1} - \alpha_{0,1}\beta_{1,0} \neq 0$*;*

ii) $sgn(d) = sgn(J\mathcal{D}_0(0, \sqrt{2\varepsilon}))$*, where* $\mathcal{D}_0 : \tau \to \Sigma$ *is the diffeomorphism induced by the flow of* Z_0 *and* $J\mathcal{D}_0$ *denotes the Jacobian of* $\mathcal{D}_0$*;*

iii) If F_{Z_0} *is transverse to* $\mathcal{P}_0(S_{Z_0} \cap \widetilde{V_1})$ *at the origin, then* $\alpha_{1,0} \neq 0$*.*

Proof. Since $\mathcal{D}_0$ is a diffeomorphism such that $\mathcal{D}_0(\sqrt{2\varepsilon}, 0) = (0, 0)$, it follows that,

$$\mathcal{D}_0(x, y) = \begin{pmatrix} a_{1,0}x + a_{0,1}(y - \sqrt{2\varepsilon}) + a_{2,0}x^2 + a_{1,1}x(y - \sqrt{2\varepsilon}) + a_{0,2}(y - \sqrt{2\varepsilon})^2 + \mathcal{O}_3(x, y - \sqrt{2\varepsilon}) \\ b_{1,0}x + b_{0,1}(y - \sqrt{2\varepsilon}) + b_{2,0}x^2 + b_{1,1}x(y - \sqrt{2\varepsilon}) + b_{0,2}(y - \sqrt{2\varepsilon})^2 + \mathcal{O}_3(x, y - \sqrt{2\varepsilon}) \end{pmatrix},$$

where $a_{i,j}, b_{i,k} \in \mathbb{R}$ are constants satisfying $a_{1,0}b_{0,1} - a_{0,1}b_{1,0} \neq 0$. Also, using the expression of $\mathcal{T}_0$ given in (5.4.1), it follows that

$$\mathcal{T}_0(x, y) = \begin{pmatrix} x \\ \sqrt{2\varepsilon} + Ky^2 + \mathcal{O}_4(y) \end{pmatrix},$$

where $K > 0$. Straightforwardly, we obtain (5.4.5) and prove items (i) and (ii).

Finally, assume that F_{Z_0} is transverse to S_{Z_0} at the origin. Denoting $Y_0(x, y, z) = (f_1, f_2, f_3)$ in this coordinate system, where $f_i = f_i(x, y, z)$, $i = 1, 2, 3$, we obtain

$$Y_0 f(x, y, z) = f_3(x, y, z).$$

Recalling that $f(x,y,z)=z$ and $X_0(x,y,z)=(0,1,y)$, we have that the correspondent sliding vector field is expressed as

$$\begin{aligned} F_{Z_0}(x,y) &= \frac{Y_0f(x,y,0)X_0(x,y,0)-X_0f(x,y,0)Y_0(x,y,0)}{Y_0f(x,y,0)-X_0f(x,y,0)} \\ &= \frac{f_3(x,y,0)(0,1,y)-y(f_1(x,y,0),f_2(x,y,0),f_3(x,y,0))}{f_3(x,y,0)-y} \\ &= \left(\frac{-yf_1(x,y,0)}{f_3(x,y,0)-y},\frac{f_3(x,y,0)-yf_2(x,y,0)}{f_3(x,y,0)-y}\right). \end{aligned}$$

Since Y_0 is transverse to Σ at p_0, it follows that $a_0=f_3(0,0,0)\neq 0$, and consequently, $F_{Z_0}(0,0)=(0,1)$.

Now, notice that $S_{Z_0}=\{(x,0);\ x\in(-\varepsilon,\varepsilon)\}$ and therefore

$$\zeta_0=\mathcal{P}_0(S_{Z_0}\cap\widetilde{V_1})=\{(\alpha_{1,0}x+\mathcal{O}_2(x),\beta_{1,0}x+\mathcal{O}_2(x));\ x\in(-\varepsilon,\varepsilon)\}, \tag{5.4.6}$$

for some $\varepsilon>0$. It follows that $T_{\mathbf{0}}\zeta_0=\text{span}\{(\alpha_{1,0},\beta_{1,0})\}$, and since F_{Z_0} is transverse to ζ_0 at the origin, we obtain that $\alpha_{1,0}\neq 0$. □

Remark 5.4.3. *Notice that $\mathcal{P}_0(S_{Z_0}\cap\widetilde{V_1})$ coincides with the curve ζ_0 given in Section 5.3.*

The proof of the following lemma is straightforward and will be omitted.

Lemma 5.4.4. *Consider the same hypotheses of Lemma 5.4.2 and assume that $\alpha_{1,0}\neq 0$. Then, the local change of coordinates at the origin of the plane Σ given by*

$$\begin{cases} u=x-\dfrac{\alpha_{0,1}}{\alpha_{1,0}}y^2, \\ v=y, \end{cases}$$

brings the full first return map $\mathcal{P}_0$ into

$$\overline{\mathcal{P}_0}(u,v)=\begin{pmatrix} \alpha_{1,0}u \\ \beta_{1,0}u+\dfrac{d}{\alpha_{1,0}}v^2 \end{pmatrix}+u^2A_1(u,v)+uv^2A_2(u,v)+v^4A_3(u,v),$$

where $A_i(u,v)$ are bounded vector-valued functions.

Notice that, the change of coordinates exhibited in Lemma 5.4.4 does not modify the structure of the problem in the coordinate system (x,y). In fact, the tangency set of Z_0 remains fixed through this change of coordinates and it is expressed as $S_{Z_0}=\{(u,0);\ u\in(-\varepsilon,\varepsilon)\}$, for some $\varepsilon>0$ sufficiently small. For the sake of simplicity, we make no difference between the coordinates (u,v) and (x,y) and so $\mathcal{P}_0$ writes as

$$\mathcal{P}_0(x,y)=\begin{pmatrix} \alpha x \\ bx+cy^2 \end{pmatrix}+x^2A_1(x,y)+xy^2A_2(x,y)+y^4A_3(x,y), \tag{5.4.7}$$

where $\alpha=\alpha_{1,0}$, $b=\beta_{1,0}$, $c=\dfrac{d}{\alpha_{1,0}}$, and A_i are bounded vector-valued functions, $i=1,2,3$.

Lemma 5.4.5. *Let $Z_0\in\Lambda_1$. Consider the full fold line map Ψ_0 and the full first return map $\mathcal{P}_0$ of Z_0 given by* (5.4.3) *and* (5.4.7)*, respectively. The following statements hold:*

i) $\alpha \neq 0$, $|\alpha| \neq 1$, $b \neq 0$ *and* $c \neq 0$;

ii) $\Psi_0(x,0) = (\alpha x + \mathcal{O}(x^2), 0)$, *for* x *small;*

iii) the origin is a hyperbolic fixed point of $\mathcal{P}_0$ *with real eigenvalues* 0 *and* α;

iv) the eigenspaces of $\mathcal{P}_0$ *corresponding to the eigenvalues* 0 *and* α *are given by* $\mathcal{E}_0 = span\{(0,1)\}$ *and* $\mathcal{E}_\alpha = span\{(\alpha, b)\}$, *respectively.*

Proof. First, notice that items (iii) and (iv) follows straightly from item (i) and the expression of $\mathcal{P}_0$ given in (5.4.7). Now, we prove items (i) and (ii). Since $Z_0 \in \Lambda_1$, it follows from Lemma 5.4.2 that $\alpha \neq 0$ and $c \neq 0$.

From (5.4.6) (with $\alpha_{1,0} = \alpha$ and $\beta_{1,0} = b$), we deduce that $T_{\mathbf{0}}\gamma_0 = \text{span}\{(1,0)\}$ and $T_{\mathbf{0}}\zeta_0 = \text{span}\{(\alpha, b)\}$, where $\gamma_0 = S_{Z_0} \cap \widetilde{V_1}$ and $\zeta_0 = \mathcal{P}_0(\gamma_0)$. From hypothesis **(T)**, we have that $\gamma_0 \pitchfork \zeta_0$ at the origin. It implies that the vectors $(1,0)$ and (α, b) are linearly independent. Hence, $b \neq 0$.

Now, from the computations done in the proof of Lemma 5.4.2, we derive that

$$F_{Z_0}(x,y) = (0,1) + (F_1, F_2),$$

where $F_1, F_2 = \mathcal{O}_1(x,y)$. Denoting $\varphi_{F_{Z_0}} = (\varphi_1, \varphi_2)$, we have that:

$$\begin{cases} \varphi_1(t; x, y) = x + F_1(x,y)t + \mathcal{O}_2(t), \\ \varphi_2(t; x, y) = y + (1 + F2(x,y))t + \mathcal{O}_2(t), \end{cases}$$

for t, x, y small enough.

Now, $\varphi_2(0;0,0) = 0$ and $\partial_t \varphi_2(0;0,0) = 1$. Thus, we use the Implicit Function Theorem to obtain a unique $\mathcal{C}^r$ function $t^*(x,y)$ such that $t^*(0,0) = 0$ and $\varphi_2(t^*(x,y); x, y) = 0$, for (x,y) small enough, with $t^*(0,0) = 0$. Also, we have that $\partial_x t^*(0,0) = 0$ and $\partial_y t^*(0,0) = -1$. Thus,

$$t^*(x,y) = -y + \mathcal{O}_2(x,y).$$

Notice that, $\gamma_0 = \{(x,0);\ x \in (-\varepsilon, \varepsilon)\}$, for $\varepsilon > 0$ sufficiently small. Therefore, the full fold line map $\Psi_0 : \gamma_0 \to \gamma_0$ writes as

$$\Psi_0(x,0) = (\varphi_1(t^*(\mathcal{P}_0(x,0)); \mathcal{P}_0(x,0)), 0).$$

Hence, it is straightforward to check that

$$\Psi_0(x,0) = (\alpha x + \mathcal{O}_2(x), 0).$$

Since $Z_0 \in \Lambda_1$, we conclude that the full fold line map Ψ_0 of Z_0 has a hyperbolic fixed point at the origin. Therefore $|\alpha| \neq 1$. □

Remark 5.4.6. *Notice that the curve* ζ_0 *is tangent to the eigenspace* $\mathcal{E}_\alpha$ *at the origin. So, it is an intrinsic degeneracy of this problem which can not be avoided.*

Using Lemma 5.4.5, we can apply some near-identity transformations to express the map $\mathcal{P}_0$ given by (5.4.7) in a more accurate normal form.

Proposition 5.4.7. *There exists a change of coordinates* $\eta : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *such that*

$$\widetilde{\mathcal{P}_0}(x,y) = \eta \circ \mathcal{P}_0 \circ \eta^{-1}(x,y) = \begin{pmatrix} \alpha x - c\alpha y^2 + cx^2 + \mathcal{O}_3(x,y) \\ x \end{pmatrix}. \tag{5.4.8}$$

In addition, $\widetilde{\mathcal{P}_0}$ *is symmetric with respect to an involution* $\mathcal{I}$ *such that*

$$Fix(\mathcal{I}) = \left\{(x,y);\ y = \frac{B}{b}x^2 + \mathcal{O}_3(x)\right\},$$

where

$$B = \frac{b\partial_x^2\pi_1 \circ \mathcal{P}_0(0,0) + \alpha(\alpha-1)\partial_x^2\pi_2 \circ \mathcal{P}_0(0,0)}{\alpha^3(\alpha-1)}. \tag{5.4.9}$$

Proof. First, we consider the change of coordinates

$$\eta_1(x,y) = \begin{pmatrix} x - Ax^2 + \mathcal{O}_3(x,y) \\ y - Bx^2 + \mathcal{O}_3(x,y) \end{pmatrix},$$

such that

$$\eta_1^{-1}(x,y) = \begin{pmatrix} x + Ax^2 \\ y + Bx^2 \end{pmatrix},$$

with B given by (5.4.9) and $A = \partial_x^2\pi_1 \circ \mathcal{P}_0(0,0)(\alpha(\alpha-1))^{-1}$. Thus, using that $\mathcal{P}_0$ is given by (5.4.7), we obtain

$$\eta_1 \circ \mathcal{P}_0 \circ \eta_1^{-1}(x,y) = \begin{pmatrix} \alpha x + G_1(x,y) \\ bx + cy^2 + G_2(x,y) \end{pmatrix},$$

where $G_1(x,y) = \mathcal{O}_3(x,y)$ and $G_2(x,y) = \mathcal{O}_3(x,y)$, and $\eta_1 \circ \mathcal{P}_0 \circ \eta_1^{-1}$ is symmetric with respect to the symmetry $\mathcal{I}_1(x,y) = (x, -y - 2Bx^2)$, which has the following set of fixed points

$$\text{Fix}(\mathcal{I}_1) = \left\{(x,y);\ y = Bx^2\right\}.$$

Now, considering the change of coordinates

$$\eta_2(x,y) = \begin{pmatrix} bx + G_2(x,y) + cy^2 \\ y \end{pmatrix},$$

and taking $\eta = \eta_2 \circ \eta_1$, the proof follows directly. □

Remark 5.4.8. *Notice that the change of coordinates* η *provided by Proposition 5.4.5 carries the fold line* S_{Z_0} *of* Z_0 *onto the set Fix*$(\mathcal{I})$*.*

The next result follows straightly from Lemma 5.4.5 and the Stable Manifold Theorem for $\mathcal{C}^r$ maps (see Theorem 10.1 in [91]).

Proposition 5.4.9. *Let* $Z_0 \in \Lambda_1$*, and consider the non-invertible full first return map* $\mathcal{P}_0$ *of* Z_0 *given by* (5.4.7)*. Therefore,* $\mathcal{P}_0$ *has a local stable invariant manifold* W_0^s *at the origin tangent to* $\mathcal{E}_0$ *and either one of the following statements hold.*

1. *If* $|\alpha| < 1$*, then* $\mathcal{P}_0$ *has a fixed point of nodal type at the origin and it has a local stable invariant manifold* W_α^s *at the origin tangent to* $\mathcal{E}_\alpha$ *(see Figure 5.12 (ii) and (iv).).*

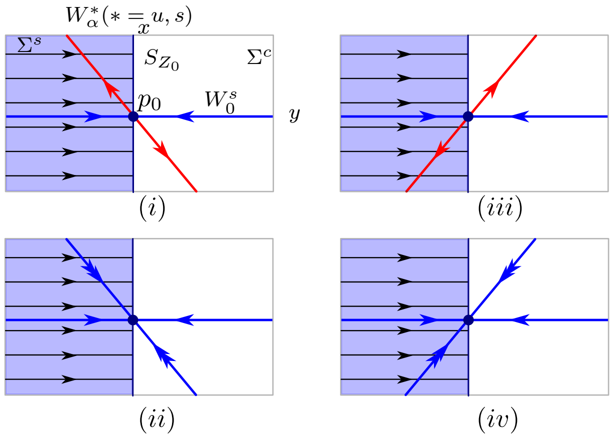


Figure 5.12: Configurations of the local invariant manifolds of $\mathcal{P}_0$ at p_0 for $b < 0$, (i) $\alpha > 1$, (ii) $0 < \alpha < 1$, (iii) $\alpha < -1$ and (iv) $-1 < \alpha < 0$. If $b > 0$, then (i) is switched by (iii) such as (ii) and (iv).

2. If $|\alpha| > 1$, then $\mathcal{P}_0$ has a fixed point of saddle type at the origin and it has a local unstable invariant manifold W^u_α at the origin tangent to $\mathcal{E}_\alpha$ (see Figure 5.12 (i) and (iii).).

Finally, we characterize the classes Λ_1^C and Λ_1^M of Λ_1 and the hypotheses (N) and (S) introduced in Section 5.3, which generate four possible types of quasi-generic loop Γ_0 passing through a fold-regular singularity p_0 of $Z_0 \in \Lambda_1$.

Proposition 5.4.10. *Let $Z_0 \in \Lambda_1$, and consider the full fold line map Ψ_0 of Z_0 given by* (5.4.3)*. The following statements hold:*

i) $Z_0 \in \Lambda_1^C$ and satisfies (S) if, and only if, $\alpha > 1$;

ii) $Z_0 \in \Lambda_1^C$ and satisfies (N) if, and only if, $0 < \alpha < 1$;

iii) $Z_0 \in \Lambda_1^M$ and satisfies (S) if, and only if, $\alpha < -1$;

iv) $Z_0 \in \Lambda_1^M$ and satisfies (N) if, and only if, $-1 < \alpha < 0$.

Proof. From Lemma 5.4.5, the full fold line map Ψ_0 of Z_0 writes as $\Psi_0(x, 0) = (\alpha x + \mathcal{O}(x^2), 0)$. In this case, the map Ψ_0 preserves the connected components $(-\varepsilon, 0) \times \{0\}$ and $(0, \varepsilon) \times \{0\}$ of S_{Z_0} if, and only if, $\alpha > 0$. The result follows from Proposition 5.4.9. $\square$

Remark 5.4.11. *Notice that, the geometry of this problem allows us to see that the first return map P_0 preserves the orientation of the y-axis, nevertheless the orientation of the x-axis is reversed if $\alpha < 0$, and it is preserved if $\alpha > 0$. Therefore, P_0 preserves orientation if, and only if, $Z_0 \in \Lambda_1^C$.*

Since the transition map T_0 does not provide any changes in the orientation of $\overline{\Sigma^c}$, it follows that $P_0 = \mathcal{D}_0 \circ T_0$ preserves orientation if, and only if, $\mathcal{D}_0$ preserves orientation. Hence, if $Z_0 \in \Lambda_1^C$, it follows from (5.4.5), (5.4.7), *and Proposition 5.4.10 that $c > 0$.*

As mentioned in Section 5.3, if $Z_0 \in \Lambda_1^C$, then the fold line map ψ_0 defines a dynamics in $\sigma^{FL}_{Z_0}$ (which is an interval of the x-axis) induced by the orbits of Z_0.

Now, let $Z_0 \in \Lambda_1^M$, and without loss of generality, assume that $b < 0$ in (5.4.7). From the proof of Lemma 5.4.5, we have that the map $\psi_0^* : V_0 \to \gamma_0 = S_{Z_0} \cap \widetilde{V_1}$ induced by the flow of F_{Z_0} is given by $\psi_0^*(x,y) = (x + \mathcal{O}_2(x,y), 0)$, and notice that, in this coordinate system, $\Sigma^s = \widetilde{V_1} \cap \{y < 0\}$ and $S_{Z_0} = \{y = 0\}$. From Proposition 5.4.10, we have that $\alpha < 0$ and thus $\mathcal{P}_0(x,0) = (\alpha x, bx) + \mathcal{O}(x^2) \in \Sigma^s$ if, and only if $x \geq 0$.

Given $x > 0$ small, we have that $\Psi_0(x,0) = \psi_0^* \circ \mathcal{P}_0(x,0)$ and $(x,0)$ are connected by orbits of Z_0. Now,

$$\mathcal{P}_0(\Psi_0(x,0)) = (\alpha^2 x, b\alpha x) + \mathcal{O}(x^2)$$

does not belong to Σ^s since $b\alpha x > 0$, therefore the points $\Psi_0^2(x,0)$ and $(x,0)$ are not connected by orbits of Z_0, and thus the iterations of Ψ_0 do not describe the dynamics of Z_0. In other words, the fold line map ψ_0 (which is the restriction of Ψ_0 to $x \geq 0$) does not induce any dynamics in the interval $\widetilde{V_1} \cap \{x \geq 0, y = 0\}$.

Nevertheless, given $x < 0$, we have that

$$\mathcal{P}_0^2((x,0)) = (\alpha^2 x, b\alpha x) + \mathcal{O}(x^2) \in \Sigma^s,$$

and hence $\psi_0^* \circ \mathcal{P}_0^2(x,0)$ and $(x,0)$ are connected by orbits of Z_0 (with a unique segment of sliding orbit). Therefore, we define the **full Möbius fold line map** $\Psi_0^M : \gamma_0 \to \gamma_0$ of Z_0 as

$$\Psi_0^M(x,0) = \psi_0^* \circ \mathcal{P}_0^2(x,0) = (\alpha^2 x + \mathcal{O}(x^2), 0),$$

and the domain

$$\sigma_{Z_0}^M = \mathcal{P}_0^{-1}(\mathcal{P}_0(S_0 \cap \widetilde{V_1}) \cap \Sigma^c) = \{x \leq 0\} \times \{0\} \cap \widetilde{V_1}.$$

Accordingly, we define the **Möbius fold line map** of Z_0 as $\psi_0^M = \Psi_0^M|_{\sigma_{Z_0}^M}$. We conclude that $\psi_0^M(\sigma_{Z_0}^M) \subset \sigma_{Z_0}^M$ and thus, this map defines a dynamical system in $\sigma_{Z_0}^M$ induced by the (real) orbits of Z_0, whether $Z_0 \in \Lambda_1^M$.

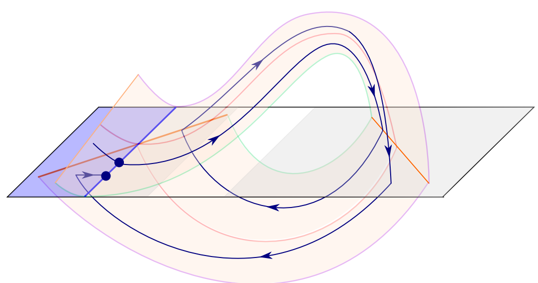


Figure 5.13: Action of the Möbius fold line map ψ_0^M of $Z_0 \in \Lambda_1^M$.

Remark 5.4.12. *Notice that, if $Z_0 \in \Lambda_1^M$, we can still define the Möbius fold line map ψ_Z^M for every Z sufficiently near Z_0, combining the ideas above with Section 5.4.1. Also, the origin is a hyperbolic fixed point of ψ_0^M if, and only if, it is a hyperbolic fixed point of the fold line map ψ_0 of Z_0.*

5.5 An illustration of Theorem M

In this section, we provide an example of Filippov system presenting a quasi-generic loop at a fold-regular singularity. In addition, we present two unfoldings of this connection, which illustrate the results stated in Theorem M.

Proposition 5.5.1. *Given $b \neq 0$ and $\alpha \in \mathbb{R}$ such that $|\alpha| \neq 1$ and $-\alpha/(1-\alpha) \notin [0,1]$, consider the Filippov system $Z_0 = (X_0, Y_0)$ with switching manifold $\Sigma = \{(x,y,z) \in \mathbb{R}^3;\ z = 0\}$, where X_0 is given by*

$$X_0(x,y,z) \quad = \quad \begin{pmatrix} 0 \\ 1 \\ y(2-3y) \end{pmatrix}, \tag{5.5.1}$$

and Y_0 is given by

$$Y_0(x,y,z) \quad = \quad \begin{pmatrix} -(1-\alpha)x + \dfrac{\beta(1-\alpha)x^2}{\alpha + (1-\beta)y} \\ -1 + \dfrac{\beta x}{\alpha + (1-\alpha)y} \\ 1 - 2\left(\dfrac{y(\alpha+(1-\alpha)y) - \beta x}{\alpha + (1-\alpha)y - \beta x}\right) \end{pmatrix}. \tag{5.5.2}$$

Therefore, $Z_0 \in \Lambda_1$, and the following statements hold.

i) Z_0 has a quasi-generic loop Γ_0 at the fold-regular singularity $(0,0,0)$, which is contained in the plane $x = 0$;

ii) X_0, Y_0 are vector fields of class $\mathcal{C}^\infty$ around Γ_0;

iii) In the plane $x = 0$, Z_0 coincides with the Filippov system $Z_0^* = (X_0, Y_0^*)$, where $Y_0^*(x,y,z) = (0,-1,1-2y)$;

iv) The fold line map of Z_0 is given by $\psi_{Z_0}(x) = \alpha x + \mathcal{O}(x^2)$

Proof. The flow of X_0 is given by

$$\varphi_{X_0}(t;x,y,z) = \begin{pmatrix} x \\ t+y \\ -(t+y)^3 + (t+y)^2 - y^2(1-y) + z \end{pmatrix}. \tag{5.5.3}$$

Thus, X_0 has a visible fold line S_{X_0} at the x-axis and an invisible fold-line at $y = 2/3$. Using (5.5.3), we obtain that $\varphi_{X_0}(1;x,0,0) = (x,1,0)$ and $\varphi_{X_0}(t;x,0,0) \in M^+$ for each $t \in (0,1)$. Hence, we define $\widetilde{S_{X_0}} = \{y = 1, z = 0\}$ and $\Pi_{X_0} : S_{X_0} \to \widetilde{S_{X_0}}$ by

$$\Pi_{X_0}(x,0,0) = (x,1,0).$$

Now, the vector field Y_0^* has flow

$$\varphi_{Y_0^*}(t;x,y,z) = \begin{pmatrix} x \\ -t+y \\ (-t+y)^2 + t - y^2 + z \end{pmatrix}.$$

A simple computation shows that $\varphi_{Y_0^*}(1;x,1,0) = (x,0,0)$ and $\varphi_{Y_0^*}(t;x,1,0) \in M^-$, for each $t \in (0,1)$. Hence, we define $\Pi_{Y_0^*} : \widetilde{S_{X_0}} \to S_{X_0}$ by

$$\Pi_{Y_0^*}(x,1,0) = (x,0,0).$$

Notice that $Z_0^* = (X_0, Y_0^*)$ has a family of homoclinic-like loops passing through points of S_{X_0}. Therefore, we must perform a slight change in Y_0^* in order to avoid such degeneracy.

Consider the map $M_1 : \mathbb{R}^3 \to \mathbb{R}^3$

$$M_1(x, y, z) = \begin{pmatrix} x \\ y - bx(y-1) \\ z \end{pmatrix},$$

with $b \neq 0$. Notice that, $|dM_1(x, y, z)| = 1 - bx$, which means that M_1 is a diffeomorphism outside the plane $x = 1/b$. Since we have a loop contained in the plane $x = 0$, it is not an obstruction for our purposes. In fact, the inverse of M_1 is given by

$$(M_1)^{-1}(x, y, z) \quad = \quad \begin{pmatrix} x \\ \dfrac{y - bx}{1 - bx} \\ z \end{pmatrix}. \tag{5.5.4}$$

Notice that $M_1(x, 1, 0) = (x, 1, 0)$ and $M_1(x, 0, 0) = (x, bx, 0)$. Thus, $M_1|_{\widetilde{S_{X_0}}} = \text{Id}$, $M_1(S_{X_0})$ is transverse to S_{X_0} at the origin and we also have that the plane $x = 0$ is M_1-invariant.

Consider

$$\overline{Y_0} = M_1^* Y_0^* = dM_1 \circ Y_0^* \circ (M_1)^{-1},$$

and recall that

$$M_1 \circ \varphi_{Y_0^*}(t; x, y, z) = \varphi_{\overline{Y_0}}(t; M_1(x, y, z)).$$

Hence, we have that $\varphi_{\overline{Y_0}}(1; x, 1, 0) = (x, bx, 0)$. Moreover, $\varphi_{\overline{Y_0}}(t; x, 1, 0) \in M^-$, for each $t \in (0, 1)$. Therefore, we conclude that the map $\Pi_{\overline{Y_0}} : \widetilde{S_{X_0}} \to \Sigma$ induced by the flow of $\overline{Y_0}$ is given by

$$\Pi_{\overline{Y_0}}(x, 1, 0) = (x, bx, 0).$$

Now, we notice that $\overline{Z_0} = (X, \overline{Y_0})$ has a homoclinic-like loop at the origin and the image $S_{X_0}^1 := \Pi_{\overline{Y_0}} \circ \Pi_X(S_{X_0})$ of the fold line through the orbits of $\overline{Z_0}$ is transverse to S_{X_0} at $(0, 0, 0)$. Also, we notice that $F_{\overline{Z_0}}(0, 0, 0) = (0, 1, 0)$, and thus, $F_{\overline{Z_0}}$ is transverse to $S_{X_0}^1$ at $(0, 0, 0)$. However, the fold line map of $\overline{Z_0}$ is given by $\psi_{\overline{Z_0}}(x) = x + \mathcal{O}(x^2)$, and hence the origin is not a hyperbolic fixed point of $\psi_{\overline{Z_0}}$. Therefore, $\overline{Z_0}$ satisfies conditions (i) and (ii) of Definition 5.3.1 but it does not satisfies condition (iii).

Now, consider the map $M_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$M_2(x, y, z) = \begin{pmatrix} x(\alpha + (1-\alpha)y) \\ y \\ z \end{pmatrix},$$

with $\alpha \neq 1$. Since, $|dM_2(x, y, z)| = \alpha + (1-\alpha)y$, we have that M_2 is a diffeomorphism outside the plane $y = -\alpha/(1-\alpha)$. Since our loop is contained in the plane $x = 0$ with $0 \leq y \leq 1$, choosing α such that $-\alpha/(1-\alpha) \notin [0, 1]$, we have that M_2 is a diffeomorphism around the loop of $\overline{Z_0}$.

In this case, the inverse of M_2 is given by

$$(M_2)^{-1}(x,y,z) = \begin{pmatrix} \dfrac{x}{\alpha+(1-\alpha)y} \\ y \\ z \end{pmatrix}. \tag{5.5.5}$$

Notice that $M_2(x,1,0) = (x,1,0)$ and $M_2(x,bx,0) = (\alpha x + b(1-\alpha)x^2, bx, 0)$. Now, a simple computation shows us that the vector field Y_0 given by (5.5.2) satisfies

$$Y_0 = M_2^* \overline{Y_0} = dM_2 \circ \overline{Y_0} \circ (M_2)^{-1}.$$

Hence $\varphi_{Y_0}(t;x,1,0) \in M^-$, for each $t \in (0,1)$, and the map $\Pi_{Y_0} : \widetilde{S_{X_0}} \to \Sigma$ induced by the flow of Y_0 is given by

$$\Pi_{Y_0}(x,1,0) = \varphi_{Y_0}(1;x,1,0) = (\alpha x - b(\alpha-1)x^2, bx, 0).$$

The proof follows directly from these facts. □

In the next result, we present a one-parameter family Z_γ, which unfolds Z_0 given by Proposition 5.5.1. We notice that this perturbation breaks the quasi-generic loop Γ_0 of Z_0, nevertheless the plane $x=0$ is Z_γ-invariant, for every γ.

Proposition 5.5.2. *Given $b \neq 0$ and $\alpha \in \mathbb{R}$ such that $|\alpha| \neq 1$ and $-\alpha/(1-\alpha) \notin [0,1]$, consider the one-parameter family of Filippov systems $Z_\gamma = (X_0, Y_\gamma)$, where X_0 is given by (5.5.1), $Y_\gamma = dM \circ Y_\gamma^* \circ M^{-1}$, Y_γ^* is the vector field given by*

$$Y_\gamma^*(x,y,z) = \begin{pmatrix} 0 \\ -1 \\ \gamma+1-2y \end{pmatrix},$$

and $M : \mathbb{R}^3 \to \mathbb{R}^3$ is the map given by

$$M(x,y,z) = \begin{pmatrix} x\left(\alpha + (1-\alpha)(y - bx(y-1))\right) \\ y - bx(y-1) \\ z \end{pmatrix}. \tag{5.5.6}$$

Therefore, Z_γ is an unfolding of the Filippov system Z_0 given by Proposition 5.5.1 at $\gamma = 0$, and there exists a solid torus $\mathcal{A}_0$ around the quasi-generic loop Γ_0 at the fold-regular singularity $(0,0,0)$ of Z_0 such that the following statements hold.

i) If $\gamma < 0$, then Z_γ has a unique sliding cycle Γ_γ in $\mathcal{A}_0$ and it is attractive ($|\alpha| < 1$) or repelling ($|\alpha| > 1$) depending on the value of α.

ii) If $\gamma = 0$, then Z_γ has a unique quasi-generic loop Γ_0 passing through a fold-regular singularity in $\mathcal{A}_0$.

iii) If $\gamma > 0$, then Z_γ has a unique crossing limit cycle Γ_γ in $\mathcal{A}_0$. Moreover, it is hyperbolic and it is attracting, when $|\alpha| < 1$, and of saddle type, when $|\alpha| > 1$.

Furthermore, Γ_γ is contained in the plane $x = 0$, for every γ sufficiently small, and Z_γ coincides with $Z_\gamma^* = (X_0, Y_\gamma^*)$ in the plane $x = 0$.

Proof. In order to prove the proposition, we must compute the full first return map $\mathcal{P}_\gamma$ of Z_γ. Notice that the flow of Y_γ^* is given by

$$\varphi_{Y_\gamma^*}(t;x,y,z) = \begin{pmatrix} x \\ -t+y \\ (-t+y)^2+(1+\gamma)t-y^2+z \end{pmatrix},$$

and, $\pi_3 \circ \varphi_{Y_\gamma^*}(t;x,y,0)=0$ if, and only if, $t=0$ or $t(y)=2y-1-\gamma$. In this case,

$$\varphi_{Y_\gamma^*}(t(y);x,y,0)=(x,1+\gamma-y,0).$$

Let $Y_\gamma = dM\circ Y_\gamma^*\circ M^{-1}$, where M is given by (5.5.6). Thus,

$$\varphi_{Y_\gamma}(t;x,y,z)=M\circ\varphi_{Y_\gamma^*}(t;M^{-1}(x,y,z)).$$

From, (5.5.4) and (5.5.5), we obtain

$$M^{-1}(x,y,0) = \begin{pmatrix} \dfrac{x}{\alpha+(1-\alpha)y} \\ \dfrac{y(\alpha+(1-\alpha)y)-bx}{\alpha+(1-\alpha)y-bx} \\ 0 \end{pmatrix}.$$

Considering $\overline{x}=\pi_1\circ M^{-1}(x,y,0)$ and $\overline{y}=\pi_2\circ M^{-1}(x,y,0)$, we have that, the map $\mathcal{P}_\gamma^-$ induced by the orbits of Y_γ is given by

$$\mathcal{P}_\gamma^-(x,y,0):=\varphi_{Y^\gamma}(t(\overline{y});x,y,0)=M(\overline{x},1+\gamma-\overline{y},0).$$

Thus,

$$\mathcal{P}_\gamma^-(x,y,0) = \begin{pmatrix} \dfrac{x(\alpha(1+\gamma-\alpha\gamma)+bx(1-\alpha-\gamma+\alpha\gamma)+(\alpha-1)^2((1+\gamma)y-y^2))}{(\alpha+(1-\alpha)y)^2} \\ 1+\gamma-y-\dfrac{b(\gamma-1)}{\alpha+(1-\alpha)y}x \\ 0 \end{pmatrix},$$

for every (x,y) is a neighborhood of $(0,1)$.

Now, using the flow of X_0 we compute the map $\mathcal{P}_0^+:\Sigma\to\Sigma$ induced by orbits of X_0, which is given by

$$\mathcal{P}_0^+(x,y,0) = \begin{pmatrix} x \\ \dfrac{1}{2}\left(1-y+\sqrt{1+2y-3y^2}\right) \\ 0 \end{pmatrix}$$

Finally, the full first return map $\mathcal{P}_\gamma:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\to\Sigma$ of Z_γ is given by

$$\mathcal{P}_\gamma(x,y,0)=\mathcal{P}_\gamma^-\circ\mathcal{P}_0^+(x,y,0),$$

and it has an explicit form. In this case, the first return map P_γ of Z_γ is given by the restriction of $\mathcal{P}_\gamma$ to $[0,\varepsilon)\times(-\varepsilon,\varepsilon)$.

An easy computation shows that

1. $\mathcal{P}_\gamma(0,0) = (0,\gamma)$;

2. $\mathcal{P}_\gamma(x,0) = \begin{pmatrix} x(\alpha(1+\gamma-\alpha\gamma)+bx(1-\alpha-\gamma+\alpha\gamma)+\gamma(\alpha-1)^2) \\ \gamma - b(\gamma-1)x \end{pmatrix}$;

3. $\mathcal{P}_0(x,y) = \begin{pmatrix} \alpha & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + h.o.t.$

4. The sliding dynamics is given by $\psi_\gamma(x) = g(x)+\mathcal{O}_2(g(x))$ where $g(x) = \pi_1 \circ \mathcal{P}_\gamma(x,0)$. In this case, $g(0) = 0$ and thus $\psi_\gamma(0) = 0$, for each $\gamma \in (-\varepsilon,\varepsilon)$.

Hence, from item (3) we have that $\mathcal{P}_0$ has a unique fixed point for $\varepsilon > 0$ sufficiently small. Since $\mathcal{P}_\gamma(0,y) = (0, 1+\gamma-1/2(1-y+\sqrt{1+2y-3y^2}))$, the equation $\mathcal{P}_\gamma(0,y) = (0,y)$ has two solutions

$$y^{\pm}(\gamma) = \frac{1}{2}\left(1+\gamma \pm \sqrt{1-3\gamma-3\gamma^2}\right).$$

Now, $y^+(0) = 1$ and $y^-(0) = 0$. Since we are looking for solutions bifurcating from $(0,0)$, we have that $\mathcal{P}_\gamma$ has a unique hyperbolic fixed point at $p_\gamma = (0, y^-(\gamma))$, for $\varepsilon > 0$ sufficiently small.

Finally, notice that $y^-(\gamma) > 0$ if, and only if $\gamma > 0$. Also, the point $\mathcal{P}_\gamma(0,0) \in \Sigma^s$ if, and only if, $\gamma < 0$. This concludes the proof. □

The bifurcation diagram of the one-parameter family Z_γ for $\alpha > 1$ and $b < 0$ is sketched in Figure 5.10 (the other cases are analogous).

Remark 5.5.3. *Notice that the one-parameter family Z^*_γ given in the Proposition 5.5.1 restricted to the plane $x = 0$, describes the critical crossing cycle bifurcation in planar Filippov systems, which was extensively studied in [40].*

Now, we present another one-parameter family $\mathcal{Z}_\gamma$ which unfolds $Z_0 \in \Lambda_1$ given by Proposition 5.5.1 at $\gamma = 0$ (with $\alpha = 2$ and $b = 1$). In this case, the perturbation breaks the homoclinic-like loop at the origin of Z_0 in such way that the plane $x = 0$ is not $\mathcal{Z}_\gamma$-invariant anymore, for $\gamma \neq 0$.

Proposition 5.5.4. *Consider the one-parameter family of Filippov systems $\mathcal{Z}_\gamma = (X_0, \mathcal{Y}_\gamma)$, where X_0 is given by* (5.5.1)*, $\mathcal{Y}_\gamma = dN \circ \mathcal{Y}^*_\gamma \circ N^{-1}$, $\mathcal{Y}^*_\gamma$ is the vector field given by*

$$\mathcal{Y}^*_\gamma(x,y,z) \quad = \quad \begin{pmatrix} \gamma \\ -1 \\ 1-2y \end{pmatrix},$$

and $N : \mathbb{R}^3 \to \mathbb{R}^3$ is the map given by

$$N(x,y,z) \quad = \quad \begin{pmatrix} x\,(2-y+x(y-1)) \\ y - x(y-1) \\ z \end{pmatrix}.$$

Then $\mathcal{Z}_\gamma$ is an unfolding of the Filippov system Z_0 given by Proposition 5.5.1 (with $\alpha = 2$ and $b = 1$) at $\gamma = 0$, and there exists a solid torus $\mathcal{A}_0$ around the quasi-generic loop Γ_0 at the fold-regular singularity $(0,0,0)$ of Z_0 such that the following statements hold.

i) If $\gamma > 0$, then $\mathcal{Z}_\gamma$ has a unique sliding cycle Γ_γ in $\mathcal{A}_0$, which is of repelling type;

ii) If $\gamma = 0$, then $\mathcal{Z}_\gamma$ has a unique quasi-generic loop Γ_0 passing through a fold-regular singularity in $\mathcal{A}_0$;

iii) If $\gamma < 0$, then $\mathcal{Z}_\gamma$ has a unique crossing limit cycle Γ_γ in $\mathcal{A}_0$, which is hyperbolic and of saddle type.

Proof. Analogously to the proof of Proposition 5.5.2, and using that have the flow of $\mathcal{Y}_\gamma^*$ is given by

$$\varphi_{\mathcal{Y}_\gamma^*}(t;x,y,z) = \begin{pmatrix} \gamma t + x \\ -t + y \\ (-t+y)^2 + t - y^2 + z \end{pmatrix},$$

we obtain that the explicit full first return map of $\mathcal{Z}_\gamma$ is given by

$$\mathcal{P}_\gamma(x,y) = \begin{pmatrix} \gamma(2-\gamma) \\ \gamma \end{pmatrix} + \begin{pmatrix} 2-2\gamma & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}_2(x,y).$$

Applying the Implicit Function Theorem to $\mathcal{P}_\gamma - \mathrm{Id}$, we obtain the existence of a hyperbolic fixed point $p_\gamma = (x(\gamma), y(\gamma))$ of $\mathcal{P}_\gamma$, which is given by

$$\begin{pmatrix} x(\gamma) \\ y(\gamma) \end{pmatrix} = \begin{pmatrix} -2\gamma \\ -\gamma \end{pmatrix} + \mathcal{O}_2(\gamma).$$

It means that $p_\gamma \in \Sigma^c$ if, and only if, $\gamma < 0$, thus $\mathcal{Z}_\gamma$ has a unique crossing limit cycle if, and only if, $\gamma < 0$.

Now, using that $\mathcal{P}_\gamma(x,0,0) = ((2-\gamma-x)(\gamma+x), \gamma+x, 0)$, one can prove that the fold line map of $\mathcal{Z}_\gamma$ is given by

$$\psi_\gamma(x) = (2-\gamma)\gamma + (2-2\gamma)x + \mathcal{O}(x^2).$$

Again, applying the Implicit Function Theorem to $\psi_\gamma - \mathrm{Id}$, we obtain the existence of a hyperbolic fixed point $s(\gamma)$ of ψ_γ which is given by

$$s(\gamma) = -2\gamma + \mathcal{O}_2(\gamma).$$

Finally, it is easy to see that $\mathcal{P}_\gamma(-\gamma,0,0) = (0,0,0)$, which means that $S^1_{X_0}$ intersects S_{X_0} at the origin for each $\gamma \in (-\varepsilon, \varepsilon)$. Thus, the domain of ψ_γ is $(-\varepsilon, 0]$, and it follows that $s(\gamma)$ corresponds to a sliding cycle of $\mathcal{Z}_\gamma$ if, and only if, $\gamma > 0$. In addition, it is a repelling sliding cycle. The proof is complete. □

Remark 5.5.5. *Notice that, considering $\alpha = 2$ and $b = 1$ in Proposition 5.5.2, we have that the one-parameter families Z_γ and $\mathcal{Z}_\gamma$ presented in Propositions 5.5.2 and 5.5.4, respectively, are topologically equivalent unfoldings of Z_0 at $\gamma = 0$.*

It is worth mentioning that, for the particular families considered in this section, we prove that the first part of Theorem M also holds for the case when Z_0 satisfies (S), despite of the stability.

5.6 Proofs of Theorems L, M, N and O

In this section, we use the maps constructed in Section 5.4 to prove Theorems L, M, N and O.

5.6.1 Proof of Theorem L

From Section 5.4.2, there exist neighborhoods $\mathcal{V}_0$ of Z_0 in Ω^r and V_0 of p_0 in Σ sufficiently small, such that, each $Z \in \mathcal{V}_0$ is associated to a full first return map $\mathcal{P}_Z : V_0 \to \Sigma$.

Let $\mathcal{A}_0$ be a solid torus containing Γ_0 such that $V_0 = \mathcal{C}_{p_0}(\mathcal{A}_0 \cap \Sigma)$ (connected component of $\mathcal{A}_0 \cap \Sigma$ containing p_0). In addition, for each $Z \in \mathcal{V}_0$, there exist coordinates (x, y, z) (which has a $\mathcal{C}^r$-dependence on Z) defined in V_0, such that Σ is given by the plane $z = 0$ and $S_Z \cap V_0$ is given by the x-axis.

Since $\mathcal{P}_{Z_0}$ has a unique hyperbolic fixed point p_0 in V_0, it follows from the Implicit Function Theorem that $\mathcal{P}_Z$ has a unique hyperbolic fixed point p_Z in V_0, for each $Z \in \mathcal{V}_0$ (reduce $\mathcal{V}_0$ if necessary). Denoting the y-coordinate of p_Z in the coordinate system (x, y, z) by p_Z^y, it follows that $p_Z \in S_Z$ if, and only if $p_Z^y = 0$.

Define $\zeta(Z) = p_Z^y$, for each $Z \in \mathcal{V}_0$. Therefore, it is straightforward to see that $\zeta(Z) = 0$ if, and only if, Z has a homoclinic-like loop at p_Z contained in $\mathcal{A}_0$. Also, it is not difficult to see that conditions (G), (T), (ii) and (iii) of Definition 5.3.1 hold for every $Z \in \mathcal{V}_0$, which means that $\zeta(Z) = 0$ if, and only if, Z has a quasi-generic loop at p_Z contained in $\mathcal{A}_0$.

Now, let $Z^* = (X^*, Y^*) \in \mathcal{V}_0$ such that $\zeta(Z^*) = 0$, and let $\mathcal{Z}_\lambda$ be a curve in Ω^r such that $\mathcal{Z}_0 = Z^*$, and $\mathcal{Z}_\lambda = (X^*, Y_\lambda)$. In this case,

$$\mathcal{P}_{\mathcal{Z}_0}(x, y) = (\alpha x, bx) + \mathcal{O}_2(x, y),$$

for some $\alpha \neq 0, \pm 1$, and $b \neq 0$. Given $v \in \mathbb{R}$, we can take Y_λ such that

$$\mathcal{P}_{\mathcal{Z}_\lambda}(x, y) = (0, -\lambda v) + (\alpha x, bx) + \mathcal{O}_2(x, y, \lambda).$$

Again, applying the Implicit Function Theorem, we can see that $\zeta(\mathcal{Z}_\gamma) = v\gamma + \mathcal{O}_2(\gamma)$, hence

$$\frac{d}{d\lambda}\zeta(\mathcal{Z}_\lambda)\bigg|_{\lambda=0} = v.$$

We conclude that 0 is a regular value of ζ. The result follows by noticing that $\Lambda_1 \cap \mathcal{V}_0 = \zeta^{-1}(0)$.

5.6.2 Proof of Theorem M

Let $\mathcal{Z} : (-\varepsilon, \varepsilon) \to \Omega^r$ be a one-parameter $\mathcal{C}^1$ family such that $\mathcal{Z}(0) = Z_0$, which is transverse to Λ_1.

From Section 5.4.2, there exist $\varepsilon > 0$ sufficiently small and a neighborhood V_0 of p_0 in Σ sufficiently small, such that, each $\mathcal{Z}(\gamma)$ is associated to a full first return map $\mathcal{P}_{\mathcal{Z}(\gamma)} : V_0 \to \Sigma$. Let $\phi_\gamma : V_0 \to \mathbb{R}^3$ be a change of coordinates (which has a $\mathcal{C}^r$ dependence on γ) such that

- Σ is brought into the plane $z = 0$;
- The fold line S_γ of $\mathcal{Z}(\gamma)$ in V_0 is brought into the x-axis;
- If we denote $S_\gamma^1 = \mathcal{P}_{\mathcal{Z}(\gamma)}(S_\gamma)$, then the point $S_\gamma \cap S_\gamma^1$ is brought into $(0, 0, 0)$.

Consider the family $\overline{\mathcal{Z}}(\gamma) = d\phi_\gamma \circ \mathcal{Z}(\gamma) \circ \phi_\gamma^{-1}$ and notice that the families $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are equivalents. Since $\mathcal{Z}$ is transverse to Λ_1 at 0, it follows that the same holds for $\overline{\mathcal{Z}}$.

Thus, the first return map $P_\gamma = P_{\overline{\mathcal{Z}}(\gamma)}$ (see Section 5.4.2) is defined for $y \geq 0$, and its extension $\mathcal{P}_\gamma$ has a fixed point

$$p_\gamma = (\mathcal{O}(\gamma), a\gamma + \mathcal{O}(\gamma^2), 0),$$

with $a \neq 0$. For instance, assume that $a > 0$.

It means that $p_\gamma \in \Sigma^c$ if, and only if $\gamma > 0$, and thus $\overline{\mathcal{Z}}(\gamma)$ has a unique hyperbolic crossing limit cycle in $\mathcal{A}_0$, if, and only if $\gamma > 0$.

Now, recall that the full fold line map $\psi_\gamma = \psi_{\overline{\mathcal{Z}}(\gamma)}$ defined in the fold line $y = 0$ introduced in Section 5.4.3 controls the existence of sliding cycles. More specifically, it associates sliding cycles with fixed points of ψ_γ belonging to a certain domain $\sigma_\gamma^{FL} = \sigma_{\overline{\mathcal{Z}}(\gamma)}^{FL}$ which is given by (5.4.4).

Since the origin is a hyperbolic fixed point of $\psi_0(x)$, it follows that $\psi_\gamma(x)$ has a unique hyperbolic fixed point x_γ. Hence $\overline{\mathcal{Z}}(\gamma)$ has at most a unique sliding cycle in $\mathcal{A}_0$.

Now, we must see whether x_γ belongs to σ_γ^{FL}. If $\gamma > 0$, then $p_\gamma \in \Sigma^c$ and thus their invariant manifolds $W^1 = W_0^s$ and $W^2 = W_\alpha^s$ (given by Theorem 5.4.9) intersect the x-axis in the points x_1^+ and x_2^+, respectively. Also, if $\gamma < 0$, then $p_\gamma \in \Sigma^s$ and W^1 and W^2 intersect the x-axis in the points x_1^- and x_2^-, respectively. Without loss of generality, assume that S_γ^1 is tangent to the line $y = kx$ at the origin, with $k < 0$. It follows that $x_1^+ < x_2^+$ and $x_1^- < x_2^-$. Now, assume that $\mathcal{Z}(0)$ satisfies (N).

In the case $\gamma > 0$, the point p_γ is in Σ^c and it is attractive. Using that $\mathcal{P}_\gamma(x_i^+, 0, 0)$ must stay in W^i and goes to p_γ, it follows that $\mathcal{P}_\gamma(x_i^+, 0, 0) \in \Sigma^c$, $i = 1, 2$, which means that if $x \leq x_2^+$ then $\mathcal{P}_\gamma(x, 0, 0)$ belongs to Σ^c and thus all these points do not belong to the domain σ_γ^{FL} (recall that $k < 0$). Nevertheless, we know that the x-axis has a unique attractive fixed point x_γ of ψ_γ and $\pi_1 \circ \mathcal{P}_\gamma(x_2^+, 0, 0) < x_2^+$ leads us to $\psi_\gamma(x_2^+) < x_2^+$, which means that $x_\gamma < x_2^+$, and thus $x_\gamma \notin \sigma_\gamma^{FL}$. We conclude that, if $\gamma > 0$, then $\overline{\mathcal{Z}}(\gamma)$ has no sliding cycles.

Now, if $\gamma < 0$, then $p_\gamma \in \Sigma^s$ and through similar arguments, it follows that $\mathcal{P}_\gamma(x_i^-, 0, 0) \in \Sigma^s$, $i = 1, 2$, and thus, if $x \geq x_2^-$ then $\mathcal{P}_\gamma(x, 0, 0)$ belongs to Σ^s and thus all these points belong to the domain σ_γ^{FL}. In this case, $\pi_1 \circ \mathcal{P}_\gamma(x_2^-, 0, 0) < x_2^-$ and thus $\psi_\gamma(x_2^-) > x_2^-$, which means that $x_\gamma > x_2^-$ and hence $x_\gamma \in \sigma_\gamma^{FL}$. We conclude that, if $\gamma < 0$, then $\overline{\mathcal{Z}}(\gamma)$ has a unique sliding cycle (see Figure 5.14).

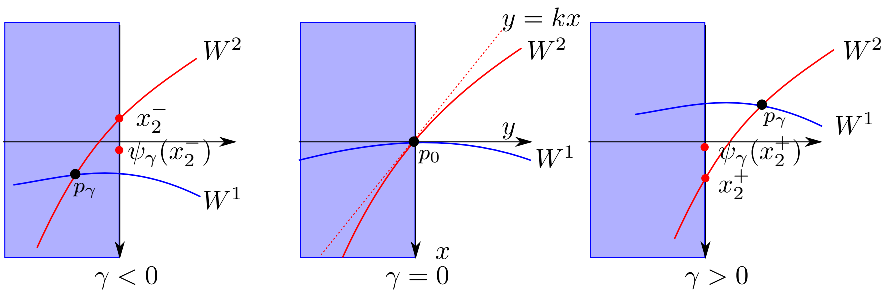


Figure 5.14: Position of the invariant manifolds W^1 and W^2 of $\mathcal{P}_\gamma$ in the case (N) as γ varies.

5.6.3 Proof of Theorem N

From the construction of the full first return map $\mathcal{P}_0$ of Z_0 in Section 5.4.2, it follows that, to prove Theorem N, it is enough to compute the basin of attraction of the origin of the map $\mathcal{P}_0$ and to analyze the sliding dynamics of Z_0.

If Z_0 satisfies (N), then the origin is a hyperbolic fixed point of $\mathcal{P}_0$ of nodal type and thus, there exists a neighborhood of the origin which is the basin of attraction of $\mathcal{P}_0$ at $(0,0)$. Since all the sliding orbits of Z_0 near the origin reaches the fold line S_{Z_0} of Z_0 and the origin is an attractive hyperbolic fixed point of the fold line map ψ_0, it follows that every orbit of Z_0 near the origin goes to the origin. Statement (i) of Theorem N follows directly.

Now, if Z_0 satisfies (S), then the origin is a hyperbolic fixed point of $\mathcal{P}_0$ of saddle type, and thus the basin of attraction of $\mathcal{P}_0$ at $(0,0)$ is given by the stable invariant manifold W^u_α. Hence all the orbits of Z_0 passing through $W^u_\alpha \cap \Sigma^c$ goes to the origin.

Also, there exists a unique sliding orbit γ_s of the sliding vector field F_{Z_0} which goes to the origin. Since the origin is a repelling hyperbolic fixed point of the fold line map ψ_0, it follows that an orbit Γ of Z_0 goes to the origin if and only $\Gamma \cap \Sigma$ contains a point of the piecewise-smooth curve $\beta = W^u_\alpha \cap \Sigma^c \cup \gamma_s$. The proof of Theorem N follows directly.

5.6.4 Proof of Theorem O

In order to prove Theorem O, we study the behavior of the iterations of the fold line S_0 of $Z_0 \in \Lambda^C_1$ through its full first return map $\mathcal{P}_0$.

Lemma 5.6.1 (Accumulation). *Let $Z_0 = (X_0, Y_0) \in \Lambda^C_1$ having a quasi-generic loop Γ_0 at p_0 and let $\mathcal{P}_0$ be the full first return map associated to Z_0 given by* (5.4.7)*. If Z_0 satisfies (S), then $S_n = \mathcal{P}^n_0(S_{X_0})$, $n \in \mathbb{N}$, is a sequence of smooth curves tangent to the eigenspace $\mathcal{E}_\alpha$ given by Proposition 5.4.9 at p_0, such that, for each $\varepsilon > 0$ sufficiently small, there exists $N_0 \in \mathbb{N}$ such that S_n is ε- close to the unstable invariant manifold W^u_α of $\mathcal{P}_0$ at p_0, for every $n \geq N_0$. Furthermore, for each $n \geq 2$, S_n is a curve having an even contact with S_{n-1} at p_0 and the following statements hold*

i) In Σ^c, S_{n-1} and S_n are given by arcs clockwise ordered, and thus $S_n \cap \Sigma^c$ and W^u_α are clockwise ordered for every $n \in \mathbb{N}$;

ii) In Σ^s, S_n is flipped back to the region delimited by S_{n-1} and S_{n-2}, for every $n \geq 2$. Thus, S_n alternates the side of W^u_α (flip property), which means that, if W^u_α and S_{n-1} are counterclockwise ordered, then S_n and W^u_α are counterclockwise ordered, and vice-versa.

Proof. From Proposition 5.4.10, we have that the parameters in (5.4.7) satisfy $\alpha > 0$ and $c > 0$. Recall that the coordinate system (x, y, z) at p_0 used to express $\mathcal{P}_0$ as (5.4.7) satisfies the following properties.

1. Σ is given by the plane $z = 0$;

2. The fold line $S_{Z_0} = S_0$ of Z_0 is given by the x-axis;

3. Without loss of generality, we assume that $b < 0$ in (5.4.7), thus the curve $S_1 = \mathcal{P}_0(S_0)$ of S_0 is tangent to the line $y = kx$ at the origin, where $k = b/\alpha < 0$.

Thus, we have the configuration in the switching manifold $(z = 0)$ illustrated in Figure 5.15.

Since Z_0 satisfies (S), it follows from Proposition 5.4.9 that the map $\mathcal{P}_0$ has a fixed point of saddle type at the origin which has a stable invariant manifold W^s_α tangent to the y-axis and a unstable invariant manifold W^u_α tangent to the line $y = kx$.

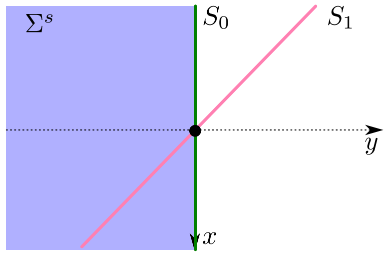


Figure 5.15: Configuration of S_0 and S_1 in Σ.

In what follows, we describe how the iterations of S_0 through $\mathcal{P}_0$ behave. From the expression of $\mathcal{P}_0$ in (5.4.7), we have that $S_n = \mathcal{P}_0^n(S_0)$ is a smooth curve passing through $(0,0)$ tangent to the line $y = kx$ at $(0,0)$, for each $n \in \mathbb{N}$. Clearly, $W^s_\alpha \cap \mathcal{P}_0^n(S_0) = \emptyset$, for each $n \in \mathbb{N}$, since S_0 and S_1 are transversal.

Now, in order to obtain the positions of the curves S_n in Σ, we must recall the construction of the map $\mathcal{P}_0$. In Section 5.4.2, $\mathcal{P}_0$ is written as the composition $\mathcal{P}_0 = \mathcal{D}_0 \circ \mathcal{T}_0$, where $\mathcal{T}_0$ is a transition map from Σ to a transversal section $\tau = \{z = \varepsilon\}$, for $\varepsilon > 0$ small, and $\mathcal{D}_0$ is an orientation-preserving diffeomorphism from τ to Σ. In addition, notice that

$$\mathcal{T}_0(x,y) = (x, \sqrt{2\varepsilon} + Ky^2 + \mathcal{O}(y^3)), \text{ for some } K > 0.$$

Without loss of generality, consider that S_1 is the line $y = kx$. Now, we describe how to obtain S_n, for $n \geq 2$.

We consider $n = 2$, since the other cases follow completely analogous. Notice that $\mathcal{T}_0(x, kx) = (x, \sqrt{2\varepsilon} + K_2x^2 + \mathcal{O}(x^3))$, where $K_2 = Kk^2 > 0$, describes a parabola tangent to the origin contained in the semi-plane $y \geq \sqrt{2\varepsilon}$ of section τ. Since the line $y = \sqrt{\varepsilon}$ is sent to the line S_1 in Σ through the diffeomorphism $\mathcal{D}_0$ (which preserves the orientation of the section τ), it follows that $\mathcal{P}_0(S_1)$ is a parabola which has a quadratic contact with S_1 at the origin. In addition, $\mathcal{P}_0(S_1) \cap \Sigma^c$ is contained in the first quadrant delimited by S_0 and S_1 and $\mathcal{P}_0(S_1) \cap \Sigma^s$ is contained in the fourth quadrant generated by S_0 and S_1 (see Figure 5.16).

Notice that, in Σ^c, the iterations S_0, S_1 and S_2 are clockwise ordered, nevertheless, in Σ^s, S_0, S_2, S_1 are counterclockwise ordered. It allows us to see that, in Σ^s, the second iteration of S_0 have flipped back to the region between S_0 and S_1. Following the same scheme, we prove items (i), and (ii).

Now, using Proposition 5.4.8 and the dominant part of $\mathcal{P}_0$, it follows that S_n accumulates onto W^u_α in the $\mathcal{C}^0$-topology. □

Notice that Lemma 5.6.1 gives rise to a region F_0, which works as a fundamental domain for $\mathcal{P}_0$ restricted to a certain region. See Figure 5.17.

Finally, we are able to prove Theorem O. Let $\mathcal{P}_0$ and $\widetilde{\mathcal{P}_0}$ be the full first return maps associated to Z_0 and $\widetilde{Z}_0$, respectively, and assume that h is a weak equivalence between Z_0 and $\widetilde{Z}_0$. Using Proposition 5.4.8, we can see that there exist coordinate systems (x,y,z) and $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ at p_0 and $\widetilde{p_0}$, respectively, such that $\mathcal{P}_0$ and $\widetilde{\mathcal{P}_0}$ are given by

$$\mathcal{P}_0(x,y) = (\alpha x - c\alpha y^2 + cx^2 + \mathcal{O}_3(x,y), x),$$

and

$$\widetilde{\mathcal{P}_0}(\widetilde{x}, \widetilde{y}) = (\widetilde{\alpha} x - \widetilde{c}\widetilde{\alpha}\widetilde{y}^2 + \widetilde{c}\widetilde{x}^2 + \mathcal{O}_3(\widetilde{x}, \widetilde{y}), \widetilde{x}),$$

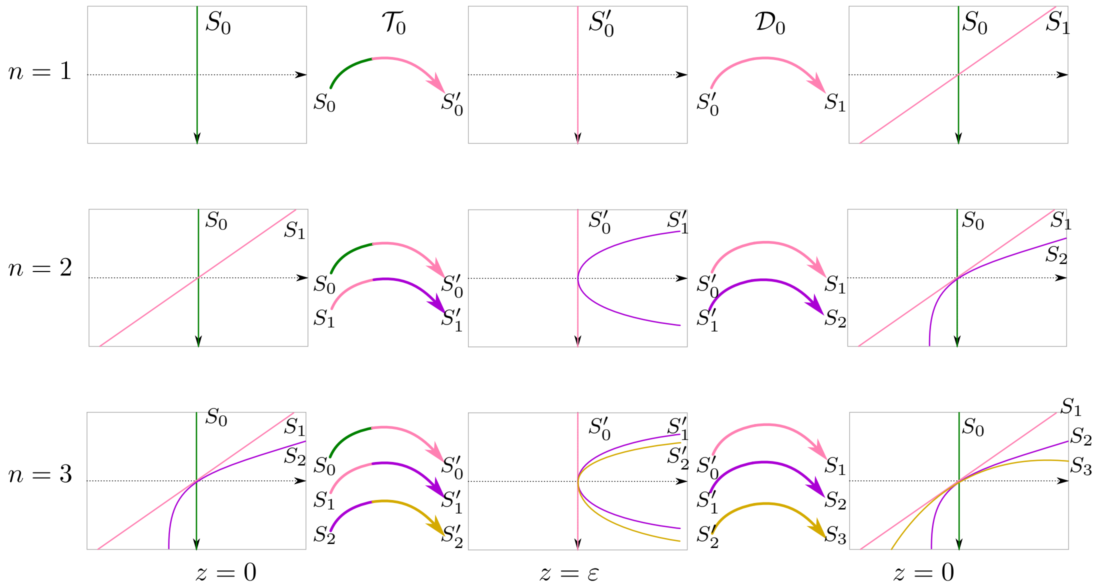


Figure 5.16: Iteration scheme of the fold line S_0 through $\mathcal{P}_0$. Denote $\mathcal{T}_0(S_i) = S_i'$.

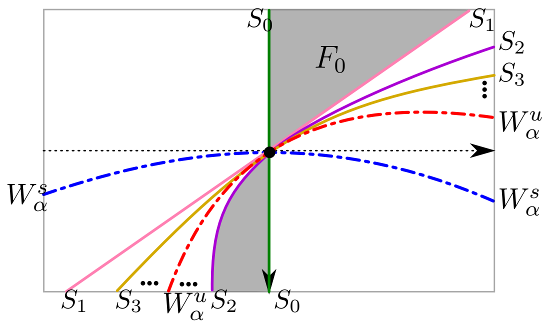


Figure 5.17: Region F_0.

respectively. Also, the fold lines S_{Z_0} and $S_{\widetilde{Z_0}}$ are given by the x-axis and the $\widetilde{x}$-axis, respectively. In this case, $\mathcal{W}(Z_0) = \alpha$ and $\mathcal{W}(\widetilde{Z_0}) = \widetilde{\alpha}$.

Consider the same notation used in the proof of Lemma 5.6.1. Let $\delta > 0$ sufficiently small, and consider the map $\mathcal{P}_0$. There exists a unique point $w \neq (0,0)$ of $W^u_\alpha \cap \{y = \delta\}$, and, for each $n \in \mathbb{N}$, take y_n as the unique point contained in $S_n \cap \{y = \delta\}$. Therefore, from the construction, there exists a sequence $(x_n, 0) \in S_0$ such that

1. $(x_n, 0) \to (0,0)$ as $n \to \infty$;

2. $y_n = \mathcal{P}_0^n(x_n, 0)$, for each $n \in \mathbb{N}$;

3. $y_n \to w$ as $n \to \infty$.

Now, for the map $\widetilde{\mathcal{P}_0}$, consider $\widetilde{w} = h(w)$, $\widetilde{x_n} = h(x_n)$ and $\widetilde{y_n} = h(y_n)$, for each $n \in \mathbb{N}$. Since h is a weak-equivalence and $w \neq (0,0)$, it follows that

1. $\widetilde{w} \neq (0,0)$;

2. $(\widetilde{x_n}, 0) \to (0,0)$ as $n \to \infty$;

3. $\widetilde{y_n} = \widetilde{\mathcal{P}_0}^n(\widetilde{x_n})$, for each $n \in \mathbb{N}$;

4. $\widetilde{y_n} \to \widetilde{w}$ as $n \to \infty$.

Notice that, since $\alpha, \widetilde{\alpha} \neq 0$, it follows that the dynamics of points near the invariant manifolds W^u_α and $W^u_{\widetilde{\alpha}}$ of $\mathcal{P}_0$ and $\widetilde{\mathcal{P}_0}$ have the same behavior of the dynamics obtained from their linear approximations. Therefore, without loss of generality, consider that

$$\mathcal{P}_0(x, y) = (\alpha x, x) \quad and \quad \widetilde{\mathcal{P}_0}(\widetilde{x}, \widetilde{y}) = (\widetilde{\alpha}\widetilde{x}, \widetilde{x}).$$

Hence, $y_n = (\alpha^n x_n, \alpha^{n-1} x_n)$ and $\widetilde{y_n} = (\widetilde{\alpha}^n \widetilde{x_n}, \widetilde{\alpha}^{n-1}\widetilde{x_n})$, for n sufficiently big. Now, since $h : S_{Z_0} \to S_{\widetilde{Z_0}}$ is a diffeomorphism, it follows that $\widetilde{x_n} = K x_n + \mathcal{O}_2(x_n)$, for some $K \neq 0$. It follows that $\alpha^n x_n \to \pi_1(w) \neq 0$ and $\widetilde{\alpha}^n x_n \to \pi_1(\widetilde{w})/K \neq 0$, as $n \to 0$.

Now, if $\alpha \neq \widetilde{\alpha}$, then it follows that either $\alpha^n x_n \to 0$ or $\widetilde{\alpha}^n x_n \to 0$, which contradicts the fact that $\alpha^n x_n \to \pi_1(w)$ and $\widetilde{\alpha}^n x_n \to \pi_1(\widetilde{w})/K$. Therefore, it follows that $\alpha = \widetilde{\alpha}$, and the proof is complete.

5.7 Conclusion and Further Directions

In this chapter, we have studied Filippov systems $Z_0 = (X_0, Y_0)$ around a homoclinic-like loop Γ_0 at a fold-regular singularity under some generic conditions and we have proven that such loops are generic in one-parameter families.

Also, we have seen that the fold line S_{Z_0} of Z_0 connects to itself through orbits of X_0, Y_0 and F_{Z_0} as a topological cylinder or a Möbius strip, giving rise to two classes of loops, Λ_1^C and Λ_1^M, respectively. For simplicity, we considered only the class Λ_1^C to avoid technicalities, nevertheless, we believe that similar results hold in the class Λ_1^M.

In the class Λ_1^C, we have seen that the first return map of Z_0 has a hyperbolic fixed point of either saddle (condition (S)) or nodal type (condition (N)). We have completely described the bifurcation diagram of Z_0 around Γ_0, provided that Z_0 satisfies (N). If Z_0 satisfies (S), we found all the bifurcating elements of Γ_0, nevertheless, the description of the bifurcation diagram remains as an open problem for this case. We conjecture that Z_0 has the same bifurcation diagram around Γ_0 for the cases (N) and (S), as can be seen in the examples provided in Section 5.5.

A natural extension of this work is to obtain bifurcation diagrams of Filippov systems around homoclinic-like loops passing through other kinds of Σ-singularities (e.g. cusp-regular and fold-fold singularities). We highlight that the connection studied herein appears in the unfolding of loops passing through a cusp-regular singularity. This study will guide us towards the comprehension of polycycles in $3D$ Filippov systems (see the planar version provided in Chapter 2).

Also, if we relax the generic conditions imposed in the quasi-generic loops, one can certainly obtain interesting global behavior for Filippov systems Z near Z_0. In fact, such degeneracy of homoclinic-like loops at a fold-regular singularity might originate other bifurcating cycles.

Chapter 6

T-Chains: A Chaotic $3D$ Foliation

In this chapter we deal with a class of $3D$ Filippov systems presenting robust connections between certain typical singularities, known as T-singularities. Such systems are locally structurally stable at these singularities and are mainly characterized by the existence of $2D$ invariant cones (named diabolos) with vertices on such points. Our main goal is to discuss the existence of chaotic dynamics when self connections between the cones occur. We highlight that the counterpart of these connections in the smooth case can happen only for highly degenerate systems.

6.1 Setting the Problem and Main Result

In this chapter, we consider Filippov systems $Z = (X, Y)$ defined on an open bounded connected region $M \subset \mathbb{R}^3$ (diffeomorphic to an open ball) with an oriented switching manifold $\Sigma = f^{-1}(0)$, where $f : M \to \mathbb{R}$ is a smooth function having 0 as a regular value. We denote the set of $\mathcal{C}^r$ vector fields by χ^r and we endow it with the $\mathcal{C}^r$ topology. Accordingly, the set of Filippov systems on M is denoted by $\Omega^r = \chi^r \times \chi^r$ and it is endowed with the product topology.

6.1.1 T-chains

As we have seen in Chapter 3, Filippov systems are locally structurally stable at certain types of fold-fold singularities and they appear in an open set of this class. Moreover, in Chapter 4, we have proven that certain fold-fold singularities still persist on the class of semi-local structurally stable Filippov systems. In light of this, the knowledge of global behavior of the dynamics of Filippov systems around connections involving these points plays a crucial role in the attempt to characterize the structurally stable Filippov systems. We formalize such connections in the following definition.

Definition 6.1.1. *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ having fold-fold singularities $p_0, q_0 \in \Sigma$ ($p_0 = q_0$ is also considered) and let $-\infty \leq a < b \leq \infty$. An oriented piecewise smooth curve $\Gamma : (a, b) \to M$ is said to be a* ***fold-fold connection*** *of Z_0 between p_0 and q_0 if it satisfies the following conditions.*

i) $\mathrm{Im}(\Gamma) \cap M^+$ *(resp.* $\mathrm{Im}(\Gamma) \cap M^-$*) is a union of orbits of X_0 (resp. Y_0).*

ii) $\mathrm{Im}(\Gamma) \cap \Sigma \subset \Sigma^c$.

iii) $\lim_{t \to a} \Gamma(t) = p_0$ *and* $\lim_{t \to b} \Gamma(t) = q_0$.

Now, we introduce the concept of fold-fold chain.

Definition 6.1.2. *Consider* $Z_0 \in \Omega^r$ *having* k *fold-fold singularities* $p_i \in \Sigma$, $i = 1, \cdots, k$. *We say that* $\gamma \subset M$ *is a* ***fold-fold chain of order*** k *of* Z_0 *if*

$$\gamma = \{p_1, \cdots, p_k\} \cup_{i=1}^{k} \operatorname{Im}(\Gamma_i),$$

where Γ_i *is a fold-fold connection between* p_i *and* p_{i+1}, $i = 1, \cdots, k$, *where* $p_{k+1} = p_1$, *and either one of the following conditions is satisfied.*

i) Γ_i *is an oriented piecewise smooth curve from* p_i *to* p_{i+1}, $i = 1, \cdots, k$.

ii) Γ_i *is an oriented piecewise smooth curve from* p_{i+1} *to* p_i, $i = 1, \cdots, k$.

Notice that, fold-fold chains generalize Σ-polycycles having only fold-fold singularities (in the planar case) to $3D$ Filippov systems. Figure 6.1 illustrates some fold-fold chains.

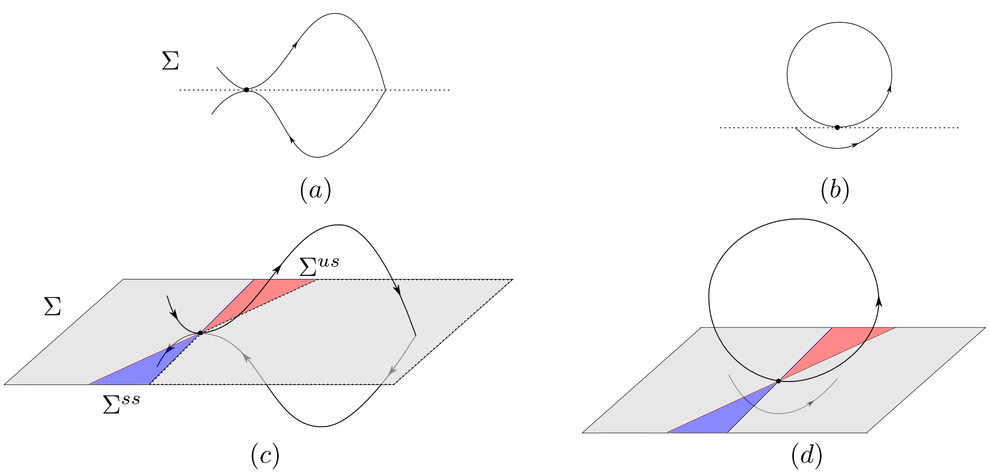


Figure 6.1: Planar Σ-polycycles passing through a visible-visible fold-fold singularity (a) and a visible-invisible fold-fold singularity (b), and tridimensional fold-fold chains passing through a visible-visible fold-fold singularity (c) and a visible-invisible fold-fold singularity (d).

As far as we know, there is a lack of works in the literature concerning this kind of object, maybe due to the difficult inherent to the problem. In fact, $3D$ Filippov systems exhibit a rich local dynamics at fold-fold singularities which is hard to comprehend, and thus, the understanding of global phenomena involving such objects becomes even harder.

As seen before, one of the most challenging types of fold-fold singularities is the elliptic one (see Section 3.5.1), also known as T-singularity. We recall that, in this case, if $Z_0 = (X_0, Y_0) \in \Omega^r$ has a T-singularity at p_0, then it is associated to a $\mathcal{C}^r$ germ of first return map $\phi_0 : (\Sigma, p_0) \to (\Sigma, p_0)$, which is given by $\phi_0 = \phi_{X_0} \circ \phi_{Y_0}$, where $\phi_{X_0}, \phi_{Y_0} : (\Sigma, p_0) \to (\Sigma, p_0)$ are the involutions induced by the orbits of X_0 and Y_0 near p_0.

For simplicity, we say that p_0 is a **stable T-singularity** of Z_0 if, and only if, p_0 is a T-singularity for which ϕ_0 has a hyperbolic fixed point of saddle type at p_0 with both local invariant manifolds $W^{u,s}_{\phi_0}(p_0)$ of ϕ_0 at p_0 contained in Σ^c. Recall that Theorem F proves that $Z_0 \in \Omega^r$ is locally structurally stable at a T-singularity p_0 if, and only if, p_0 is a stable T-singularity of Z_0.

Also, if Z_0 has a stable T-singularity at p_0, then there exists a (local) invariant cone $\mathcal{N}(p_0)$ with vertex at p_0 which is filled up with crossing orbits of Z_0. In addition, $\mathcal{N}(p_0)$ is piecewise smooth and $\mathcal{N}(p_0) \cap \Sigma = W^u_{\phi_0}(p_0) \cup W^s_{\phi_0}(p_0)$. Denote the stable and unstable branches of $\mathcal{N}(p_0)$ by W^s_{cross} and W^u_{cross}, respectively. The existence of such cone $\mathcal{N}(p_0)$ has been exhibited in Section 3.6, and it is also referred as the diabolo associated to Z_0 at p_0 (see [24]). See Figure 6.2.

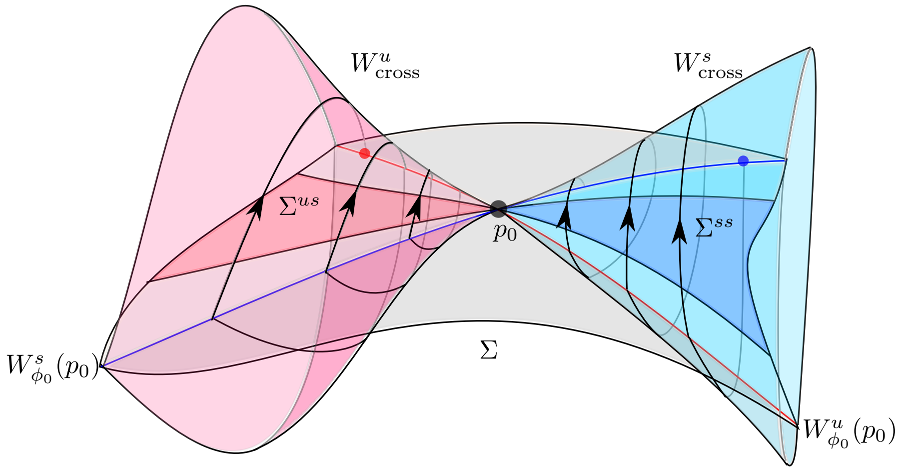


Figure 6.2: Nonsmooth diabolo $\mathcal{N}(p_0)$ at a stable T-singularity p_0 of Z_0.

In light of this discussion, we have seen that a Filippov system has local crossing invariant manifolds (stable and unstable) at a stable T-singularity, which persist under small perturbation. Therefore, a natural question arises in such scenario: what kind of dynamics is originated from the global extension of these local invariant manifolds?

Definition 6.1.3. *Consider $Z_0 \in \Omega^r$. We say that $\gamma \subset M$ is a **T-chain** of Z_0 if γ is a fold-fold chain of order 1 of Z_0 having a unique stable T-singularity of Z_0.*

In this chapter, we study the dynamics of Filippov systems around T-chains through a semi-local analysis at this global connection. We highlight that T-chains are the simplest fold-fold chains having stable T-singularities and we restrict our studied to this case because, even in this situation, a Filippov system displays a very complicated dynamics in the presence of such object.

6.1.2 Robustness Conditions

Let $Z_0 = (X_0, Y_0) \in \Omega^r$ having a stable T-singularity at p_0. For $\star = u, s$, let $\tau^\star$ be a section such that $\tau^\star \cap W^\star_{\text{cross}} = \mathcal{C}^\star$ is a piecewise smooth closed curve homotopic to a circle which is nonsmooth only at (the two) points belonging to $\mathcal{C}^\star \cap \Sigma$. Assume that $\tau^\star$ is transverse to the flow of Z_0 at the points of $\mathcal{C}^\star$ and that $\tau^\star$, $W^\star_{\text{cross}}$ and Σ are in general position (see Figure 6.3). Also, consider that $\tau^\star$ is contained in a neighborhood V_{3D} of p_0 in M, for which the local first return map $\phi_0 : V \to \Sigma$ associated to Z_0 at p_0 is defined in $V = V_{3D} \cap \Sigma$.

Remark 6.1.4. *By saying that $\tau^\star$ is transverse to the flow of Z_0 at the points of $\mathcal{C}^\star$, we mean that X_0 (resp. Y_0) is transverse to $\tau^\star$ at each point $q \in W^\star_{cross} \cap \overline{M^+}$ (resp. $q \in W^\star_{cross} \cap \overline{M^-}$).*

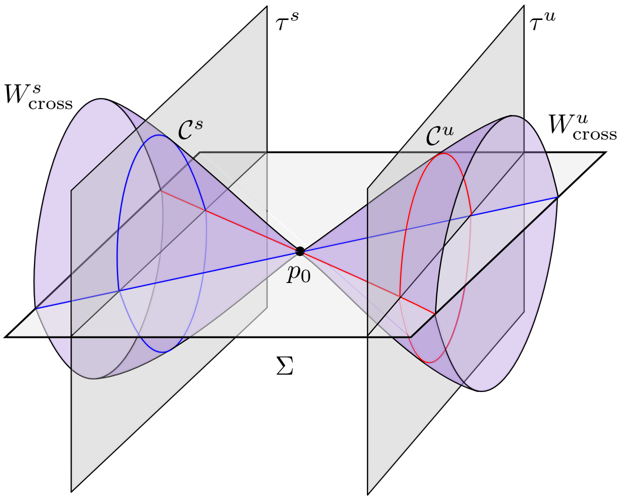


Figure 6.3: Sections τ^u and τ^s.

Assume that Z_0 satisfies the following set of conditions **(TC)**:

(TC_1) Z_0 has a stable T-singularity at $p_0 \in \Sigma$;

(TC_2) There exists a germ of diffeomorphism $\mathcal{D} : \tau^u \to \tau^s$ at $\mathcal{C}^u$, induced by orbits of X_0 and Y_0 such that, for each $q \in \text{Dom}(\mathcal{D})$ (domain of $\mathcal{D}$), q and $\mathcal{D}(q)$ are connected by a crossing orbit of Z_0 and $\mathcal{D}(\mathcal{C}^u) = \widehat{\mathcal{C}^u}$ is a topological circle contained in τ^s;

(TC_3) There exists a Z_0-invariant topological cylinder $\mathcal{R}$ (2-dimensional) connecting $\mathcal{C}^u$ and $\widehat{\mathcal{C}^u}$, which is filled up with crossing orbits of Z_0. Assume that $\mathcal{R} \cap \Sigma$ is given by two compact distinct curves $\mathcal{R}^u$, $\mathcal{R}^s$ which contains the points $W^u_{\phi_0}(p_0) \cap \tau^u$ and $W^s_{\phi_0}(p_0) \cap \tau^u$, respectively. Also, consider that each crossing orbit contained in $\mathcal{R}$ does not intersect $\mathcal{R}^\star$ consecutively, for $\star = u, s$.

The set of hypotheses (TC) allows us to extend the crossing invariant manifold $W^u_{\text{cross}}(p_0)$ of Z_0 through a cylinder $\mathcal{R}$ in such a way that it intersects the section τ^s at a topological circle $\widehat{C^u}$. Below, we show that such conditions allow us to extend the local first return map ϕ_0 of Z_0 at p_0 into a first return map in Σ around $\{p_0\} \cup \mathcal{R}^u \cap \mathcal{R}^s$, in such way that the local invariant manifolds $W^u_{\phi_0}$ and $W^s_{\phi_0}$ are extended by $\mathcal{R}^u$ and $\mathcal{R}^s$.

Lemma 6.1.5 (Extension)**.** *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ satisfying the set of conditions (TC) and let $\phi_0 = \phi_{X_0} \circ \phi_{Y_0} : V \to \Sigma$ be its local first return map at the stable T-singularity p_0. There exists a small connected neighborhood W of $\{p_0\} \cup \mathcal{R}^u \cup \mathcal{R}^s$ in Σ such that*

i) $V \subset W$ *and* $W \setminus V \subset \Sigma^c$;

ii) *There exists an involution* $\Phi_{Y_0} : W \to W$ *induced by orbits of* Y_0, *i.e., for each* $p \in W$, p *and* $\Phi_{Y_0}(p)$ *are connected through an orbit of* Y_0 *contained in* $\overline{M^-}$;

iii) *There exists an involution* $\Phi_{X_0} : W \to \Sigma$ *induced by orbits of* X_0;

iv) $\Phi_0 = \Phi_{X_0} \circ \Phi_{Y_0}$ *is a reversible mapping which is an extension of* ϕ_0. *In addition,* Φ_0 *has a unique hyperbolic fixed point at* p_0 *which is of saddle type.*

Furthermore, for $\star = u, s$, the global invariant manifold $W^{\star}_{\Phi_0}$ of Φ_0 at p_0 is an extension of $W^{\star}_{\phi_0}(p_0)$ and contains the curve $\mathcal{R}^{\star}$.

Proof. We prove only item (ii), since item (iii) is proved in an analogous way and item (iv) is a direct consequence of items (ii) and (iii). Item (i) will follows from the construction.

Since Y_0 is transverse to τ^u and τ^s at $\mathcal{C}^u \cap \overline{M^-}$ and $\widehat{\mathcal{C}^u} \cap \overline{M^-}$, respectively, it follows from condition (TC_3) that, for each point $p \in \mathcal{R}^u$, there exists either $q \in \mathcal{R}^s$ or $q \in V$ such that p and q are connected by a unique orbit of Y_0 contained in $\overline{M^-}$. Also, if $q \in V$, then such orbit intersects $\mathcal{C}^u$ or $\widehat{\mathcal{C}^u}$. See Figure 6.4.

Therefore, for each $p \in \mathcal{R}^u$, we use the Implicit Function Theorem to define a $\mathcal{C}^r$ map $\Phi^{Y_0}_p : W_p \to \Sigma$ induced by the orbits of Y_0 in such a way that, $x \in W_p$ and $\Phi^{Y_0}_p(x)$ are connected by a unique orbit of Y_0 contained in $\overline{M^-}$. An analogous argument shows that the same holds for points of $\mathcal{R}^s$.

Clearly, given $p_1, p_2 \in \mathcal{R}^s \cup \mathcal{R}^s$, if $x \in W_{p_1} \cap W_{p_2}$, then $\Phi^{Y_0}_{p_1}(x) = \Phi^{Y_0}_{p_2}(x)$.

From compactness of $\mathcal{R}^u \cup \mathcal{R}^s$, there exist a small neighborhood $W_{\mathcal{R}} \subset \Sigma^c$ of $\mathcal{R}^u \cup \mathcal{R}^s$ and a $\mathcal{C}^r$ map $\Phi_{Y_0} : V \cup W_{\mathcal{R}} \to V \cup W_{\mathcal{R}}$ induced by orbits of Y_0. From construction, we have that Φ_{Y_0} is an involution and $\Phi_{Y_0}|_V = \phi_{Y_0}$. Take $W = V \cup W_{\mathcal{R}}$.

Clearly, the local invariant manifolds $W^{u,s}_{\phi_0}$ are extended through $\mathcal{R}^{u,s}$ to invariant manifolds of Φ_0, respectively. □

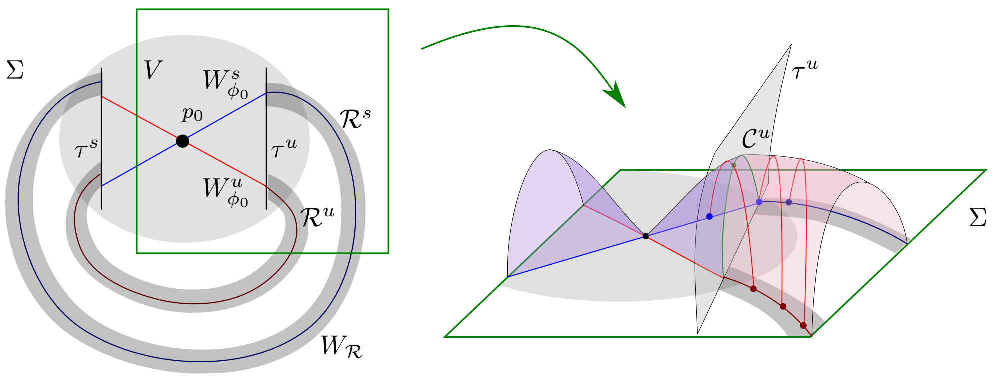


Figure 6.4: Neighborhood W for which the extended first return map Φ_0 is defined and behavior of the orbits of X_0 at points of $\mathcal{R}^u$.

Notice that T-chains of Z_0 at p_0 are characterized as intersections between the topological circles $\widehat{C^u}$ and $\mathcal{C}^s$.

Proposition 6.1.6. *Let $Z_0 \in \Omega^r$ satisfying (TC). The following statements hold.*

i) If $\widehat{C^u} \cap \mathcal{C}^s = \emptyset$, then Z_0 has no T-chains at p_0;

ii) If $\widehat{C^u} = \mathcal{C}^s$, then Z_0 has an invariant (piecewise smooth) pinched torus at p_0 foliated by T-chains at p_0;

iii) If $\widehat{C^u} \cap \mathcal{C}^s = q_1, \cdots, q_K \subset M^+ \cup M^-$ and $\widehat{\mathcal{C}^u} \pitchfork \mathcal{C}^s$ at q_i, $i \in 1, \cdots, K$, then Z_0 has K distinct T-chains at p_0 and $K = 2k$, for some $k \in \mathbb{N}$.

The proof of Proposition 6.1.6 is straightforward and it will be omitted.

Remark 6.1.7. *Notice that the topological circles $\mathcal{C}^s$ and $\widehat{\mathcal{C}^u}$ are smooth at the points q_i, $1 \leq i \leq 2k$, since $q_i \notin \Sigma$. Therefore, the notion of transversality is well-defined in condition (iii) of Proposition 6.1.6.*

We notice that if item (ii) of Proposition 6.1.6 is satisfied then the (reversible) first return map Φ_0 obtained in Lemma 6.1.5 has a homoclinic connection at p_0. Clearly, such situation is not robust, since a small perturbation breaks the condition $\widehat{\mathcal{C}^u} = \mathcal{C}^s$. It is worth mentioning that results on bifurcation of reversible maps around homoclinic orbits can be used to understand what happens with these manifolds under small perturbations, nevertheless, this situation is highly degenerated and thus, it can give rise to very complicated phenomena. In [30], the authors have studied bifurcations of homoclinic orbits of some planar reversible maps.

In order to avoid further degeneracies, we consider the following **robustness condition** on Z_0:

(R) $\mathcal{C}^s \cap \widehat{\mathcal{C}^u} = \{q_1, \cdots, q_{2k}\}$, for some $k \in \mathbb{N}$, where $q_i \notin \Sigma$ and $\mathcal{C}^s \pitchfork \widehat{\mathcal{C}^u}$ at q_i, $i \in 1, \cdots, 2k$.

Without loss of generality, we consider that $k = 1$ throughout this chapter. Also, we highlight that the condition $q_i \notin \Sigma$ in (R) and item (iii) of Proposition 6.1.6 is only technical and can be dropped by extending the notion of transversality of $\mathcal{C}^s$ and $\widehat{\mathcal{C}^u}$ at points of Σ.

Therefore, if Z_0 satisfies (TC) and (R), then Z_0 has two distinct T-chains at p_0. We notice that, in this case, each T-chain can be seen as a crossing homoclinic orbit of Z_0, since it reaches p_0 only at infinite time (see Figure 6.5). In addition, conditions (TC) and (R) are persistent under small perturbations of Z_0, thus we have that such T-chains of Z_0 are robust in Ω^r (i.e. can not be destroyed for Z near Z_0).

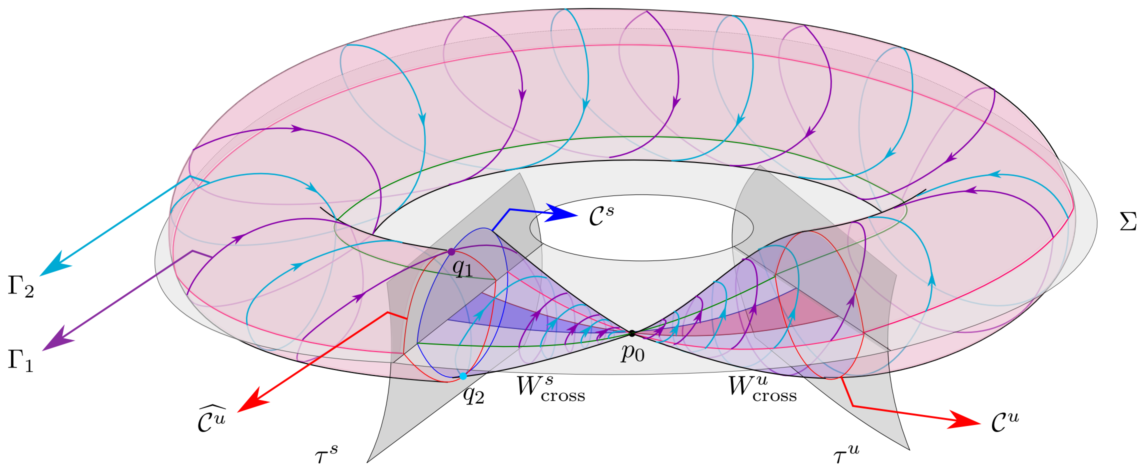


Figure 6.5: A Filippov system Z_0 satisfying hypotheses (TC) and (R) having two T-chains Γ_1 and Γ_2 passing through q_1 and q_2, respectively.

6.1.3 Main Result

Let $Z_0 \in \Omega^r$ satisfying (TC) and (R). From 6.1.5, we have that Z_0 is associated to a first return map $\Phi_0 = \Phi_{X_0} \circ \Phi_{Y_0}$ induced by orbits of X_0 and Y_0. Recall that Φ_{X_0} and Φ_{Y_0} describes the foliation generated by X_0 and Y_0 in $\overline{M^+}$ and $\overline{M^-}$, respectively, in the sense that x and $\phi_{X_0}(x)$ (resp. $\phi_{Y_0}(x)$) are connected by an orbit of X_0 (resp. Y_0) contained in $\overline{M^+}$ (resp. $\overline{M^-}$), for every $x \in W$. It follows that the foliation generated by all the orbits of Z_0 is described by Φ_0.

Hence, in order to understand the dynamics of the points in the foliation generated by Z_0 around the T-chains γ_1 and γ_2 at p_0, we must study the dynamics of the first return map Φ_0. Although, notice that the map Φ_0 does not care about how two pieces of orbits of X_0 and Y_0 are concatenated. In fact, the dynamics of a point through Φ_0 can represent a piecewise smooth curve having pieces of orbits of X_0 and Y_0 which are concatenated in opposite directions. In light of this, we introduce the following definition.

Definition 6.1.8. *We say that a piecewise smooth curve γ is a* ***pseudo-orbit*** *of $Z_0 = (X_0, Y_0)$ if it satisfies the following conditions*

i) $\gamma \cap \overline{M^+}$ is tangent to X_0;

ii) $\gamma \cap \overline{M^-}$ is tangent to Y_0;

iii) There exists at least a point $p \in \gamma$ such that $X_0 f(p) Y_0 f(p) < 0$.

Hence, the orbits of Φ_0 are associated to crossing orbits and pseudo-orbits of Z_0, and vice-versa. Also, notice that if $(\Phi_0^n(x))_{n\in\mathbb{N}}$ corresponds to a crossing orbit of Z_0, then the evolution of x through Φ_0 might not coincide with the evolution in time of the corresponding orbit of Z_0.

It is worth saying that pseudo-orbits of Z_0 do not have dynamical meaning, nevertheless, they have to be preserved by topological equivalences preserving Σ. Thus, the dynamics of Φ_0 plays an important role to determine the structure of Z_0 around T-chains.

Finally, we state the main result of this chapter.

Theorem P. *Let $Z_0 \in \Omega^r$ satisfying (TC) and (R) and let γ_1 and γ_2 be the two T-chains at p_0. Then, for an arbitrarily small neighborhood of p_0, there exist $n_1, n_2 \in \mathbb{N}$ such that $\Phi_0^{n_1}$ and $\Phi_0^{n_2}$ admit Smale horseshoes Δ_{γ_1} and Δ_{γ_2}, respectively. Furthermore, $\Delta_{\gamma_1} \cap \Delta_{\gamma_2} = \emptyset$ and, for $i = 1, 2$, the hyperbolic invariant set Λ_i in the horseshoe Δ_{γ_i} contains a point of $\gamma_i \cap \Sigma$.*

Remark 6.1.9. *In [61], one finds a detailed description of Smale horseshoes for a diffeomorphism and some basic properties. Also, in [108], the author provides an elucidative construction of Smale horseshoes.*

Theorem P shows us that, if Z_0 satisfies (TC) and (R), then the dynamics originated by its orbits and pseudo-orbits is **chaotic** (see [108] for more details). A direct consequence of Theorem P is stated below.

Proposition 6.1.10. *Let $Z_0 \in \Omega^r$ satisfying conditions (TC) and (R) and let Λ_1 and Λ_2 be the hyperbolic sets given by Theorem P. For each $i = 1, 2$, the following statements hold.*

1. *There exists an infinity of closed orbits (or pseudo-orbits) Γ of Z_0 such that $\Gamma \cap \Sigma \subset \Lambda_i$;*

2. *There exists an infinity of non-closed orbits (or pseudo-orbits) Γ of Z_0, such that $\Gamma \cap \Sigma \subset \Lambda_i$;*

3. *There exists an orbit (or pseudo-orbit) Γ_d of Z_0 such that $\Gamma_d \cap \Sigma$ is dense in Λ_i.*

The proof of Proposition 6.1.10 follows directly from Definition 6.1.8, Theorem P and Theorem 2.1.4 of [108].

Section 6.2 is devoted to prove Theorem P. In Section 6.3 we present a model realizing a robust fold-fold connection. Some further directions of this problem are given in Section 6.4.

6.2 Proof of Theorem P

First we discuss about the local structure of the stable T-singularity p_0. Without loss of generality, we consider the following assumptions:

- The switching manifold is given by $\Sigma = \{z = 0\}$ and $p_0 = (0, 0, 0)$;
- The sections τ^u and τ^s are contained in the planes $\{y = \varepsilon\}$ and $\{y = -\varepsilon\}$, for some $\varepsilon > 0$ sufficiently small;
- $S_{X_0} \cap V$ and $S_{Y_0} \cap V$ are contained in the lines $x = K_1 y$ and $x = K_2 y$, respectively, for some coefficients $K_1 < 0$ and $K_2 > 0$;
- The orbits of Y_0 in V_{3D} go from $\{x < K_2 y\}$ to $\{x > K_2 y\}$ and the orbits of X_0 in V_{3D} goes from $\{x > K_1 y\}$ to $\{x < K_1 y\}$.

Such assumptions imply that, if $p \in \{x < K_2 y\} \cap \{x < K_1 y\}$, then the orbit $(\phi_0^n(p))_{n\in\mathbb{N}}$ of the local first return map $\phi_0 : V \to \Sigma$ represents a crossing orbit of Z_0 and its evolution through time coincides with the order generated by $(\phi_0^n(p))_{n\in\mathbb{N}}$.

Recall that, since the origin is a stable T-singularity of Z_0, ϕ_0 has local invariant manifolds $W^{u,s}_{\phi_0}(0, 0)$ at $(0, 0)$, which is a hyperbolic fixed point of saddle type of ϕ_0. Without loss of generality, we assume that these invariant manifolds are contained in the union of lines $\{x = K_3 y\} \cup \{x = K_4 y\}$, where $K_3 < K_1$ and $K_4 > K_2$.

It follows from the orientation of the orbits of X_0 and Y_0 and the position of $\tau^{u,s}$ that

$$W^s_{\phi_0}(0, 0) \subset \{x = K_4 y\} \quad \textit{and} \quad W^u_{\phi_0}(0, 0) \subset \{x = K_3 y\}.$$

Figure 6.6 illustrates the situation considered above.

Remark 6.2.1. *Notice that, if $x > 0$, then the orientation of the crossing orbits through a point p of $W^{u,s}_{\phi_0}(0, 0)$ is reverse with respect to the order given by the orbit $(\phi_0^n(p))_{n\in\mathbb{N}}$ of ϕ_0 through p.*

Using the Extension Lemma 6.1.5, we obtain the first return map $\Phi_0 : W \to \Sigma$ induced by orbits of X_0 and Y_0, which extends $\phi_0 : V \to \Sigma$.

From conditions (TC), we have that X_0 (resp. Y_0) is transverse to τ^s at points of $(\mathcal{C}^s \cup \widehat{\mathcal{C}^u}) \cap \overline{M^+}$ (resp. $(\mathcal{C}^s \cup \widehat{\mathcal{C}^u}) \cap \overline{M^-}$). Also, $\pi_2(X_0(p)), \pi_2(Y_0(p)) > 0$ in such points.

Remark 6.2.2. *Notice that S_{X_0} and S_{Y_0} must intersect τ^s at points lying in the interior of the bounded regions of $\{y = -\varepsilon\}$ delimited by the circles $\mathcal{C}^s$ and $\widehat{\mathcal{C}^u}$.*

Now, since $(\mathcal{C}^s \cup \widehat{\mathcal{C}^u}) \cap \overline{M^+}$ is a compact set, and $\pi_2(X_0(p)) > 0$ for every $p \in (\mathcal{C}^s \cup \widehat{\mathcal{C}^u}) \cap \overline{M^+}$, it follows from the Implicit Function Theorem that there exist an open neighborhood N^+ of $(\mathcal{C}^s \cup \widehat{\mathcal{C}^u}) \cap \overline{M^+}$ in the plane $\{y = -\varepsilon\}$ and a $\mathcal{C}^r$ diffeomorphism $\varphi_+ : N^+ \cap M^+ \to \Sigma$ such that, for each $x \in N^+ \cap M^+$, x and $\varphi_+(x)$ are connected by a unique piece of orbit

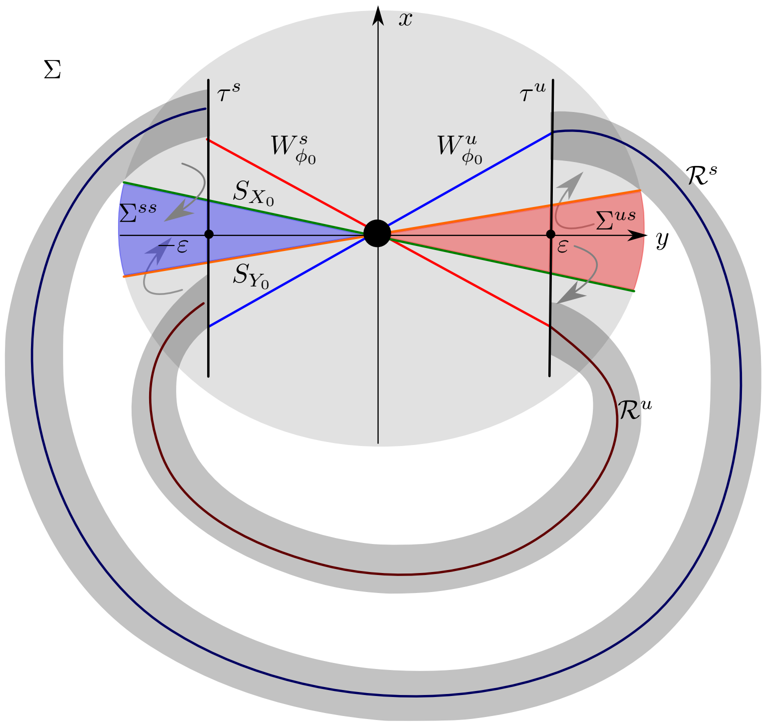


Figure 6.6: Switching manifold of Z_0: Position of the invariant manifolds and tangency sets.

of X_0 contained in M^+ which is oriented from x to $\varphi_+(x)$. Analogously, we obtain a $\mathcal{C}^r$ diffeomorphism $\varphi_- : N^- \cap M^- \to \Sigma$ defined in a neighborhood N^- of $(\mathcal{C}^s \cup \widehat{\mathcal{C}^u}) \cap \overline{M^-}$ in the plane $\{y = -\varepsilon\}$, such that, for every $x \in N^- \cap M^-$, there exists a unique piece of orbit of Y_0 contained in M^- connecting x and $\varphi_-(x)$ oriented from x to $\varphi_-(x)$. See Figure 6.7.

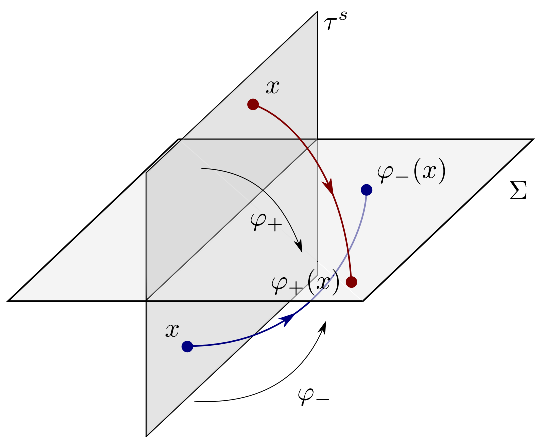


Figure 6.7: Sections τ^u and τ^s.

Let $N = (N^+ \cap M^+) \cup (N^- \cap M^-)$ and $\varphi : N \to \Sigma$ be the $\mathcal{C}^r$ diffeomorphism defined by

$$\varphi(p) = \begin{cases} \varphi_+(p), & \text{if } p \in N^+ \cap M^+, \\ \varphi_-(p), & \text{if } p \in N^- \cap M^-. \end{cases}$$

Now, for $\star = u, s$, let $p^\star_{\text{loc}}$ be the unique point of $\mathcal{C}^s$ contained in the local invariant manifold $W^\star_{\phi_0}(p_0)$. Recall that, there exists a crossing orbit of Z_0 from p^s_{loc} to $\Phi_0(p^s_{\text{loc}}) \in W^s_{\phi_0}(p_0)$ and $\pi_2(\Phi_0(p^s_{\text{loc}})) > -\varepsilon$. Also we have that $-\varepsilon < \pi_2(\Phi_{X_0}(p^u_{\text{loc}})) < \pi_2(\Phi_0(p^s_{\text{loc}}))$. See Figure 6.8.

From the definition of Φ_0, it follows that:

- if $p \in \mathcal{C}^s \cap M^+$, then $\varphi(p) \in W^s_{\phi_0}(p_0)$;
- if $p \in \mathcal{C}^s \cap M^-$, then $\varphi(p) \in W^u_{\phi_0}(p_0)$.

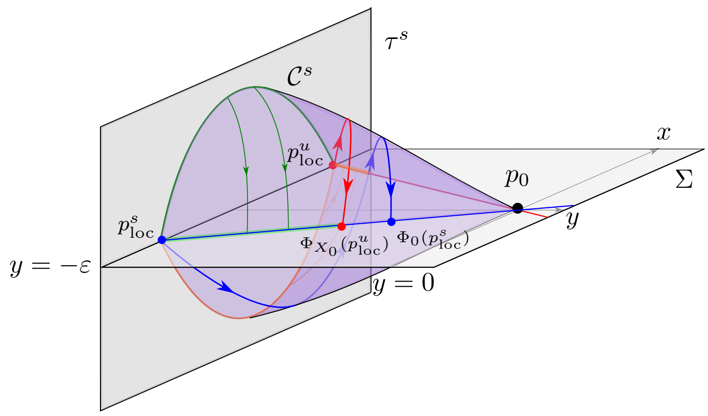


Figure 6.8: Evolution of the flow of Z_0 through the points p^u_{loc} and p^s_{loc}.

Recall that, for each $p \in W^u_{\phi_0}(0,0) \cap \{x < 0\}$, $(\phi_0^n(p))_{n\in\mathbb{N}}$ represents a crossing orbit of Z_0 which is oriented in the order given by the iterations of p through ϕ_0. Since $\mathcal{R}^u$ extends $W^u_{\phi_0}(0,0) \cap \{x < 0\}$, the same property holds for $p \in \mathcal{R}^u$.

For $\star = u, s$, let $p^\star_{\mathcal{R}}$ be the unique point of $\widehat{\mathcal{C}^u}$ contained in the curve $\mathcal{R}^\star$. There exists a crossing orbit of Z_0 from $\Phi_0^{-1}(p^u_{\mathcal{R}}) \in \mathcal{R}^u$ and $p^u_{\mathcal{R}}$, and $\pi_2(\Phi_0^{-1}(p^u_{\mathcal{R}})) < -\varepsilon$. Also we have that $\pi_2(\Phi_0^{-1}(p^u_{\mathcal{R}})) < \pi_2(\Phi_{Y_0}(p^s_{\mathcal{R}})) < -\varepsilon$. See Figure 6.9. Therefore,

- if $p \in \widehat{\mathcal{C}^u} \cap M^+$, then $\varphi(p) = \Phi_0(\tilde{p})$, for some $\tilde{p} \in W^u_{\Phi_0}$;
- if $p \in \widehat{\mathcal{C}^u} \cap M^-$, then $\varphi(p) = \Phi_0^{-1}(\tilde{p})$, for some $\tilde{p} \in W^s_{\Phi_0}$.

Finally, from hypothesis (R), we have that $\mathcal{C}^s \cap \widehat{\mathcal{C}^u} = \{q_1, q_2\}$, where $q_i \notin \Sigma$ and $\mathcal{C}^s \pitchfork \widehat{\mathcal{C}^u}$ at q_i, $i = 1, 2$. Without loss of generality, assume that $q_1 \in M^+$. Since φ is a diffeomorphism, $\varphi(\widehat{\mathcal{C}^u} \cap M^+) \subset W^u_{\Phi_0}$ and $\varphi(\mathcal{C}^s \cap M^+) \subset W^s_{\Phi_0}$, it follows that the invariant manifolds $W^s_{\Phi_0}$ and $W^u_{\Phi_0}$ intersect transversally at the point $\widehat{q}_1 = \varphi(q_1)$. Analogously, we have that $W^s_{\Phi_0}$ and $W^u_{\Phi_0}$ intersect transversally at the point $\widehat{q}_2 = \varphi(q_2)$. Also, $(\Phi_0^n(\widehat{q}_1))_{n\in\mathbb{N}}$ and $(\Phi_0^n(\widehat{q}_2))_{n\in\mathbb{N}}$ define two distinct orbits of Φ_0.

Therefore, Theorem P follows straightly from Theorem 6.5.5 from [61], which is stated below.

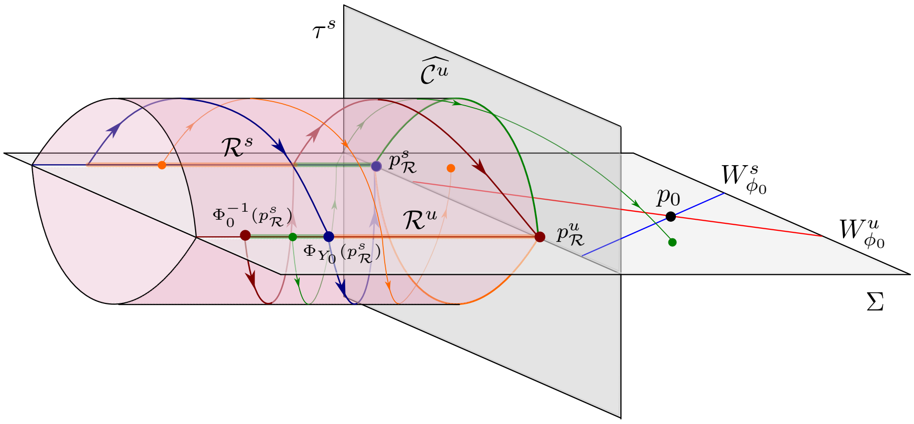


Figure 6.9: Evolution of the flow of Z_0 through the points $p^u_{\mathcal{R}}$ and $p^s_{\mathcal{R}}$.

Theorem 6.2.3 (Theorem 6.5.5 of [61])**.** *Let M be a smooth manifold, $U \subset M$ open, $f : U \to M$ an embedding, and $p \in U$ a hyperbolic fixed point with a transverse homoclinic point q. Then in an arbitrarily small neighborhood of p there exists a horseshoe for some iterate of f. Furthermore the hyperbolic invariant set in this horseshoe contains an iterate of q.*

6.3 A Model Presenting a T-connection

In this section we construct a Filippov system having two robust fold-fold connections. It is worth mentioning that such model can be used to produce examples of T-chains satisfying conditions (TC) and (R).

6.3.1 Filippov System Z_1

Consider the Filippov system

$$Z_1(x, y, z) = \begin{cases} X_1(x, y, z) = (1, -1, y), & \text{if } z > 0, \\ Y_1(x, y, z) = (-1, 2, -x), & \text{if } z < 0. \end{cases}$$

Notice that $(0, 0, 0)$ is a T-singularity and $S_{X_1} = \{y = 0\}$ and $S_{Y_1} = \{x = 0\}$ are fold lines which divide the switching manifold $\Sigma = \{z = 0\}$ in four quadrants (see Figure 6.10). In addition

- $\Sigma^{ss} = \{(x, y, 0);\ x, y < 0\}$;
- $\Sigma^{us} = \{(x, y, 0);\ x, y > 0\}$;
- $\Sigma^{c} = \{(x, y, 0);\ xy < 0\}$;

A straightforward computation shows that the flow of X_1 and Y_1 are given by

$$\varphi_{X_1}(t; (x, y, z)) = \begin{pmatrix} x + t \\ y - t \\ z - \dfrac{(y - t)^2}{2} + \dfrac{y^2}{2} \end{pmatrix},$$

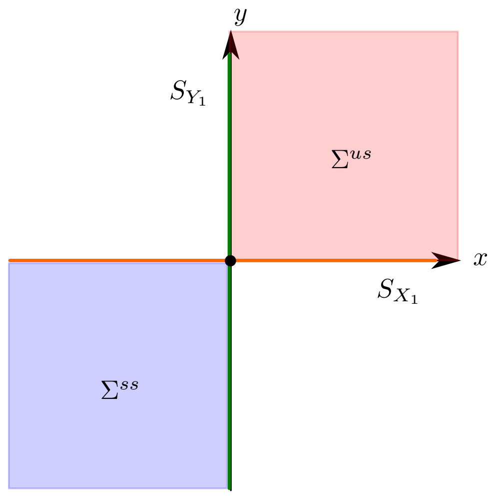


Figure 6.10: Switching manifold associated to Z_1.

and

$$\varphi_{Y_1}(t;(x,y,z)) = \begin{pmatrix} x-t \\ y+2t \\ z+\dfrac{(x-t)^2}{2}-\dfrac{x^2}{2} \end{pmatrix},$$

respectively. It allows us to see that

$$\varphi_{X_1}(t;(x,y,0)) \in \Sigma \iff t=0 \text{ or } t=2y,$$

and

$$\varphi_{Y_1}(t;(x,y,0)) \in \Sigma \iff t=0 \text{ or } t=2x.$$

Thus, we can associate involutions $\phi_{X_1}, \phi_{Y_1} : \Sigma \to \Sigma$ associated to the fold lines S_{X_1} and S_{Y_1} of the vector fields X_1 and Y_1 respectively. They are given by

$$\phi_{X_1}(x,y) = \begin{pmatrix} x+2y \\ -y \end{pmatrix}, \text{ and } \phi_{Y_1}(x,y) = \begin{pmatrix} -x \\ 4x+y \end{pmatrix}.$$

Now, use the involutions to construct the first return map $\phi_1 : \Sigma \to \Sigma$ given by:

$$\phi_1(x,y) = \phi_{Y_1} \circ \phi_{X_1}(x,y) = \begin{pmatrix} -x-2y \\ 4x+7y \end{pmatrix}.$$

Notice that ϕ_1 is globally defined on Σ. The eigenvalues of ϕ_1 are given by

$$\lambda_1^{\pm} = 3 \pm 2\sqrt{2},$$

and their respective eigenvectors are

$$v_1^{\pm} = (-1 \pm \sqrt{2}/2, 1).$$

Since ϕ_1 is linear, the lines

$$D_1^{\pm} = \{\alpha(-1 \pm \sqrt{2}/2, 1);\ \alpha \in \mathbb{R}\},$$

are global invariant manifolds of ϕ_1. Furthermore,

$$\phi_{X_1}(D_1^-) = D_1^+ \text{ and } \phi_{Y_1}(D_1^+) = D_1^-.$$

These facts ensure the existence of global crossing invariant manifolds in the form of a nonsmooth diabolo $\mathcal{N}_1$ which intersects Σ in $D_1^{\pm}$. See Figure 6.11.

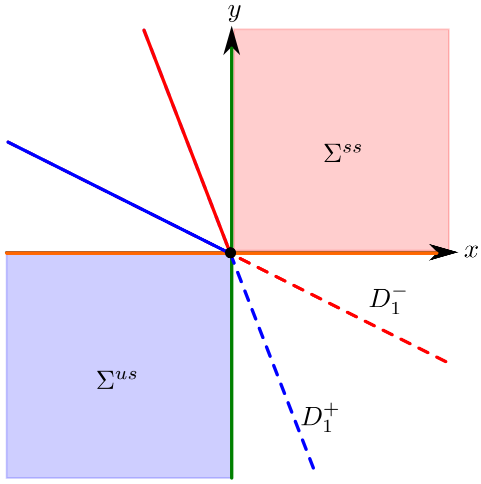


Figure 6.11: Illustration of $\mathcal{N}_1 \cap \Sigma$.

6.3.2 Filippov System Z_2

Consider the Filippov system

$$Z_2(x,y,z) = \begin{cases} X_2(x,y,z) = (1,-1,y-2), & \text{if } z > 0, \\ Y_2(x,y,z) = (-1,3,-(x-2)), & \text{if } z < 0. \end{cases}$$

Notice that $(2,2,0)$ is a T-singularity and $S_{X_2} = \{y = 2\}$ and $S_{Y_2} = \{x = 2\}$ are fold lines which divide the switching manifold $\Sigma = \{z = 0\}$ in four quadrants (see Figure 6.12). In addition

- $\Sigma^{us} = \{(x,y,0);\ x,y > 2\}$;
- $\Sigma^{ss} = \{(x,y,0);\ x,y < 2\}$;
- $\Sigma^{c} = \{(x,y,0);\ (x-2)(y-2) < 0\}$;

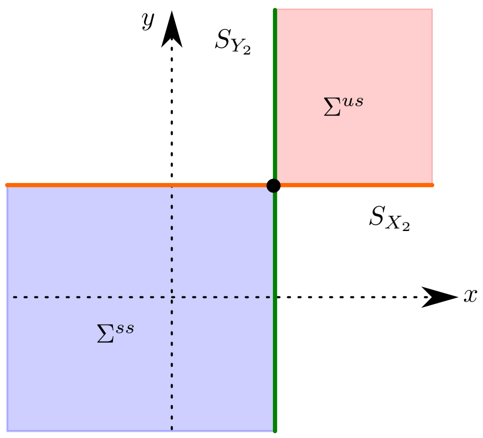


Figure 6.12: Switching manifold associated to Z_2.

A straightforward computation shows that the flows of X_2 and Y_2 are given by

$$\varphi_{X_2}(t;(x,y,z)) = \begin{pmatrix} x+t \\ y-t \\ z-\dfrac{(y-t)^2}{2}+\dfrac{y^2}{2}-2t \end{pmatrix},$$

and

$$\varphi_{Y_2}(t;(x,y,z)) = \begin{pmatrix} x-t \\ y+3t \\ z+\dfrac{(x-t)^2}{2}-\dfrac{x^2}{2}+2t \end{pmatrix},$$

respectively. It allows us to see that

$$\varphi_{X_2}(t;(x,y,0)) \in \Sigma \Longleftrightarrow t=0 \text{ or } t=2(y-2),$$

and

$$\varphi_{Y_2}(t;(x,y,0)) \in \Sigma \Longleftrightarrow t=0 \text{ or } t=2(x-2).$$

As before, we can associate involutions $\varphi_{X_2}, \varphi_{Y_2} : \Sigma \to \Sigma$ associated to the fold lines S_{X_2} and S_{Y_2} of the vector fields X_2 and Y_2, respectively. They are given by

$$\phi_{X_2}(x,y) = \begin{pmatrix} x+2(y-2) \\ 2-(y-2) \end{pmatrix}, \text{ and } \phi_{Y_2}(x,y) = \begin{pmatrix} 2-(x-2) \\ y+6(x-2) \end{pmatrix}.$$

Now, use the involutions to construct the first return map $\phi_2 : \Sigma \to \Sigma$ given by

$$\phi_2(x,y) = \phi_{Y_2} \circ \phi_{X_2}(x,y) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ 6 & 11 \end{pmatrix} \begin{pmatrix} x-2 \\ y-2 \end{pmatrix}.$$

The eigenvalues of ψ_2 at the fixed point $(2,2)$ are given by

$$\lambda_2^{\pm} = 5 \pm 2\sqrt{6},$$

and their respective eigenvectors are

$$v_2^{\pm} = (-1 \pm \sqrt{6}/3, 1).$$

Since ϕ_2 is linear, the lines

$$D_2^{\pm} = \{(2,2) + \alpha(-1 \pm \sqrt{6}/3, 1);\ \alpha \in \mathbb{R}\},$$

are global invariant manifolds of ϕ_2. Furthermore,

$$\phi_{X_2}(D_2^-) = D_2^+ \text{ and } \phi_{Y_2}(D_2^+) = D_2^-.$$

These facts ensure the existence of a nonsmooth diabolo $\mathcal{N}_2$ which intersects Σ in $D_2^{\pm}$. See Figure 6.13.

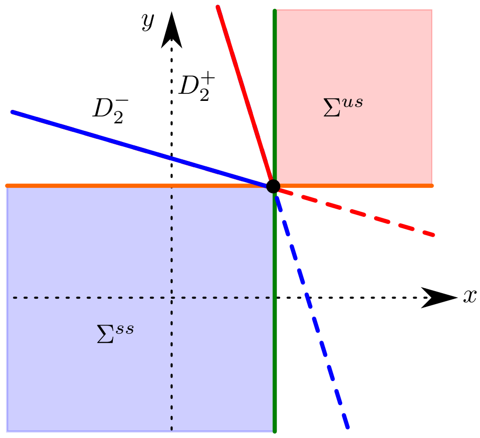


Figure 6.13: Illustration of $\mathcal{N}_2 \cap \Sigma$.

6.3.3 A Filippov System with a Cross Shaped Switching Manifold

Consider a new switching manifold $\Pi = g^{-1}(0)$, where

$$g(x, y, z) = -(y + 17/10x - 5/2).$$

Let $W_1^u(0,0,0)$ be the unstable branch of the nonsmooth diabolo $\mathcal{N}_1$ of Z_1 at the origin. In this case, $W_1^u(0,0,0) \cap \{z > 0\}$ is given by the parametrized set

$$\mathcal{W}_1^{u,+} = \{(\alpha(-1+\sqrt{2}/2)\alpha + t, \alpha - t, -(\alpha-t)^2/2 + \alpha^2/2);\ 0 \leq t \leq 2\alpha \text{ and } \alpha \geq 0\}.$$

Similarly, denoting the stable branch of the nonsmooth diabolo $\mathcal{N}_2$ of Z_2 at $(2,2,0)$ by $W_2^s(2,2,0)$, we obtain that $W_2^s(2,2,0) \cap \{z > 0\}$ is given by

$$\mathcal{W}_2^{s,+} = \{(2+(-1-\sqrt{2/3})\alpha+t, 2+\alpha-t, (2+\alpha)^2/2-(2+\alpha-t)^2/2-2t);\ 0 \leq t \leq 2\alpha \text{ and } \alpha \geq 0\}$$

A straight computation, shows that

$$\mathcal{C}_+^u = \mathcal{W}_1^{u,+} \cap \Pi = \{\gamma_+^u(\alpha);\ 25/191(-14+17\sqrt{2}) \leq \alpha \leq 25/191(14+17\sqrt{2})\},$$

where

$$\gamma_+^u(\alpha) = (-5/7(-5+\sqrt{2}\alpha), 1/14(-50+17\sqrt{2}\alpha), 1/196(-1250+850\sqrt{2}\alpha-191\alpha^2)).$$

Also

$$\mathcal{C}_-^s = \mathcal{W}_2^{s,+} \cap \Pi = \{\gamma_+^s(\alpha);\ 29/431(-21+17\sqrt{6}) \leq \alpha \leq 29/431(21+17\sqrt{6})\},$$

where

$$\gamma_+^s(\alpha) = (5/21(-9+2\sqrt{6}\alpha), 1/21(129-17\sqrt{6}\alpha), 1/294(-2523+986\sqrt{6}\alpha-431\alpha^2)).$$

In addition

$$\mathcal{C}_+^u \cap \mathcal{C}_-^s = p_+^* = \left(-5/49(-67+8\sqrt{51}), 1/49(-447+68\sqrt{51}), (-330577+48248\sqrt{51})/4802\right), \tag{6.3.1}$$

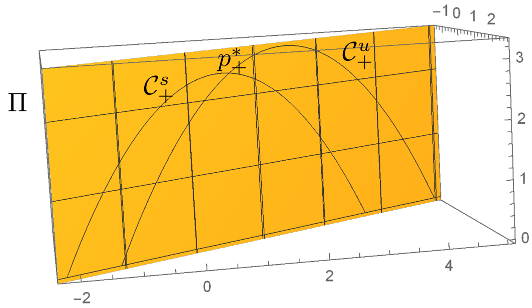


Figure 6.14: Sketch of $\mathcal{C}_+^u$ and $\mathcal{C}_-^s$.

and $\mathcal{C}_+^u \pitchfork \mathcal{C}_-^s$ at p_+^*. See Figure 6.14.

Also, notice that the vector fields X_1, X_2, Y_1 and Y_2 are transverse to Π at every point of Π. Thus, the piecewise smooth system

$$Z_0(x,y,z) = \begin{cases} Z_1(x,y,z), & \text{if } g(x,y,z) < 0, \\ Z_2(x,y,z), & \text{if } g(x,y,z) > 0. \end{cases}$$

with a cross-shaped switching manifold has an isolated crossing orbit connecting the T-singularities $(0,0,0)$ and $(2,2,0)$ of Z_0 passing through p_+^* given in (6.3.1). See Figure 6.15.

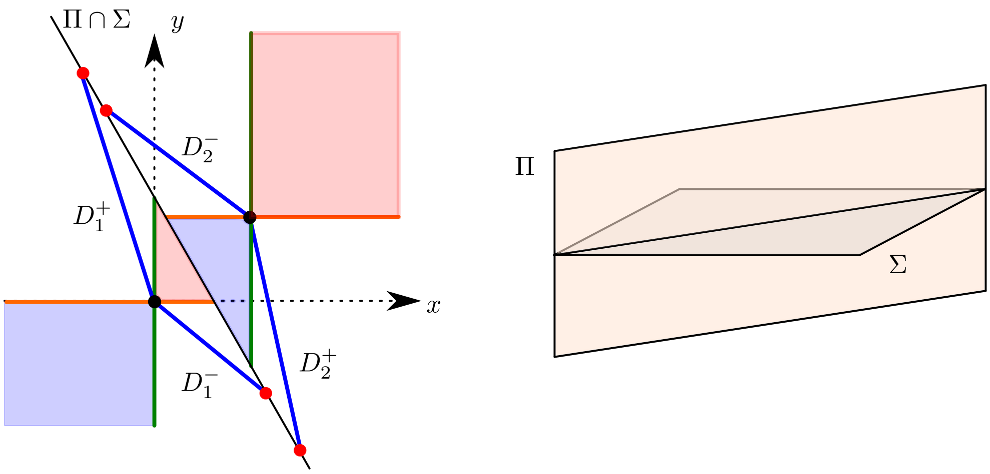


Figure 6.15: (a) Switching manifold Σ associated to Z_0. (b) Cross-shaped switching manifold $\Sigma \cup \Pi$.

Analogous conclusions can be shown for $z < 0$. In this case, Z_0 has another isolated crossing orbit connecting the T-singularities $(0,0,0)$ and $(2,2,0)$ of Z_0 passing through a point $p_-^* \in \Pi$ contained in $z < 0$.

6.3.4 A Filippov system presenting fold-fold connections

Consider the $\mathcal{C}^1$-regularization function

$$\varphi(x) = \begin{cases} -1, & \text{if } x < -1, \\ \sin(\pi/2x), & \text{if } |x| \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Thus, regularizing Filippov systems (X_1, X_2, Π) and (Y_1, Y_2, Π) with respect to the switching manifold Π. We obtain

$$\mathcal{X}_\varepsilon(x,y,z) = \frac{X_2(x,y,z) + X_1(x,y,z)}{2} + \varphi\left(\frac{f(x,y,z)}{\varepsilon}\right)\frac{X_2(x,y,z) - X_1(x,y,z)}{2}$$

and

$$\mathcal{Y}_\varepsilon(x,y,z) = \frac{Y_2(x,y,z) + Y_1(x,y,z)}{2} + \varphi\left(\frac{f(x,y,z)}{\varepsilon}\right)\frac{Y_2(x,y,z) - Y_1(x,y,z)}{2}.$$

Thus, for $\varepsilon > 0$, $\mathcal{Z}_\varepsilon = (\mathcal{X}_\varepsilon, \mathcal{Y}_\varepsilon)$ is a Filippov system with switching manifold $\Sigma = \{z = 0\}$ and $\mathcal{Z}_\varepsilon \in \Omega^1$.

Remark 6.3.1. *If we consider a regularizing function φ which is of class $\mathcal{C}^r$, then $\mathcal{Z}_\varepsilon \in \Omega^r$.*

It follows that $\mathcal{Z}_\varepsilon$ has two (stable) T-singularities at $p_1 = (0,0,0)$ and $p_2 = (2,2,0)$. Now, since the invariant manifolds $W_1^u(p_1)$ and $W_2^s(p_2)$ intersect Π transversally in two topological circles $\mathcal{C}^u$ and $\mathcal{C}^s$, we have that:

1. The unstable crossing invariant manifold $W_\varepsilon^u(p_1)$ of $\mathcal{Z}_\varepsilon$ at p_1 intersects the transversal section $\Pi_{-\varepsilon} = \{g(x,y,z) = -\varepsilon\}$ in a topological circle $\mathcal{C}_\varepsilon^u$;

2. The stable crossing invariant manifold $W_\varepsilon^s(p_2)$ of $\mathcal{Z}_\varepsilon$ at p_2 intersects the transversal section $\Pi_\varepsilon = \{g(x,y,z) = \varepsilon\}$ in a topological circle $\mathcal{C}_\varepsilon^s$.

Consider a small annulus D_ε^u around $\mathcal{C}_\varepsilon^u$ contained in the plane $\Pi_{-\varepsilon}$. Now, $\mathcal{X}_\varepsilon$ and $\mathcal{Y}_\varepsilon$ are transverse to $\Pi_{-\varepsilon}$, Π_ε and $z = 0$ and there is no singularities of $\mathcal{Z}_\varepsilon$ in the cylindrical region delimited by D_ε^u and the regularization zone $R_\varepsilon = \{(x,y,z);\ |g(x,y,z)| < \varepsilon\}$. It means that the flow of $\mathcal{Z}_\varepsilon$ is tubular inside this region (it has only crossing orbits which goes from D_ε^u to Π_ε.

Therefore, $W_\varepsilon^u(p_1)$ extend itself in the regularization zone through crossing orbits of $\mathcal{Z}_\varepsilon$, and it intersects Π_ε in a topological circle $\widehat{\mathcal{C}_\varepsilon^u}$.

Since $\mathcal{C}^u$ and $\mathcal{C}^s$ intersect transversally at the points p_-^* and p_+^*, it follows that, for $\varepsilon > 0$ sufficiently small, $\widehat{\mathcal{C}_\varepsilon^u}$ and $\mathcal{C}_\varepsilon^s$ intersect themselves at two points $q_1(\varepsilon) \in \Sigma^-$ and $q_2(\varepsilon) \in \Sigma^+$.

It means that, there exists $\varepsilon_0 > 0$ sufficiently small such that, for each $\varepsilon \leq \varepsilon_0$, the one-parameter family $\mathcal{Z}_\varepsilon \in \Omega^1$ has two robust fold-fold connections between the T-singularities p_1 and p_2.

6.4 Conclusion and Further Directions

In this chapter, we have presented a robust global phenomenon in $3D$ Filippov systems Z which generates a chaotic behavior in the foliation associated to Z (composed by orbits and pseudo-orbits of Z).

In general the notion of chaos in Filippov system is still poorly understood due to the richness and complexity of the dynamics generated by discontinuities. In fact, there are works exploring this subject (see, for instance, [20, 77, 78]), nevertheless most of them uses classical concepts in order to characterize chaotic behavior, which is sufficient for the specific situations treated in such works. In light of this, a generalization of the concept of chaos is needed for Filippov systems, taking into account the non-uniqueness of solutions at Σ-singularities and sliding orbits, in order to characterize complicated behavior which can not be reduced to a classical setting.

It is known that a Filippov system presents non-deterministic chaos at a stable T-singularity due to the local behavior of the sliding vector field at such point (see [25]). So, we ask ourselves how the sliding dynamics interacts with the hyperbolic invariant sets associated to the Smale horseshoe of the first return map found in this chapter. It is an arduous task which might bring new ways towards the comprehension of chaos for Filippov systems.

Chapter 7

Critical Velocity in Kink-Defect Interaction Models

In this work we study a model of interaction of kinks of the sine-Gordon equation with a weak defect. We obtain rigorous results concerning the so-called critical velocity derived in [47] by a geometric approach. More specifically, we prove that a heteroclinic orbit in the energy level 0 of a 2-dof Hamiltonian H_ε is destroyed giving rise to heteroclinic connections between certain elements (at infinity) for exponentially small (in ε) energy levels. In this setting Melnikov theory does not apply because there are exponentially small phenomena.

7.1 Introduction

Given an evolutionary partial differential equation, a traveling wave is a solution which travels with constant speed and shape. There are several types of traveling waves which are important in modeling physical phenomena. In particular, we give special attention to kinks, also referred as solitons. A soliton is a spatially localized traveling wave which usually appears as a result of a balance between a nonlinearity and dispersion.

In fact, kinks are traveling waves which travel from one asymptotic state to another. In the last years, solitons have attracted the focus of researchers due to their significant role in many scientific fields as optical fibers, fluid dynamics, plasma physics and others (see [51, 64, 106] and references therein).

In this work, we study a model of interaction between kinks (traveling waves) of the sine-Gordon equation and a weak defect. The defect is modeled as a small perturbation given by a Dirac delta function. Such interaction has also been studied for the nonlinear Schrödinger equation in [58, 59].

We consider the finite-dimensional reduction of the equation given by a 2-degrees of freedom Hamiltonian H proposed in [36, 47]. Following a geometric approach, we give conditions on the energy of the system to admit kink-like solutions.

7.1.1 The model

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation given by

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0,$$

which presents a family of kinks $u_{\mathtt{k}}(x,t)$ given by

$$u_{\mathtt{k}}(x,t) = 4\arctan\left(\exp\left(\frac{x - vt - x_0}{\sqrt{1-v^2}}\right)\right), \tag{7.1.1}$$

where the parameter v represents the velocity of the kink.

In this work, we perturb this equation by a localized nonlinear defect at the origin

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = \varepsilon\delta(x)\sin(u), \tag{7.1.2}$$

where $\delta(x)$ is the Dirac delta function. This equation was studied in [36, 47] where the authors consider finite-dimensional reductions of it to understand the kink-like dynamics. As a first step, they consider solutions u of small amplitude of (7.1.2), which can be approximated by solutions of the linear partial differential equation

$$\partial_t^2 u - \partial_x^2 u + u = \varepsilon\delta(x)u, \tag{7.1.3}$$

which has a family of wave solutions $u_{\mathtt{im}}(x,t)$ given by

$$u_{\mathtt{im}}(x,t) = a(t)e^{-\varepsilon|x|/2}, \tag{7.1.4}$$

where $a(t) = a_0\cos(\Omega t + \theta_0)$, $\Omega = \sqrt{1-\varepsilon^2/4}$ and $\mathtt{im}$ stands for impurity. The solution $u_{\mathtt{im}}$ is not a traveling wave, but it is spatially localized at $x = 0$.

In order to study the interaction of kinks of the sine-Gordon equation with the defect considered in (7.1.2), [36, 47] use variational approximation techniques to obtain the equations which describe the evolution of the kink position X and the defect mode amplitude a. To derive such equations, they consider the ansatz

$$u(x,t) = 4\arctan(\exp(x - X(t))) + a(t)e^{-\varepsilon|x|/2}. \tag{7.1.5}$$

Notice that (7.1.5) combines the traveling property of the family of kinks (7.1.1) with the localized shape of (7.1.4). If

$$X(t) = \frac{vt - x_0}{\sqrt{1-v^2}} \text{ and } a(t) \equiv 0,$$

then (7.1.5) becomes the original family of kinks (7.1.1) of (7.1.2) for $\varepsilon = 0$.

Using the ansatz (7.1.5) in (7.1.2) and considering terms up to order 2 in ε, [36, 47] obtain the system of Euler-Lagrange equations

$$\begin{cases} 8\ddot{X} + \varepsilon U'(X) + \varepsilon a F'(X) = 0, \\ \ddot{a} + \Omega^2 a + \frac{1}{2}\varepsilon^2 F(X) = 0, \end{cases} \tag{7.1.6}$$

where

$$U(X) = -2\,\mathrm{sech}^2(X),\ \ F(X) = -2\tanh(X)\,\mathrm{sech}(X) \text{ and } \Omega = \sqrt{1 - \frac{\varepsilon^2}{4}}, \tag{7.1.7}$$

which describes approximately the evolution of the kink position X and the defect mode amplitude a. More details of this approach and its applications can be found in [36, 47, 72]. It is worth mentioning that the finite dimensional reduction of PDE problems to

ODE systems via an adequate ansatz and variational methods has been considered in an extensive range of works (see [35, 42, 48, 49, 50, 107, 109]).

It remains as an open problem to prove that the solutions of the reduced system rigorously approximate the PDE solutions. Nevertheless there are numerical evidences ensuring this reasoning (see [88, 89]). In particular, in [105], the authors analyze numerically the simulations done in [47] for the perturbed sine-Gordon equation (7.1.2).

From (7.1.5), if $X(t)$ and $a(t)$ satisfy $X(t) \to \pm\infty$, $\dot{X}(t) \to C^{\pm}$ and $a(t) \to 0$ as $t \to \pm\infty$, then $u(x,t)$ can be seen as an approximation for a kink of (7.1.2), since it transitions from an asymptotic state to another when $x - X(t) \to \pm\infty$. In this case, we say that $(X(t), a(t))$ is a **kink-like solution**, or simply a kink, of (7.1.6), and we say that $v_i = C^-$ and $v_f = C^+$ are the **initial velocity** and **final velocity** of the kink.

If $X(t)$ satisfy $X(t) \to \pm\infty$, $\dot{X}(t) \to C^{\pm}$ and $a(t)$ is asymptotic to a periodic function with small amplitude when $t \to +\infty$ of $t \to -\infty$, then $u(x,t)$ can be seen as an approximation for a kink of (7.1.2) with asymptotically periodic oscillations. In this case, we say that $(X(t), a(t))$ is an **oscillating kink-like solution**, or simply an oscillating kink, of (7.1.6), and their initial and final velocities are defined in the same way. In addition, if $(X(t), a(t))$ is an oscillating kink such that $a(t) \to 0$ as $t \to -\infty$ and $a(t)$ is asymptotically periodic as $t \to +\infty$, then it is said to be a **quasi kink-like solution**, or quasi kink.

In this paper we perform a rigorous study of such solutions of the finite-dimensional reduction (7.1.6) of the partial differential equation (7.1.2).

7.1.2 The reduced model

Consider the change of variables $(X, \dot{X}, a, \dot{a}) \to (X, Z, b, B)$, where

$$X = X, Z = \frac{8\dot{X}}{\sqrt{\varepsilon}},\ b = \sqrt{\frac{2\Omega}{\varepsilon}}\varepsilon^{-1/4}a, B = \sqrt{\frac{\varepsilon}{2\Omega}}\varepsilon^{-1/4}\frac{2}{\varepsilon}\dot{a},$$

and the time rescaling $\tau = \sqrt{\varepsilon}t$. Then, denoting $' = d/d\tau$, the evolution equations of (7.1.6) are equivalent to:

$$\begin{cases} X' = \dfrac{Z}{8}, \\ Z' = -U'(X) - \dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}F'(X)b, \\ b' = \dfrac{\Omega}{\sqrt{\varepsilon}}B, \\ B' = -\dfrac{\Omega}{\sqrt{\varepsilon}}b - \dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}F(X), \end{cases} \quad \text{with } \Omega = \sqrt{1 - \frac{\varepsilon^2}{4}}. \tag{7.1.8}$$

Notice that (7.1.8) is a Hamiltonian system with respect to

$$H(X, Z, b, B; \varepsilon) = \frac{Z^2}{16} + U(X) + \frac{\Omega}{2\sqrt{\varepsilon}}(B^2 + b^2) + \frac{\varepsilon^{3/4}}{\sqrt{2\Omega}}F(X)b, \tag{7.1.9}$$

which can be split as $H = H_{\mathrm{p}} + H_{\mathrm{osc}} + R$, where

$$\begin{cases} H_{\mathrm{p}}(X,Z) = \dfrac{Z^2}{16} + U(X), \\ H_{\mathrm{osc}}(b,B) = H_{\mathrm{osc}}(b,B;\varepsilon) = \dfrac{\Omega}{2\sqrt{\varepsilon}}(B^2+b^2), \\ R(X,b) = R(X,b;\varepsilon) = \dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}} F(X) b. \end{cases}$$

Thus the Hamiltonian H is the sum of a pendulum-like Hamiltonian H_{p} with an oscillator H_{osc} coupled by the term R.

Remark 7.1.1. *Applying the change of variables $Y = 4\arctan(e^X)$, the Hamiltonian system* (7.1.8) *is brought into*

$$\begin{cases} \dot{Y} = 2\sin(Y/2)Z/8, \\ \dot{Z} = 2\sin(Y/2)\left(\sin(Y) - \dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}\cos(Y)b\right), \\ \dot{b} = \dfrac{\Omega}{\sqrt{\varepsilon}}B, \\ \dot{B} = -\dfrac{\Omega}{\sqrt{\varepsilon}}b - \dfrac{\varepsilon^{3/4}}{\sqrt{2\Omega}}\sin(Y). \end{cases}$$

When $Y = 0$ and $Y = 2\pi$, this system has parabolic critical points and periodic orbits which have invariant manifolds.

The hyperplanes $Y = 0$ and $Y = 2\pi$ correspond to $X = -\infty$ and $X = +\infty$ of (7.1.8) *respectively. For this reason, even if they are not solutions of the system, they can be seen as asymptotic solutions at infinity. Thus, abusing notation, we denote $f(\pm\infty)$ as $\lim\limits_{X\to\pm\infty} f(X)$ when it is well defined.*

System (7.1.8) inherits many properties of the sine-Gordon equation. In fact, the functions U and F have exponential decay when $|X| \to +\infty$, therefore, for large values of X the system becomes decoupled. Nevertheless, when $X = \mathcal{O}(1)$, the equations are coupled and the Hamiltonians H_{p} and H_{osc} may exchange energy, and this will result in interesting global phenomena.

If $F = 0$ (i.e. $R = 0$), then each energy level $H = h \geq 0$ of system (7.1.6) contains a unique kink solution and all the other solutions will be oscillating kinks (with the same oscillation in both tails). In this paper, we prove that the kink solution in $H = h$ breaks down for low energies (see Theorem Q) and we obtain a **critical energy** h_c (with associated critical initial velocity $v_c = 4\sqrt{h_c}$) such that the energy level $H = h$ (h small) contains a quasi kink (continuation of an unperturbed kink) if and only if $h \geq h_c$. In addition we give an asymptotic formula for h_c (see Theorem S) which happens to be exponentially small in the parameter ε. We also find an energy $0 < h_s < h_c$ such that the energy level $H = h$ (h small) has oscillating kinks if and only if $h \geq h_s$ (see Theorem R).

In [47], the authors present numerical and formal arguments for the existence of the critical velocity v_c and they conjecture that the final velocity v_f of a quasi kink lying in an energy level $h \geq h_c$ (h small) is given by $v_f \approx (v_i - v_c)^{1/2}$, where $v_i \geq v_c$ is its initial velocity. Our results prove the validity of the asymptotic formula for v_c and the conjecture for v_f (see Theorem T).

We emphasize that the rigorous approach presented in this work is necessary to validate the conclusions obtained in [47]. In fact, their results rely on the computation of a Melnikov integral as a first order for the total loss of energy ΔE over the separatrix of (7.1.8) with $\delta = 0$ (or more precisely of the transfer of energy from the separatrix to the oscillator). Nevertheless, Melnikov theory cannot be applied in this case due the exponentially smallness in the parameter ε of the Melnikov function. In this paper we prove that it is indeed a first order of ΔE. Note that this is not always the case: in general problems presenting exponentially small phenomena, often the Melnikov integral is not the dominant part of the total loss of energy over a separatrix of a Hamiltonian system (see [8]).

In this paper, we relate the loss of energy ΔE, and thus the existence of kinks, quasi kinks and oscillating kinks, with the exponentially small transversal intersection of the invariant manifolds $W^{u,s}$ of certain objects (critical points and periodic orbits) at infinity.

7.2 Mathematical Formulation and Main Goal

7.2.1 The unperturbed Problem

Consider system (7.1.8) for $F = 0$. Then $H = H_{\mathrm{p}} + H_{\mathrm{osc}}$ is just two uncoupled integrable systems.

In the XZ-plane, the solutions are contained in the level curves $H_{\mathrm{p}}(X, Z) = \kappa$. This system can be transformed into a degenerate (parabolic) pendulum by a change of coordinates (see Remark 7.1.1). For $\kappa < 0$, $H_{\mathrm{p}} = \kappa$ is diffeomorphic to a circle. For $\kappa \geq 0$, $H_{\mathrm{p}} = \kappa$ contains the points $q_\kappa^\pm = (\pm\infty, 4\sqrt{\kappa})$ which behave as "fixed points" and are connected by a heteroclinic orbit Υ_κ given by the graph of

$$Z_\kappa(X) = 4\sqrt{\kappa - U(X)} = 4\sqrt{\kappa + \frac{2}{\cosh^2(X)}}, \quad X \in \mathbb{R}. \tag{7.2.1}$$

Notice that Υ_0 is a separatrix. Analogously, $(\pm\infty, -4\sqrt{\kappa}) \in \{H_{\mathrm{p}} = \kappa\}$ are fixed points at infinity connected by the heteroclinic orbit given by the graph of $-Z_\kappa(X)$. See Figure 7.1. From now on, we focus on the heteroclinic orbits contained in $Z > 0$, since all the results of this paper can be obtained for the orbits in $Z < 0$ in an analogous way.

In the bB-plane, the solutions of (7.1.8) for $F = 0$ are

$$P_\kappa = \{H_{\mathrm{osc}} = \kappa\} = \left\{(b, B);\ b^2 + B^2 = 2\kappa\sqrt{\varepsilon}/\Omega\right\} \text{ (see Figure 7.2).} \tag{7.2.2}$$

Combining (7.2.1) and (7.2.2) in the energy level $H = h$, we define

$$\Lambda_{\kappa_1,\kappa_2}^\pm = q_{\kappa_1}^\pm \times P_{\kappa_2} = \left\{(\pm\infty, 4\sqrt{\kappa_1}, b, B);\ b^2 + B^2 = 2\kappa_2/\omega\right\},$$

for every $\kappa_1, \kappa_2 \geq 0$ such that $\kappa_1 + \kappa_2 = h$. Notice that

- If $\kappa_2 = 0$, then $\Lambda_{h,0}^\pm$ is a degenerate saddle (parabolic) point of (7.1.8);
- If $\kappa_2 > 0$, then $\Lambda_{\kappa_1,\kappa_2}^\pm$ are degenerate saddle (parabolic) periodic orbits of (7.1.8).

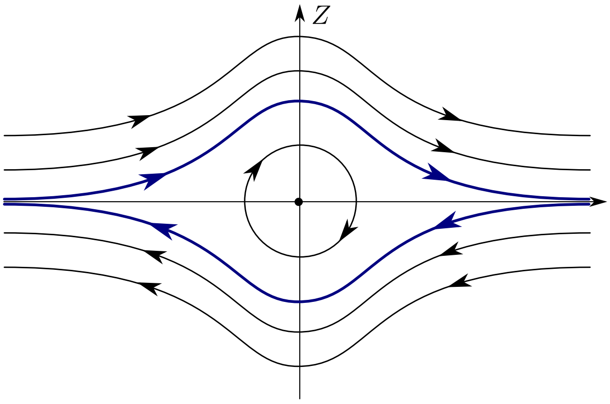


Figure 7.1: Projection of the phase space of the unperturbed system in the XZ-plane.

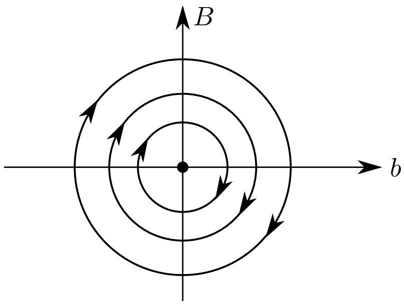


Figure 7.2: Projection of the phase space of the unperturbed system in the bB-plane.

For simplicity, we denote the limit cases $\kappa_1 = 0$ and $\kappa_2 = 0$ by

$$\begin{aligned} \Lambda_h^{\pm} &= \Lambda_{0,h}^{\pm} = \{(\pm\infty, 0, b, B), b^2 + B^2 = 2h/\omega\}, \\ p_h^{\pm} &= \Lambda_{h,0}^{\pm} = (\pm\infty, 4\sqrt{h}, 0, 0), \end{aligned}$$

respectively. We stress that $p_h^{\pm}$ are points and $\Lambda_h^{\pm}$ are periodic orbits, both contained in the planes $X = \pm\infty$ and in the energy level $H = h$.

These invariant objects have invariant manifolds. Denote

$$W(\kappa_1, \kappa_2) = \Upsilon_{\kappa_1} \times P_{\kappa_2} = \left\{(X, Z, b, B);\ Z = 4\sqrt{\kappa_1 - U(X)} \text{ and } b^2 + B^2 = 2\kappa_2\sqrt{\varepsilon}/\Omega\right\}, \tag{7.2.3}$$

for each $\kappa_1, \kappa_2 \geq 0$ such that $\kappa_1 + \kappa_2 = h$.

(1D-0) $W(0,0) = W_0^u(p_0^-) = W_0^s(p_0^+)$ is a 1-dimensional heteroclinic connection (separatrix) between the points p_0^- and p_0^+;

(1D-κ_1) $W(h,0) = W_0^u(p_h^-) = W_0^s(p_h^+)$ is a 1-dimensional heteroclinic connection between the points p_h^- and p_h^+;

(2D-0) If $h > 0$, then $W(0,h) = W_0^u(\Lambda_h^-) = W_0^s(\Lambda_h^+)$ is a 2-dimensional heteroclinic manifold (separatrix) between Λ_h^- and Λ_h^+;

(2D-κ_1) If $\kappa_1, \kappa_2 > 0$, then $W(\kappa_1, \kappa_2)$ is a 2-dimensional heteroclinic manifold between $\Lambda_{\kappa_1,\kappa_2}^-$ and $\Lambda_{\kappa_1,\kappa_2}^+$.

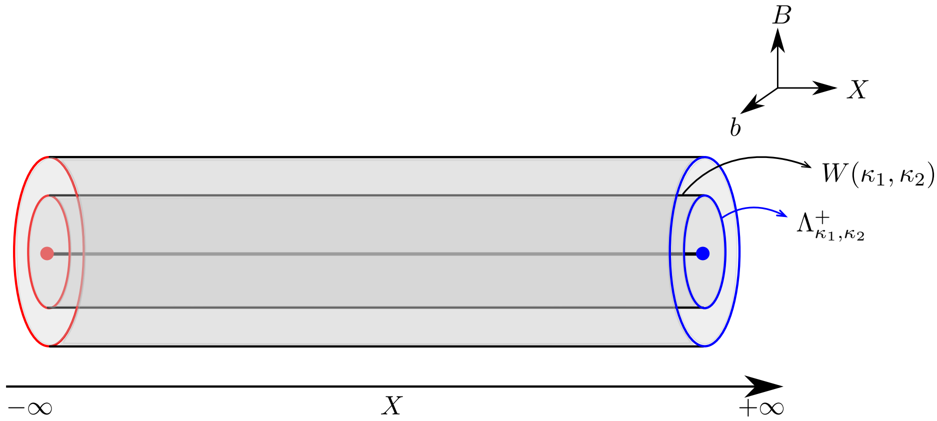


Figure 7.3: Projection of the heteroclinic manifolds $W(\kappa_1, \kappa_2)$ in the bXB-space. In the figure, the most external cylinder is the projection of $W(0, h)$ and the straight line represents the projection of $W(h, 0)$.

For $h > 0$ fixed, the level energy $H = h$ is a 3-dimensional manifold. Eliminating the variable Z by the Hamiltonian conservation, the manifolds $W(\kappa_1, \kappa_2)$ project into the the bXB-space as horizontal cylinders centered along the X-axis.

In this unperturbed case, there is no exchange of energy between the pendulum and the oscillator through the heteroclinic connections of $W(\kappa_1, \kappa_2)$, i.e. H_{p} and $H^{\varepsilon}_{\mathrm{osc}}$ are first integrals. In the perturbed case (7.1.8) ($F \neq 0$) the coupling term R (see (7.1.9)) goes to 0 as $X \to \pm\infty$, thus, the system is uncoupled at $X = \pm\infty$. As a consequence, $\Lambda^{\pm}_{\kappa_1,\kappa_2}$ are orbits of system (7.1.8) in the sense of Remark 7.1.1. Nevertheless, the system may exchange energy between the pendulum and the oscillator when X varies, through the appearance of heteroclinic connections between different $\Lambda^{-}_{\kappa_1,\kappa_2}$ and $\Lambda^{+}_{\kappa_1',\kappa_2'}$ such that $\kappa_1 + \kappa_2 = \kappa_1' + \kappa_2' = h$.

Recall that a quasi-kink (see Section 7.1.1) is a solution $(X(t), Z(t), b(t), B(t))$ which has initial velocity $v_i > 0$ and final velocity $v_f > 0$ and satisfies the asymptotic conditions

$$\lim_{t\to-\infty} X(t) = -\infty, \quad \lim_{t\to-\infty} Z(t) = v_i, \quad \lim_{t\to-\infty} b(t) = \lim_{t\to-\infty} B(t) = 0, \tag{7.2.4}$$

$$\lim_{t\to+\infty} X(t) = +\infty, \quad \lim_{t\to+\infty} Z(t) = v_f. \tag{7.2.5}$$

and $(b(t), B(t))$ are asymptotic to periodic functions as $X \to +\infty$. For such solutions

$$h_i = H(X(t), Z(t), b(t), B(t)) = \frac{v_i^2}{16},$$

for every $t \in \mathbb{R}$.

Thus, considering $h_i = v_i^2/16$ and $\kappa_f = v_f^2/16$, we have that, the quasi-kink solution $(X(t), Z(t), b(t), B(t))$ satisfying (7.2.4) and (7.2.5) is a heteroclinic connection between the 1-dimensional unstable manifold of $p^-_{h_i}$ and the 2-dimensional stable manifold of $\Lambda^+_{\kappa_f, h_i - \kappa_f}$.

7.2.2 Main results

Our aim is to look for solutions traveling from $X = -\infty$ to $X = +\infty$. More concretely, we prove the existence of $v_c > 0$ such that the solutions X of (7.1.6) incoming with velocity v_i escape the defect location and continue traveling towards $X+ = \infty$ with (asymptotic) final velocity v_f, provided $v_i \geq v_c$.

Therefore, the critical energy h_c is characterized as the lowest energy level $h_c = v_c^2/16$ such that for any $h \geq h_c$, there exist $\kappa_1, \kappa_2 > 0$ with $\kappa_1 + \kappa_2 = h$ such that $W_\varepsilon^u(p_h^-) \subset W_\varepsilon^s(\Lambda_{\kappa_1,\kappa_2}^+)$.

Notice that $W_\varepsilon^u(p_h^-) \subset W_\varepsilon^s(\Lambda_{\kappa_1,\kappa_2}^+)$ implies that the final velocity of the corresponding orbit $X(t)$ (which has initial velocity $4\sqrt{h}$) is given by $v_f = 4\sqrt{\kappa_1}$.

To analyze the existence of heteroclinic orbits between the invariant objects at $X = \pm\infty$ we consider the section $X = 0$, which is transversal to the flow. Restricting to the energy level $H = h$, eliminating the variable Z and using (7.1.7), this section becomes the disk

$$\Sigma_h = \left\{(0, b, B); b^2 + B^2 \leq \frac{(4 + 2h)\sqrt{\varepsilon}}{\Omega}\right\}. \tag{7.2.6}$$

We compute intersections between unstable and stable manifolds in Σ_h.

In the unperturbed case $F = 0$, the one-dimensional heteroclinic connection between the "infinity points" p_h^+ and p_h^-, $W(h, 0) = W_0^u(p_h^-) = W_0^s(p_h^+)$ intersect Σ_h at the point $(0, 0)$. In the following theorem, we show that it breaks down when $F \neq 0$ (see Figure 7.4).

Theorem Q (Breakdown of kinks)**.** *Consider system* (7.1.8)*. There exists* $\varepsilon_0 > 0$ *and* $h_0 > 0$ *sufficiently small such that, for every* $0 < \varepsilon < \varepsilon_0$ *and* $0 \leq h \leq h_0$*, the invariant manifolds* $W_\varepsilon^{u,s}(p_h^\mp)$ *intersect* Σ_0 *(given in* (7.2.6)*). Denoting by* $P_h^{u,s}$ *the first intersection points,*

$$\begin{aligned} |P_0^u - P_0^s| &= d_0(\varepsilon) = \frac{2\pi\varepsilon^{3/4}}{\sqrt{\Omega}} e^{-\Omega\sqrt{2/\varepsilon}} + \mathcal{O}\left(\varepsilon^{7/4} e^{-\Omega\sqrt{2/\varepsilon}}\right), \ \textit{where } \Omega = \sqrt{1 - \frac{\varepsilon^2}{4}} \\ |P_h^u - P_h^s| &= d_0(\varepsilon) + \mathcal{O}(\varepsilon^{7/4}\sqrt{h}). \end{aligned} \tag{7.2.7}$$

The first statement of this theorem is proven in Section 7.3.2 and the second one is a consequence of Theorem 7.3.12 stated in Section 7.3.3 below.

Remark 7.2.1. *In the asymptotic formula* (7.2.7)*, we could write* $\Omega = 1$*. Nevertheless, we keep* $\Omega = \sqrt{1 - \varepsilon^2/4}$ *in order to compare our results with [47]. The same remark holds for Theorems B, C and D below.*

When $F = 0$, the energy level h has a family of heteroclinic manifolds $W(\kappa_1, \kappa_2)$, with $\kappa_1 + \kappa_2 = h$, $\kappa_1, \kappa_2 > 0$, connecting the periodic orbits $\Lambda_{\kappa_1,\kappa_2}^\pm$. Each one intersects Σ_h at a circle centered at $(0, 0)$ with radius $\sqrt{2\kappa_2\sqrt{\varepsilon}/\Omega}$, which generates a disk of radius $\sqrt{2h\sqrt{\varepsilon}/\Omega}$ when we vary $0 < \kappa_2 \leq h$ (see (7.2.1) and (7.2.2)).

We show that, for the perturbed case, $W_\varepsilon^u(\Lambda_{\kappa_1,\kappa_2}^-)$ and $W_\varepsilon^s(\Lambda_{\kappa_1,\kappa_2}^+)$ also intersect Σ_h in closed curves near circles of radius $\sqrt{2\kappa_2\sqrt{\varepsilon}/\Omega}$ centered in P_h^u and P_h^s. Thus, varying $0 \leq \kappa_2 \leq h$, we can see that $W_\varepsilon^{u,s}(\Lambda_{\kappa_1,\kappa_2}^\pm)$ intersect Σ_h in topological disks $\mathcal{D}_h^u$ and $\mathcal{D}_h^s$ near the disks of radius $\sqrt{2h\sqrt{\varepsilon}/\Omega}$ centered in P_h^u and P_h^s, respectively (see Figure 7.4).

The existence of heteroclinic connections continuation of the unperturbed ones corresponds to intersections between the disks $\mathcal{D}_h^u$ and $\mathcal{D}_h^s$. Even if in the energy level $h = 0$, there is no (first round) heteroclinic connections between the points at $X = \pm\infty$ (p_0^- and p_0^+), the heteroclinic connections between the periodic orbits $\Lambda_{\kappa_1,\kappa_2}^\pm$ may certainly exist when $h > 0$, since the two disks may intersect for some values of h. The lowest energy level $h_s > 0$ for which these heteroclinic connections exist is reached when the boundaries of these disks are tangent (see Figure 7.4). Equivalently, when $W_\varepsilon^u(\Lambda_h^-)$ intersects $W_\varepsilon^s(\Lambda_h^+)$ in the energy level $h_s = h_s(\varepsilon)$.

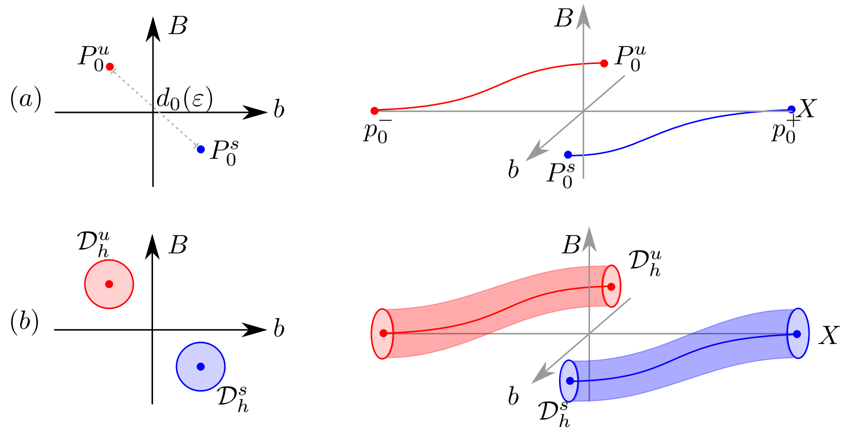


Figure 7.4: Splitting of the invariant manifolds contained in the energy level h (in the section Σ_h) on the left and their projections in the bXB-space on the right, for (a) $h = 0$ and (b) $h > 0$ small.

Theorem R (Existence of oscillating kinks)**.** *Fix $h_0 > 0$. There exists $\varepsilon_0 > 0$ sufficiently small such that, for every $0 < \varepsilon < \varepsilon_0$ and $0 \le h \le h_0$, the invariant manifolds $W_\varepsilon^u(\Lambda_h^-)$, $W_\varepsilon^s(\Lambda_h^+)$ intersect Σ_h (given in* (7.2.6)*). The first intersection is given by closed curves, which we denote by $\partial\mathcal{D}_h^{u,s}$. Then, there exists*

$$h_s(\varepsilon) = \frac{\varepsilon\pi^2 e^{-2\Omega\sqrt{2/\varepsilon}}}{2}(1 + \mathcal{O}(\varepsilon)), \ \textit{with } \Omega = \sqrt{1 - \frac{\varepsilon^2}{4}},$$

such that the following statements hold for system (7.1.8)*.*

1. *If $0 \le h < h_s(\varepsilon)$, the closed curves $\partial\mathcal{D}_h^{u,s}$ do not intersect each other.*
2. *If $h_s(\varepsilon) \le h \le h_0$, the closed curves $\partial\mathcal{D}_h^{u,s}$ intersect at least once.*

Furthermore, given $\mu > 1$, there exists $\varepsilon_\mu > 0$ and

$$h_\mu(\varepsilon) = \frac{\varepsilon\pi^2 e^{-2\Omega\sqrt{2/\varepsilon}}}{2}(\mu + \mathcal{O}(\varepsilon))^2 \ge h_s(\varepsilon),$$

such that, for $0 < \varepsilon < \varepsilon_\mu$ and $h_\mu(\varepsilon) \le h \le h_0$, the closed curves $\partial\mathcal{D}_h^{u,s}$ have at least two intersections.

Thus, we can see that there is a family of heteroclinic connections between elements of $X = \pm\infty$ which are contained in the energy level h, for $h > h_s$.

Actually, we prove that, in the energy level $H = h_s$, $\partial\mathcal{D}_{h_s}^u$ and $\partial\mathcal{D}_{h_s}^s$ intersect (tangentially) at least once, and for this reason, $\partial\mathcal{D}_{h_s}^u \cap \partial\mathcal{D}_{h_s}^s$ may have more than one point. Also, our methods show that, for $h > h_s$, $\partial\mathcal{D}_h^u \cap \partial\mathcal{D}_h^s$ has at least two points and $\mathcal{D}_h^u \cap \mathcal{D}_h^s$ has at least one connected component with positive Lebesgue measure (see Figure 7.5).

The Critical Energy Level h_c

From our approach and the definitions of Section 7.1.2, the critical energy level occurs for the smallest h such that $W_\varepsilon^u(p_h^-) \subset W_\varepsilon^s(\Lambda_{\kappa_1,\kappa_2}^+)$, for some κ_1, κ_2 satisfying $\kappa_1 + \kappa_2 = h$. Thus, h_c occurs when $W_\varepsilon^u(p_{h_c}^-) \subset W_\varepsilon^s(\Lambda_{h_c}^+)$.

Geometrically speaking, h_c is characterized as the energy level such that $P_{h_c}^u$ belongs to the boundary of the (topological) disk $\mathcal{D}_{h_c}^s$ "centered" in $P_{h_c}^s$ (see Figure 7.5). In the next theorem, we compute $h_c = h_c(\varepsilon)$.

Theorem S (Existence of quasi-kinks)**.** *Consider system* (7.1.8). *There exist* $\varepsilon_0 > 0$, $h_0 > 0$ *and a function*

$$h_c(\varepsilon) = 2\pi^2 \varepsilon e^{-2\Omega\sqrt{2/\varepsilon}}(1 + \mathcal{O}(\varepsilon)), \quad \text{with } 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad \Omega = \sqrt{1 - \frac{\varepsilon^2}{4}},$$

such that, for every $0 < \varepsilon < \varepsilon_0$ *and* $0 < h < h_0$*, the invariant manifolds* $W^u_\varepsilon(p^-_h)$, $W^s_\varepsilon(\Lambda^+_h)$ *intersect* Σ_h *(given in* (7.2.6)*). The first intersection of* $W^u_\varepsilon(p^-_h)$, $W^s_\varepsilon(\Lambda^+_h)$ *with* Σ_h *is given by a point and a closed curve, denoted by* P^u_h *and* $\partial\mathcal{D}^s_h$*, respectively. Then,* $P^u_h \in \partial\mathcal{D}^s_h$ *if, and only if* $h = h_c(\varepsilon)$*.*

See Figure 7.6. Theorem S also holds if we change p^-_h and Λ^+_h by p^+_h and Λ^-_h, respectively.

Now, given $h \geq h_c$, we compute the radius $\kappa_2 = \kappa_2(h)$ of the periodic orbit $\Lambda^+_{\kappa_1,\kappa_2}$ such that p^-_h connects to $\Lambda^+_{\kappa_1,\kappa_2}$ through a heteroclinic orbit.

Theorem T. *There exist* $\varepsilon_0 > 0$, $h_0 > 0$ *sufficiently small such that, for each* $0 < \varepsilon < \varepsilon_0$ *and* $h_c(\varepsilon) \leq h < h_c(\varepsilon) + 2\pi^2\varepsilon e^{-2\Omega\sqrt{2/\varepsilon}} h_0$*, where* $h_c(\varepsilon)$ *is given by Theorem S and* $\Omega = \sqrt{1 - \varepsilon^2/4}$*, there exists a function*

$$\kappa : \left(h_c(\varepsilon), h_c(\varepsilon) + 2\pi^2\varepsilon e^{-2\Omega\sqrt{2/\varepsilon}} h_0\right) \to \mathbb{R},$$

such that:

1. $0 < \kappa(h) < h$ *and* $\lim_{h \to h_c(\varepsilon)^+} \kappa(h) = 0$*;*
2. *For system* (7.1.8)*,* $W^u_\varepsilon(p^-_h) \subset W^s_\varepsilon(\Lambda^+_{\kappa(h), h-\kappa(h)})$*;*
3. *There exists an orbit of* (7.1.8) *with input velocity* $v_i = 4\sqrt{h}$ *and output velocity* $v_f = 4\sqrt{\kappa(h)}$*. Furthermore, define* $v_c = 4\sqrt{h_c}$*, then*

   $$v_f = \sqrt{2 v_c c_\varepsilon}\sqrt{v_i - v_c} + \mathcal{O}((v_i - v_c)^{3/2}),$$

   where $c_\varepsilon = 1 + \mathcal{O}(\varepsilon)$*.*

The last item of Theorem T proves the conjecture $v_f \approx \mathcal{O}\left((v_i - v_c)^{1/2}\right)$ raised in [47].

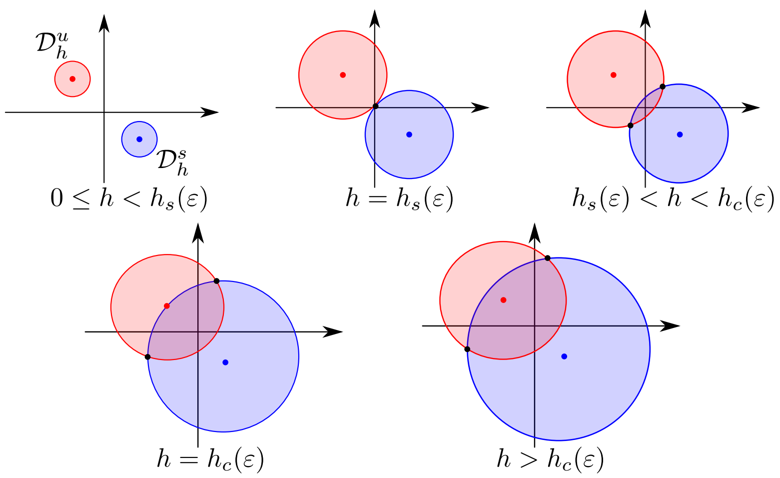


Figure 7.5: Relative position of the disks $\mathcal{D}^u_h$ and $\mathcal{D}^s_h$ in the section Σ_h in function of the energy level h.

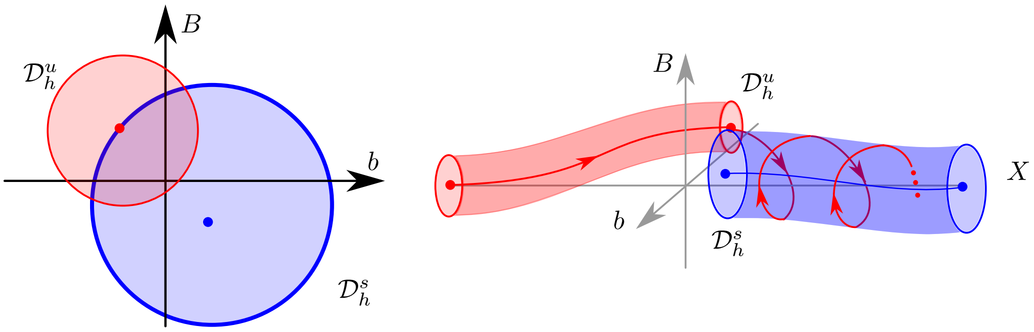


Figure 7.6: Heteroclinic orbit (quasi-kink) between a critical point at $X = -\infty$ and a periodic orbit at $X = +\infty$ in the critical energy level $h = h_c$.

7.3 Proofs of Theorems Q, R, S and T

Applying the change of coordinates $\Gamma = B + ib$ and $\Theta = B - ib$ to (7.1.8) we obtain

$$\begin{cases} X' = \dfrac{Z}{8}, \\ Z' = -U'(X) - \dfrac{\delta}{\sqrt{2\Omega}} F'(X) \dfrac{(\Gamma - \Theta)}{2i}, \\ \Gamma' = \omega i \Gamma - \dfrac{\delta}{\sqrt{2\Omega}} F(X), \\ \Theta' = -\omega i \Theta - \dfrac{\delta}{\sqrt{2\Omega}} F(X), \end{cases} \quad \text{with} \quad \begin{cases} \delta = \varepsilon^{3/4}, \\ \omega = \dfrac{\Omega}{\sqrt{\varepsilon}}, \\ \Omega = \sqrt{1 - \dfrac{\varepsilon^2}{4}}. \end{cases} \tag{7.3.1}$$

This system is Hamiltonian with respect to

$$\mathcal{H}(X, Z, \Gamma, \Theta) = \frac{Z^2}{16} + U(X) + \frac{\delta}{\sqrt{2\Omega}} F(X) \frac{\Gamma - \Theta}{2i} + \frac{\omega}{2} \Gamma \Theta. \tag{7.3.2}$$

and the symplectic form $dX \wedge dZ + \dfrac{1}{2i} d\Gamma \wedge d\Theta$.

7.3.1 Decoupled System ($F = 0$)

We parameterize the invariant manifolds $W(\kappa_1, \kappa_2)$ (see (7.2.3)) of the decoupled system (7.3.1) (with $\delta = 0$) in the coordinates (X, Z, Γ, Θ).

Lemma 7.3.1. *The one-dimensional invariant manifold $W(h, 0) = W_0^u(p_h^-) = W_0^s(p_h^+)$ is parameterized in the coordinate system (X, Z, Γ, Θ) by*

$$N_{h,0}(v) = (X_h(v), Z_h(v), 0, 0), \;\; v \in \mathbb{R} \tag{7.3.3}$$

such that:

1. *If $h = 0$, then*

$$\begin{cases} X_0(v) = \operatorname{arcsinh}\left(\dfrac{\sqrt{2}}{2} v\right), \\ Z_0(v) = 8(X_0)'(v) = \dfrac{8}{\sqrt{v^2 + 2}}. \end{cases} \tag{7.3.4}$$

2. If $h > 0$, then

$$\begin{cases} X_h(v) = \operatorname{arcsinh}\left(\sqrt{\frac{2+h}{h}}\sinh\left(v\sqrt{h}/2\right)\right), \\ Z_h(v) = 8(X_h)'(v) = \dfrac{4\cosh(v\sqrt{h}/2)}{\sqrt{\frac{1}{2+h} + \frac{\sinh^2(v\sqrt{h}/2)}{h}}}. \end{cases} \tag{7.3.5}$$

A simple application of the L'Hospital rule shows us that $X_h(v) \to X_0(v)$, point-wisely, as $h \to 0$. Nevertheless, the decay of X_h at ∞ is significantly different from X_0 (for $h = 0$, the decay is polynomial and for $h > 0$ is exponential). Notice that $N_{0,0}(v)$ has poles at the points $\pm\sqrt{2}i$, whereas the poles of $N_{h,0}(v)$ are all contained in the imaginary axis and the closest to the real line are $\pm\sqrt{2}i + \mathcal{O}(h)$.

Lemma 7.3.2. *The two-dimensional invariant manifold $W(\kappa_1, \kappa_2) = W_0^u(\Lambda^-_{\kappa_1,\kappa_2}) = W_0^s(\Lambda^+_{\kappa_1,\kappa_2})$, with $\kappa_1 \geq 0$, $\kappa_2 > 0$ and $\kappa_1 + \kappa_2 = h$ is parameterized in the coordinate system (X, Z, Γ, Θ) by*

$$N_{\kappa_1,\kappa_2}(v, \tau) = (X_{\kappa_1}(v), Z_{\kappa_1}(v), \Gamma_{\kappa_2}(\tau), \Theta_{\kappa_2}(\tau)), \tag{7.3.6}$$

with $v \in \mathbb{R}$ and $\tau \in \mathbb{T}$, such that

$$\Gamma_{\kappa_2}(\tau) = \sqrt{\frac{2\kappa_2}{\omega}}e^{i\tau}, \text{ and } \Theta_{\kappa_2}(\tau) = \sqrt{\frac{2\kappa_2}{\omega}}e^{-i\tau}, \tag{7.3.7}$$

and X_{κ_1}, Z_{κ_1} are given in (7.3.4) ($\kappa_1 = 0$) and (7.3.5) ($\kappa_1 > 0$).

Remark 7.3.3. *Notice that, if $\kappa_2 = 0$, then N_{κ_1,κ_2} depends on one variable and if $\kappa_2 > 0$, then it depends on two variables.*

Roughly speaking, in the case $\kappa_1 > 0$, the parameterization of the invariant manifolds $W(\kappa_1, \kappa_2)$ have the dependence on v expressed in terms of $e^{v\sqrt{\kappa_1}/2}$. Thus, if we consider v in compact domains, these functions can be easily understood by expanding them in a Taylor series in κ_1. Nevertheless, we must control them for values of v at infinity and κ_1 near of 0, which generates an undetermined situation. For this reason, we have a singular dependence of N_{κ_1,κ_2} at the parameter $\kappa_1 = 0$.

Notice that $N_{\kappa_1,\kappa_2}(v, \tau) \to N_{\kappa_1,0}(v)$ as $\kappa_2 \to 0$ uniformly, and thus the dependence of N_{κ_1,κ_2} is regular at $\kappa_2 = 0$.

Remark 7.3.4. *The $N_{\kappa_1,\kappa_2}(v, \tau)$, with v or τ fixed, do not parameterize the solutions of (7.3.1). Nevertheless, if $\delta = 0$, and $\phi_t^0(\cdot)$ is the flow of (7.3.1), we have*

$$\phi_t^0(N_{\kappa_1,\kappa_2}(v, \tau)) = N_{\kappa_1,\kappa_2}(v + t, \tau + \omega t),$$

therefore they are invariant by the flow.

7.3.2 Proof of Theorem Q (First statement)

The first step to compute the splitting of the separatrix $W(0, 0)$ (parameterized by $N_{0,0}(v)$ in (7.3.3)) in the energy level $h = 0$ is to consider parameterizations

$$N^\star_{0,0}(v) = (X_0(v), Z^\star_0(v), \Gamma^\star_0(v), \Theta^\star_0(v)), \ \star = u, s \tag{7.3.8}$$

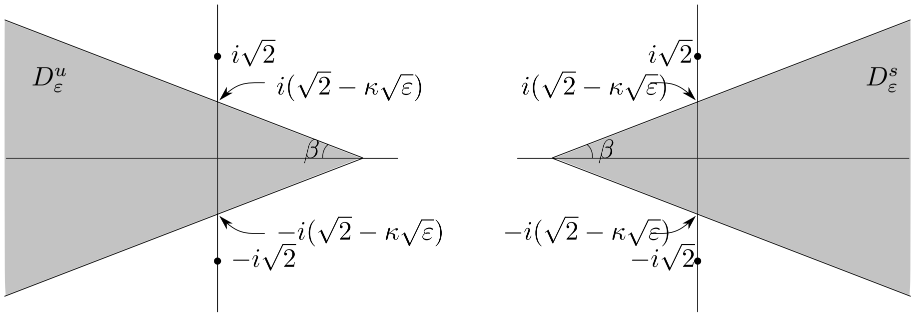


Figure 7.7: Complex domains D^u_ε and D^s_ε.

of the invariant manifolds $W^u_\varepsilon(p_0^-)$ and $W^s_\varepsilon(p_0^+)$ near $N_{0,0}$, in the complex domains

$$\begin{aligned} D^u_\varepsilon &= \{v \in \mathbb{C};\ |\mathrm{Im}(v)| < -\tan\beta\,\mathrm{Re}(v) + \sqrt{2} - \sqrt{\varepsilon}\}, \\ D^s_\varepsilon &= \{v \in \mathbb{C};\ -v \in D^u_\varepsilon\}, \end{aligned} \tag{7.3.9}$$

where $0 < \beta < \pi/4$ is a fixed angle independent of ε (see Figure 7.7). The parameterization $N_{0,0}(v)$ in (7.3.4) has singularities only at $\pm\sqrt{2}i$, thus $N_{0,0}$ is analytic in $D^{u,s}_\varepsilon$.

We state all the results for the unstable case, since it is analogous for the stable one. Based on a fixed point argument, we prove the following theorem in Section 7.4.

Theorem 7.3.5. *Given $\nu > 0$. There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, the one-dimensional manifold $W^u_\varepsilon(p_0^-)$ is parameterized by*

$$N^u_{0,0}(v) = (X_0(v), Z^u_0(v), \Gamma^u_0(v), \Theta^u_0(v)),$$

with $v \in D^u_\varepsilon$, where X_0 is given in (7.3.4), *$Z^u_0(v)$ is obtained from $\mathcal{H}(N^u_{0,0}(v)) = 0$ ($\mathcal{H}$ given in* (7.3.2)*) and*

$$\left\{ \begin{array}{lcl} \Gamma^u_0(v) &=& Q^0(v) + \gamma^u_0(v), \\ \Theta^u_0(v) &=& -Q^0(v) + \theta^u_0(v), \end{array} \right. \tag{7.3.10}$$

with

$$Q^0(v) = -i\frac{\delta}{\omega\sqrt{2\Omega}}F(X_0(v)). \tag{7.3.11}$$

Furthermore, $\gamma^u_0(v), \theta^u_0(v)$ are analytic functions such that $\theta^u_0(v) = \overline{\gamma^u_0(v)}$, for every $v \in \mathbb{R} \cap D^u_\varepsilon$, and there exists a constant $M > 0$ independent of ε such that

1. $|\gamma^u_0(v)|, |\theta^u_0(v)| \leq M\dfrac{\delta}{\omega^2}\dfrac{1}{|v|^2}$, *for each $v \in D^u_\varepsilon$, $|\mathrm{Re}(v)| \leq \nu$;*

2. $|\gamma^u_0(v)|, |\theta^u_0(v)| \leq M\dfrac{\delta}{\omega^2}\dfrac{1}{|v^2+2|^2}$, *for each $v \in D^u_\varepsilon$, $|\mathrm{Re}(v)| \geq \nu$;*

with $\delta = \varepsilon^{3/4}$, $\omega = \Omega/\sqrt{\varepsilon}$ and $\Omega = \sqrt{1-\varepsilon^2/4}$.

Remark 7.3.6. *Notice the points $p_0^\pm$ behave as degenerate-saddles at infinity, and thus the existence of local invariant manifolds for the perturbed system is not standard. Nevertheless, these singularities at infinity behave as parabolic points (see Remark 7.1.1) and Theorem 7.3.5 gives the existence of their invariant manifolds.*

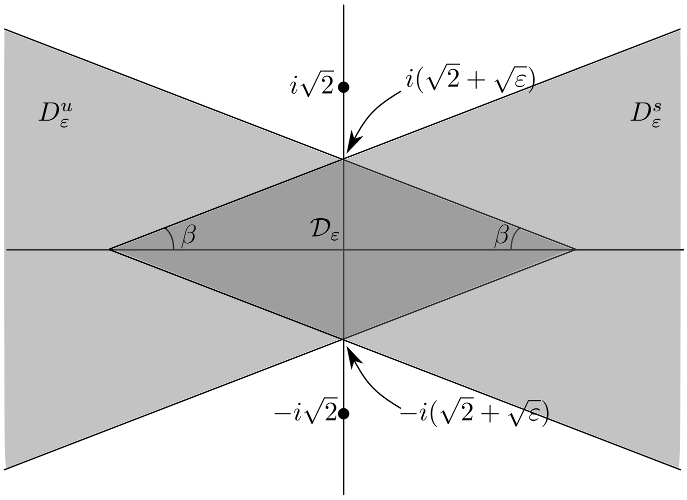


Figure 7.8: Domain $\mathcal{D}_\varepsilon$.

By Theorem 7.3.5, both parameterizations $N^{u,s}_{0,0}(v)$ are defined in the complex domain $\mathcal{D}_\varepsilon = D^u_\varepsilon \cap D^s_\varepsilon$, which contains 0 (see Figure 7.8). To compute the difference between the invariant manifolds in the section Σ_0 (see (7.2.6)), we analyze $\Delta\xi(v)$ given by

$$\Delta\xi(v) = \begin{pmatrix} \Gamma^u_0(v) - \Gamma^s_0(v) \\ \Theta^u_0(v) - \Theta^s_0(v) \end{pmatrix},$$

for $v \in \mathcal{I}_\varepsilon = \mathcal{D}_\varepsilon \cap \mathbb{R}$. We prove that $\Delta\xi$ satisfies

$$\Delta\xi' = \begin{pmatrix} \omega i & 0 \\ 0 & -\omega i \end{pmatrix} \Delta\xi + B(v)\Delta\xi,$$

where the entries of the matrix B are small functions of order $\mathcal{O}(\delta^2)$.

Notice that, if $B \equiv 0$, then $\Delta\xi$ is the analytic function

$$\Delta\xi(v) = \begin{pmatrix} e^{\omega i(v-v_0)}\Delta\xi(v_0) \\ e^{-\omega i(v-v_1)}\Delta\xi(v_1) \end{pmatrix},$$

for fixed $v_0, v_1 \in \mathcal{D}_\varepsilon$. Thus, choosing $v_0 = -i(\sqrt{2}-\sqrt{\varepsilon})$ and $v_1 = i(\sqrt{2}-\sqrt{\varepsilon})$, we have that $|\Delta\xi(v)| \le Me^{-\sqrt{2}\omega} \le 2Me^{-\sqrt{\frac{2}{\varepsilon}}}$, for $v \in \mathcal{I}_\varepsilon$, and therefore it is exponentially small with respect to ε.

Roughly speaking, we prove in Section 7.5.2 that this reasoning will also be true when $B \neq 0$, by using ideas from [9], and we prove the following theorem.

Theorem 7.3.7. *Consider system* (7.3.1)*. Given any compact interval $\mathcal{I} \subset \mathbb{R}$ containing 0, there exists $\varepsilon_0 > 0$ sufficiently small such that, for every $0 < \varepsilon < \varepsilon_0$, the parameterizations $N^\star_{0,0}(v)$, $\star = u, s$, given in* (7.3.8)*, are defined for $v \in \mathcal{I}$ and satisfy*

$$\begin{cases} \Gamma^u_0(0) - \Gamma^s_0(0) &= -i\dfrac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega} + \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}), \\ \Theta^u_0(0) - \Theta^s_0(0) &= i\dfrac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega} + \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}), \end{cases} \quad \Omega = \sqrt{1-\frac{\varepsilon^2}{4}},\ \omega = \frac{\Omega}{\sqrt{\varepsilon}} \text{ and } \delta = \varepsilon^{3/4}.$$

First statement of Theorem Q follows as a corollary of Theorem 7.3.7.

7.3.3 Parameterization of the Invariant Manifolds $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ and $W^s_\varepsilon(\Lambda^+_{\kappa_1,\kappa_2})$

In this section we find parameterizations of the invariant manifolds $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ and $W^s_\varepsilon(\Lambda^+_{\kappa_1,\kappa_2})$, for $\kappa_1, \kappa_2 \geq 0$ and $\kappa_1 + \kappa_2 = h > 0$. Even if one theorem could contain all the results for $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$, we state three separate theorems, Theorem 7.3.8 ($\kappa_1 = 0$), Theorem 7.3.10 ($\kappa_2 = 0$) and Theorem 7.3.11 ($\kappa_1, \kappa_2 > 0$), to clarify the exposition (and the corresponding proofs).

Zero Energy for the Pendulum (Separatrix Case $\kappa_1 = 0$ and $\kappa_2 = h > 0$)

We look for parameterizations of the 2-dimensional invariant manifolds $W^u_\varepsilon(\Lambda^-_h)$ and $W^s_\varepsilon(\Lambda^+_h)$,

$$N^\star_{0,h}(v,\tau) = \Big(X_0(v), Z_0(v) + Z^\star_{0,h}(v,\tau), \Gamma_h(\tau) + \Gamma^\star_{0,h}(v,\tau), \Theta_h(\tau) + \Theta^\star_{0,h}(v,\tau)\Big), \star = u, s$$

as perturbations of $W(0,h)$ (see Lemma 7.3.2).

For our purpose, it is not necessary to extend $N^\star_{0,h}$ to a domain which is $\sqrt{\varepsilon}$-close to the singularities of Z_0. Thus, it is sufficient to consider the domains

$$\begin{aligned} D^u &= \left\{v \in \mathbb{C};\ |\mathrm{Im}(v)| \leq -\tan(\beta)\,\mathrm{Re}(v) + \sqrt{2}/2\right\}, \\ D^s &= \{v \in \mathbb{C};\ -v \in D^u\}, \end{aligned} \tag{7.3.12}$$

for some $0 < \beta < \pi/4$ fixed. We also consider

$$\mathbb{T}_\sigma = \{\tau \in \mathbb{C};\ |\mathrm{Im}(\tau)| < \sigma \text{ and } \mathrm{Re}(\tau) \in \mathbb{T}\}. \tag{7.3.13}$$

We prove the following theorem in Section 7.6.

Theorem 7.3.8. *Fix $\sigma > 0$ and $h_0 > 0$. There exists $\varepsilon_0 > 0$ sufficiently small such that, for $0 < \varepsilon \leq \varepsilon_0$ and $0 < h \leq h_0$, $W^u_\varepsilon(\Lambda^-_h)$ is parameterized by*

$$N^u_{0,h}(v,\tau) = (X_0(v), Z_0(v) + Z^u_{0,h}(v,\tau), \Gamma_h(\tau) + \Gamma^u_{0,h}(v,\tau), \Theta_h(\tau) + \Theta^u_{0,h}(v,\tau)),$$

with $v \in D^u$ (see (7.3.12)) and $\tau \in \mathbb{T}_\sigma$, where $X_0, Z_0, \Gamma_h, \Theta_h$ are given by (7.3.4) and (7.3.7),

$$\begin{cases} Z^u_{0,h}(v,\tau) = Z_{0,h}(v,\tau) + z^u_{0,h}(v,\tau), \\ \Gamma^u_{0,h}(v,\tau) = Q^0(v) + \gamma^u_{0,h}(v,\tau), \\ \Theta^u_{0,h}(v,\tau) = -Q^0(v) + \theta^u_{0,h}(v,\tau), \end{cases} \tag{7.3.14}$$

where Q^0 is given by (7.3.11), and

$$Z_{0,h}(v,\tau) = \frac{\delta}{\omega\sqrt{2\Omega}} F'(X_0(v)) \frac{\Gamma_h(\tau) + \Theta_h(\tau)}{2}. \tag{7.3.15}$$

Furthermore, $z^u_{0,h}$ is a real-analytic function and $\gamma^u_{0,h}, \theta^u_{0,h}$ are analytic functions satisfying

$$\theta^u_{0,h}(v,\tau) = \overline{\gamma^u_{0,h}(v,\tau)},\ (v,\tau) \in \mathbb{R}^2 \cap D^u \times \mathbb{T}_\sigma,$$

such that there exists a constant $M > 0$ independent of ε and h such that, for $(v,\tau) \in D^u \times \mathbb{T}_\sigma$,

$$|z^u_{0,h}(v,\tau)|, |\gamma^u_{0,h}(v,\tau)|, |\theta^u_{0,h}(v,\tau)| \leq M \frac{\delta}{\omega} \frac{1}{|\sqrt{v^2+2}|} \tag{7.3.16}$$

with $\delta = \varepsilon^{3/4}$, $\omega = \Omega/\sqrt{\varepsilon}$ and $\Omega = \sqrt{1 - \varepsilon^2/4}$.

Remark 7.3.9. *We stress that the bounds in* (7.3.16) *are only valid for* $v^2+2>1/2$ *and therefore do not give any information about the behavior of* $N^u_{0,h}(v,\tau)$ *near the singularities* $v=\pm i\sqrt{2}$. *We use this norm to control the functions at* $X=\pm\infty$.

Positive Energy for the Pendulum

This section is devoted to study the invariant manifolds of the periodic orbits $\Lambda^{\mp}_{\kappa_1,\kappa_2}$ for $\kappa_1>0$. First, we consider the case $\kappa_1=h$ and $\kappa_2=0$. In this case $\Lambda^{\mp}_{h,0}=p^{\mp}_h$ is a critical point. We apply the same ideas of Section 7.3.2 to parameterize $W^u_\varepsilon(p^-_h)$ as

$$N^u_{h,0}(v)=(X_h(v),Z^u_{h,0}(v),\Gamma^u_{h,0}(v),\Theta^u_{h,0}(v)),$$

where $X_h(v)$ has been introduced in (7.3.5). The main difference is that we need to take into account the singular dependence on the parameter h at $h=0$.

As in Theorem 7.3.8, for our purposes it is sufficient to parameterize the manifolds in the domains $D^{u,s}$ (see (7.3.12)). We prove the following theorem in Section 7.7.

Theorem 7.3.10. *There exist* $\varepsilon_0>0$ *and* $h_0>0$ *sufficiently small such that, for* $0<\varepsilon\le\varepsilon_0$ *and* $0<h\le h_0$, $W^u_\varepsilon(p^-_h)$ *is parameterized by*

$$N^u_{h,0}(v)=(X_h(v),Z^u_{h,0}(v),\Gamma^u_{h,0}(v),\Theta^u_{h,0}(v)),\ \ v\in D^u,$$

where X_h *is given by* (7.3.5), $Z^u_{h,0}(v)$ *is obtained from* $\mathcal{H}(N^u_{h,0}(v))=h$ *(*$\mathcal{H}$ *given in* (7.3.2)*) and*

$$\left\{\begin{array}{rcl}\Gamma^u_{h,0}(v)&=&Q^h(v)+\gamma^u_{h,0}(v),\\ \Theta^u_{h,0}(v)&=&-Q^h(v)+\theta^u_{h,0}(v),\end{array}\right. \tag{7.3.17}$$

with

$$Q^h(v)=-i\frac{\delta}{\omega\sqrt{2\Omega}}F(X_h(v)). \tag{7.3.18}$$

Furthermore, $\gamma^u_{h,0}(v),\theta^u_{h,0}(v)$ *are analytic functions satisfying* $\theta^u_{h,0}(v)=\overline{\gamma^u_{h,0}(v)}$ *for* $v\in\mathbb{R}\cap D^u$ *such that there exists a constant* $M>0$ *independent of* ε *such that for* $v\in D^u$

$$\left|\gamma^u_{h,0}(v)\right|,\left|\theta^u_{h,0}(v)\right|\le M\frac{\delta}{\omega^2}\frac{1}{|v^2+2|}, \tag{7.3.19}$$

with $\delta=\varepsilon^{3/4}$, $\omega=\Omega/\sqrt{\varepsilon}$ *and* $\Omega=\sqrt{1-\varepsilon^2/4}$.

Finally we deal with the case $\kappa_1,\kappa_2>0$. Next theorem, proven in Section 7.8, gives the parameterizations of $W^u_\delta(\Lambda^-_{\kappa_1,\kappa_2})$.

Theorem 7.3.11. *Fix* $\sigma>0$. *There exist* $\varepsilon_0>0$ *and* $h_0>0$ *sufficiently small such that, for* $0<\varepsilon\le\varepsilon_0$, $0<h\le h_0$, *and* $\kappa_1>0$, $\kappa_2\ge 0$ *with* $\kappa_1+\kappa_2=h$, *the invariant manifold* $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ *is parameterized by*

$$N^u_{\kappa_1,\kappa_2}(v,\tau)=(X_{\kappa_1}(v),Z_{\kappa_1}(v)+Z^u_{\kappa_1,\kappa_2}(v,\tau),\Gamma_{\kappa_2}(\tau)+\Gamma^u_{\kappa_1,\kappa_2}(v,\tau),\Theta_{\kappa_2}(\tau)+\Theta^u_{\kappa_1,\kappa_2}(v,\tau)),$$

for $(v,\tau)\in D^u\times\mathbb{T}_\sigma$, *where* X_{κ_1}, $Z_{\kappa_1},\Gamma_{\kappa_2},\Theta_{\kappa_2}$ *are given by* (7.3.5) *and* (7.3.7),

$$\left\{\begin{array}{l}Z^u_{\kappa_1,\kappa_2}(v,\tau)=Z_{\kappa_1,\kappa_2}(v,\tau)+z^u_{\kappa_1,\kappa_2}(v,\tau),\\ \Gamma^u_{\kappa_1,\kappa_2}(v,\tau)=Q^{\kappa_1}(v)+\gamma^u_{\kappa_1,\kappa_2}(v,\tau),\\ \Theta^u_{\kappa_1,\kappa_2}(v,\tau)=-Q^{\kappa_1}(v)+\theta^u_{\kappa_1,\kappa_2}(v,\tau),\end{array}\right. \tag{7.3.20}$$

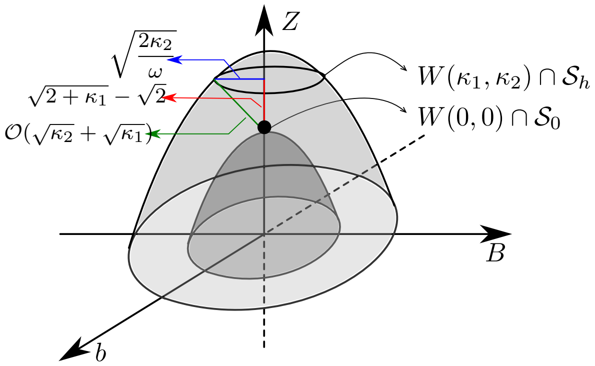


Figure 7.9: Comparison between $W(\kappa_1, \kappa_2)$ and $W(0,0)$ in the section $\mathcal{S}_h$ ($X = 0$, $H = h$) projected in the bZB-space .

where Q^{κ_1} *is given in* (7.3.18) *and*

$$Z_{\kappa_1,\kappa_2}(v,\tau) = \frac{\delta}{\omega\sqrt{2\Omega}} F'(X_{\kappa_1}(v)) \frac{\Gamma_{\kappa_2}(\tau)+\Theta_{\kappa_2}(\tau)}{2}.$$

Furthermore, $z^u_{\kappa_1,\kappa_2}$ *is a real-analytic function and* $\gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2}$ *are analytic functions satisfying* $\theta^u_{\kappa_1,\kappa_2}(v,\tau) = \overline{\gamma^u_{\kappa_1,\kappa_2}(v,\tau)}$ *for* $(v,\tau) \in \mathbb{R}^2 \cap D^u \times \mathbb{T}_\sigma$ *such that there exists a constant* $M > 0$ *independent of* ε, κ_1 *and* κ_2 *such that, for* $(v,\tau) \in D^u \times \mathbb{T}_\sigma$ *(see* (7.3.12)*),*

$$|z^u_{\kappa_1,\kappa_2}(v,\tau)|, |\gamma^u_{\kappa_1,\kappa_2}(v,\tau)|, |\theta^u_{\kappa_1,\kappa_2}(v,\tau)| \leq M\frac{\delta}{\omega}\frac{1}{|v^2+2|^{\frac{1}{2}}} \tag{7.3.21}$$

with $\delta = \varepsilon^{3/4}$, $\omega = \Omega/\sqrt{\varepsilon}$ *and* $\Omega = \sqrt{1-\varepsilon^2/4}$.

7.3.4 Approximation of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ by $W^u_\varepsilon(p_0^-)$ in the section Σ_h

Recall that for the unperturbed case, we have that

$$W(\kappa_1,\kappa_2) \cap \Sigma_h = \{(Z,b,B);\ Z = 4\sqrt{2+\kappa_1} \text{ and } b^2+B^2 = 2\kappa_2/\omega\}.$$

Thus, in the section Σ_h, the sets $W(\kappa_1,\kappa_2)$ and $W(0,0)$ are $(\kappa_1+\sqrt{\kappa_2})$-close (see Figure 7.9). Since the perturbed invariant manifolds are close to the unperturbed ones (see Theorems 7.3.8, 7.3.10, 7.3.11), in the next theorem we approximate $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ by $W^u_\varepsilon(p_0^-)$ for κ_1,κ_2 small. Using energy conservation and the fact that Γ and Θ are complex conjugate for real values of the variables, it is enough to compare the invariant manifolds only in the variable Γ. We define the projection $\pi_\Gamma(X,Z,\Gamma,\Theta) = \Gamma$.

Theorem 7.3.12. *Consider* $\kappa_1,\kappa_2 \geq 0$, $\kappa_1+\kappa_2 = h$, *and the parameterization of* $N^u_{\kappa_1,\kappa_2}$ *of* $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ *obtained in Theorems 7.3.5, 7.3.8, 7.3.10, and 7.3.11. Then, there exist* $\varepsilon_0 > 0$ *and* $h_0 > 0$ *sufficiently small such that for* $0 < h \leq h_0$ *and* $0 < \varepsilon \leq \varepsilon_0$

$$\pi_\Gamma N^u_{\kappa_1,\kappa_2}(0,\tau) - \pi_\Gamma N^u_{0,0}(0) = \Gamma_{\kappa_2}(\tau) + \mathcal{O}\left(\frac{\delta\sqrt{\kappa_1}}{\omega^2} + \frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}\right),\ \tau \in \mathbb{T},$$

where $\Gamma_{\kappa_2}(\tau)$ *has been introduced in* (7.3.7), $\delta = \varepsilon^{3/4}$, $\omega = \Omega/\sqrt{\varepsilon}$ *and* $\Omega = \sqrt{1-\varepsilon^2/4}$.

The proof of this theorem is done in Sections 7.9.1, 7.9.2 and 7.9.3. The result of this theorem for $\kappa_1 = h$ and $\kappa_2 = 0$ implies the second statement of Theorem Q (note that we are abusing notation since, in this case, the function $N^u_{\kappa_1,\kappa_2}$ does not depend on τ).

7.3.5 Proof of Theorem R

Theorems 7.3.8 and 7.3.12 provide, for $0 \le h \le h_0$, $\varepsilon \le \varepsilon_0$, the existence of the invariant manifolds $W^u_\varepsilon(\Lambda_h^-)$ and $W^s_\varepsilon(\Lambda_h^+)$ which are parameterized by

$$N_{0,h}^{u,s}(v,\tau) = \begin{pmatrix} X_0(v) \\ Z_0(v) + Z_{0,h}(v,\tau) + z_{0,h}^{u,s}(v,\tau) \\ \Gamma_h(\tau) + \Gamma_0^{u,s}(v) + F^{u,s}(v,\tau,h,\varepsilon) \\ \Theta_h(\tau) + \Theta_0^{u,s}(v) + \overline{F^{u,s}(v,\tau,h,\varepsilon)} \end{pmatrix}, \ (v,\tau) \in (D^{u,s} \cap \mathbb{R}) \times \mathbb{T},$$

where X_0, Z_0 are given in (7.3.4), $Z_{0,h}$ and $z_{0,h}^{u,s}$ are given by (7.3.14), Γ_h and Θ_h are given in (7.3.7), $\Gamma_0^{u,s}, \Theta_0^{u,s}$ are given in (7.3.10) and $F^{u,s}$ are analytic functions such that

$$F^{u,s}(v,\tau,h,\varepsilon) = \mathcal{O}\left(\frac{\delta\sqrt{h}}{\omega^{3/2}}\right).$$

Consider the section Σ_h (which corresponds to $v = 0 \in D^u \cap D^s$). Then, $W^u_\varepsilon(\Lambda_h^-)$ and $W^s_\varepsilon(\Lambda_h^+)$ intersect along a heteroclinic orbit if and only if there exist τ^u, τ^s in $[-\pi,\pi)$ such that $N^u_{0,h}(0,\tau^u) = N^s_{0,h}(0,\tau^s)$. Moreover, using energy conservation, $N^u_{0,h}(0,\tau^u) = N^s_{0,h}(0,\tau^s)$ if, and only if,

$$\begin{cases} \Gamma_h(\tau^u) + \Gamma_0^u(0) + F^u(0,\tau^u,h,\varepsilon) & = & \Gamma_h(\tau^s) + \Gamma_0^s(0) + F^s(0,\tau^s,h,\varepsilon) \\ \Theta_h(\tau^u) + \Theta_0^u(0) + \overline{F^u(0,\tau^u,h,\varepsilon)} & = & \Theta_h(\tau^s) + \Theta_0^s(0) + \overline{F^u(0,\tau^s,h,\varepsilon)}. \end{cases}$$

Since $\tau^u, \tau^s \in \mathbb{R}$, using Theorem 7.3.7, the expression of Γ_h in (7.3.7), the equations above are equivalent to

$$\begin{cases} \sqrt{\dfrac{2h}{\omega}}(\cos(\tau^u) - \cos(\tau^s)) + M_1(\varepsilon) + F_1(\tau^u,\tau^s,h,\varepsilon) = 0, \\ \sqrt{\dfrac{2h}{\omega}}(\sin(\tau^u) - \sin(\tau^s)) - \dfrac{2\pi\delta}{\sqrt{\Omega}} e^{-\sqrt{2}\omega} + M_2(\varepsilon) + F_2(\tau^u,\tau^s,h,\varepsilon) = 0, \end{cases} \tag{7.3.22}$$

where $0 < \varepsilon \le \varepsilon_0$, $0 < h \le h_0$ and M_1, M_2, F_1, F_2 are real-analytic functions such that

$$M_1, M_2 = \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}) \text{ and } F_1, F_2 = \mathcal{O}\left(\frac{\delta\sqrt{h}}{\omega^{3/2}}\right).$$

We change the parameter $h \ge 0$

$$h = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}\mu^2, \text{ for } \mu \ge 0.$$

Then, since $0 < h \le h_0$, it is sufficient to consider

$$0 < \mu \le \mu_0 = \frac{1}{\delta_0\pi}\sqrt{\frac{2\Omega_0 h_0}{\omega_0}} e^{\sqrt{2}\omega_0},$$

where $\Omega_0 = \sqrt{1-\varepsilon_0^2/4}$, $\omega_0 = \Omega_0/\sqrt{\varepsilon_0}$ and $\delta_0 = \varepsilon_0^{3/4}$. Considering $\varepsilon_0 > 0$ sufficiently small, we can assume that $\mu_0 > 1$. Replacing h in (7.3.22) and multiplying the equation by $\dfrac{\sqrt{\Omega}}{\pi\delta}e^{\sqrt{2}\omega} > 0$, we may rewrite (7.3.22) as

$$\begin{cases} \mu(\cos(\tau^u) - \cos(\tau^s)) + \widetilde{M}_1(\varepsilon) + \widetilde{F}_1(\tau^u, \tau^s, \mu, \varepsilon) = 0, \\ \mu(\sin(\tau^u) - \sin(\tau^s)) - 2 + \widetilde{M}_2(\varepsilon) + \widetilde{F}_2(\tau^u, \tau^s, \mu, \varepsilon) = 0, \end{cases} \tag{7.3.23}$$

where $\widetilde{M}_1, \widetilde{M}_2, \widetilde{F}_1, \widetilde{F}_2$, are real-analytic functions such that

$$\widetilde{M}_1, \widetilde{M}_2 = \mathcal{O}(\omega\delta^2) \text{ and } \widetilde{F}_1, \widetilde{F}_2 = \mathcal{O}\left(\frac{\delta}{\omega}\mu\right).$$

Define the function $G = (G_1, G_2) : [-\pi, \pi]^2 \times (0, \mu_0] \times [0, \varepsilon_0] \to \mathbb{R}^2$ corresponding to the left-hand side of system (7.3.23). Recalling that $\delta = \varepsilon^{3/4}$ and $\omega = \Omega/\sqrt{\varepsilon}$, it is clear that

$$G(\tau^u, \tau^s, \mu, \varepsilon) = \begin{pmatrix} \mu(\cos(\tau^u) - \cos(\tau^s)) + \mathcal{O}(\varepsilon) \\ \mu(\sin(\tau^u) - \sin(\tau^s)) - 2 + \mathcal{O}(\varepsilon) \end{pmatrix}.$$

The equation $G(\tau^u, \tau^s, \mu, 0) = (0, 0)$ has a unique family of solutions

$$\mathcal{S}_0 = \{(\alpha, -\alpha, 1/\sin(\alpha), 0);\ \arcsin(1/\mu_0) \le \alpha \le \pi - \arcsin(1/\mu_0)\}.$$

We find zeroes of G using the Implicit Function Theorem around every solution of the family $\mathcal{S}_0$. Denote $\alpha_0 = \arcsin(1/\mu_0)$ and fix $0 < \alpha_0 \le \alpha \le \pi - \alpha_0$. Then,

1. $G(\alpha, -\alpha, 1/\sin(\alpha), 0) = (0, 0)$,
2. $\det\left(\dfrac{\partial(G_1, G_2)}{\partial(\mu, \tau^s)}\right)(\alpha, -\alpha, 1/\sin(\alpha), 0) = 2\sin(\alpha) \neq 0.$

Thus, it follows from the Implicit Function Theorem that there exist $\varepsilon_\alpha > 0$ and unique functions $\tau_\alpha^s : (\alpha - \varepsilon_\alpha, \alpha + \varepsilon_\alpha) \times [0, \varepsilon_\alpha) \to [-\pi, \pi]$, $\mu_\alpha : (\alpha - \varepsilon_\alpha, \alpha + \varepsilon_\alpha) \times [0, \varepsilon_\alpha) \to (0, \mu_0]$ such that

$$G(\tau^u, \tau_\alpha^s(\tau^u, \varepsilon), \mu_\alpha(\tau^u, \varepsilon), \varepsilon) = (0, 0).$$

Furthermore

$$\begin{cases} \tau_\alpha^s(\tau^u, \varepsilon) = -\alpha + \mathcal{O}(\tau^u - \alpha, \varepsilon), \\ \mu_\alpha(\tau^u, \varepsilon) = 1/\sin(\alpha) + \mathcal{O}(\tau^u - \alpha, \varepsilon) \end{cases}, \quad \tau^u \in (\alpha - \varepsilon_\alpha, \alpha + \varepsilon_\alpha).$$

Consider the compact set $K = [\alpha_0, \pi - \alpha_0]$. We can find $n \in \mathbb{N}$, $\alpha_1, \cdots, \alpha_n$ with respectives $\varepsilon_{\alpha_1}, \cdots, \varepsilon_{\alpha_n}$, previously found, such that the intervals $(\alpha_i - \varepsilon_{\alpha_i}, \alpha_i + \varepsilon_{\alpha_i})$, $i = 1, \cdots, n$ form a finite cover of K. Using the uniqueness of solutions obtained from the Implicit Function Theorem, it is possible to conclude that there exist $\varepsilon_1 > 0$ sufficiently small and functions

$$\begin{cases} \tau_*^s(\tau^u, \varepsilon) = -\tau^u + \mathcal{O}(\varepsilon), \\ \mu_*(\tau^u, \varepsilon) = 1/\sin(\tau^u) + \mathcal{O}(\varepsilon), \end{cases}$$

defined for every $\varepsilon < \varepsilon_1$ and $\tau^u \in K$, such that

$$G\left(\tau^u, \tau_*^s(\tau^u, \varepsilon), \mu_*(\tau^u, \varepsilon), \varepsilon\right) = (0, 0).$$

This implies that there exists at least one heteroclinic connection in the energy level

$$h = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}(\mu_*(\tau^u,\varepsilon))^2,\ \ \tau^u \in K.$$

Moreover, $(\mu_*(\tau^u,0))^2 \geq (\mu_*(\pi/2,0))^2 = 1$, for every $\tau^u \in K$. Thus $(\mu_*(\tau^u,\varepsilon))^2 \geq 1+\mathcal{O}(\varepsilon)$ for $\tau^u \in K$ and $\varepsilon < \varepsilon_1$. Therefore, since $\mu_*(\pi/2,\varepsilon) = 1+\mathcal{O}(\varepsilon)$, there must exist a curve $\tau^u_{\min}(\varepsilon)$, such that

$$(\mu_*(\tau^u,\varepsilon))^2 \geq (\mu_*(\tau^u_{\min}(\varepsilon),\varepsilon))^2,$$

for $\tau^u \in K$, $\varepsilon < \varepsilon_1$, and $\mu_*(\tau^u_{\min}(\varepsilon),\varepsilon) = 1+\mathcal{O}(\varepsilon)$.

Thus, defining

$$h_s(\varepsilon) = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}(\mu_*(\tau^u_{\min}(\varepsilon),\varepsilon))^2 = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}(1+\mathcal{O}(\varepsilon)),$$

system (7.3.1) has one heteroclinic orbit between the periodic orbits Λ_h^- and Λ_h^+ in the energy level $0 < h \leq h_0$ if, and only if $h \geq h_s(\varepsilon)$.

It only remains to prove the last statement of Theorem R.Given $\mu_1 > 1$, let $\tau_1^u = \arcsin(\mu_1^{-1}) \in [\alpha_0,\pi/2) \subset K$, and consider the function $g(\tau^u,\varepsilon) = \mu_*(\tau^u,\varepsilon) - \mu_*(\tau_1^u,\varepsilon)$. Applying the Implicit Function Theorem to $g = 0$ at the point $(\pi - \tau_1^u, 0)$, there exist $\varepsilon_{\mu_1} > 0$ and a unique curve $\tau_2^u = \tau_2^u(\tau_1^u,\varepsilon)$, defined for $0 \leq \varepsilon < \varepsilon_{\mu_1}$, such that $\mu_*(\tau_2^u,\varepsilon) = \mu_*(\tau_1^u,\varepsilon)$ and $\tau_2^u(\tau_1^u,\varepsilon) = \pi - \tau_1^u + \mathcal{O}(\varepsilon)$. Moreover, taking ε_{μ_1} small enough $\tau_1^u \neq \tau_2^u$ for $\varepsilon < \varepsilon_{\tau_1^u}$. Thus, in the energy level

$$h_{\mu_1} = \frac{\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{2\Omega}(\mu_*(\tau_1^u,\varepsilon))^2,$$

where $\mu_*(\tau_1^u,\varepsilon) = \mu_1 + \mathcal{O}(\varepsilon)$, there exist two heteroclinic connections corresponding to τ_1^u and τ_2^u.

This completes the proof of Theorem R.

Remark 7.3.13. *Notice that $g(\pi/2,0) = \partial_{\tau^u} g(\pi/2,0) = 0$ and $\partial^2_{\tau^u} g(\pi/2,0) \neq 0$. Unfortunately, the characterization of the bifurcation of zeros for $\varepsilon > 0$ becomes impossible, since there is no information on $\partial_\varepsilon g(\pi/2,0)$, and its computation requires complicated second order expansions which are beyond the objectives of this work. Nevertheless, under some non-degenericity condition, for example $\partial_\varepsilon g(\pi/2,0) \neq 0$, it is posible to detect a saddle-node bifurcation.*

7.3.6 Proof of Theorem S

Following the same lines of Section 7.3.5, we use Theorems 7.3.8 (for the invariant manifold $W^s_\varepsilon(\Lambda_h^+)$), 7.3.10 (for the invariant manifold $W^u_\varepsilon(p_h^-)$) and 7.3.12 (to compare them to $W^s_\varepsilon(p_0^+)$ and $W^u_\varepsilon(p_0^-)$). Then, we can see that $W^u_\delta(p_h^-) \subset W^s_\delta(\Lambda_h^+)$, if and only if

$$\begin{cases} -\sqrt{\dfrac{2h}{\omega}}\cos(\tau^s) + M_1(\varepsilon) + F_1(\tau^s,h,\varepsilon) = 0, \\ -\sqrt{\dfrac{2h}{\omega}}\sin(\tau^s) - \dfrac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega} + M_2(\varepsilon) + F_2(\tau^s,h,\varepsilon) = 0, \end{cases} \tag{7.3.24}$$

has solutions $\tau^u, \tau^s \in [-\pi, \pi]$, $0 < \varepsilon \leq \varepsilon_0$, $0 < h \leq h_0$ where h_0 is given in Theorem 7.3.10.. The functions M_j, F_j are real-analytic and satisfy

$$M_j = \mathcal{O}(\omega \delta^3 e^{-\sqrt{2}\omega}) \text{ and } F_j = \mathcal{O}\left(\frac{\delta\sqrt{h}}{\omega^{3/2}} + \frac{\delta\sqrt{h}}{\omega^2}\right), \; j = 1, 2$$

In order to look for solutions of (7.3.24), we consider the change

$$h = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}\mu^2, \quad 0 < \mu \leq \mu_0 = \frac{\sqrt{\Omega_0 h_0}}{\delta_0 \pi \sqrt{2\omega_0} e^{-\sqrt{2}\omega_0}}$$

Considering $\varepsilon_0 > 0$ sufficiently small, we can assume that $\mu_0 > 1$. Replacing h in (7.3.24) and multiplying it by $\dfrac{\sqrt{\Omega}}{2\pi\delta e^{-\sqrt{2}\omega}} > 0$, we may rewrite this system as

$$\begin{cases} -\mu\cos(\tau^s) + \widetilde{M}_1(\varepsilon) + \widetilde{F}_1(\tau^s, \mu, \varepsilon) = 0, \\ -\mu\sin(\tau^s) - 1 + \widetilde{M}_2(\varepsilon) + \widetilde{F}_2(\tau^s, \mu, \varepsilon) = 0, \end{cases} \tag{7.3.25}$$

where $\widetilde{M}_j, \widetilde{F}_j$, are real-analytic functions such that

$$\widetilde{M}_j = \mathcal{O}(\omega\delta^2) \text{ and } \widetilde{F}_j = \mathcal{O}\left(\frac{\delta}{\omega}\mu\right), \; j = 1, 2.$$

Define the function $G : [-\pi, \pi] \times (0, \mu_0] \times [0, \varepsilon_0] \to \mathbb{R}^2$ as as the left-hand side of system (7.3.25). Recalling that $\delta = \varepsilon^{3/4}$ and $\omega = \Omega/\sqrt{\varepsilon}$, we can see that

$$G(\tau^s, \mu, \varepsilon) = \begin{pmatrix} -\mu\cos(\tau^s) + \mathcal{O}(\varepsilon) \\ -\mu\sin(\tau^s) - 1 + \mathcal{O}(\varepsilon) \end{pmatrix}.$$

Since,

1. $G(-\pi/2, 1, 0) = (0, 0)$,
2. $\det\left(\dfrac{\partial(G_1, G_2)}{\partial(\tau^s, \mu)}\right)(-\pi/2, 1, 0) = 1$,

we can apply the Implicit Function Theorem to obtain $\varepsilon_* > 0$ and functions $\tau^s_* : [0, \varepsilon_*) \to [-\pi, \pi]$, $\mu_* : [0, \varepsilon_*) \to (0, \mu_0]$ such that $G(\tau^s_*(\varepsilon), \mu_*(\varepsilon), \varepsilon) = 0$ for $0 \leq \varepsilon \leq \varepsilon_*$. Furthermore, $\tau^s_*(\varepsilon) = -\pi/2 + \mathcal{O}(\varepsilon)$ and $\mu_*(\varepsilon) = 1 + \mathcal{O}(\varepsilon)$.

Defining

$$h_c(\varepsilon) = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon))^2 = \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(1 + \mathcal{O}(\varepsilon))$$

and reducing ε_0 to ε_*, Theorem S follows directly from these facts.

7.3.7 Proof of Theorem T

Following the same lines of Section 7.3.5, we use Theorems 7.3.10 (for the invariant manifold $W^u_\varepsilon(p_h^-)$), 7.3.11 (for the invariant manifold $W^s_\varepsilon(\Lambda^+_{\kappa_1,\kappa_2})$), and 7.3.12 (to compare them to $W^s_\varepsilon(p_0^+)$ and $W^u_\varepsilon(p_0^-)$). We can see that $W^u_\delta(p_h^-) \subset W^s_\delta(\Lambda^+_{\kappa_1,\kappa_2})$, if and only if

$$\begin{cases} -\sqrt{\dfrac{2\kappa_2}{\omega}}\cos(\tau^s) + M_1(\varepsilon) + F_1(\tau^s, h, \varepsilon) = 0, \\ -\sqrt{\dfrac{2\kappa_2}{\omega}}\sin(\tau^s) - \dfrac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega} + M_2(\varepsilon) + F_2(\tau^s, h, \varepsilon) = 0, \end{cases} \tag{7.3.26}$$

has a solution $\tau^s \in [-\pi, \pi]$ for $\varepsilon \le \varepsilon_0$, $h \le h_0$. The functions M_j, F_j are real-analytic and

$$M_j = \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}) \text{ and } F_j = \mathcal{O}\left(\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}} + \frac{\delta\sqrt{\kappa_1}}{\omega^2} + \frac{\delta\sqrt{h}}{\omega^2}\right), \; j = 1, 2, \; \kappa_1 + \kappa_2 = h.$$

We consider the change of parameters and variables

$$\begin{aligned} h &= \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon) + \mu)^2, \\ \kappa_2 &= \frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon) + \mu - \xi)^2, \\ \tau^s &= \tau^s_*(\varepsilon) + \tau, \end{aligned}$$

where $(\mu_*(\varepsilon), \tau^s_*(\varepsilon))$ is the solution of (7.3.25). Since $\kappa_2 \le h$, $\mu_*(\varepsilon) = 1 + \mathcal{O}(\varepsilon)$ and we are looking for solutions with $\mu, \xi, \tau \approx 0$, we have that $\xi \ge 0$ and $(\mu_*(\varepsilon)+\mu-\xi)^2 \le (\mu_*(\varepsilon)+\mu)^2$.

Replacing h, κ_2 and κ_1 and multiplying it by $\dfrac{\sqrt{\Omega}}{2\pi\delta e^{-\sqrt{2}\omega}} > 0$, system (7.3.26) as

$$\begin{cases} -(\mu_*(\varepsilon) + \mu - \xi)\cos(\tau^s_*(\varepsilon) + \tau) + \widetilde{M}_1(\varepsilon) + \widetilde{F}_1(\tau, \mu, \xi, \varepsilon) = 0, \\ -(\mu_*(\varepsilon) + \mu - \xi)\sin(\tau^s_*(\varepsilon) + \tau) - 1 + \widetilde{M}_2(\varepsilon) + \widetilde{F}_2(\tau, \mu, \xi, \varepsilon) = 0, \end{cases}$$

where $\widetilde{M}_j, \widetilde{F}_j$, are real-analytic functions such that $\widetilde{M}_j = \mathcal{O}(\omega\delta^2)$ and

$$\widetilde{F}_j = \mathcal{O}\left(\frac{\delta}{\omega}\left((\mu_*(\varepsilon) + \mu - \xi) + \frac{\sqrt{(\mu_*(\varepsilon) + \mu)^2 + (\mu_*(\varepsilon) + \mu - \xi)^2}}{\omega^{1/2}} + \frac{(\mu_*(\varepsilon) + \mu)}{\omega^{1/2}}\right)\right), \; j = 1, 2.$$

Define the function $G : [-\chi_0, \chi_0] \times [0, \chi_0] \times [-\chi_0, \chi_0] \times [0, \chi_0] \to \mathbb{R}^2$ as the left hand side of system (7.3.25) and fix $\chi_0 > 0$ small enough. Recalling that $\delta = \varepsilon^{3/4}$ and $\omega = \Omega/\sqrt{\varepsilon}$, we can see that

$$G(\tau, \mu, \xi, \varepsilon) = \begin{pmatrix} -(\mu_*(\varepsilon) + \mu - \xi)\cos(\tau^s_*(\varepsilon) + \tau) + \mathcal{O}(\varepsilon) \\ -(\mu_*(\varepsilon) + \mu - \xi)\sin(\tau^s_*(\varepsilon) + \tau) - 1 + \mathcal{O}(\varepsilon) \end{pmatrix}.$$

From Section 7.3.6, $\mu_*(0) = 1$ and $\tau^s_*(0) = -\pi/2$. Thus $G(\tau, \mu, \xi, 0) = (0, 0)$ has a solution $\tau = 0$ and $\mu = \xi$. Since, we are looking for solutions with $\mu, \xi \approx 0$, we consider the solution $\mu = \xi = 0$. Then, since

1. $G(0, 0, 0, 0) = (0, 0)$,

2. $\det\left(\dfrac{\partial(G_1,G_2)}{\partial(\tau,\xi)}\right)(0,0,0,0)=1,$

we can apply the Implicit Function Theorem to obtain $\varepsilon_0>0$ and unique functions $\overline{\tau}:[0,\varepsilon_0)\times[0,\varepsilon_0)\to[-\chi_0,\chi_0]$, $\overline{\xi}:[0,\varepsilon_0)\times[0,\varepsilon_0)\to[-\chi_0,\chi_0]$ such that $G(\overline{\tau}(\mu,\varepsilon),\mu,\overline{\xi}(\mu,\varepsilon),\varepsilon)=0$. Furthermore $\overline{\tau}(\mu,\varepsilon)=\mathcal{O}(\mu,\varepsilon)$ and $\overline{\xi}(\mu,\varepsilon)=\mathcal{O}(\mu,\varepsilon)$. For $\varepsilon=0$, we have that $\xi=\mu$ and $\tau=0$ is a solution of $G(\tau,\mu,\xi,\varepsilon)=(0,0)$. Thus $\overline{\xi}(\mu,0)=\mu$ and, for ε small enough, $\overline{\xi}(\mu,\varepsilon)=\mu+\mathcal{O}(\varepsilon)$.

Finally, if $\xi=0$, then $\kappa_2=h$, $\kappa_1=0$ and therefore (7.3.26) becomes (7.3.24). Thus, considering the different scalings done in the systems and the uniqueness of solutions of (7.3.24) obtained in Section 7.3.6, we conclude that $\overline{\xi}(0,\varepsilon)=\overline{\tau}(0,\varepsilon)\equiv 0$.

These facts, allows us to see that

$$\overline{\xi}(\mu,\varepsilon)=c_\varepsilon\mu+\mathcal{O}(\mu^2),\quad \text{with } c_\varepsilon=1+\mathcal{O}(\varepsilon).$$

Hence, for $\mu\geq 0$ sufficiently small, in the energy level

$$h_\mu=\frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon)+\mu)^2,$$

there exists a unique heteroclinic connection between $p^-_{h_\mu}$ and $\Lambda^-_{h_\mu}(\kappa_1^\mu,\kappa_2^\mu)$, where

$$\kappa_2^\mu=\frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon)+\mu-\overline{\xi}(\mu,\varepsilon))^2,$$

and $\kappa_1^\mu=h_\mu-\kappa_2^\mu$. Moreover, if $-\mu_*(\varepsilon)<\mu<0$ there is no heteroclinic connections in the energy level h_μ.

Setting $v_i=\sqrt{h_\mu}$, $v_f=\sqrt{\kappa_1^\mu}$ and $v_c=\sqrt{h_c}$, where

$$h_c(\varepsilon)=\frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}(\mu_*(\varepsilon))^2,$$

it means that a soliton starting with velocity $v_i<v_c$ is trapped and will surround the defect location, otherwise, if $v_i\geq v_c$, then it will escape the defect location and propagate itself with some output velocity v_f. In what follows we give an asymptotic formula to the output velocity v_f of orbits with incoming velocity $v_i\approx v_c$. We omit the dependence of v_i,v_f on μ in order to simplify the notation.

For $\mu\geq 0$ sufficiently small, we have

$$\begin{aligned}
v_f^2 &= \kappa_1^\mu\\
&= h_\mu-\kappa_2^\mu\\
&= \frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}\Big((\mu_*(\varepsilon)+\mu)^2-(\mu_*(\varepsilon)+\mu-\overline{\xi}(\mu,\varepsilon))^2\Big)\\
&= \frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}\Big(\overline{\xi}(\mu,\varepsilon)(2(\mu_*(\varepsilon)+\mu)-\overline{\xi}(\mu,\varepsilon))\Big)\\
&= \frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}(c_\varepsilon\mu+\mathcal{O}(\mu^2))(2\mu_*(\varepsilon)+(2-c_\varepsilon)\mu+\mathcal{O}(\mu^2))\\
&= \frac{2\pi^2\omega\delta^2e^{-2\sqrt{2}\omega}}{\Omega}(2\mu_*(\varepsilon)c_\varepsilon\mu+\mathcal{O}(\mu^2)).
\end{aligned}$$

Notice that

$$v_i - v_c \;=\; \sqrt{\frac{2\pi^2\omega\delta^2 e^{-2\sqrt{2}\omega}}{\Omega}\mu}.$$

Thus

$$v_f^2 \;=\; 2v_c c_\varepsilon (v_i - v_c) + \mathcal{O}((v_i - v_c)^2).$$

Finally, we obtain that

$$v_f \;=\; \sqrt{2v_c c_\varepsilon}\sqrt{v_i - v_c} + \mathcal{O}((v_i - v_c)^{3/2}).$$

Theorem T follow directly from these facts.

7.4 Proof of Theorem 7.3.5

The strategy to prove the existence of $W_\varepsilon^u(p_0^-)$ and $W_\varepsilon^s(p_0^+)$ when $\delta \neq 0$ (see (7.3.1)), is to look for a parameterization $N_{0,0}^u(v)$ of $W_\varepsilon^u(p_0^\pm)$ as a perturbation of $N_{0,0}(v)$.

As in the unperturbed case $W(0,0)$ is parameterized as a graph over X (see (7.3.4)), we look for $N_{0,0}^u$ as

$$N_{0,0}^u(v) = (X_0(v), Z_0^u(v), \Gamma_0^u(v), \Theta_0^u(v)). \tag{7.4.1}$$

Next lemma, which is straightforward, gives the equation $N_{0,0}^u(v)$ has to satisfy to be invariant by the flow of (7.3.1).

Lemma 7.4.1. *The invariant manifold $W_\delta^u(p_0^-)$, with $\delta \neq 0$, is parameterized by $N_{0,0}^u(v)$ if and only if $(\Gamma_0^u(v), \Theta_0^u(v))$ satisfy*

$$\begin{cases} \dfrac{d\Gamma}{dv}(v) - \omega i\Gamma(v) = -\dfrac{\delta}{\sqrt{2\Omega}}F(X_0(v)) + \left(\dfrac{Z_0(v)}{\widetilde{\eta}_0(v,\Gamma,\Theta)} - 1\right)\left(\omega i\Gamma(v) - \dfrac{\delta}{\sqrt{2\Omega}}F(X_0(v))\right), \\ \dfrac{d\Theta}{dv}(v) + \omega i\Theta(v) = -\dfrac{\delta}{\sqrt{2\Omega}}F(X_0(v)) + \left(\dfrac{Z_0(v)}{\widetilde{\eta}_0(v,\Gamma,\Theta)} - 1\right)\left(-\omega i\Theta(v) - \dfrac{\delta}{\sqrt{2\Omega}}F(X_0(v))\right), \\ \lim\limits_{v\to-\infty}\Gamma(v) = \lim\limits_{v\to-\infty}\Theta(v) = 0. \end{cases} \tag{7.4.2}$$

where

$$\widetilde{\eta}_0(v,\Gamma,\Theta) = 4\sqrt{-U(X_0(v)) - \frac{\delta}{\sqrt{2\Omega}}F(X_0(v))\frac{\Gamma(v)-\Theta(v)}{2i} - \frac{\omega}{2}\Gamma(v)\Theta(v)},$$

with X_0 given in (7.3.4), *U, F given in* (7.1.7), *and $Z_0^u(v) = \widetilde{\eta}_0(v, \Gamma_0^u(v), \Theta_0^u(v))$.*

The term $\frac{\delta}{\sqrt{2\Omega}}F(X_0(v))$ decays as $1/v$ as $v \to \infty$. To have integrability, we consider the change of variables (7.3.10) to system (7.4.2). Then, (γ_0^u, θ_0^u) satisfy

$$\begin{cases} \dfrac{d}{dv}\gamma - \omega i\gamma = \omega i\gamma(\eta_0(v,\gamma,\theta) - 1) - (Q^0)'(v), \\ \dfrac{d}{dv}\theta + \omega i\theta = -\omega i\theta(\eta_0(v,\gamma,\theta) - 1) + (Q^0)'(v), \\ \lim\limits_{v\to-\infty}\gamma(v) = \lim\limits_{v\to-\infty}\theta(v) = 0, \end{cases} \tag{7.4.3}$$

where Q^0 is given by (7.3.11) and

$$\eta_0(v,\gamma,\theta) = \left(1+\frac{4\delta^2}{\Omega\omega}\left(\frac{F(X_0(v))}{Z_0(v)}\right)^2 - 8\omega\frac{\gamma\theta}{(Z_0(v))^2}\right)^{-1/2}. \tag{7.4.4}$$

To prove Theorem 7.3.5, it is sufficient to find a solution of (7.4.3).

Proposition 7.4.2. *Fix $\nu > 0$. There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the equation* (7.4.3) *has a solution $(\gamma_0^u(v), \theta_0^u(v))$ defined in the domain $D_\varepsilon^u \subset \mathbb{C}$ (see* (7.3.9)*) such that $\theta_0^u(v) = \overline{\gamma_0^u(v)}$, for every $v \in D_\varepsilon^u \cap \mathbb{R}$. Furthermore, both γ_0^u, θ_0^u satisfy bounds* (1) *and* (2) *of Theorem 7.3.5.*

We look for a fixed point (γ_0^u, θ_0^u) of the operator

$$\mathcal{G}_{\omega,0} = \mathcal{G}_\omega \circ \mathcal{F}_0, \tag{7.4.5}$$

where

$$\mathcal{G}_\omega(\gamma,\theta)(v) = \begin{pmatrix} \displaystyle\int_{-\infty}^{v} e^{\omega i(v-r)}\gamma(r)dr \\ \displaystyle\int_{-\infty}^{v} e^{-\omega i(v-r)}\theta(r)dr \end{pmatrix}, \tag{7.4.6}$$

$$\mathcal{F}_0(\gamma,\theta)(v) = \begin{pmatrix} \omega i\gamma(v)(\eta_0(v,\gamma(v),\theta(v))-1)-(Q^0)'(v) \\ -\omega i\theta(v)(\eta_0(v,\gamma(v),\theta(v))-1)+(Q^0)'(v) \end{pmatrix}, \tag{7.4.7}$$

and Q^0, η_0 are given in (7.3.11) and (7.4.4), respectively.

7.4.1 Banach Spaces and Technical Lemmas

In this section, we introduce a Banach space which will be used to find a fixed point of $\mathcal{G}_{\omega,0}$.

Consider the complex domain D_ε^u given in (7.3.9). For each analytic function $f : D_\varepsilon^u \to \mathbb{C}$, $\nu > 0$, $\alpha \geq 0$, we consider:

$$\|f\|_{\alpha,\nu} = \sup_{v\in D_\varepsilon^u\cap\{\mathrm{Re}(v)\leq-\nu\}} |v^2 f(v)| + \sup_{v\in D_\varepsilon^u\cap\{\mathrm{Re}(v)>-\nu\}} |(v^2+2)^\alpha f(v)|.$$

For any $\nu > 0$, and $\alpha > 0$ fixed, the function space

$$\mathcal{X}_{\alpha,\nu} = \{f : D_\varepsilon^u \to \mathbb{C};\ f \text{ is an analytic function such that, } \|f\|_{\alpha,\nu} < \infty\}$$

is a Banach space with respect to the norm $\|\cdot\|_{\alpha,\nu}$.

We also consider the product space

$$\mathcal{X}_{\alpha,\nu}^2 = \left\{(f,g) \in \mathcal{X}_{\alpha,\nu} \times \mathcal{X}_{\alpha,\nu};\ g(v) = \overline{f(v)} \text{ for every } v \in D_\varepsilon^u \cap \mathbb{R}\right\}$$

endowed with the norm

$$\|(f,g)\|_{\alpha,\nu} = \|f\|_{\alpha,\nu} + \|g\|_{\alpha,\nu}.$$

Proposition 7.4.3. *Given $\nu > 0$, $\alpha > 0$ fixed, and $(f,g) \in \mathcal{X}_{\alpha,\nu}^2$, we have that $\mathcal{G}_\omega(f,g) \in \mathcal{X}_{\alpha,\nu}^2$. Furthermore, there exists a constant $M > 0$ independent of ε such that*

$$\|\mathcal{G}_\omega(f,g)\|_{\alpha,\nu} \leq \frac{M}{\omega}\|(f,g)\|_{\alpha,\nu},$$

for every $(f,g) \in \mathcal{X}_{\alpha,\nu}^2$.

The proof of Proposition 7.4.3 follows from [53].

Proposition 7.4.4. *Let* η_0 *be the function given in* (7.4.4)*, and* $\mathcal{F}_0$ *given in* (7.4.7)*. Given* $\nu > 0$ *and* $K > 0$*, there exist* $\varepsilon_0 > 0$ *and* $M > 0$ *such that:*

For $0 < \varepsilon \le \varepsilon_0$ *and* $(\gamma_j, \theta_j) \in \mathcal{B}_0(R) \subset \mathcal{X}^2_{2,\nu}$ *where* $R = K\dfrac{\delta}{\omega^2}$ *and* $j = 1, 2$*, the following statements hold for* $v \in D^u_\varepsilon$*.*

1. $|\eta_0(v, \gamma_j(v), \theta_j(v)) - 1| \le M\delta^2$;

2. $|\eta_0(v, \gamma_1(v), \theta_1(v)) - \eta_0(v, \gamma_2(v), \theta_2(v))| \le M\delta\omega^2\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_{2,\nu}$;

3. $\mathcal{F}_0(\gamma_j, \theta_j) \in \mathcal{X}^2_{2,\nu}$;

4. $\|\mathcal{F}_0(\gamma_1, \theta_1) - \mathcal{F}_0(\gamma_2, \theta_2)\|_{2,\nu} \le M\delta^2\omega\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_{2,\nu}$.

Proof. Replacing the expressions of F, X_0 and Z_0 given in (7.1.7) and (7.3.4) in (7.4.4), we obtain

$$\eta_0(v, \gamma, \theta) = \left(1 + \frac{\delta^2}{4\Omega\omega}\frac{v^2}{v^2+2} - (v^2+2)\omega\frac{\gamma\theta}{8}\right)^{-1/2}.$$

Taking $\gamma, \theta \in \mathcal{B}_0(R)$, the first statement of the proposition comes from the following inequalities

$$\begin{aligned} \left|\frac{\delta^2}{4\Omega\omega}\frac{v^2}{v^2+2} - (v^2+2)\omega\frac{\gamma\theta}{8}\right| &\le M\frac{\delta^2}{\omega}, \qquad \text{if } \operatorname{Re}(v) \le -\nu \\ \left|\frac{\delta^2}{4\Omega\omega}\frac{v^2}{v^2+2} - (v^2+2)\omega\frac{\gamma\theta}{8}\right| &\le M\delta^2, \qquad \text{if } \operatorname{Re}(v) \ge -\nu. \end{aligned}$$

We observe that

$$\begin{aligned} |\eta_0(v, \gamma_1, \theta_1) - \eta_0(v, \gamma_2, \theta_2)| \;\le\; & M\omega|(v^2+2)\gamma_1(v)||\theta_1(v) - \theta_2(v)| \\ & + M\omega|(v^2+2)\theta_2(v)||\gamma_1(v) - \gamma_2(v)| \end{aligned} \tag{7.4.8}$$

Thus, if $\operatorname{Re}(v) \le -\nu$, then

$$\left|(v^2+2)\gamma_1(v)(\theta_1(v) - \theta_2(v))\right| \le R\left|\frac{v^2+2}{v^2}\right|\frac{\|\theta_1 - \theta_2\|_{2,\nu}}{|v|^2} \le M\frac{\delta}{\omega^2}\|\theta_1 - \theta_2\|_{2,\nu}, \tag{7.4.9}$$

whereas, if $\operatorname{Re}(v) \ge -\nu$,

$$|(v^2+2)\gamma_1(v)(\theta_1(v) - \theta_2(v))| \;\le\; M\frac{\delta}{\sqrt{\varepsilon}}\|\theta_1 - \theta_2\|_{2,\nu}. \tag{7.4.10}$$

Recalling that $\omega = \Omega/\sqrt{\varepsilon}$ and joining (7.4.9) and (7.4.10), we obtain that estimate (7.4.10) holds in D^u_ε. The other term in (7.4.8) is bounded in an analogous way. Thus, statement (2) holds.

If $(\gamma_j, \theta_j) \in \mathcal{X}^2_{2,\nu}$, then $\eta_0(v, \gamma_j, \theta_j) \in \mathbb{R}$, for each $v \in D^u_\varepsilon \cap \mathbb{R}$, thus, it is clear that $\mathcal{F}_0(\gamma_j, \theta_j) \in \mathcal{X}^2_{2,\nu}$.

Finally, for $v \in D^u_\varepsilon$,

$$
\begin{aligned}
|\pi_1 \circ \mathcal{F}_0(\gamma_1,\theta_1)(v) - \pi_1 \circ \mathcal{F}_0(\gamma_2,\theta_2)(v)| &= \omega\,|\gamma_1(v)(\eta_0(v,\gamma_1,\theta_1)-1) - \gamma_2(v)(\eta_0(v,\gamma_2,\theta_2)-1)| \\
&\leq M\delta^2\left(\frac{1}{\omega}+1\right)\omega|\gamma_1(v)-\gamma_2(v)| \\
&\quad + M\delta\omega^3\|(\gamma_1,\theta_1)-(\gamma_2,\theta_2)\|_{2,\nu}|\gamma_2(v)|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\pi_1 \circ \mathcal{F}_0(\gamma_1,\theta_1) - \pi_1 \circ \mathcal{F}_0(\gamma_2,\theta_2)\|_{2,\nu} &\leq M\delta^2\left(\frac{1}{\omega}+1\right)\omega\|\gamma_1-\gamma_2\|_{2,\nu} \\
&\quad + MR\delta\omega^3\|(\gamma_1,\theta_1)-(\gamma_2,\theta_2)\|_{2,\nu} \\
&\leq M\delta^2\omega\|(\gamma_1,\theta_1)-(\gamma_2,\theta_2)\|_{2,\nu}.
\end{aligned}
$$

We can prove the same bound for the second coordinate of $\mathcal{F}_0$ analogously. □

Proposition 7.4.5. *Consider the operator $\mathcal{G}_{\omega,0} = \mathcal{G}_\omega \circ \mathcal{F}_0$, where $\mathcal{G}_\omega$ and $\mathcal{F}_0$ are given in* (7.4.6) *and* (7.4.7)*. Given $\nu > 0$, there exists a constant $M > 0$ independent of ε, such that*

$$\|\mathcal{G}_{\omega,0}(0,0)\|_{2,\nu} \leq M\frac{\delta}{\omega^2}.$$

Proof. Recall that $\mathcal{F}_0(0,0) = (-(Q^0)'(v), (Q^0)'(v))$, where Q^0 is given by (7.3.11). Thus $\pi_1 \circ \mathcal{F}_0(0,0)(v) = \pi_2 \circ \mathcal{F}_0(0,0)(v)$, for each $v \in D^u_\varepsilon \cap \mathbb{R}$ and

$$\|\mathcal{F}_0(0,0)\|_{2,\nu} = 2\frac{\delta}{\omega\sqrt{2\Omega}}\|F(X_0)'\|_{2,\nu}.$$

A straightforward computation shows that

$$F(X_0(v))' = \frac{2\sqrt{2}(v^2-2)}{(v^2+2)^2}$$

Then,

$$
\begin{aligned}
|v^2 F(X_0(v))'| \leq M &\qquad \text{for } \operatorname{Re}(v) \leq -\nu, \\
|(v^2+2)^2 F(X_0(v))'| \leq M|v^2+2| \leq M &\qquad \text{for } \operatorname{Re}(v) \geq -\nu.
\end{aligned}
$$

The result follows directly from these bounds and Proposition 7.4.3. □

7.4.2 The Fixed Point argument

Finally, we are able to prove the existence of a fixed point of $\mathcal{G}_{\omega,0}$.

Proposition 7.4.6. *Given $\nu > 0$ fixed. There exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, the operator $\mathcal{G}_{\omega,0}$ has a fixed point (γ_0^u, θ_0^u) in $\mathcal{X}^2_{2,\nu}$. Furthermore, there exists a constant $M > 0$ independent of ε such that*

$$\|(\gamma_0^u, \theta_0^u)\|_{2,\nu} \leq M\frac{\delta}{\omega^2}.$$

Proof. From Proposition 7.4.5, there exists a constant $b_1 > 0$ independent of h and ε such that

$$\|\mathcal{G}_{\omega,0}(0,0)\|_{2,\nu} \leq \frac{b_1}{2}\frac{\delta}{\omega^2}.$$

Given $(\gamma_1,\theta_1),(\gamma_2,\theta_2) \in \mathcal{B}_0(b_1\delta/\omega^2) \subset \mathcal{X}^2_{2,\nu}$, we can use Propositions 7.4.3, 7.4.4 (with $K = b_1$) and the linearity of the operator $\mathcal{G}_\omega$ to see that

$$\begin{aligned}\|\mathcal{G}_{\omega,0}(\gamma_1,\theta_1) - \mathcal{G}_{\omega,0}(\gamma_2,\theta_2)\|_{2,\nu} &\leq \frac{M}{\omega}\|\mathcal{F}_0(\gamma_1,\theta_1) - \mathcal{F}_0(\gamma_2,\theta_2)\|_{2,\nu} \\ &\leq M\delta^2\|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_{2,\nu}.\end{aligned}$$

Thus, choosing ε_0 sufficiently small, we have that $\mathrm{Lip}(\mathcal{G}_{\omega,0}) \leq 1/2$. Also, it follows that $\pi_1 \circ \mathcal{G}_{\omega,0}(\gamma,\theta)(v) = \pi_2 \circ \mathcal{G}_{\omega,0}(\gamma,\theta)(v)$, for each $v \in D^u_\varepsilon \cap \mathbb{R}$ and $(\gamma,\theta) \in \mathcal{B}_0(b_1\delta/\omega^2)$.

Therefore $\mathcal{G}_{\omega,0}$ sends the ball $\mathcal{B}_0(b_1\delta/\omega^2)$ into itself and it is a contraction. Thus, it has a unique fixed point $(\gamma^u_0,\theta^u_0) \in \mathcal{B}_0(b_1\delta/\omega^2)$. □

Proposition 7.4.2 is a consequence of Proposition 7.4.6.

7.5 Proof of Theorem 7.3.7

7.5.1 The Difference Map

In Proposition 7.4.6, we have found complex functions $\Gamma^\star_0 = Q^0 + \gamma^\star_0$ and $\Theta^\star_0 = -Q^0 + \theta^\star_0$ defined in the complex domains $D^\star_\varepsilon$, respectively, such that,

$$N^\star_{0,0}(v) = (X_0(v), Z^\star_0(v), \Gamma^\star_0(v), \Theta^\star_0(v)),$$

are parameterizations of $W^\star_\delta(p^\mp_0)$ of (7.3.1). Both (Γ^u_0,Θ^u_0) and (Γ^s_0,Θ^s_0) are defined in the complex domain

$$\mathcal{D}_\varepsilon = D^u_\varepsilon \cap D^s_\varepsilon.$$

Note that $0 \in \mathcal{I}_\varepsilon := \mathcal{D}_\varepsilon \cap \mathbb{R}$. To prove that the heteroclinic connection between p^-_0 and p^+_0 of (7.3.1) is broken for $\varepsilon > 0$ sufficiently small, it is sufficient to show that

$$\left|N^u_{0,0}(v) - N^s_{0,0}(v)\right| \geq |(\Gamma^u_0,\Theta^u_0)(v) - (\Gamma^s_0,\Theta^s_0)(v)| > 0,$$

for some $v \in \mathcal{I}_\varepsilon$. To this end, we study the difference map

$$\Delta\xi(v) = \begin{pmatrix} \Gamma^u_0(v) - \Gamma^s_0(v) \\ \Theta^u_0(v) - \Theta^s_0(v)\end{pmatrix} = \begin{pmatrix} \gamma^u_0(v) - \gamma^s_0(v) \\ \theta^u_0(v) - \theta^s_0(v)\end{pmatrix}, \tag{7.5.1}$$

where $(\gamma^\star_0,\theta^\star_0)$, $\star = u,s$, are given by Proposition 7.4.6.

Proposition 7.5.1. *The difference map $\Delta\xi$ satisfies the differential equation:*

$$\Delta\xi' = A\Delta\xi + B(v)\Delta\xi, \tag{7.5.2}$$

where

$$A = \begin{pmatrix} \omega i & 0 \\ 0 & -\omega i\end{pmatrix} \text{ and } B(v) = \begin{pmatrix} b_{1,1}(v) & b_{1,2}(v) \\ b_{2,1}(v) & b_{2,2}(v)\end{pmatrix}, \tag{7.5.3}$$

and there exists a constant M independent of ε, such that for $v \in \mathcal{D}_\varepsilon$,

$$|b_{j,k}(v)| \leq M\omega\delta^2, \quad j,k = 1,2. \tag{7.5.4}$$

Proof. Recall that both $(\gamma_0^{u,s}, \theta_0^{u,s})$ satisfy (7.4.3) and therefore

$$\begin{pmatrix} \gamma' - \omega i\gamma \\ \theta' + \omega i\theta \end{pmatrix} = \mathcal{F}_0(\gamma, \theta),$$

where $\mathcal{F}_0$ is given in (7.4.7). Therefore $\Delta\xi$ satisfies

$$\Delta\xi' \;=\; A\Delta\xi + G(v),$$

where $G(v) = g(v, \gamma_0^u(v), \theta_0^u(v)) - g(v, \gamma_0^s(v), \theta_0^s(v))$, with

$$g(v, z_1, z_2) = \begin{pmatrix} i\omega z_1(\eta_0(v, z_1, z_2) - 1) \\ -i\omega z_2(\eta_0(v, z_1, z_2) - 1) \end{pmatrix}.$$

Notice that $G(v)$ is a known function, since $(\gamma_0^{u,s}, \theta_0^{u,s})$ are given by Proposition 7.4.6. We apply the Integral Mean Value Theorem to obtain

$$g(v, \gamma_0^u, \theta_0^u) - g(v, \gamma_0^s, \theta_0^s) \;=\; \begin{pmatrix} b_{1,1}(v) & b_{1,2}(v) \\ b_{2,1}(v) & b_{2,2}(v) \end{pmatrix} \cdot \begin{pmatrix} \gamma_0^u - \gamma_0^s \\ \theta_0^u - \theta_0^s \end{pmatrix},$$

where $b_{j,k}$ are analytic functions, $j, k = 1, 2$. Estimate (7.5.4) follows from Propositions 7.4.4 and 7.4.6.

□

7.5.2 Exponentially Small Splitting of $W_\varepsilon^u(p_0^-)$ and $W_\varepsilon^s(p_0^+)$

We study the solutions of (7.5.2). Notice that, if $B = 0$, then any analytic solution of (7.5.2) which is bounded in $\mathcal{D}_\varepsilon$ is exponentially small with respect to ε for real values $v \in \mathcal{I}_\varepsilon$. In this section, we follow ideas from [9] to prove that the same holds for solutions of the full equation (7.5.2) using that B (given in (7.5.3)) is small for ε small enough.

We are interested in obtaining an asymptotic expression for $\Delta\xi$ given in (7.5.1). From Proposition 7.4.6, we have that $(\gamma_0^{u,s}, \theta_0^{u,s})$ is obtained as a fixed point of $\mathcal{G}_{\omega,0}^{u,s}$. Thus, the difference map can be expressed as

$$\Delta\xi = \mathcal{G}_{\omega,0}^u(\gamma_0^u, \theta_0^u) - \mathcal{G}_{\omega,0}^s(\gamma_0^s, \theta_0^s).$$

Therefore, as $\gamma_0^{u,s}, \theta_0^{u,s}$ are small, it suggests that the dominant part of $\Delta\xi$ should be given by $\mathcal{M} = \mathcal{G}_{\omega,0}^u(0,0) - \mathcal{G}_{\omega,0}^s(0,0)$. For this reason, we decompose

$$\Delta\xi = \mathcal{M} + \Delta\xi_1, \tag{7.5.5}$$

where $\mathcal{M} = (\mathcal{M}_\Gamma, \mathcal{M}_\Theta)$ is given by the Melnikov integrals

$$\begin{aligned} \mathcal{M}_\Gamma(v) &= ie^{i\omega v}\int_{-\infty}^{\infty} e^{-i\omega r}\frac{2\delta(r^2-2)}{\omega\sqrt{\Omega}(r^2+2)^2}dr = c_1^0 e^{i\omega v}, \\ \mathcal{M}_\Theta(v) &= -ie^{-i\omega v}\int_{-\infty}^{\infty} e^{i\omega r}\frac{2\delta(r^2-2)}{\omega\sqrt{\Omega}(r^2+2)^2}dr = c_2^0 e^{-i\omega v} \end{aligned} \tag{7.5.6}$$

and $\Delta\xi_1 = (\Delta_\Gamma^1, \Delta_\Theta^1)$.

A straightforward computation proves the following lemma.

Lemma 7.5.2. *The constants c_1^0 and c_2^0 are given by*

$$c_1^0 = -i\frac{2\pi\delta}{\sqrt{\Omega}}e^{-\sqrt{2}\omega}, \text{ and } c_2^0 = \overline{c_1^0}. \tag{7.5.7}$$

Theorem 7.3.7 is equivalent to the following theorem. The remainder of Section 7.5.2 is devoted to prove it.

Theorem 7.5.3. *There exists $\varepsilon_0 > 0$ sufficiently small such that for $v \in \mathcal{I}_\varepsilon \subset \mathbb{R}$, $0 < \varepsilon \leq \varepsilon_0$,*

$$\Delta\xi(v) = \mathcal{M}(v) + \mathcal{O}(\omega\delta^3 e^{-\sqrt{2}\omega}),$$

where $\mathcal{M} = (\mathcal{M}_\Gamma, \mathcal{M}_\Theta)$ is the Melnikov vector defined in (7.5.6).

A Fixed Point Argument for the error $\Delta\xi_1$

We write $\Delta\xi_1$ in (7.5.5) as solution of a fixed point equation in the functional space

$$\mathcal{E} = \left\{ f : \mathcal{D}_\varepsilon \to \mathbb{C}^2;\ f \text{ is analytic and } \|f\|_{\mathcal{E}} < \infty \right\},$$

where

$$\|f\|_{\mathcal{E}} = \sum_{j=1}^{2} \sup_{v\in\mathcal{D}_\varepsilon} |(v^2+2)^2 \pi_j \circ f(v)|.$$

We also consider the linear operator $\mathcal{H}_0$ given by

$$\mathcal{H}_0(g)(v) = \begin{pmatrix} e^{\omega i v}\displaystyle\int_{v^*}^{v} e^{-i\omega r}\pi_1(B(r)\cdot g(r))dr \\ e^{-\omega i v}\displaystyle\int_{\overline{v^*}}^{v} e^{i\omega r}\pi_2(B(r)\cdot g(r))dr \end{pmatrix},$$

where $v^* = -(\sqrt{2}-\sqrt{\varepsilon})i$ and B is the matrix given (7.5.3).

Using (7.5.4), the operator $\mathcal{H}_0$ is well-defined from $\mathcal{E}_\varepsilon$ to itself. To simplify the notation, we introduce the function

$$I(k_1,k_2)(v) = e^{Av}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} e^{i\omega v}k_1 \\ e^{-i\omega v}k_2 \end{pmatrix}, \tag{7.5.8}$$

where $k_j \in \mathbb{C}$, $j = 1,2$, $v \in \mathcal{D}_\varepsilon$ and A is the matrix given by (7.5.3). Notice that $\mathcal{M}(v) = I(c_1^0, c_2^0)(v)$.

Lemma 7.5.4. *The difference map $\Delta\xi$ belongs to $\mathcal{E}_\varepsilon$ and $\|\Delta\xi\|_{\mathcal{E}} \leq M\varepsilon$. Furthermore, there exist $(c_1, c_2) \in \mathbb{C}^2$ such that:*

$$\Delta\xi_1(v) = I(c_1 - c_1^0, c_2 - c_2^0)(v) + \mathcal{H}_0(\Delta\xi_1)(v) + \mathcal{H}_0(\mathcal{M})(v), \tag{7.5.9}$$

and $|c_j - c_j^0| \leq M\delta^3 e^{-\sqrt{2}\omega}$, $j = 1,2$, where M is a constant independent of ε.

Proof. Since $(\gamma_0^{u,s}, \theta_0^{u,s}) \in \mathcal{X}_{2,\nu}^2$, it is clear to see that $\Delta\xi \in \mathcal{E}_\varepsilon$. In addition, from Proposition 7.4.6,

$$\|\Delta\xi\|_{\mathcal{E}} \leq 2(\|(\gamma_0^u, \theta_0^u)\|_{2,\nu} + \|(\gamma_0^s, \theta_0^s)\|_{2,\nu}) \leq M\frac{\delta}{\omega^2},$$

where M is a constant independent of ε.

Since $\Delta\xi$ is a solution of (7.5.2), the method of variation of parameters implies that, given $v_1, v_2 \in \mathcal{D}_\varepsilon$, there exist $c_1, c_2 \in \mathbb{C}$ such that

$$\Delta\xi(v) = \begin{pmatrix} e^{\omega iv}c_1 + e^{\omega iv}\int_{v_1}^{v} e^{-i\omega r}\pi_1(B(r)\cdot\Delta\xi(r))dr \\ e^{-\omega iv}c_2 + e^{-\omega iv}\int_{v_2}^{v} e^{i\omega r}\pi_2(B(r)\cdot\Delta\xi(r))dr \end{pmatrix}.$$

We take $v_1 = v^*$, $v_2 = \overline{v^*}$, with $v^* = -(\sqrt{2}-\sqrt{\varepsilon})i$. Thus,

$$\Delta\xi(v) = I(c_1,c_2)(v) + \mathcal{H}_0(\Delta\xi)(v).$$

Using that $\Delta\xi = \mathcal{M} + \Delta\xi_1$, $\mathcal{M}(v) = I(c_1^0, c_2^0)(v)$ and $\mathcal{H}_0$ is linear,

$$\Delta\xi_1(v) = I(c_1 - c_1^0, c_2 - c_2^0)(v) + \mathcal{H}_0(\Delta\xi_1)(v) + \mathcal{H}_0(\mathcal{M})(v).$$

Now, we bound $|c_j - c_j^0|$, $j = 1, 2$. By (7.5.5) and Proposition 7.4.6,

$$\begin{aligned} \|\Delta\xi_1\|_{2,\nu} &= \|\Delta\xi - \mathcal{M}\|_{2,\nu} \\ &= \|(\gamma_0^u,\theta_0^u) - (\gamma_0^s,\theta_0^s) - (\mathcal{G}^u_{\omega,0}(0,0) - \mathcal{G}^s_{\omega,0}(0,0))\|_{2,\nu} \\ &= \|\mathcal{G}^u_{\omega,0}(\gamma_0^u,\theta_0^u) - \mathcal{G}^u_{\omega,0}(0,0) - (\mathcal{G}^s_{\omega,0}(\gamma_0^s,\theta_0^s) - \mathcal{G}^s_{\omega,0}(0,0))\|_{2,\nu} \\ &\leq M\delta^2(\|(\gamma_0^u,\theta_0^u)\|_{2,\nu} + \|(\gamma_0^s,\theta_0^s)\|_{2,\nu}) \\ &\leq M\frac{\delta^3}{\omega^2}. \end{aligned}$$

Thus,

$$|\pi_j(\Delta\xi_1(v))| \leq M\frac{\delta^3}{\omega^2|v^2+2|^2} \leq M\delta^3, \text{ for each } v \in \mathcal{D}_\varepsilon,\ j = 1,2.$$

In particular, replacing $v = v^*$ in the first component of (7.5.9), we obtain that

$$|e^{\omega iv^*}(c_1 - c_1^0)| \leq M\delta^3 \Leftrightarrow |c_1 - c_1^0| \leq M\delta^3 e^{\omega\sqrt{\varepsilon}}e^{-\sqrt{2}\omega} \leq 2M\delta^3 e^{-\sqrt{2}\omega}.$$

Analogously, taking $v = \overline{v^*}$ in the second component of (7.5.9), we obtain that $|c_2 - c_2^0| \leq 2M\delta^3 e^{-\sqrt{2}\omega}$. □

Exponentially Smallness of $\Delta\xi_1$

Consider the functional space

$$\mathcal{Z} = \{f : \mathcal{D}_\varepsilon \to \mathbb{C}^2;\ f \text{ is analytic and } \|f\|_{\mathcal{Z}} < +\infty\},$$

where

$$\|f\|_{\mathcal{Z}} = \sum_{j=1}^{2}\sup_{v\in\mathcal{D}_\varepsilon}\left|e^{\omega(\sqrt{2}-|\mathrm{Im}(v)|)}\pi_j \circ f(v)\right|. \tag{7.5.10}$$

In order to prove Theorem 7.5.3, it is enough to check that $\Delta\xi_1$ belongs to $\mathcal{Z}$ and that $\|\Delta\xi_1\|_{\mathcal{Z}} \leq M\omega\delta^3$. Our strategy to achieve these results is to prove that both $I(c_1 - c_1^0, c_2 - c_2^0)$ and $\mathcal{H}_0(\mathcal{M})$ belong to $\mathcal{Z}$ and that the operator $\mathrm{Id} - \mathcal{H}_0$ is invertible in $\mathcal{Z}$.

Lemma 7.5.5. *There exists $\varepsilon_0 > 0$, such that the linear operator* $\mathrm{Id} - \mathcal{H}_0$ *is invertible in* $\mathcal{Z}$ *for* $\varepsilon \le \varepsilon_0$. *Furthermore, there exists* $M > 0$ *independent of* ε *such that* $\|\mathcal{H}_0\|_{\mathcal{Z}} \le M\omega\delta^2$ *and hence*

$$\|(\mathrm{Id} - \mathcal{H}_0)^{-1}\|_{\mathcal{Z}} \le (1 - \|\mathcal{H}_0\|_{\mathcal{Z}})^{-1} \le 1 + M\omega\delta^2. \tag{7.5.11}$$

Proof. Since $\mathcal{H}_0$ is a linear operator, to prove this lemma, it is sufficient to show that $\|\mathcal{H}_0\|_{\mathcal{Z}} \le M\omega\delta^2 < 1$.

Let $h \in \mathcal{Z}$ and denote by M any constant independent of ε. Using (7.5.4) and (7.5.10), we have that for $v \in \mathcal{D}_\varepsilon$ and $j = 1, 2$,

$$|\pi_j(B(v) \cdot h(v))| \le \sum_{k=1}^{2} |b_{j,k}(v)\pi_k(h(v))| \le M\omega\delta^2 e^{-\omega(\sqrt{2} - |\mathrm{Im}(v)|)} \|h\|_{\mathcal{Z}}.$$

Thus

$$\begin{aligned} |e^{\omega(\sqrt{2} - |\mathrm{Im}(v)|)}\pi_1(\mathcal{H}_0(h)(v))| &= \left| e^{\sqrt{2}\omega} \int_{v^*}^{v} e^{-i\omega(r - v - i|\mathrm{Im}(v)|)} \pi_1(B(r) \cdot h(r)) dr \right| \\ &\le M\omega\delta^2 e^{-\sqrt{2}\omega} e^{\sqrt{2}\omega} \|h\|_{\mathcal{Z}} \int_{v^*}^{v} \left| e^{-i\omega(r - v - i|\mathrm{Im}(v)|)} \right| e^{\omega|\mathrm{Im}(r)|} dr \\ &\le M\omega\delta^2 \|h\|_{\mathcal{Z}} \int_{v^*}^{v} e^{\omega(\mathrm{Im}(r) + |\mathrm{Im}(r)| - \mathrm{Im}(v) - |\mathrm{Im}(v)|)} dr. \end{aligned}$$

Since $\mathrm{Im}(v^*) \le \mathrm{Im}(r) \le \mathrm{Im}(v)$, we have that $\mathrm{Im}(r) + |\mathrm{Im}(r)| - \mathrm{Im}(v) - |\mathrm{Im}(v)| \le 0$, then

$$\left| \int_{v^*}^{v} e^{\omega(\mathrm{Im}(r) + |\mathrm{Im}(r)| - \mathrm{Im}(v) - |\mathrm{Im}(v)|)} dr \right| \le M.$$

Analogously, we have that

$$|e^{\omega(\sqrt{2} - |\mathrm{Im}(v)|)}\pi_2(\mathcal{H}_0(h)(v))| \le M\omega\delta^2 \|h\|_{\mathcal{Z}},$$

and thus $\|\mathcal{H}_0(h)\|_{\mathcal{Z}} \le M\omega\delta^2 \|h\|_{\mathcal{Z}}$. Since, $\|\mathcal{H}_0\|_{\mathcal{Z}} < 1$, for ε sufficiently small, the linear operator $\mathrm{Id} - \mathcal{H}_0$ is invertible and satisfies (7.5.11). □

Now, recall that $\mathcal{M} = I(c_1^0, c_2^0)$, where I is given by (7.5.8) and c_1^0, c_2^0 are given by (7.5.7). Moreover, from Lemma 7.5.4, we have that

$$(\mathrm{Id} - \mathcal{H}_0)\Delta\xi_1 = I(c_1 - c_1^0, c_2 - c_2^0) + \mathcal{H}_0(I(c_1^0, c_2^0)).$$

Since $\mathrm{Id} - \mathcal{H}_0$ is invertible in $\mathcal{Z}$, it only remains to show that $I(c_1 - c_1^0, c_2 - c_2^0)$ and $I(c_1^0, c_2^0)$ belong to $\mathcal{Z}$.

Lemma 7.5.6. *Given* $k_1, k_2 \in \mathbb{C}$, *then the function* I *given in* (7.5.8) *satisfies*

$$\|I(k_1, k_2)\|_{\mathcal{Z}} \le M e^{\sqrt{2}\omega}(|k_1| + |k_2|),$$

where M *is a constant independent of* ε.

To prove Lemma 7.5.6 it is enough to recall the definitions of $\|\cdot\|_{\mathcal{Z}}$ in (7.5.10) and I in (7.5.8).

Lemma 7.5.7. *The error vector* $\Delta\xi_1$ *given in* (7.5.5) *belongs to* $\mathcal{Z}$ *and it is determined by*

$$\Delta\xi_1 = (\mathrm{Id} - \mathcal{H}_0)^{-1} \left(I(c_1 - c_1^0, c_2 - c_2^0 \right) + (\mathrm{Id} - \mathcal{H}_0)^{-1} \left(\mathcal{H}_0(\mathcal{M}) \right). \tag{7.5.12}$$

Furthermore, there exists a constant $M > 0$ *independent of* ε *such that*

$$\|\Delta\xi_1\|_{\mathcal{Z}} \le M\omega\delta^3. \tag{7.5.13}$$

Proof. From Lemmas 7.5.4 and 7.5.2, we have that $|c_j - c_j^0|\leq M\delta^3 e^{-\sqrt{2}\omega}$, and $|c_j^0|\leq M\delta e^{-\sqrt{2}\omega}$, $j=1,2$. Therefore, it follows from Lemma 7.5.6 that $I(c_1-c_1^0, c_2-c_2^0)\in\mathcal{Z}$, and $\mathcal{M}=I(c_1^0,c_2^0)\in\mathcal{Z}$. Furthermore

$$\|I(c_1-c_1^0,c_2-c_2^0)\|_{\mathcal{Z}}\leq M\delta^3 \text{ and } \|\mathcal{M}\|_{\mathcal{Z}}\leq M\delta.$$

As $\mathrm{Id}-\mathcal{H}_0$ is invertible in $\mathcal{Z}$ by Lemma 7.5.5, formula (7.5.12) is equivalent to (7.5.9). Therefore, $\Delta\xi_1\in\mathcal{Z}$ and, using again Lemma 7.5.5,

$$\begin{aligned}\|\Delta\xi_1\|_{\mathcal{Z}} &\leq \|(\mathrm{Id}-\mathcal{H}_0)^{-1}\|_{\mathcal{Z}}(\|I(c_1-c_1^0,c_2-c_2^0)\|_{\mathcal{Z}}+\|\mathcal{H}_0(\mathcal{M})\|_{\mathcal{Z}})\\ &\leq M\delta^3+M\|\mathcal{H}_0\|_{\mathcal{Z}}\|\mathcal{M}\|_{\mathcal{Z}}\\ &\leq M\omega\delta^3.\end{aligned}$$

□

Proof of Theorem 7.5.3. Finally, we prove that $\Delta\xi_1$ is exponentially small and we obtain an asympotic formula for $\Delta\xi$. From (7.5.13) and the definition of the norm (7.5.10), we have

$$|e^{\omega(\sqrt{2}-|\mathrm{Im}(v)|)}\pi_j\circ\Delta\xi_1(v)|\leq M\omega\delta^3, \text{ for } v\in\mathcal{D}_\varepsilon, \text{ and } j=1,2.$$

In particular, if $v\in\mathcal{I}_\varepsilon=\mathcal{D}_\varepsilon\cap\mathbb{R}$, $|\Delta\xi_1(v)|\leq M\omega\delta^3e^{-\sqrt{2}\omega}$, for $j=1,2$. The result follows directly from this bound and (7.5.5).

□

7.6 Proof of Theorem 7.3.8

In this section we look for parameterizations of the invariant manifolds $W_\varepsilon^u(\Lambda_h^-)$ of the periodic orbits Λ_h^- of the form

$$N_{0,h}^u(v,\tau)=(X_0(v),Z_0(v)+Z_{0,h}^u(v,\tau),\Gamma_h(\tau)+\Gamma_{0,h}^u(v,\tau),\Theta_h(\tau)+\Theta_{0,h}^u(v,\tau)), \quad (7.6.1)$$

where Z_0,Γ_h,Θ_h are given in (7.3.4) and (7.3.7), as a perturbation of $N_{0,h}(v,\tau)$ (see (7.3.6)).

Lemma 7.6.1. *The invariant manifold $W_\delta^u(\Lambda_h^-)$, with $\delta\neq 0$, can be parameterized by $N_{0,h}^u(v,\tau)$ in (7.6.1) if $(Z_{0,h}^u(v,\tau),\Gamma_{0,h}^u(v,\tau),\Theta_{0,h}^u(v,\tau))$ satisfy the following system of partial differential equations*

$$\begin{cases}\partial_vZ+\omega\partial_\tau Z+\dfrac{Z_0'(v)}{Z_0(v)}Z = -\dfrac{Z}{Z_0(v)}\partial_vZ-\dfrac{\delta}{\sqrt{2\Omega}}F'(X_0(v))\dfrac{\Gamma-\Theta}{2i}\\ \qquad -\dfrac{\delta}{\sqrt{2\Omega}}F'(X_0(v))\dfrac{\Gamma_h(\tau)-\Theta_h(\tau)}{2i},\\ \partial_v\Gamma+\omega\partial_\tau\Gamma = -\dfrac{Z}{Z_0(v)}\partial_v\Gamma+\omega i\Gamma-\dfrac{\delta}{\sqrt{2\Omega}}F(X_0(v)),\\ \partial_v\Theta+\omega\partial_\tau\Theta = -\dfrac{Z}{Z_0(v)}\partial_v\Theta-\omega i\Theta-\dfrac{\delta}{\sqrt{2\Omega}}F(X_0(v)),\\ \lim\limits_{v\to-\infty}Z(v,\tau)=\lim\limits_{v\to-\infty}\Gamma(v,\tau)=\lim\limits_{v\to-\infty}\Theta(v,\tau)=0, \text{ for each } \tau\in[0,2\pi],\end{cases} \quad (7.6.2)$$

and $Z_{0,h}^u,\Gamma_{0,h}^u,\Theta_{0,h}^u$ are 2π-periodic in the variable τ.

In contrast to the 1-dimensional case, for technical reasons, we do not use that $\mathcal{H}(W_\varepsilon^u(\Lambda_h^-)) = h$ to obtain $Z = Z(X, \Gamma, \Theta)$. Thus, we deal with the problem in dimension 3.

As in the 1-dimensional case (7.4.2), if we set $Z = \Gamma = \Theta = 0$, the right-hand side of (7.6.2) decays as $1/|v|$ as $v \to -\infty$. To have quadratic decay as $|v| \to \infty$ to have integrability, we perform with the change (7.3.14) to system (7.6.2). Then, $(z_{0,h}^u, \gamma_{0,h}^u, \theta_{0,h}^u)$ satisfy

$$\left\{\begin{array}{rcl}
\partial_v z + \omega\partial_\tau z + \dfrac{Z_0'(v)}{Z_0(v)} z &=& f_1^h(v,\tau) - \dfrac{z + Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v z - \dfrac{\partial_v Z_{0,h}(v,\tau)}{Z_0(v)} z \\
&& -\dfrac{\delta}{\sqrt{2\Omega}} F'(X_0(v)) \dfrac{\gamma - \theta}{2i} \\
\partial_v \gamma + \omega\partial_\tau \gamma - \omega i \gamma &=& f_2^h(v,\tau) - \dfrac{(Q^0)'(v)}{Z_0(v)} z - \dfrac{z + Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v \gamma, \\
\partial_v \theta + \omega\partial_\tau \theta + \omega i \theta &=& -f_2^h(v,\tau) + \dfrac{(Q^0)'(v)}{Z_0(v)} z - \dfrac{z + Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v \theta, \\
\lim\limits_{v\to-\infty} z(v,\tau) = \lim\limits_{v\to-\infty} \gamma(v,\tau) = \lim\limits_{v\to-\infty} \theta(v,\tau) = 0, &&
\end{array}\right. \tag{7.6.3}$$

where

$$\begin{aligned}
f_1^h(v,\tau) = &- \partial_v Z_{0,h}(v,\tau) - \frac{Z_0'(v)}{Z_0(v)} Z_{0,h}(v,\tau) \\
&- \frac{\delta}{\sqrt{2\Omega}} F'(X_0(v)) \frac{Q^0(v)}{i} - \frac{Z_{0,h}(v,\tau)\partial_v Z_{0,h}(v,\tau)}{Z_0(v)},
\end{aligned} \tag{7.6.4}$$

$$f_2^h(v,\tau) = - (Q^0)'(v) - \frac{Z_{0,h}(v,\tau)(Q^0)'(v)}{Z_0(v)}, \tag{7.6.5}$$

and Q^0, $Z_{0,h}$ are given by (7.3.11), (7.3.15), respectively.

We consider equation (7.6.3) with $(v,\tau) \in D^u \times \mathbb{T}_\sigma$ (see (7.3.12) and (7.3.13)), and asymptotic conditions $\lim\limits_{\mathrm{Re}(v)\to-\infty} z(v,\tau) = \lim\limits_{\mathrm{Re}(v)\to-\infty} \gamma(v,\tau) = \lim\limits_{\mathrm{Re}(v)\to-\infty} \theta(v,\tau) = 0$, for every $\tau \in \mathbb{T}_\sigma$.

Proposition 7.6.2. *Fix $\sigma > 0$ and $h_0 > 0$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $0 < \varepsilon \le \varepsilon_0$ and $0 \le h \le h_0$, equation* (7.6.3) *has a solution $(z_{0,h}^u, \gamma_{0,h}^u, \theta_{0,h}^u)$ defined in $D^u \times \mathbb{T}_\sigma$ such that $z_{0,h}^u$ is real-analytic, $\gamma_{0,h}^u, \theta_{0,h}^u$ are analytic, $\theta_{0,h}^u(v,\tau) = \overline{\gamma_{0,h}^u(v,\tau)}$ for each $(v,\tau) \in \mathbb{R}^2$, and*

$$\lim_{\mathrm{Re}(v)\to-\infty} z_{0,h}^u(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \gamma_{0,h}^u(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \theta_{0,h}^u(v,\tau) = 0,$$

for every $\tau \in \mathbb{T}_\sigma$. Furthermore, $(z_{0,h}^u, \gamma_{0,h}^u, \theta_{0,h}^u)$ satisfy the bounds in (7.3.16).

We devote the rest of this section to prove Proposition 7.6.2. Equation (7.6.3) can be written as the functional equation

$$\mathcal{L}_\omega(z,\gamma,\theta) = \mathcal{P}_h(z,\gamma,\theta),$$

where $\mathcal{L}_\omega$ and $\mathcal{P}_h$ are the operators

$$\mathcal{L}_\omega(z,\gamma,\theta) = \begin{pmatrix} \partial_v z + \omega\partial_\tau z + \dfrac{Z_0'(v)}{Z_0(v)} z \\ \partial_v \gamma + \omega\partial_\tau \gamma - \omega i \gamma \\ \partial_v \theta + \omega\partial_\tau \theta + \omega i \theta \end{pmatrix}, \tag{7.6.6}$$

$$\mathcal{P}_h(z,\gamma,\theta) = \begin{pmatrix} f_1^h(v,\tau) - \dfrac{z+Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v z - \dfrac{\partial_v Z_{0,h}}{Z_0(v)} z - \dfrac{\delta}{\sqrt{2\Omega}} F'(X_0(v))\dfrac{\gamma-\theta}{2i} \\ f_2^h(v,\tau) - \dfrac{(Q^0)'(v)}{Z_0(v)} z - \dfrac{z+Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v\gamma \\ -f_2^h(v,\tau) + \dfrac{(Q^0)'(v)}{Z_0(v)} z - \dfrac{z+Z_{0,h}(v,\tau)}{Z_0(v)}\partial_v\theta \end{pmatrix}. \tag{7.6.7}$$

7.6.1 Banach spaces and technical results

For analytic functions $f : D^u \to \mathbb{C}$ and $g : D^u \times \mathbb{T}_\sigma \to \mathbb{C}$ and $\alpha > 0$, we define

$$\|f\|_\alpha = \sup_{v\in D^u} |(v^2+2)^{\alpha/2} f(v)|,$$
$$\|g\|_{\alpha,\sigma} = \sum_{k\in\mathbb{Z}} \|g^{[k]}\|_\alpha e^{|k|\sigma},$$

where $g(v,\tau) = \sum_{k\in\mathbb{Z}} g^{[k]}(v) e^{ik\tau}$.

Remark 7.6.3. *Notice that there exists a constant $d > 0$ independent of ε such that the distance between each $v \in D^u$ (given in* (7.3.12)*) and the poles $\pm i\sqrt{2}$ of $N_{0,h}(v,\tau)$ (given in* (7.3.6)*) is greater than d. The weight $|v^2+2|^{\alpha/2}$ in the norm $\|\cdot\|_\alpha$ is chosen to control the behavior as $\operatorname{Re} v \to -\infty$ and to have it well-defined for $v = 0 \in D^u$. In fact, at infinity this norm is equivalent to the norm with weight $|v|^\alpha$.*

We also define

$$[\![g]\!]_{\alpha,\sigma} = \max\{\|g\|_{\alpha,\sigma}, \|\partial_\tau g\|_{\alpha,\sigma}, \|\partial_v g\|_{\alpha+1,\sigma}\} \tag{7.6.8}$$

and the Banach spaces

$$\begin{aligned} \mathcal{X}_{\alpha,\sigma} &= \{g : D^u \times \mathbb{T}_\sigma \to \mathbb{C} \text{ is an analytic function, such that } \|g\|_{\alpha,\sigma} < \infty\}, \\ \mathcal{Y}_{\alpha,\sigma} &= \{g : D^u \times \mathbb{T}_\sigma \to \mathbb{C} \text{ is an analytic function, such that } [\![g]\!]_{\alpha,\sigma} < \infty\}. \end{aligned}$$

Consider the product spaces

$$\begin{aligned} \mathcal{X}^3_{\alpha,\sigma} &= \Big\{(f,g,h) \in \mathcal{X}_{\alpha,\sigma} \times \mathcal{X}_{\alpha,\sigma} \times \mathcal{X}_{\alpha,\sigma};\ f \text{ is real-analytic, } g(v,\tau) = \overline{h(v,\tau)}, \\ &\qquad \text{for every } v \in D^u \cap \mathbb{R}, \tau \in \mathbb{T}\Big\}, \\ \mathcal{Y}^3_{\alpha,\sigma} &= \Big\{(f,g,h) \in \mathcal{Y}_{\alpha,\sigma} \times \mathcal{Y}_{\alpha,\sigma} \times \mathcal{Y}_{\alpha,\sigma};\ f \text{ is real-analytic, } g(v,\tau) = \overline{h(v,\tau)}, \\ &\qquad \text{for every } v \in D^u \cap \mathbb{R}, \tau \in \mathbb{T}\Big\}, \end{aligned}$$

endowed with the norms

$$\begin{aligned} \|(f,g,h)\|_{\alpha,\sigma} &= \|f\|_{\alpha,\sigma}+\|g\|_{\alpha,\sigma}+\|h\|_{\alpha,\sigma}, \\ [\![(f,g,h)]\!]_{\alpha,\sigma} &= [\![f]\!]_{\alpha,\sigma}+[\![g]\!]_{\alpha,\sigma}+[\![h]\!]_{\alpha,\sigma}, \end{aligned}$$

respectively. We present some properties of the norm $\|\cdot\|_{\alpha,\sigma}$, which are proven in [8].

Lemma 7.6.4. *Given real-analytic functions $f : \mathbb{C} \to \mathbb{C}$, $g, h : D^u \times \mathbb{T}_\sigma \to \mathbb{C}$, the following statements hold*

1. *If $\alpha_1 \geq \alpha_2 \geq 0$, then*
$$\|h\|_{\alpha_2,\sigma} \leq \|h\|_{\alpha_1,\sigma}.$$
2. *If $\alpha_1, \alpha_2 \geq 0$, and $\|g\|_{\alpha_1,\sigma}, \|h\|_{\alpha_2,\sigma} < \infty$, then*
$$\|gh\|_{\alpha_1+\alpha_2,\sigma} \leq \|g\|_{\alpha_1,\sigma}\|h\|_{\alpha_2,\sigma}.$$
3. *If $\|g\|_{\alpha,\sigma}$, $\|h\|_{\alpha,\sigma} \leq R_0/4$, where R_0 is the convergence ratio of f' at 0, then*
$$\|f(g)-f(h)\|_{\alpha,\sigma} \leq M\|g-h\|_{\alpha,\sigma}.$$

7.6.2 The Operators $\mathcal{L}_\omega$ and $\mathcal{G}_\omega$

Let f, g, and h be analytic functions defined in $D^u \times \mathbb{T}_\sigma$. We define

$$\begin{aligned} F^{[k]}(f)(v) &= \int_{-\infty}^{v} \frac{e^{\omega ik(r-v)}Z_0(r)}{Z_0(v)} f^{[k]}(r)dr, \\ G^{[k]}(g)(v) &= \int_{-\infty}^{v} e^{\omega i(k-1)(r-v)} g^{[k]}(r)dr, \\ H^{[k]}(h)(v) &= \int_{-\infty}^{v} e^{\omega i(k+1)(r-v)} h^{[k]}(r)dr, \end{aligned} \tag{7.6.9}$$

and consider the linear operator $\mathcal{G}_\omega$ given by

$$\mathcal{G}_\omega(f,g,h) = \begin{pmatrix} \sum_k F^{[k]}(f)(v)e^{ik\tau} \\ \sum_k G^{[k]}(g)(v)e^{ik\tau} \\ \sum_k H^{[k]}(h)(v)e^{ik\tau} \end{pmatrix}. \tag{7.6.10}$$

Lemma 7.6.5. *Fix $\alpha \geq 1$ and $\sigma > 0$, the operator*

$$\mathcal{G}_\omega : \mathcal{X}^3_{\alpha+1,\sigma} \to \mathcal{Y}^3_{\alpha,\sigma}$$

given in (7.6.10) *is well-defined and the following statements hold:*

1. *$\mathcal{G}_\omega$ is an inverse of the operator $\mathcal{L}_\omega : \mathcal{Y}^3_{\alpha,\sigma} \to \mathcal{X}^3_{\alpha+1,\sigma}$ given in* (7.6.6), *i.e.* $\mathcal{G}_\omega \circ \mathcal{L}_\omega = \mathcal{L}_\omega \circ \mathcal{G}_\omega = \mathrm{Id}$;
2. $[\![\mathcal{G}_\omega(f,g,h)]\!]_{\alpha,\sigma} \leq M\|(f,g,h)\|_{\alpha+1,\sigma}$;
3. *If $f^{[0]} = g^{[1]} = h^{[-1]} = 0$, then* $[\![\mathcal{G}_\omega(f,g,h)]\!]_{\alpha,\sigma} \leq \dfrac{M}{\omega}[\![(f,g,h)]\!]_{\alpha,\sigma}$.

The proof of Lemma 7.6.5 can be found in [8].

To find a solution of (7.6.3), it is sufficient to find a fixed point of the operator

$$\overline{\mathcal{G}}_{\omega,h} = \mathcal{G}_\omega \circ \mathcal{P}_h, \tag{7.6.11}$$

where $\mathcal{G}_\omega$ is given by (7.6.10) and $\mathcal{P}_h$ is given by (7.6.7).

7.6.3 The Operator $\mathcal{P}_h$

We show some properties of the operator $\mathcal{P}_h$ defined in (7.6.7).

Lemma 7.6.6. *Fix $\sigma > 0$, $h_0 > 0$. For $0 \leq h \leq h_0$, the operator $\mathcal{P}_h$ defined in* (7.6.7) *satisfies*

$$\|\mathcal{P}_h(0,0,0)\|_{2,\sigma} \leq M\frac{\delta}{\omega}.$$

Proof. Notice that $\mathcal{P}_h(0,0,0) = (f_1^h, f_2^h, -f_2^h)$, where f_1^h and f_2^h are given by (7.6.4), and (7.6.5) respectively, and involve the functions $F'(X_0)$, Z_0, Z_0', Q^0, Q_0', $Z_{0,h}, \partial_v Z_{0,h}$. By (7.1.7), (7.3.4), (7.3.11) and (7.3.15), we can see that

$$\begin{aligned}
\|Q^0\|_{1,\sigma}, \|(Q^0)'\|_{2,\sigma} &\leq M\frac{\delta}{\omega}, \\
\|Z_{0,h}\|_{1,\sigma}, \|\partial_v Z_{0,h}\|_{2,\sigma} &\leq M\frac{\delta\sqrt{h}}{\omega^{3/2}}, \\
\|Z_0\|_{1,\sigma}, \|Z_0'\|_{2,\sigma}, \|F'(X_0)\|_{1,\sigma} &\leq M.
\end{aligned}$$

It follows from these bounds and Lemma 7.6.4 that

$$\|f_1^h\|_{2,\sigma} \leq M \max\left\{\frac{\delta\sqrt{h}}{\omega^{3/2}}, \frac{\delta^2}{\omega}, \frac{\delta^2}{\omega^3}h\right\} = M\max\left\{\frac{\delta\sqrt{h}}{\omega^{3/2}}, \frac{\delta^2}{\omega}\right\},$$

$$\|f_2^h\|_{2,\sigma} \leq M\max\left\{\frac{\delta}{\omega}, \frac{\delta^2}{\omega^{5/2}}\sqrt{h}\right\} = M\frac{\delta}{\omega}.$$

□

Lemma 7.6.7. *Fix $\sigma > 0$, $h_0 > 0$ and $K > 0$. If $0 \leq h \leq h_0$, the operator*

$$\mathcal{P}_h : \mathcal{Y}_{1,\sigma}^3 \to \mathcal{X}_{2,\sigma}^3$$

is well defined. Moreover, given $(z_j, \gamma_j, \theta_j) \in \mathcal{B}_0(K\delta/\omega) \subset \mathcal{Y}_{1,\sigma}^3$, $j = 1, 2$,

$$\|\mathcal{P}_h(z_1,\gamma_1,\theta_1) - \mathcal{P}_h(z_2,\gamma_2,\theta_2)\|_{2,\sigma} \leq M\left(\delta + \frac{\delta}{\omega^{3/2}}\sqrt{h}\right)[\![(z_1,\gamma_1,\theta_1) - (z_2,\gamma_2,\theta_2)]\!]_{1,\sigma},$$

where M is a constant independent of ε and h.

Proof. It is straightforward to see that $\mathcal{P}_h$ is well defined. Denote $\mathcal{P}_h^j = \pi_j \circ \mathcal{P}_h$. We show the bound of the difference for $\mathcal{P}_h^1$ and $\mathcal{P}_h^2$, since the bound of $\mathcal{P}_h^3$ can be obtained in exactly the same way as $\mathcal{P}_h^2$.

Notice that

$$\begin{aligned}
\mathcal{P}_h^1(z_1,\gamma_1,\theta_1) - \mathcal{P}_h^1(z_2,\gamma_2,\theta_2) \quad = \quad & -\frac{\delta}{\sqrt{2\Omega}}F'(X_0(v))\frac{(\gamma_1-\gamma_2)-(\theta_1-\theta_2)}{2i} \\
& -\frac{\partial_v Z_{0,h}(v,\tau)}{Z_0(v)}(z_1-z_2) - \partial_v z_2\frac{z_1-z_2}{Z_0(v)} \\
& -\frac{z_1+Z_{0,h}(v,\tau)}{Z_0(v)}(\partial_v z_1 - \partial_v z_2).
\end{aligned}$$

Using the bounds contained in the proof of Lemma 7.6.6 and that Z_0 is lower bounded in D^u by a positive constant independent of ε, one can see that

$$\left\|\mathcal{P}_h^1(z_1,\gamma_1,\theta_1)-\mathcal{P}_h^1(z_2,\gamma_2,\theta_2)\right\|_{2,\sigma}\leq M\max\left\{\delta,\frac{\delta}{\omega^{3/2}}\sqrt{h}\right\}[\![(z_1,\gamma_1,\theta_1)-(z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}.$$

Now,

$$\begin{aligned}\mathcal{P}_h^2(z_1,\gamma_1,\theta_1)-\mathcal{P}_h^2(z_2,\gamma_2,\theta_2) &= -\frac{(Q^0)'(v)}{Z_0(v)}(z_1-z_2)-\partial_v\gamma_2\frac{z_1-z_2}{Z_0(v)}\\ &\quad -\frac{z_1+Z_{0,h}(v,\tau)}{Z_0(v)}(\partial_v\gamma_1-\partial_v\gamma_2)\end{aligned}$$

which, proceeding analogously,

$$\left\|\mathcal{P}_h^2(z_1,\gamma_1,\theta_1)-\mathcal{P}_h^2(z_2,\gamma_2,\theta_2)\right\|_{2,\sigma}\leq M\max\left\{\frac{\delta}{\omega},\frac{\delta}{\omega^{3/2}}\sqrt{h}\right\}[\![(z_1,\gamma_1,\theta_1)-(z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}.$$

□

7.6.4 The Fixed Point Theorem

Now, we write Proposition 7.6.2 in terms of Banach spaces and we prove it through a fixed point argument applied to the operator $\overline{\mathcal{G}}_{\omega,h}$ given by (7.6.11).

Proposition 7.6.8. *Fix $\sigma>0$ and $h_0>0$. There exists $\varepsilon_0>0$ such that for $0<\varepsilon\leq\varepsilon_0$, the operator $\overline{\mathcal{G}}_{\omega,h}$ in (7.6.11) has a fixed point $(z_{0,h}^u,\gamma_{0,h}^u,\theta_{0,h}^u)\in\mathcal{Y}_{1,\sigma}^3$. Furthermore, there exists a constant $M>0$ independent of ε and h such that*

$$[\![(z_{0,h}^u,\gamma_{0,h}^u,\theta_{0,h}^u)]\!]_{1,\sigma}\leq M\frac{\delta}{\omega}.$$

Proof. From Lemmas 7.6.5 and 7.6.6, there exists a constant $b_2>0$ independent of ε and h such that

$$[\![\overline{\mathcal{G}}_{\omega,h}(0,0,0)]\!]_{1,\sigma}\leq M\|\mathcal{P}_h(0,0,0)\|_{2,\sigma}\leq\frac{b_2}{2}\frac{\delta}{\omega}.$$

Consider the operator $\overline{\mathcal{G}}_{\omega,h}=\mathcal{G}_\omega\circ\mathcal{P}_h:\mathcal{B}_0(b_2\delta/\omega)\subset\mathcal{Y}_{1,\sigma}\to\mathcal{Y}_{1,\sigma}$. Notice that Lemmas 7.6.5 and 7.6.7 imply that it is well defined in these spaces.

To show that $\overline{\mathcal{G}}_{\omega,h}$ sends $\mathcal{B}_0(b_2\delta/\omega)$ into itself, consider $K=b_2$ in Lemma 7.6.7 and $(z_j,\gamma_j,\theta_j)\in\mathcal{B}_0(b_2\delta/\omega)$, $j=1,2$. It follows from Lemmas 7.6.5, 7.6.7 and the fact that $\mathcal{G}_\omega$ is a linear operator that

$$\begin{aligned}[\![\overline{\mathcal{G}}_{\omega,h}(z_1,\gamma_1,\theta_1)-\overline{\mathcal{G}}_{\omega,h}(z_2,\gamma_2,\theta_2)]\!]_{1,\sigma} &\leq M\left\|\mathcal{P}_h(z_1,\gamma_1,\theta_1)-\mathcal{P}_h(z_2,\gamma_2,\theta_2)\right\|_{2,\sigma},\\ &\leq M\delta\,[\![(z_1,\gamma_1,\theta_1)-(z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}.\end{aligned}$$

Choosing ε_0 sufficiently small such that $\mathrm{Lip}(\overline{\mathcal{G}}_{\omega,h})<1/2$, $\overline{\mathcal{G}}_{\omega,h}$ sends $\mathcal{B}_0(b_2\delta/\omega)$ into itself and it is a contraction. Thus, it has a unique fixed point $(z_{0,h}^u,\gamma_{0,h}^u,\theta_{0,h}^u)\in\mathcal{B}_0(b_2\delta/\omega)$.
□

7.7 Proof of Theorem 7.3.10

The strategy used to prove Theorem 7.3.10 is analogous to the one of Theorem 7.3.5 taking into account that all the expressions appearing become singular as $h \to 0$.

We write

$$N^u_{h,0}(v) = (X_h(v), Z^u_{h,0}(v), \Gamma^u_{h,0}(v), \Theta^u_{h,0}(v)).$$

Lemma 7.7.1. *Given $h > 0$, the invariant manifold $W^u_\delta(p_h^-)$, with $\delta \neq 0$, is parameterized by $N^u_{h,0}(v)$ if and only if $(\Gamma^u_{h,0}(v), \Theta^u_{h,0}(v))$ satisfy*

$$\begin{cases} \dfrac{d\Gamma}{dv}(v) = \dfrac{Z_h(v)}{\widetilde{\eta}_h(v,\Gamma,\Theta)}\left(\omega i\Gamma(v) - \dfrac{\delta}{\sqrt{2\Omega}}F(X_h(v))\right), \\ \dfrac{d\Theta}{dv}(v) = \dfrac{Z_h(v)}{\widetilde{\eta}_h(v,\Gamma,\Theta)}\left(-\omega i\Theta(v) - \dfrac{\delta}{\sqrt{2\Omega}}F(X_h(v))\right). \\ \lim\limits_{v\to-\infty}\Gamma(v) = \lim\limits_{v\to-\infty}\Theta(v) = 0, \end{cases} \tag{7.7.1}$$

and

$$\widetilde{\eta}_h(v,\Gamma,\Theta) = 4\sqrt{h - U(X_h(v)) - \frac{\delta}{\sqrt{2\Omega}}F(X_h(v))\frac{\Gamma(v)-\Theta(v)}{2i} - \frac{\omega}{2}\Gamma(v)\Theta(v)},$$

with X_h given in (7.3.4)*, U, F given in* (7.1.7)*, and $Z^u_{h,0}(v) = \widetilde{\eta}_h(v, \Gamma^u_{h,0}(v), \Theta^u_{h,0}(v))$.*

As in Section 7.4, we compute an explicit term of $(\Gamma^u_{h,0}, \Theta^u_{h,0})$. Thus, the solution of (7.7.1) can be written as (7.3.17) and $(\gamma^u_{h,0}, \theta^u_{h,0})$ satisfy

$$\begin{cases} \dfrac{d}{dv}\gamma - \omega i\gamma = \omega i\gamma(\eta_h(v,\gamma,\theta) - 1) - (Q^h)'(v), \\ \dfrac{d}{dv}\theta + \omega i\theta = -\omega i\theta(\eta_h(v,\gamma,\theta) - 1) + (Q^h)'(v), \\ \lim\limits_{v\to-\infty}\gamma(v) = \lim\limits_{v\to-\infty}\theta(v) = 0, \end{cases} \tag{7.7.2}$$

where Q^h is given in (7.3.18) and

$$\eta_h(v,\gamma,\theta) = \left(1 + \frac{4\delta^2}{\Omega\omega}\left(\frac{F(X_h(v))}{Z_h(v)}\right)^2 - 8\omega\frac{\gamma\theta}{(Z_h(v))^2}\right)^{-1/2}. \tag{7.7.3}$$

We prove Theorem 7.3.10 by finding a solution of (7.7.2) in the next proposition.

Proposition 7.7.2. *There exists $\varepsilon_0 > 0$ and $h_0 > 0$ such that for $0 < h \leq h_0$ and $0 < \varepsilon \leq \varepsilon_0$, equation* (7.7.2) *has a solution $(\gamma^u_{h,0}(v), \theta^u_{h,0}(v))$ defined in D^u (see* (7.3.12)*) such that $\theta^u_{h,0}(v) = \overline{\gamma^u_{h,0}(v)}$ for every $v \in \mathbb{R}$. Furthermore, $(\gamma^u_{h,0}, \theta^u_{h,0})$ satisfy the bound* (7.3.19)*.*

To prove Proposition 7.7.2, it is sufficient to find a fixed point $(\gamma^u_{h,0}, \theta^u_{h,0})$ of the operator

$$\mathcal{G}_{\omega,h} = \mathcal{G}_\omega \circ \mathcal{F}_h, \tag{7.7.4}$$

where $\mathcal{G}_\omega$ is given in (7.4.6) and

$$\mathcal{F}_h(\gamma,\theta)(v) = \begin{pmatrix} \omega i\gamma(v)(\eta_h(v,\gamma(v),\theta(v)) - 1) - (Q^h)'(v) \\ -\omega i\theta(v)(\eta_h(v,\gamma(v),\theta(v)) - 1) + (Q^h)'(v) \end{pmatrix}, \tag{7.7.5}$$

and Q^h, η_h are given in (7.3.18) and (7.7.3), respectively.

The rest of this section is devoted to find a fixed point of (7.7.4).

7.7.1 Banach spaces and technical lemmas

By (7.1.7), (7.3.5) and (7.3.18)

$$Q^h(v) = \frac{2\delta i}{\omega\sqrt{2\Omega}}\left(\frac{\sqrt{\frac{2+h}{h}}\sinh(v\sqrt{h}/2)}{1+\frac{2+h}{h}\sinh^2(v\sqrt{h}/2)}\right), \qquad (7.7.6)$$

which has poles at

$$s_{h,k}^{\pm,j} = i\frac{2}{\sqrt{h}}\left(\delta_{j,1}\pi \pm \arcsin\left(\sqrt{\frac{h}{2+h}}\right) + 2k\pi\right),$$

where $\delta_{j,1}$ is the delta of Kronecker, $j = 0, 1$ and $k \in \mathbb{Z}$. All these singularities are contained in the imaginary axis and satisfy

$$s_{h,k}^{\pm,j} = i\left(\pm\sqrt{2} + \mathcal{O}(h) + \frac{2}{\sqrt{h}}(\delta_{j,1}\pi + 2k\pi)\right).$$

Thus, for h sufficiently small $\left|s_{h,k}^{\pm,j}\right| \geq 3\sqrt{2}/4$, $j = 0, 1$ and $k \in \mathbb{Z}$.

Therefore, we can consider the same domain D^u in (7.3.12). It satisfies the following property, whose proof is straightforward.

Lemma 7.7.3. *If $v \in D^u$ is such that $|\mathrm{Re}(v)| \geq \chi_0$, for some $\chi_0 > 0$, then*

$$|\mathrm{Im}(v)| \leq \frac{\chi_0 + 1}{\chi_0}|\mathrm{Re}(v)|.$$

For $\alpha \geq 0$, we consider the Banach space

$$\mathcal{X}_\alpha = \{f : D^u \to C;\ f \text{ is analytic and } \|f\|_\alpha < \infty\}$$

endowed with the norm

$$\|f\|_\alpha = \sup_{v\in D^u}|(v^2+2)^{\alpha/2}f(v)|,$$

and the product space

$$\mathcal{X}_\alpha^2 = \left\{(f,g) \in \mathcal{X}_\alpha \times \mathcal{X}_\alpha;\ g(v) = \overline{f(v)} \text{ for every } v \in \mathbb{R}\right\}$$

endowed with the norm $\|(f,g)\|_\alpha = \|f\|_\alpha + \|g\|_\alpha$. Remark 7.6.3 and Lemma 7.6.4 also apply to $\|\cdot\|_\alpha$.

Lemma 7.7.4. *Given $0 < h_0 \leq 1$, there exists a constant $M^* > 0$ such that, for each $v \in D^u$ and $0 < h \leq h_0$,*

$$\left|\sinh(v\sqrt{h}/2)\right| \geq M^*\sqrt{h}|v|, \qquad \left|\cosh(v\sqrt{h}/2)\right| \geq M^*.$$

The following Lemma is proved in [7].

Lemma 7.7.5. *Let $1/2 < \beta < \pi/4$ be fixed. The following statements hold*

1. *There exists $\beta_0 > 0$ sufficiently small such that $D^u \subset D^u(\beta_0)$, where*

$$D^u(\beta_0) = \left\{v \in \mathbb{C};\ |\mathrm{Im}(v)| \leq -\tan(\beta+\beta_0)\,\mathrm{Re}(v) + 2\sqrt{2}/3\right\}.$$

2. *Given $\alpha > 0$, if $f : D^u(\beta_0) \to \mathbb{C}$ is a real-analytic function such that*

$$m_\alpha(f) = \sup_{v \in D^u(\beta_0)} |(v^2+2)^{\alpha/2} f(v)| < \infty,$$

then, for any $n \in \mathbb{N}$

$$\|f^{(n)}\|_{\alpha+n} \leq M m_\alpha(f).$$

In the remaining of this paper, all the Landau symbols $\mathcal{O}(f(v,h,\varepsilon))$ denote a function dependent on v, h and ε such that there exists a constant $M > 0$ independent of h and ε such that $|\mathcal{O}(f(v,h,\varepsilon))| \leq M|f(v,h,\varepsilon)|$, for every (v,h,ε) in the domain considered.

Lemma 7.7.6. *There exist $h_0 \in (0,1)$ and a constant $M > 0$ such that, for $v \in D^u$ and $0 < h \leq h_0$,*

1. $|F(X_h(v))| \leq \dfrac{M}{|\sqrt{v^2+2}|}$;

2. $|F(X_h(v))'| \leq \dfrac{M}{|v^2+2|}$.

where X_h given in (7.3.5) *and $F(X)$ in* (7.1.7).

Proof. By (7.3.18) and (7.7.6), we have that

$$F(X_h(v)) = -2\sqrt{\frac{h}{2+h}} \frac{1}{\sinh(v\sqrt{h}/2)} \left(\frac{1}{1 + \frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}} \right).$$

Then, Lemma 7.7.4 implies

$$|F(X_h(v))| \leq M\sqrt{h} \frac{1}{\sqrt{h}|v|} \left(\frac{1}{\left|1 + \frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}\right|} \right).$$

Notice that

$$\left|1 + \frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}\right| \geq 1 - \frac{h}{2+h}\left|\frac{1}{\sinh^2(v\sqrt{h}/2)}\right|$$

and, by Lemma 7.7.4,

$$\frac{h}{2+h}\left|\frac{1}{\sinh^2(v\sqrt{h}/2)}\right| \leq \frac{h}{2+h}\frac{1}{(M^*)^2 h|v|^2} \leq \frac{1}{2(M^*)^2|v|^2}.$$

Thus, for $|v| \geq (M^*)^{-1}$,

$$\left|1 + \frac{h}{2+h}\frac{1}{\sinh^2(v\sqrt{h}/2)}\right| \geq 1/2.$$

We also know that, if $|v| \geq (M^*)^{-1}$, $|\sqrt{v^2+2}| \leq \sqrt{1+2M^*}|v|$. Hence

$$|(\sqrt{v^2+2})F(X_h(v))| \leq M\frac{|\sqrt{v^2+2}|}{|v|} \leq M.$$

Now, assume that $|v|\leq (M^*)^{-1}$. Hence $|v\sqrt{h}/2|\leq M$ and expanding $\sinh(z)$ at 0 we obtain

$$\begin{aligned} F(X_h(v)) &= -2\frac{\sqrt{\frac{2+h}{h}}\left(v\sqrt{h}/2+\mathcal{O}(h^{3/2}v^3)\right)}{1+\frac{2+h}{h}\left(hv^2/4+\mathcal{O}(h^2v^4)\right)} \\ &= -2\frac{\sqrt{2+h}(v/2+\mathcal{O}(h))}{1+v^2/2+\mathcal{O}(h)}. \end{aligned}$$

Since $v\in D^u$, we have that there exists $M>0$ such that

$$|1+v^2/2+\mathcal{O}(h)|\geq |1+v^2/2|-\mathcal{O}(h)\geq M-\mathcal{O}(h).$$

Therefore, for $h>0$ sufficiently small, we have that $|F(X_h(v))|\leq M$, for $|v|\leq (M^*)^{-1}$, and since $|\sqrt{v^2+2}|$ is inferiorly and superiorly bounded by nonzero constants in this domain, we have that

$$|(\sqrt{v^2+2})F(X_h(v))|\leq M \quad \text{for} \quad |v|\leq (M^*)^{-1}.$$

This concludes the proof of the first item. One can obtain item 2 using Lemma 7.7.5. □

Lemma 7.7.7. *Given $0<h_0\leq 1$, there exists a constant $M>0$ such that, for $v\in D^u$ and $0<h\leq h_0$,*

$$\left|\frac{1}{Z_h^2(v)}\frac{1}{v^2+2}\right|\leq M$$

where Z_h in (7.3.5).

The proof is analogous to the one of Lemma 7.7.6.

7.7.2 The Fixed Point Theorem

Now, we study the operator $\mathcal{G}_{\omega,h}$ in order to find a fixed point in $\mathcal{X}_2^2$. Recall the definition of $\mathcal{G}_{\omega,h}=\mathcal{G}_\omega\circ\mathcal{F}_h$ in (7.7.4), and notice that $\mathcal{G}_\omega$ is the same operator of the case $h=0$. Thus, Proposition 7.4.3 still holds for functions in the Banach space $\mathcal{X}_2^2$.

Proposition 7.7.8. *Given $(f,g)\in\mathcal{X}_2^2$, we have that $\mathcal{G}_\omega(f,g)\in\mathcal{X}_2^2$. Furthermore, there exists a constant $M>0$ independent of ε such that*

$$\|\mathcal{G}_\omega(f,g)\|_2\leq\frac{M}{\omega}\|(f,g)\|_2.$$

We proceed by studying the operator $\mathcal{F}_h$ in (7.7.5).

Proposition 7.7.9. *There exists $h_0>0$, $\varepsilon_0>0$ and a constant $M>0$ such that for, $0<\varepsilon\leq\varepsilon_0$ and $0<h\leq h_0$,*

$$\|\mathcal{G}_{\omega,h}(0,0)\|_2\leq M\frac{\delta}{\omega^2}.$$

Proof. Notice that $\mathcal{F}_h(0,0)=(-(Q^h)'(v),(Q^h)'(v))$ (see (7.3.18)), which implies

$$\|\mathcal{F}_h(0,0)\|_2=2\frac{\delta}{\omega\sqrt{2\Omega}}\|F(X_h)'\|_2.$$

Thus, it is enough to apply Lemma 7.7.6 and Proposition 7.7.8. □

Proposition 7.7.10. *There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, $0 < h \leq h_0$:*

Let η_h be given in (7.7.3) and take $(\gamma_j, \theta_j) \in \mathcal{B}_0(R) \subset \mathcal{X}_2^2$ with $j = 1, 2$ and $R = K\dfrac{\delta}{\omega^2}$, where K is a constant independent of h and ε, the following statements hold.

1. $|\eta_h(v, \gamma_j(v), \theta_j(v)) - 1| \leq M\dfrac{\delta^2}{\omega}$;

2. $|\eta_h(v, \gamma_1(v), \theta_1(v)) - \eta_h(v, \gamma_2(v), \theta_2(v))| \leq M\dfrac{\delta}{\omega}\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_0$;

3. $\|\mathcal{F}_h(\gamma_1, \theta_1) - \mathcal{F}_h(\gamma_2, \theta_2)\|_2 \leq M\delta^2\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_2$;

Proof. Lemmas 7.7.6 and 7.7.7 and the fact that $(\gamma, \theta) \in \mathcal{B}_0(R)$ imply

$$\left|\frac{4\delta^2}{\Omega\omega}\left(\frac{F(X_h(v))}{Z_h(v)}\right)^2 - 8\omega\frac{\gamma\theta}{(Z_h(v))^2}\right| \;\leq\; M\frac{\delta^2}{\omega}.$$

Thus, using (7.7.3), it follows that

$$|\eta_h(v, \gamma, \theta) - 1| \leq M\left|\frac{4\delta^2}{\Omega\omega}\left(\frac{F(X_h(v))}{Z_h(v)}\right)^2 - 8\omega\frac{\gamma\theta}{(Z_h(v))^2}\right| \leq M\frac{\delta^2}{\omega}$$

and using also Lemma 7.7.7, we have

$$\begin{aligned}
|\eta_h(v, \gamma_1, \theta_1) - \eta_h(v, \gamma_2, \theta_2)| \;&\leq\; M\omega\left|\frac{\gamma_1\theta_1 - \gamma_2\theta_2}{(Z_h(v))^2}\right| \\
&\leq\; MR\omega\left(\frac{|\theta_1 - \theta_2|}{|(Z_h(v))^2(v^2+2)|} + \frac{|\gamma_1 - \gamma_2|}{|(Z_h(v))^2(v^2+2)|}\right) \\
&\leq\; M\frac{\delta}{\omega}\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_0
\end{aligned}$$

Finally, it follows from items (1) and (2) of this proposition and (7.7.5) that

$$\begin{aligned}
\|\pi_1 \circ \mathcal{F}_h(\gamma_1, \theta_1) - \pi_1 \circ \mathcal{F}_h(\gamma_2, \theta_2)\|_2 \;&\leq\; \omega\,\|\eta_h(v, \gamma_1, \theta_1) - 1\|_0\,\|\gamma_1 - \gamma_2\|_2 \\
&\quad + \omega\|\gamma_2\|_2\|\eta_h(v, \gamma_1, \theta_1) - \eta_h(v, \gamma_2, \theta_2)\|_0 \\
&\leq\; M\delta^2\|\gamma_1 - \gamma_2\|_2 + M\omega R\frac{\delta}{\omega}\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_0 \\
&\leq\; M\delta^2\|(\gamma_1, \theta_1) - (\gamma_2, \theta_2)\|_2.
\end{aligned}$$

Analogously, we obtain the same inequality for the second component of $\mathcal{F}_h$. □

Finally, we are able to prove Proposition 7.7.2 (and thus Theorem (7.3.10)) by a fixed point argument.

Proposition 7.7.11. *There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 < h \leq h_0$ and $\varepsilon \leq \varepsilon_0$, the operator $\mathcal{G}_{\omega,h}$ (given in (7.7.4)) has a fixed point $(\gamma^u_{h,0}, \theta^u_{h,0})$ in $\mathcal{X}_2^2$ which satisfies*

$$\|(\gamma^u_{h,0}, \theta^u_{h,0})\|_2 \leq M\frac{\delta}{\omega^2}.$$

Proof. From Proposition 7.7.9, there exists a constant $b_3 > 0$ independent of h and ε such that
$$\|\mathcal{G}_{\omega,h}(0,0)\|_2 \leq \frac{b_3}{2}\frac{\delta}{\omega^2},$$
Now, given (γ_1,θ_1) and (γ_2,θ_2) in $\mathcal{B}_0(b_3\delta/\omega^2)$, we can use Propositions 7.7.10 (with $K = b_3$) and 7.7.8 and the linearity of the operator $\mathcal{G}_\omega$ to see that

$$\begin{aligned}\|\mathcal{G}_{\omega,h}(\gamma_1,\theta_1) - \mathcal{G}_{\omega,h}(\gamma_2,\theta_2)\|_2 &\leq \frac{M}{\omega}\|\mathcal{F}_h(\gamma_1,\theta_1) - \mathcal{F}_h(\gamma_2,\theta_2)\|_2 \\ &\leq M\frac{\delta^2}{\omega}\|(\gamma_1,\theta_1) - (\gamma_2,\theta_2)\|_2.\end{aligned}$$

Choosing ε_0 sufficiently small, we have that $\mathrm{Lip}(\mathcal{G}_{\omega,h}) \leq 1/2$. Therefore $\mathcal{G}_{\omega,h}$ sends the ball $\mathcal{B}_0(b_3\delta/\omega^2)$ into itself and it is a contraction. Thus, it has a unique fixed point $(\gamma^u_{h,0},\theta^u_{h,0}) \in \mathcal{B}_0(b_3\delta/\omega^2)$.

□

7.8 Proof of Theorem 7.3.11

In this section we prove the existence of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$, with $\delta \neq 0$. As in the previous sections, we look for parameterizations $N^u_{\kappa_1,\kappa_2}$ of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ as graphs

$$N^{u,s}_{\kappa_1,\kappa_2}(v,\tau) = (X_{\kappa_1}(v), Z_{\kappa_1}(v) + Z^{u,s}_{\kappa_1,\kappa_2}(v,\tau), \Gamma_{\kappa_2}(\tau) + \Gamma^{u,s}_{\kappa_1,\kappa_2}(v,\tau), \Theta_{\kappa_2}(\tau) + \Theta^{u,s}_{\kappa_1,\kappa_2}(v,\tau)), \tag{7.8.1}$$

where $X_{\kappa_1}, Z_{\kappa_1}$ are given in (7.3.4) and $\Gamma_{\kappa_2}, \Theta_{\kappa_2}$ are given in (7.3.7).

Following the same lines of Section 7.7 we have a characterization of $N^u_{\kappa_1,\kappa_2}$.

Lemma 7.8.1. *Write $Z^u_{\kappa_1,\kappa_2}(v,\tau) = Z_{\kappa_1,\kappa_2}(v,\tau) + z^u_{\kappa_1,\kappa_2}(v,\tau)$, $\Gamma^u_{\kappa_1,\kappa_2}(v,\tau) = Q^{\kappa_2}(v) + \gamma^u_{\kappa_1,\kappa_2}(v,\tau)$, $\Theta^u_{\kappa_1,\kappa_2}(v,\tau) = -Q^{\kappa_1}(v) + \theta^u_{\kappa_1,\kappa_2}(v,\tau)$, where Q^{κ_1} is given by (7.3.18) and*

$$Z_{\kappa_1,\kappa_2}(v,\tau) = \frac{\delta}{\omega\sqrt{2\Omega}}F'(X_{\kappa_1}(v))\frac{\Gamma_{\kappa_2}(\tau) + \Theta_{\kappa_2}(\tau)}{2},$$

with $\Gamma_{\kappa_1}, \Theta_{\kappa_1}$ given by (7.3.7). Then, $N^u_{\kappa_1,\kappa_2}(v,\tau)$, given in (7.8.1), with $\kappa_1,\kappa_2 \geq 0$ and $\kappa_1 + \kappa_2 = h$, parameterizes $W^u(\Lambda^-_{\kappa_1,\kappa_2})$ provided $(z^u_{\kappa_1,\kappa_2}, \gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2})$ satisfy

$$\left\{\begin{aligned} \partial_v z + \omega\partial_\tau z + \frac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)}z &= f_1^{\kappa_1,\kappa_2}(v,\tau) - \frac{z + Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v z - \frac{\partial_v Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}z \\ &\quad -\frac{\delta}{\sqrt{2\Omega}}F'(X_{\kappa_1}(v))\frac{\gamma - \theta}{2i}, \\ \partial_v\gamma + \omega\partial_\tau\gamma - \omega i\gamma &= f_2^{\kappa_1,\kappa_2}(v,\tau) - \frac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}z - \frac{z + Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v\gamma, \\ \partial_v\theta + \omega\partial_\tau\theta + \omega i\theta &= -f_2^{\kappa_1,\kappa_2}(v,\tau) + \frac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}z - \frac{z + Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)}\partial_v\theta, \\ \lim_{v\to-\infty} z(v,\tau) &= \lim_{v\to-\infty}\gamma(v,\tau) = \lim_{v\to-\infty}\theta(v,\tau) = 0, \end{aligned}\right. \tag{7.8.2}$$

where

$$f_1^{\kappa_1,\kappa_2}(v,\tau) = -\partial_v Z_{\kappa_1,\kappa_2}(v,\tau) - \frac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)} Z_{\kappa_1,\kappa_2}(v,\tau) - \frac{\delta}{\sqrt{2\Omega}} F'(X_{\kappa_1}(v)) \frac{Q^{\kappa_1}(v)}{i} \quad (7.8.3)$$
$$- \frac{Z_{\kappa_1,\kappa_2}(v,\tau)\partial_v Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)},$$

$$f_2^{\kappa_1,\kappa_2}(v,\tau) = -(Q^{\kappa_1})'(v) - \frac{Z_{\kappa_1,\kappa_2}(v,\tau)(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)}. \quad (7.8.4)$$

We consider the equation (7.8.2) with $(v,\tau) \in D^u \times \mathbb{T}_\sigma$ with the asymptotic conditions $\lim_{\mathrm{Re}(v)\to-\infty} z(v) = \lim_{\mathrm{Re}(v)\to-\infty} \gamma(v) = \lim_{\mathrm{Re}(v)\to-\infty} \theta(v) = 0$, for every $\tau \in \mathbb{T}_\sigma$.

Theorem (7.3.11) is a consequence of the following proposition.

Proposition 7.8.2. *Fix $\sigma > 0$. There exist $h_0 > 0$ and $\varepsilon_0 > 0$ sufficiently small such that for $0 < \varepsilon \leq \varepsilon_0$, $0 < h \leq h_0$ and $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = h$, system* (7.8.2) *has an analytic solution $(z^u_{\kappa_1,\kappa_2}, \gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2})$ defined in $D^u \times \mathbb{T}_\sigma$ (see* (7.3.12) *and* (7.3.13)*) such that $z^u_{\kappa_1,\kappa_2}$ is real-analytic, $\theta^u_{\kappa_1,\kappa_2}(v,\tau) = \overline{\gamma^u_{\kappa_1,\kappa_2}(v,\tau)}$ for each $(v,\tau) \in D^u \times \mathbb{T}_\sigma \cap \mathbb{R}^2$ and*

$$\lim_{\mathrm{Re}(v)\to-\infty} z^u_{\kappa_1,\kappa_2}(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \gamma^u_{\kappa_1,\kappa_2}(v,\tau) = \lim_{\mathrm{Re}(v)\to-\infty} \theta^u_{\kappa_1,\kappa_2}(v,\tau) = 0,$$

for every $\tau \in \mathbb{T}_\sigma$. Furthermore, $(z^u_{\kappa_1,\kappa_2}, \gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2})$ satisfies the bounds in (7.3.21).

Equation (7.8.2) can be written as the functional equation

$$\mathcal{L}_{\omega,\kappa_1}(z,\gamma,\theta) = \mathcal{P}_{\kappa_1,\kappa_2}(z,\gamma,\theta),$$

where $\mathcal{L}_{\omega,\kappa_1}$ and $\mathcal{P}_{\kappa_1,\kappa_2}$ are the functional operators given by

$$\mathcal{L}_{\omega,\kappa_1}(z,\gamma,\theta) = \begin{pmatrix} \partial_v z + \omega\partial_\tau z + \dfrac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)} z \\ \partial_v \gamma + \omega\partial_\tau \gamma - \omega i \gamma \\ \partial_v \theta + \omega\partial_\tau \theta + \omega i \theta \end{pmatrix},$$

and

$$\mathcal{P}_{\kappa_1,\kappa_2}(z,\gamma,\theta) = \begin{pmatrix} f_1^{\kappa_1,\kappa_2}(v,\tau) - \dfrac{z + Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)} \partial_v z - \dfrac{\partial_v Z_{\kappa_1,\kappa_2}}{Z_{\kappa_1}(v)} z - \dfrac{\delta}{\sqrt{2\Omega}} F'(X_{\kappa_1}(v)) \dfrac{\gamma - \theta}{2i} \\ f_2^{\kappa_1,\kappa_2}(v,\tau) - \dfrac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)} z - \dfrac{z + Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)} \partial_v \gamma \\ -f_2^{\kappa_1,\kappa_2}(v,\tau) + \dfrac{(Q^{\kappa_1})'(v)}{Z_{\kappa_1}(v)} z - \dfrac{z + Z_{\kappa_1,\kappa_2}(v,\tau)}{Z_{\kappa_1}(v)} \partial_v \theta \end{pmatrix}. \quad (7.8.5)$$

We show the existence of an inverse $\mathcal{G}^{\kappa_1}_\omega$ of $\mathcal{L}_{\omega,\kappa_1}$ in the Banach spaces $\mathcal{X}^3_{\alpha,\sigma}$ and $\mathcal{Y}^3_{\alpha,\sigma}$ introduced in Section 7.6.1.

Given analytic functions f, g, and h defined in $D^u \times \mathbb{T}_\sigma$, consider

$$F^{[k]}_{\kappa_1}(f)(v) = \int_{-\infty}^{v} \frac{e^{\omega i k(r-v)} Z_{\kappa_1}(r)}{Z_{\kappa_1}(v)} f^{[k]}(r)\, dr,$$

and $G^{[k]}(g)$, $H^{[k]}(h)$ given in (7.6.9). Then, we define the linear operator $\mathcal{G}_\omega^{\kappa_1}$

$$\mathcal{G}_\omega^{\kappa_1}(f,g,h) = \begin{pmatrix} \sum_k F_{\kappa_1}^{[k]}(f)(v)e^{ik\tau} \\ \sum_k G^{[k]}(g)(v)e^{ik\tau} \\ \sum_k H^{[k]}(h)(v)e^{ik\tau} \end{pmatrix}. \tag{7.8.6}$$

Lemma 7.8.3. *Fix $\alpha \geq 1$ and $\sigma > 0$. There exists $\kappa_1^0 > 0$ sufficiently small, such that, for $0 < \varepsilon \leq \varepsilon_0$ and $0 < \kappa_1 \leq \kappa_1^0$, the operator*

$$\mathcal{G}_\omega^{\kappa_1} : \mathcal{X}_{\alpha+1,\sigma}^3 \to \mathcal{Y}_{\alpha,\sigma}^3$$

is well-defined and satisfies:

1. *$\mathcal{G}_\omega^{\kappa_1}$ is an inverse of the operator $\mathcal{L}_{\omega,\kappa_1} : \mathcal{Y}_{\alpha,\sigma}^3 \to \mathcal{X}_{\alpha+1,\sigma}^3$, i.e. $\mathcal{G}_\omega^{\kappa_1} \circ \mathcal{L}_{\omega,\kappa_1} = \mathcal{L}_{\omega,\kappa_1} \circ \mathcal{G}_\omega^{\kappa_1} = \mathrm{Id}$;*
2. *$[\![\mathcal{G}_\omega^{\kappa_1}(f,g,h)]\!]_{\alpha,\sigma} \leq M\|(f,g,h)\|_{\alpha+1,\sigma}$;*
3. *If $f^{[0]} = g^{[1]} = h^{[-1]} = 0$, then $[\![\mathcal{G}_\omega^{\kappa_1}(f,g,h)]\!]_{\alpha,\sigma} \leq \dfrac{M}{\omega}[\![(f,g,h)]\!]_{\alpha,\sigma}$,*

where M is a constant independent of κ_1 and ε.

The proof of the following lemma is analogous to that in Lemma 7.9.3 below.

Lemma 7.8.4. *Let F, $X_{\kappa_1}, Z_{\kappa_1}$ be given by (7.1.7) and (7.3.5). There exist $\kappa_1^0 > 0$ and a constant $M > 0$ such that, for $v \in D^u$ and $0 < \kappa_1 \leq \kappa_1^0$,*

1. *$|F(X_{\kappa_1}(v))''| \leq \dfrac{M}{|v^2+2|^{3/2}}$;*
2. *$\left|\dfrac{Z'_{\kappa_1}(v)}{Z_{\kappa_1}(v)}\right| \leq \dfrac{M}{|\sqrt{v^2+2}|}$.*

Lemma 7.8.5. *Fix $\sigma > 0$ and $K > 0$. There exist $\varepsilon_0 > 0$ and $h_0 > 0$ sufficiently small such that, for $0 < \varepsilon < \varepsilon_0$, $0 \leq h \leq h_0$ and $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = h$, the operator $\mathcal{P}_{\kappa_1,\kappa_2} : \mathcal{Y}_{1,\sigma}^3 \to \mathcal{X}_{2,\sigma}^3$, is well defined and there exists a constant $M > 0$ such that*

$$\|\mathcal{P}_{\kappa_1,\kappa_2}(0,0,0)\|_{2,\sigma} \leq M\frac{\delta}{\omega}.$$

Moreover, given $(z_j,\gamma_j,\theta_j) \in \mathcal{B}_0(K\delta/\omega) \subset \mathcal{Y}_{1,\sigma}^3$, $j = 1,2$,

$$\|\mathcal{P}_{\kappa_1,\kappa_2}(z_1,\gamma_1,\theta_1) - \mathcal{P}_{\kappa_1,\kappa_2}(z_2,\gamma_2,\theta_2)\|_{2,\sigma} \leq M\left(\delta + \frac{\delta}{\omega^{3/2}}\sqrt{h}\right)[\![(z_1,\gamma_1,\theta_1) - (z_2,\gamma_2,\theta_2)]\!]_{1,\sigma}.$$

Proof. Recall that $\mathcal{P}_{\kappa_1,\kappa_2}(0,0,0) = (f_1^{\kappa_1,\kappa_2}, f_2^{\kappa_1,\kappa_2}, -f_2^{\kappa_1,\kappa_2})$, where $f_1^{\kappa_1,\kappa_2}, f_2^{\kappa_1,\kappa_2}$ are given in (7.8.3) and (7.8.4), respectively, and involve the functions $F'(X_{\kappa_1})$, $Z'_{\kappa_1}/Z_{\kappa_1}$, Q^{κ_1}, $(Q^{\kappa_1})'$, $Z_{\kappa_1,\kappa_2}, \partial_v Z_{\kappa_1,\kappa_2}$ which can be computed using the expressions in (7.1.7), (7.3.5), (7.3.11), and (7.3.15). By Lemmas 7.7.6, 7.7.7 and 7.8.4, we have

$$\|Q^{\kappa_1}\|_{1,\sigma}, \|(Q^{\kappa_1})'\|_{2,\sigma} \leq M\frac{\delta}{\omega},$$
$$\|Z_{\kappa_1,\kappa_2}\|_{1,\sigma}, \|\partial_v Z_{\kappa_1,\kappa_2}\|_{2,\sigma} \leq M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}},$$
$$\|Z'_{\kappa_1}/Z_{\kappa_1}\|_{1,\sigma}, \|F'(X_{\kappa_1})\|_{1,\sigma} \leq M.$$

Therefore, using also Lemma 7.6.4, one has

$$\|f_1^{\kappa_1,\kappa_2}\|_{2,\sigma} \leq M \max\left\{\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}, \frac{\delta^2}{\omega}, \frac{\delta^2}{\omega^3}\kappa_2\right\} = M \max\left\{\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}, \frac{\delta^2}{\omega}\right\},$$
$$\|f_2^{\kappa_1,\kappa_2}\|_{2,\sigma} \leq M \max\left\{\frac{\delta}{\omega}, \frac{\delta^2}{\omega^{5/2}}\sqrt{\kappa_2}\right\} = M\frac{\delta}{\omega}.$$

Thus, $\|\mathcal{P}_{\kappa_1,\kappa_2}(0,0,0)\|_{2,\sigma} \leq M\delta/\omega$.

Following the lines of the proof of Lemma 7.6.7 one can complete the proof of Lemma 7.8.5. □

Now, we write Proposition 7.8.2 in terms of Banach spaces. Then, it can be proved in the same way as Proposition 7.6.8 by considering the operator $\overline{\mathcal{G}}_{\omega,\kappa_1,\kappa_2} = \mathcal{G}_\omega^{\kappa_1} \circ \mathcal{P}_{\kappa_1,\kappa_2}$.

Proposition 7.8.6. *Fix $\sigma > 0$. There exist $h_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $0 < h \leq h_0$ and $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = h$, the operator $\overline{\mathcal{G}}_{\omega,\kappa_1,\kappa_2} = \mathcal{G}_\omega^{\kappa_1} \circ \mathcal{P}_{\kappa_1,\kappa_2}$, with $\mathcal{G}_\omega^{\kappa_1}$ and $\mathcal{P}_{\kappa_1,\kappa_2}$ given in* (7.8.6) *and* (7.8.5)*, respectively, has a fixed point $(z^u_{\kappa_1,\kappa_2}, \gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2}) \in \mathcal{Y}^3_{1,\sigma}$. Furthermore, there exists a constant $M > 0$ independent of ε, κ_1 and κ_2 such that*

$$[\![(z^u_{\kappa_1,\kappa_2}, \gamma^u_{\kappa_1,\kappa_2}, \theta^u_{\kappa_1,\kappa_2})]\!]_{1,\sigma} \leq M\frac{\delta}{\omega}.$$

This completes the proof of Theorem 7.3.11.

7.9 Proof of Theorem 7.3.12

We compare the parameterizations of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ obtained in Sections 7.6, 7.7 and 7.8, respectively, with the parameterization (7.4.1) of $W^u_\varepsilon(p_0^-)$ obtained in Section 7.4.

7.9.1 Approximation of $W^u_\varepsilon(\Lambda^-_h)$ by $W^u_\varepsilon(p_0^-)$

We compare the parameterizations $N^u_{0,h}$ and $N^u_{0,0}$ of $W^u_\varepsilon(\Lambda^-_h)$ and $W^u_\varepsilon(p_0^-)$, obtained in Theorems 7.3.8 and 7.3.5, respectively.

Proposition 7.9.1. *Let $\Gamma^u_0(v)$, $\Theta^u_0(v)$ and $\Gamma^u_{0,h}(v,\tau)$, $\Theta^u_{0,h}(v,\tau)$ be given in* (7.3.10) *and* (7.3.14)*, respectively. Given $h_0 > 0$, there exists $\varepsilon_0 > 0$ and a constant $M > 0$, such that for $v \in D^u \cap \mathbb{R}$, $\tau \in \mathbb{T}$, $0 \leq \varepsilon \leq \varepsilon_0$ and $0 \leq h \leq h_0$,*

$$\left|\partial_\tau(\Gamma^u_{0,h}(v,\tau) - \Gamma^u_0(v))\right|, \left|\Gamma^u_{0,h}(v,\tau) - \Gamma^u_0(v)\right| \leq M\frac{\delta\sqrt{h}}{\omega^{3/2}},$$
$$\left|\partial_\tau(\Theta^u_{0,h}(v,\tau) - \Theta^u_0(v))\right|, \left|\Theta^u_{0,h}(v,\tau) - \Theta^u_0(v)\right| \leq M\frac{\delta\sqrt{h}}{\omega^{3/2}}.$$

Proof. Considering $h = 0$ in Theorem 7.3.8, it follows that $N^u_{0,0}(v,\tau)$ is also a parameterization of $W^u_\varepsilon(p^-_0)$. Since $W^u_\varepsilon(p^-_0)$ is parameterized by both $N^u_{0,0}(v)$ (from Theorem 7.3.5) and $N^u_{0,0}(v,\tau)$ (from Theorem 7.3.8) and both have the same first coordinate, these parameterizations coincide. Therefore $\gamma^u_{0,0}$ and $\theta^u_{0,0}$ given in Theorem 7.3.8 with $h = 0$ depend only on the variable v and we can write

$$\Gamma^u_0(v) = Q^0(v) + \gamma^u_{0,0}(v),$$
$$\Theta^u_0(v) = -Q^0(v) + \theta^u_{0,0}(v).$$

Based on these arguments, we can use Theorem 7.3.8 and Proposition 7.6.8 to see that

$$\begin{pmatrix} \Gamma^u_{0,h}(v,\tau) - \Gamma^u_0(v) \\ \Theta^u_{0,h}(v,\tau) - \Theta^u_0(v) \end{pmatrix} = \begin{pmatrix} \gamma^u_{0,h}(v,\tau) - \gamma^u_{0,0}(v) \\ \theta^u_{0,h}(v,\tau) - \theta^u_{0,0}(v) \end{pmatrix},$$

where $(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})$ and $(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h})$ are fixed points of the operators $\overline{\mathcal{G}}_{\omega,0}$ and $\overline{\mathcal{G}}_{\omega,h}$ given in (7.6.11), respectively.

Denoting

$$\mathcal{E} = (z^u_{0,h} - z^u_{0,0}, \gamma^u_{0,h} - \gamma^u_{0,0}, \theta^u_{0,h} - \theta^u_{0,0}),$$

we compute $\|\mathcal{E}\|_{1,\sigma}$.

Notice that

$$\begin{aligned} \mathcal{E} &= (z^u_{0,h} - z^u_{0,0}, \gamma^u_{0,h} - \gamma^u_{0,0}, \theta^u_{0,h} - \theta^u_{0,0}) \\ &= \overline{\mathcal{G}}_{\omega,h}(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h}) - \overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) \\ &\quad + \overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \overline{\mathcal{G}}_{\omega,0}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}). \end{aligned}$$

For $0 \leq h \leq h_0$, $(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h}) \in \mathcal{B}_0(M\delta/\omega)$ and $\overline{\mathcal{G}}_{\omega,h}$ is Lipschitz in this ball with $\mathrm{Lip}(\overline{\mathcal{G}}_{\omega,h}) \leq M\delta$. Then,

$$[\![\overline{\mathcal{G}}_{\omega,h}(z^u_{0,h}, \gamma^u_{0,h}, \theta^u_{0,h}) - \overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})]\!]_{1,\sigma} \leq M\delta[\![\mathcal{E}]\!]_{1,\sigma}.$$

Choosing ε_0 sufficiently small such that $\mathrm{Lip}(\overline{\mathcal{G}}_{\omega,h}) < 1/2$, we obtain

$$[\![\mathcal{E}]\!]_{1,\sigma} \leq M[\![\overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \overline{\mathcal{G}}_{\omega,0}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})]\!]_{1,\sigma}.$$

Now, denoting $\mathcal{P}_h(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \mathcal{P}_0(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) = \Delta^0_h$, where $\mathcal{P}_h$ is given in (7.6.7), and using that $\left[\!\left[(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})\right]\!\right]_{1,\sigma} \leq M\delta/\omega$, we have that $\|\Delta^0_h\|_{2,\sigma} \leq M\frac{\delta\sqrt{h}}{\omega^{3/2}}$.

It follows from the linearity of $\mathcal{G}_\omega$ and Lemma 7.6.5 that

$$\left[\!\left[\overline{\mathcal{G}}_{\omega,h}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0}) - \overline{\mathcal{G}}_{\omega,0}(z^u_{0,0}, \gamma^u_{0,0}, \theta^u_{0,0})\right]\!\right]_{1,\sigma} \leq M\frac{\delta\sqrt{h}}{\omega^{3/2}}.$$

Thus, we conclude that $[\![\mathcal{E}]\!]_{1,\sigma} \leq M\dfrac{\delta\sqrt{h}}{\omega^{3/2}}$. □

7.9.2 Approximation of $W^u_\varepsilon(p^-_h)$ by $W^u_\varepsilon(p^-_0)$

We compare the parameterizations $N^u_{0,0}$ and $N^u_{h,0}$ of $W^u_\varepsilon(p^-_0)$ and $W^u_\varepsilon(p^-_h)$, obtained in Theorems 7.3.5 and 7.3.10, respectively.

Proposition 7.9.2. *Let* $\Gamma_0^u(v)$, $\Theta_0^u(v)$ *and* $\Gamma_{h,0}^u(v)$, $\Theta_{h,0}^u(v)$ *be given in* (7.3.10) *and* (7.3.17), *respectively. There exist* $\varepsilon_0 > 0$, $h_0 > 0$ *and a constant* $M > 0$ *such that, for* $0 < \varepsilon \leq \varepsilon_0$ *and* $0 \leq h \leq h_0$,

1. $\left|\Gamma_{h,0}^u(0) - \Gamma_0^u(0)\right| \leq M\dfrac{\delta\sqrt{h}}{\omega^2}$;

2. $\left|\Theta_{h,0}^u(0) - \Theta_0^u(0)\right| \leq M\dfrac{\delta\sqrt{h}}{\omega^2}$.

Technical Lemmas

To prove Proposition 7.9.2, we first state some lemmas.

Lemma 7.9.3. *Let* X_0, Z_0, X_h, Z_h, Q^0, *and* Q^h *be given in* (7.3.4), (7.3.5), (7.3.11) *and* (7.3.18) *and fix* $M_0 > 0$. *There exist* $h_0 > 0$ *and a constant* $M > 0$ *such that, for* $0 \leq h \leq h_0$ *and* $v \in D^u$ *with* $|h^{1/4}v| \leq M_0$,

1. $|F(X_h(v)) - F(X_0(v))| \leq \dfrac{M\sqrt{h}}{|\sqrt{v^2+2}|}$;

2. $|Z_h(v) - Z_0(v)| \leq \dfrac{M\sqrt{h}}{|\sqrt{v^2+2}|}$;

3. $\left|\dfrac{1}{Z_h(v)} - \dfrac{1}{Z_0(v)}\right| \dfrac{1}{|\sqrt{v^2+2}|} \leq M\sqrt{h}$;

4. $\left|(Q^h)'(v) - (Q^0)'(v)\right| \leq \dfrac{M\delta\sqrt{h}}{\omega|v^2+2|}$.

Proof. Using the formulas (7.1.7), (7.3.4) and (7.3.5), we obtain

$$F(X_h(v)) - F(X_0(v)) = -2\left(\frac{\sqrt{\frac{2+h}{h}}\sinh(v\sqrt{h}/2)}{1+\frac{2+h}{h}\sinh^2(v\sqrt{h}/2)} - \sqrt{2}\frac{v}{v^2+2}\right).$$

Since $|vh^{1/4}| \leq M_0$, it follows that $|v\sqrt{h}/2| \leq Mh^{1/4} \ll 1$.
Expanding $\sinh(z)$ at 0, we have

$$\begin{aligned}
\frac{\sqrt{\dfrac{2+h}{h}}\sinh(v\sqrt{h}/2)}{1+\frac{2+h}{h}\sinh^2(v\sqrt{h}/2)} &= \frac{\sqrt{\dfrac{2+h}{h}}\left(\dfrac{v\sqrt{h}}{2} + \mathcal{O}(h^{3/2}|v|^3)\right)}{1+\dfrac{2+h}{h}\left(\dfrac{v^2h}{4} + \mathcal{O}(h^2|v|^4)\right)} \\
&= \frac{\sqrt{2}v + \mathcal{O}(\sqrt{h}|v|)}{v^2+2+\mathcal{O}(\sqrt{h}|v|^2)} \\
&= \frac{\sqrt{2}v}{v^2+2}\left(1+\mathcal{O}(\sqrt{h})\right).
\end{aligned}$$

Item (1) follows directly from this expression, considering h sufficiently small. Items (2) and (3) can be computed in an analogous way.

Formulas (7.3.11) and (7.3.18) imply

$$\left|(Q^h)'(v) - (Q^0)'(v)\right| \leq M\frac{\delta}{\omega}|F'(X_h(v))Z_h(v) - F'(X_0(v))Z_0(v)|.$$

Thus, it is enough to apply the bounds in items (1) and (2) to obtain item (4).

□

Lemma 7.9.4. *Let η_0 and η_h be given in* (7.4.4) *and* (7.7.3)*, respectively, and consider the functions (γ_0^u, θ_0^u) obtained in Proposition 7.4.6. Fix $M_0 > 0$. There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, $0 \leq h \leq h_0$ and $v \in D^u$ with $|h^{1/4}v| \leq M_0$,*

$$|\eta_h(v, \gamma_0^u, \theta_0^u) - \eta_0(v, \gamma_0^u, \theta_0^u)| \leq \frac{M\delta\sqrt{h}}{\omega}.$$

Proof. Using the expression of η_h in (7.7.3) and that $\|(\gamma_0^u, \theta_0^u)\|_2 \leq M\delta/\omega^2 \ll 1$, it follows from Lemmas 7.7.6, 7.7.7 and 7.9.3 that

$$\begin{aligned} |\eta_h(v, \gamma_0^u, \theta_0^u) - \eta_0(v, \gamma_0^u, \theta_0^u)| &\leq M\frac{\delta}{\omega}\left|\left(\frac{F(X_h)}{Z_h}\right)^2 - \left(\frac{F(X_0)}{Z_0}\right)^2\right| + M\omega\,|\gamma_0^u\theta_0^u|\left|\frac{1}{Z_h^2} - \frac{1}{Z_0^2}\right| \\ &\leq \frac{M\delta\sqrt{h}}{\omega}. \end{aligned}$$

□

Proof of Proposition 7.9.2

The domain D^u defined in (7.3.12) is contained in the domain D^u_ε defined in (7.3.9). Therefore, the restriction of the fixed point obtained in Section 7.4 can be seen as an element of the space $\mathcal{X}_2^2$ with the same bound.

Proposition 7.9.5. *Consider (γ_0^u, θ_0^u) and $(\gamma_{h,0}^u, \theta_{h,0}^u)$ obtained in Theorems 7.4.6 and 7.7.11, respectively, and the operator $\mathcal{G}_{\omega,h}$ given by* (7.7.4)*. Then, there exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that for $0 \leq h \leq h_0$ and $0 < \varepsilon \leq \varepsilon_0$,*

$$\left\|\mathcal{G}_{\omega,h}(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u)\right\|_0 \leq M\frac{\delta^2}{\omega}\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0.$$

Proof. By Proposition 7.7.10, we have

$$\left|\eta_h(v, \gamma_{h,0}^u, \theta_{h,0}^u) - \eta_h(v, \gamma_0^u, \theta_0^u)\right| \leq M\frac{\delta}{\omega}\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0.$$

Thus, using the expression of $\mathcal{F}_h$ in (7.7.5) and Proposition 7.7.10,

$$\begin{aligned} \left\|\pi_1(\mathcal{F}_h(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{F}_h(\gamma_0^u, \theta_0^u))\right\|_0 &\leq \omega\left\|\eta_h(v, \gamma_{h,0}^u, \theta_{h,0}^u) - 1\right\|_0\left\|\gamma_{h,0}^u - \gamma_0^u\right\|_0 \\ &\quad + \omega\left\|\gamma_0^u\right\|_0\left\|\eta_h(v, \gamma_{h,0}^u, \theta_{h,0}^u) - \eta_h(v, \gamma_0^u, \theta_0^u)\right\|_0 \\ &\leq M\delta^2\left\|\gamma_{h,0}^u - \gamma_0^u\right\|_0 \\ &\quad + M\delta\left\|\gamma_0^u\right\|_2\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0 \\ &\leq M\left(\delta^2 + \frac{\delta^2}{\omega^2}\right)\left\|(\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u)\right\|_0. \end{aligned}$$

The same bound can be obtained for the second coordinate of $\mathcal{F}_h$. Thus

$$\left\|\mathcal{F}_h(\gamma^u_{h,0},\theta^u_{h,0})-\mathcal{F}_h(\gamma^u_0,\theta^u_0)\right\|_0\le M\delta^2\left\|(\gamma^u_{h,0},\theta^u_{h,0})-(\gamma^u_0,\theta^u_0)\right\|_0.$$

Now, denote $\Delta^j_h=\pi_j\left(\mathcal{F}_h(\gamma^u_{h,0},\theta^u_{h,0})-\mathcal{F}_h(\gamma^u_0,\theta^u_0)\right)$, $j=1,2$, and $\Delta_h=(\Delta^1_h,\Delta^2_h)$. Then,

$$\left|\pi_1\left(\mathcal{G}_{\omega,h}(\gamma^u_{h,0},\theta^u_{h,0})-\mathcal{G}_{\omega,h}(\gamma^u_0,\theta^u_0)\right)(v)\right| = \left|\int_{-\infty}^0 e^{\omega is}\Delta^1_h(s+v)ds\right|.$$

Since $\Delta_h\in\mathcal{X}^2_2$, we can change the path of integration to obtain

$$\begin{aligned}\left|\int_{-\infty}^0 e^{\omega is}\Delta^1_h(s+v)ds\right| &= \left|\int_{-\infty}^0 e^{\omega ie^{-i\beta}\xi}\Delta^1_h(e^{-i\beta}\xi+v)e^{i\beta}d\xi\right|\\ &\le \int_{-\infty}^0 e^{\omega\sin(\beta)\xi}|\Delta^1_h(e^{-i\beta}\xi+v)|d\xi\\ &\le \|\Delta_h\|_0\int_{-\infty}^0 e^{\omega\sin(\beta)\xi}d\xi\\ &\le \frac{M}{\omega}\|\Delta_h\|_0.\end{aligned}$$

The same argument holds for the second coordinate of $\mathcal{G}_{\omega,h}(\gamma^u_{h,0},\theta^u_{h,0})-\mathcal{G}_{\omega,h}(\gamma^u_0,\theta^u_0)$. □

Lemma 7.9.6. *Let $\mathcal{F}_0$ and $\mathcal{F}_h$ be given in* (7.4.7) *and* (7.7.5)*, respectively, and consider the functions (γ^u_0,θ^u_0) obtained in Theorem 7.4.6. Given $M_0>0$ fixed, there exist ε_0, $h_0>0$ and a constant $M>0$ such that for $0\le h\le h_0$, $0<\varepsilon\le\varepsilon_0$ and $v\in D^u$ with $|h^{1/4}v|\le M_0$,*

$$|\pi_j\circ\mathcal{F}_h(\gamma^u_0,\theta^u_0)(v)-\pi_j\circ\mathcal{F}_0(\gamma^u_0,\theta^u_0)(v)|\le\frac{M\delta\sqrt{h}}{\omega|v^2+2|},\quad j=1,2.$$

Proof. Lemmas 7.9.3 and 7.9.4 imply

$$\begin{aligned}|\pi_1(\mathcal{F}_h(\gamma^u_0,\theta^u_0)(v)-\mathcal{F}_0(\gamma^u_0,\theta^u_0))(v)| &\le |(Q^h)'(v)-(Q^0)'(v)|\\ &\quad+\omega\,|\gamma^u_0|\,|\eta_h(v,\gamma^u_0,\theta^u_0)-\eta_0(v,\gamma^u_0,\theta^u_0)|\\ &\le M\frac{\delta\sqrt{h}}{\omega|v^2+2|}.\end{aligned}$$

The same holds for the second coordinate. □

Proposition 7.9.7. *Consider the functions (γ^u_0,θ^u_0) obtained in Proposition 7.4.6 and the operators $\mathcal{G}_{\omega,0}$ and $\mathcal{G}_{\omega,h}$ given in* (7.4.5) *and* (7.7.4)*, respectively. There exist $\varepsilon_0>$, $h_0>0$ and a constant $M>0$ such that, for $0<\varepsilon\le\varepsilon_0$ and $0<h\le h_0$*

$$\|\mathcal{G}_{\omega,h}(\gamma^u_0,\theta^u_0)-\mathcal{G}_{\omega,0}(\gamma^u_0,\theta^u_0)\|_0\le\frac{M\delta\sqrt{h}}{\omega^2}.$$

Proof. It follows from the proof of Proposition 7.7.11 that the Lipschitz constant of $\mathcal{G}_{\omega,h}$ in a ball $\mathcal{B}_0(K\delta/\omega^2)$, for some $K>0$ fixed, satisfies $\mathrm{Lip}(\mathcal{G}_{\omega,h})\le M\delta^2/\omega$. Moreover, $\|\mathcal{G}_{\omega,h}(0,0)\|_2\le M\delta/\omega^2$ and $\|(\gamma^u_0,\theta^u_0)\|_2\le M\delta/\omega^2$. Thus

$$\|\mathcal{G}_{\omega,h}(\gamma^u_0,\theta^u_0)\|_2\le\|\mathcal{G}_{\omega,h}(\gamma^u_0,\theta^u_0)-\mathcal{G}_{\omega,h}(0,0)\|_2+\|\mathcal{G}_{\omega,h}(0,0)\|_2\le M\frac{\delta}{\omega^2}.$$

Moreover, $\|\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)\|_2 = \|(\gamma_0^u,\theta_0^u)\|_2 \leq M\delta/\omega^2$.

Let $v\in D^u$ and first assume that $|h^{1/4}v|\geq 1$, hence

$$\begin{aligned}|\pi_j(\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)(v)-\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)(v))| &\leq \frac{\|\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)\|_2}{|v^2+2|}+\frac{\|\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)\|_2}{|v^2+2|}\\ &\leq M\frac{\delta}{\omega^2||v|^2-2|}\\ &\leq M\frac{\delta}{\omega^2(1/\sqrt{h}-2)}\\ &\leq M\frac{\delta}{\omega^2}\sqrt{h},\end{aligned}$$

for $h>0$ sufficiently small, $j=1,2$.

Now, assume that $|h^{1/4}v|<1$, and denote $\Delta_h^j=\pi_j(\mathcal{F}_h(\gamma_0^u,\theta_0^u)-\mathcal{F}_0(\gamma_0^u,\theta_0^u))$, $j=1,2$.

Consider the path $s=e^{-i\beta}\xi$ (since $\Delta_h\in\mathcal{X}_2^2$) and let $\xi_0(v)\in\mathbb{R}$ be such that $v_0(v)=v+e^{-i\beta}\xi_0(v)$ is the unique point of intersecion between the curve $\gamma(\xi)=v+e^{-i\beta}\xi$ and the circle S_h of radius $h^{-1/4}$ centered at the origin.

$$\begin{aligned}|\pi_1(\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)-\mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u))(v)| &= \left|\int_{-\infty}^0 e^{\omega is}\Delta_h^1(s+v)ds\right|\\ &= \left|\int_{-\infty}^0 e^{-\omega ie^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right|\\ &\leq \left|\int_{-\infty}^{\xi_0(v)} e^{-\omega ie^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right|\\ &\quad+\left|\int_{\xi_0(v)}^0 e^{-\omega ie^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right|\end{aligned}$$

Notice that the points in the path $\gamma(\xi)=v+e^{-i\beta}\xi$ satisfy that $|\gamma(\xi)h^{1/4}|\geq 1$ for every $\xi\leq\xi_0(v)$ and $|\gamma(\xi)h^{1/4}|<1$ for every $\xi_0(v)<\xi<0$. Also, let $v_0^*(v)=e^{i\beta}v_0(v)$, and notice that $\mathrm{Im}(v_0^*(v))=\mathrm{Im}(v)$ and $|h^{1/4}v_0^*(v)|=1$.

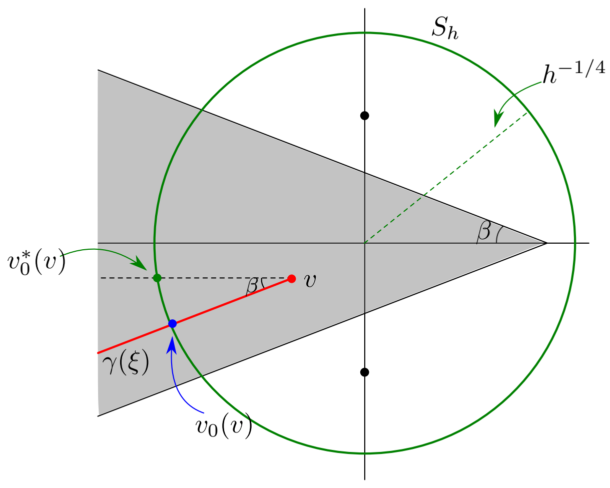


Figure 7.10: Definition of the points $v_0(v)$ and $v_0^*(v)$.

Thus the first integral satisfies that

$$\begin{aligned}
\left|\int_{-\infty}^{\xi_0(v)} e^{-\omega i e^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right| &= \left|\int_{-\infty}^{v_0^*(v)} e^{\omega i(v-r)}\Delta_h^1(r)dr\right| \\
&= \left|e^{\omega i(v-v_0^*(v))}\int_{-\infty}^{v_0^*(v)} e^{\omega i(v_0^*(v)-r)}\Delta_h^1(r)dr\right| \\
&= |\pi_1(\mathcal{G}_{\omega,h}(\gamma_0^u,\theta_0^u)(v_0^*(v)) - \mathcal{G}_{\omega,0}(\gamma_0^u,\theta_0^u)(v_0^*(v)))| \\
&\le M\frac{\delta\sqrt{h}}{\omega^2}.
\end{aligned}$$

Now, since $|\gamma(\xi)h^{1/4}|< 1$ for every $\xi_0(v) < \xi < 0$, we can use Lemma 7.9.6 to see that the second integral satisfies

$$\begin{aligned}
\left|\int_{\xi_0(v)}^{0} e^{-\omega i e^{-i\beta}\xi}\Delta_h^1(v+e^{-i\beta}\xi)e^{-i\beta}d\xi\right| &\le \int_{\xi_0(v)}^{0} e^{\omega\sin(\beta)\xi}|\Delta_h^1(v+e^{-i\beta}\xi)|d\xi \\
&\le \frac{M\delta\sqrt{h}}{\omega}\int_{-\infty}^{0} e^{\omega\sin(\beta)\xi}\frac{1}{|(v+e^{-i\beta}\xi)^2+2|}d\xi \\
&\le \frac{M\delta\sqrt{h}}{\omega|v^2+2|}\int_{-\infty}^{0} e^{\omega\sin(\beta)\xi}d\xi \\
&\le \frac{M\delta\sqrt{h}}{\omega^2|v^2+2|}.
\end{aligned}$$

The result follows from these bounds. □

Now, define $\mathcal{E}(v) = (\gamma_{h,0}^u(v) - \gamma_0^u(v), \theta_{h,0}^u(v) - \theta_0^u(v))$ and notice that

$$\begin{pmatrix} \Gamma_{h,0}^u(0) - \Gamma_0^u(0) \\ \Theta_{h,0}^u(0) - \Theta_0^u(0) \end{pmatrix} = \begin{pmatrix} Q^h(0) - Q^0(0) \\ -Q^h(0) + Q^0(0) \end{pmatrix} + \mathcal{E}(0)^T.$$

Using (7.3.11) and (7.3.18), we have $Q^h(0) = Q^0(0) = 0$. Hence, to prove Proposition 7.9.2, it is enough to bound $\|\mathcal{E}\|_0$. Since $(\gamma_{h,0}^u, \theta_{h,0}^u)$ and (γ_0^u, θ_0^u) are fixed points of $\mathcal{G}_{\omega,h}$ and $\mathcal{G}_{\omega,0}$, respectively,

$$\begin{aligned}
\mathcal{E} &= (\gamma_{h,0}^u, \theta_{h,0}^u) - (\gamma_0^u, \theta_0^u) \\
&= \mathcal{G}_{\omega,h}(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u) + \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u) - \mathcal{G}_{\omega,0}(\gamma_0^u, \theta_0^u).
\end{aligned}$$

It follows from Propositions 7.9.5 and 7.9.7 that

$$\begin{aligned}
\|\mathcal{E}\|_0 &\le \|\mathcal{G}_{\omega,h}(\gamma_{h,0}^u, \theta_{h,0}^u) - \mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u)\|_0 + \|\mathcal{G}_{\omega,h}(\gamma_0^u, \theta_0^u) - \mathcal{G}_{\omega,0}(\gamma_0^u, \theta_0^u)\|_0 \\
&\le M\delta^2\|\mathcal{E}\|_0 + \frac{M\delta\sqrt{h}}{\omega^2}.
\end{aligned}$$

Thus, for ε_0 sufficiently small, we have that $\|\mathcal{E}\|_0 \le 2\frac{M\delta\sqrt{h}}{\omega^2}$. This completes the proof.

7.9.3 Approximation of $W^u_\varepsilon(\Lambda^-_{\kappa_1,\kappa_2})$ by $W^u_\varepsilon(p^-_0)$

In this section, we obtain an approximation of $N^u_{\kappa_1,\kappa_2}$ by $N^u_{0,0}$, by approximating $N^u_{\kappa_1,\kappa_2}$ by $N^u_{\kappa_1,0}$ and $N^u_{\kappa_1,0}$ by $N^u_{0,0}$.

Proceeding as for Proposition 7.9.1 and Lemma 7.8.5, one can obtain the next result.

Proposition 7.9.8. *Let $\Gamma^u_{\kappa_1,0}(v)$, $\Theta^u_{\kappa_1,0}(v)$ and $\Gamma^u_{\kappa_1,\kappa_2}(v,\tau)$, $\Theta^u_{\kappa_1,\kappa_2}(v,\tau)$ be given in* (7.3.17) *and* (7.3.20)*, respectively. There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that, for $v \in D^u \cap \mathbb{R}$, $\tau \in \mathbb{T}$, $0 \le \varepsilon \le \varepsilon_0$, $0 \le h \le h_0$ $\kappa_1, \kappa_2 \ge 0$ with $\kappa_1 + \kappa_2 = h$,*

$$\left|\partial_\tau(\Gamma^u_{\kappa_1,\kappa_2}(v,\tau) - \Gamma^u_{\kappa_1,0}(v))\right|, \left|\Gamma^u_{\kappa_1,\kappa_2}(v,\tau) - \Gamma^u_{\kappa_1,0}(v)\right| \le M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}},$$
$$\left|\partial_\tau(\Theta^u_{\kappa_1,\kappa_2}(v,\tau) - \Theta^u_{\kappa_1,0}(v))\right|, \left|\Theta^u_{\kappa_1,\kappa_2}(v,\tau) - \Theta^u_{\kappa_1,0}(v)\right| \le M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}.$$

Notice that Proposition 7.9.2 allows us to approximate $N^u_{\kappa_1,0}$ by $N^u_{0,0}$, for κ_1 sufficiently small. Thus, we can combine this fact with Proposition 7.9.8 to obtain the following proposition.

Proposition 7.9.9. *Let $\Gamma^u_0(v)$, $\Theta^u_0(v)$ and $\Gamma^u_{\kappa_1,\kappa_2}(v,\tau)$, $\Theta^u_{\kappa_1,\kappa_2}(v,\tau)$ be given in* (7.3.10) *and* (7.3.20)*, respectively. There exist $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $M > 0$ such that, for $v \in D^u \cap \mathbb{R}$, $\tau \in \mathbb{T}$, $0 \le \varepsilon \le \varepsilon_0$, $0 \le h \le h_0$ and $\kappa_1, \kappa_2 \ge 0$ with $\kappa_1 + \kappa_2 = h$,*

$$\begin{aligned}&\left|\Gamma^u_{\kappa_1,\kappa_2}(v,\tau) - \Gamma^u_0(v)\right|, \left|\Theta^u_{\kappa_1,\kappa_2}(v,\tau) - \Theta^u_0(v)\right| \le M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}} + M\frac{\delta\sqrt{\kappa_1}}{\omega^2},\\ &\left|\partial_\tau(\Gamma^u_{\kappa_1,\kappa_2}(v,\tau) - \Gamma^u_0(v))\right|, \left|\partial_\tau(\Theta^u_{\kappa_1,\kappa_2}(v,\tau) - \Theta^u_0(v))\right| \le M\frac{\delta\sqrt{\kappa_2}}{\omega^{3/2}}.\end{aligned} \tag{7.9.1}$$

7.10 Conclusion and Further Directions

In this chapter we have studied a 2-dof Hamiltonian H arising from an approximation of the solutions of the partial differential equation (7.1.3). More specifically, we have found conditions on the energy of H in order to detect certain heteroclinic connections (corresponding to quasi kink-like and oscillating kink-like solutions). It provides a rigorous treatment for the computation of the critical velocity done in [47]. Also, we provided an asymptotic formula for the final velocity of a quasi kink-like solution which was conjectured in [47].

As we have mentioned in Section 7.1, there are many works studying the efficacy of this toy-model to approximate the solutions of (7.1.3), nevertheless a rigorous study of it remains as an open problem.

Also, in [47], they mention the existence of n-bounce resonant solutions of H, which correspond to heteroclinic connections passing n times through the transversal section Σ_h considered in this work (see (7.2.6)). A rigorous study of the existence of such solutions is a difficult task which deserves attention.

Finally, a similar approach can be performed to validate the formula of critical velocity obtained for another models in [48, 49].

Chapter 8

On the Breakdown of Breathers for Reversible Klein-Gordon Equations

Breathers are nontrivial time-periodic and spatially localized solutions of evolutionary Partial Differential Equations (PDE's). In this chapter, we associate breathers of certain reversible Klein-Gordon equations with homoclinic orbits of a singularly perturbed Hamiltonian at the origin (which is a critical point) and we provide an asymptotic formula for the splitting of the invariant manifolds at the origin, which happens to be exponentially small with respect to the perturbation parameter.

8.1 Introduction

As far as the authors know, breathers were introduced by [1] in the context of the *sine-Gordon partial differential equation*

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0. \tag{8.1.1}$$

This kind of solutions has shown to be very important in physical applications and thus the proof of their existence or breakdown is a fundamental problem in the study of the dynamics of evolutionary PDE's. Moreover, this problem is strongly related to the analysis of invariant manifolds of PDE's.

It is known that (8.1.1) admits an explicit family of breathers

$$u_m(x,\tau) = 4\arctan\left(\frac{m}{\omega}\frac{\sin(\omega t)}{\cosh(mx)}\right), \quad m,\omega>0,\ m^2+\omega^2=1. \tag{8.1.2}$$

Nevertheless, in general, the existence of such solutions for nonlinear wave equations is rare (see [31, 94]).

As far as the authors know, there are few results concerning breathers which have been rigorously proved. In [31], Denzler has shown that the breathers of the sine-Gordon equation do not persist under any nontrivial perturbation of the form

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = \varepsilon\Delta(u) + \mathcal{O}(\varepsilon^2),$$

where Δ is an analytic function in a small neighborhood of $u=0$. In [71], Lu has shown that reversible nonlinear Klein-Gordon equations admit small amplitude breathers with exponentially small tails. In [31], the author has related breathers to homoclinic orbits at the origin of an infinite dimensional dynamical system, then the solutions constructed in [71] correspond to homoclinic orbits which tend to exponentially small solutions in some center manifold in the phase space of an infinite dimensional dynamical system.

In [93], Kruskal and Segur use formal asymptotic expansions to justify the nonexistence of small amplitude breathers in a certain Klein-Gordon equation, but no rigorous proof is given. In the present chapter, we use rigorous analysis to obtain some results concerning this problem.

There are several physical problems which make use of *nonlinear Klein-Gordon equations* in one-dimensional space

$$\partial_t^2 u - \partial_x^2 u + h(u) = 0,$$

where h is a real-analytic function such that $h(0) = 0$ and $h'(0) > 0$. For example, we can point their use in the study of magnetic chains and quantum field theory as mentioned in [93]. The existence or not of breathers is a relevant topic in these physical models, and thus to know whether a Klein-Gordon equation admits breathers is a fundamental problem which remains open.

In this work, we consider a class of reversible Klein-Gordon equations

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \tag{8.1.3}$$

where f is a real-analytic odd function which satisfies $f(u) = \mathcal{O}(u^5)$, and we associate the existence of breathers of (8.1.3) with the existence of homoclinic orbits (with respect to the variable x) of (8.1.3) at the origin (which is a critical point). In fact, a solution $u(x,t)$ of (8.1.3) is a breather if, and only if, $u(x,t)$ is periodic in the variable t, and for each t fixed, $u_t(x) := u(x,t)$ is a homoclinic solution of (8.1.3). We are interested in reversible breathers of the form

$$u(x,t) = \sum_{n\geq 1} u_n(x)\sin(n\omega t), \tag{8.1.4}$$

which are $\frac{2\pi}{\omega}-$periodic in the variable t and real-analytic in the variable x with $\omega \approx 1$. In this case, the origin has one-dimensional stable and unstable invariant manifolds $W^s(0)$ and $W^u(0)$.

In this work, we provide a rigorous treatment to compute the distance between $W^s(0)$ and $W^u(0)$ when they intersect a transversal section for the first time.

Such class of Klein-Gordon equations has also been considered in [62], and the authors have proved the non-existence of small breathers $u(x,\tau)$ of (8.1.3) which are odd in the variables x and τ. It is worth saying that the known family of breathers of the sine-Gordon equation given by (8.1.2) is even in the variable x, thus the oddness assumption on the variable x considered in [62] excludes such solutions. Therefore, our study concerns about a larger (and more complicated) class of solutions of (8.1.3) than the ones considered in [62].

8.1.1 Model

Considering the ansatz (8.1.4) and the parameterization of time $\tau = \omega t$, with $\omega \neq 0$, we obtain that

$$u(x,\tau) = \sum_{n\geq 1} u_n(x)\sin(n\tau), \tag{8.1.5}$$

and the Fourier coefficients $u_n \in \mathbb{R}$, for every $n \in \mathbb{N}$ (due to real-analyticity hypothesis).

Denote

$$g(u) = \frac{1}{3}u^3 + f(u), \tag{8.1.6}$$

and observe that, since g is odd, we have that $g(u(x,-\tau)) = g(-u(x,\tau)) = -g(u(x,\tau))$, and thus $g(u(x,\tau))$ admits a sine Fourier expansion.

Define the projections

$$\Pi_n[f](y) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y,\tau)\sin(n\tau)d\tau \quad and \quad \widehat{\Pi}[f](y,\tau) = \sum_{n\geq 2}\Pi_n(f)(y)\sin(n\tau), \tag{8.1.7}$$

and denote

$$g_n(u) = \Pi_n\left[g(u(x,\tau))\right].$$

Thus, replacing (8.1.5) in (8.1.3) and using (8.1.6), we obtain

$$(\partial_x^2 + n^2\omega^2 - 1)u_n = -g_n(u), \ n \in \mathbb{N}. \tag{8.1.8}$$

From the definition of breathers, $u(x,\tau)$ given in (8.1.5) is a breather of (8.1.3) if and only if

$$\lim_{x\to\pm\infty} u(x,\tau) = 0, \ \forall \tau \in \mathbb{T},$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Therefore, we have a bijection between reversible breathers of (8.1.3) and homoclinic connections at 0 of (8.1.8) seen as a dynamical system taking x as time. A simple analysis shows us that the eigenvalues of the linearization of (8.1.8) at 0 are given by $\mu_n^{\pm} = \pm\sqrt{1-n^2\omega^2}$, for each $n \geq 1$. Thus, for $\omega > 0$ fixed such that $\omega \neq n^{-2}$, for every $n \in \mathbb{N}$, we have that there exists $N_0 = N_0(\omega) \geq 0$ such that $\mu_n^{\pm} \in \mathbb{R}$, for each $1 \leq n \leq N_0$ and $\mu_n^{\pm}$ are purely imaginary for every $n > N_0$. It means that the singular point 0 has a hyperbolic eigenspace $E^h(0)$ of dimension $2N_0$ (with N_0 unstable directions and N_0 stable ones) and a central eigenspace $E^c(0)$ of codimension $2N_0$. Notice that $N_0(\omega) = 0$ for each $\omega \geq 1$.

It means that, if $\omega < 1$, then 0 has stable and unstable local invariant manifolds $W^s(0)$ and $W^u(0)$, respectively, both of dimension $N_0 > 0$, and a central manifold $W^c(0)$ with codimension $2N_0$ (it has infinite dimension). Thus, the breathers of (8.1.3) are characterized as intersections of $W^s(0)$ and $W^u(0)$.

It is known from the study of splitting of separatrices that, the difficulty of the problem increases when the dimensions of the invariant manifolds increase. In order to attack the simplest version of the problem (which already presents major difficulties), we consider that $\omega < 1$ and $\omega \approx 1$ to have $N_0 = 1$.

Now, we set a singular perturbation problem to compute the distance between the invariant manifolds $W^s(0)$ and $W^u(0)$. Define

$$\varepsilon = \sqrt{1-\omega^2}, \ \text{for } \omega < 1,$$

and observe that $0 < \varepsilon < 1$. Consider the following scaling of the variables and time

$$u = \varepsilon v \text{ and } y = \varepsilon x.$$

Thus $u(x,\tau) = \varepsilon v(\varepsilon x,\tau)$ satisfies (8.1.3) if, and only if, $v(y,\tau)$ satisfies

$$\partial_y^2 v - \frac{\omega^2}{\varepsilon^2}\partial_\tau^2 v - \frac{1}{\varepsilon^2}v + \frac{1}{3}v^3 + \frac{1}{\varepsilon^3}f\left(\varepsilon v\right) = 0, \tag{8.1.9}$$

which is a Hamiltonian Partial Differential Equation with respect to

$$\mathcal{H}(v, \partial_y v) = \frac{1}{\pi} \int_{\mathbb{T}} \left(\frac{(\partial_y v)^2}{2} + \frac{(\omega \partial_\tau v)^2}{2\varepsilon^2} - \frac{v^2}{2\varepsilon^2} + \frac{v^4}{12} + \frac{F(\varepsilon v)}{\varepsilon^4} \right) d\tau, \tag{8.1.10}$$

where F is an analytic function such that $F(z) = \mathcal{O}(z^6)$ and $F'(z) = f(z)$.

Also, denoting $v_n(y) = \Pi_n[v](y)$ and $\cdot = d/dy$, we obtain

$$\ddot{v}_n = -\frac{(n^2\omega^2 - 1)}{\varepsilon^2} v_n - \frac{1}{\varepsilon^3} \Pi_n \left[g(\varepsilon v) \right]. \tag{8.1.11}$$

Now, define

$$\lambda_n = \sqrt{n^2(1-\varepsilon^2) - 1},$$

for each $n \geq 2$ and notice that $\lambda_n \in \mathbb{R}$ and $\mu_n^+ = i\lambda_n$ (with $\omega^2 = 1 - \varepsilon^2$). Thus, system (8.1.11) may be written as

$$\begin{cases} \ddot{v}_1 = v_1 - \dfrac{1}{\varepsilon^3} \Pi_1 \left[g(\varepsilon v) \right], \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\varepsilon^2} v_n - \dfrac{1}{\varepsilon^3} \Pi_n \left[g(\varepsilon v) \right], \ n \geq 2, \end{cases} \tag{8.1.12}$$

and notice that $\varepsilon^{-3} g(\varepsilon v) = \mathcal{O}(1)$, therefore, $\varepsilon^{-3} \Pi_n \left[g(\varepsilon v) \right] = \mathcal{O}(1)$, for every $n \geq 1$. Then, the one-dimensional stable and unstable invariant manifolds $W^s(0)$ and $W^u(0)$ of the singular point 0 are characterized as the solutions v^s and v^u of (8.1.12) satisfying the asymptotic conditions

$$\lim_{y \to +\infty} v^s(y, \tau) = \lim_{y \to -\infty} v^u(y, \tau) = 0, \ \forall \tau \in \mathbb{T}. \tag{8.1.13}$$

Notice that the singular perturbation problem (8.1.12) may be written as

$$\begin{cases} \ddot{v}_1 = v_1 - \varepsilon^{-3} \Pi_1 \left[g(\varepsilon v) \right], \\ \varepsilon^2 \ddot{v}_n = -\lambda_n^2 v_n - \varepsilon^{-1} \Pi_n \left[g(\varepsilon v) \right], \ n \geq 2. \end{cases} \tag{8.1.14}$$

The singular limit of (8.1.14) ($\varepsilon = 0$) defines a slow-manifold $\mathcal{M}$, which is a plane given by

$$\mathcal{M} = \{ v(y, \tau); \ \widehat{\Pi}[v] = 0 \}.$$

with dynamics

$$\ddot{v}_1 = v_1 - \frac{1}{3} \Pi_1 [(v_1^3 \sin^3(\tau))], \ v \in \mathcal{M}. \tag{8.1.15}$$

Hence, $\mathcal{M}$ has dimension 1 and the limit problem has a drastic reduction of dimensions in comparison to the original one.

Recalling the expression of the projection Π_1 (see (8.1.7)), we obtain that the dynamics of the slow variable v_1 in $\mathcal{M}$ is governed by the Duffing equation

$$\ddot{v}_1 = v_1 - \frac{v_1^3}{4}. \tag{8.1.16}$$

It is known that (8.1.16) has a unique homoclinic orbit at 0 with $v_1 > 0$, which is given by

$$v_1^h(y) = \frac{2\sqrt{2}}{\cosh(y)}. \tag{8.1.17}$$

See Figure 8.1.

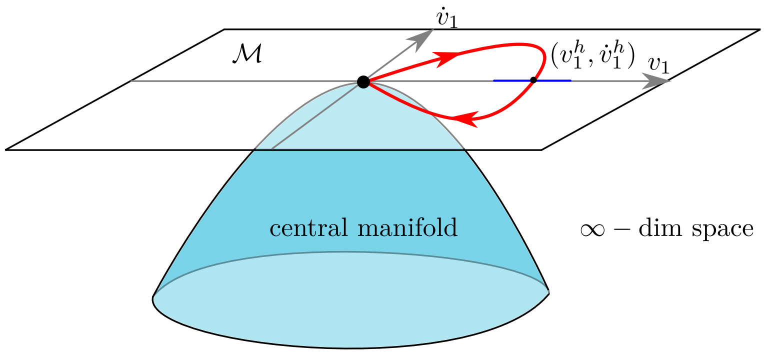


Figure 8.1: Representation of the homoclinic orbit v_1^h in the slow manifold $\mathcal{M}$.

Remark 8.1.1. *The Duffing equation has another homoclinic orbit at* 0 *with* $v_1 < 0$*, which is symmetric to* v_1^h*. In this paper, we treat only* v_1^h*.Nevertheless all the results hold for the another homoclinic orbit in an analogous way.*

Therefore, we have that in the limit problem (8.1.15), the invariant manifolds $W^s(0)$ and $W^u(0)$ of (8.1.12) coincide. This paper is devoted to obtain an asymptotic formula for the difference between $W^s(0)$ and $W^u(0)$, for $\varepsilon > 0$, at certain section Σ.

8.2 Main Theorem

Since (8.1.12) is a Hamiltonian system with respect to $\mathcal{H}$ given in (8.1.10) and and $(v^s, \partial_y v^s)$ and $(v^u, \partial_y v^u)$ are contained in the zero energy level of $\mathcal{H}$. Thus, we consider the section

$$\Sigma = \{(v, \partial_y v);\ \mathcal{H}(v, \partial_y v) = 0 \quad \textit{and} \quad \Pi_1 [\partial_y v] = 0\},$$

to measure the distance between $W^u(0)$ and $W^s(0)$. Since $\mathcal{H}(v, \partial_y v) = 0$ and $\Pi_1 [\partial_y v] = 0$ for all points of Σ, we use the coordinates $(\widehat{\Pi} [v] , \widehat{\Pi} [\partial_y v])$ to parameterize Σ.

Throughout this paper, given an odd function $f(y, \tau)$ periodic in τ, we consider the point-wise ℓ_1-norm

$$\|f\|_{\ell_1}(y) = \sum_{n\geq 1} |\Pi_n[f](y)| \,.$$

We state the main result of this paper.

Theorem U (Main Theorem)**.** *Consider system* (8.1.12)*. Then, there exist* $\varepsilon_0 > 0$ *and a complex constant* $C_{\rm in}$ *independent of* ε *such that, for every* $\varepsilon \leq \varepsilon_0$*, the following statements hold.*

1. *The invariant manifolds* $W^u(0)$ *and* $W^s(0)$ *are parameterized by real-analytic solutions* $v^u(y, \tau)$ *and* $v^s(y, \tau)$ *of* (8.1.12) *satisfying* (8.1.13) *such that* $\Pi_1 [\partial_y v^{u,s}] (0) = 0$*, respectively. Moreover,* $\Pi_{2l} [v] \equiv 0$*, for every* $l \in \mathbb{N}$*.*

2. *Let* $p^\star(\tau) = (\widehat{\Pi} [v^\star] (0, \tau), \widehat{\Pi} [\partial_y v^\star(0, \tau)]) \in \Sigma$*, for* $\star = u, s$*, and consider* $d(\tau; \varepsilon) = p^u(\tau) - p^s(\tau)$*, therefore*

$$d(\tau; \varepsilon) = \begin{pmatrix} \dfrac{2}{\varepsilon} e^{-\frac{\pi\lambda_3}{2\varepsilon}} \left(\mathrm{Re}(C_{\rm in}) \sin(3\tau) + \mathcal{O}_{\ell_1}\left(\dfrac{1}{\log(\varepsilon^{-1})}\right)\right) \\ \dfrac{2\lambda_3}{\varepsilon^2} e^{-\frac{\pi\lambda_3}{2\varepsilon}} \left(\mathrm{Im}(C_{\rm in}) \sin(3\tau) + \mathcal{O}_{\ell_1}\left(\dfrac{1}{\log(\varepsilon^{-1})}\right)\right) \end{pmatrix}, \tag{8.2.1}$$

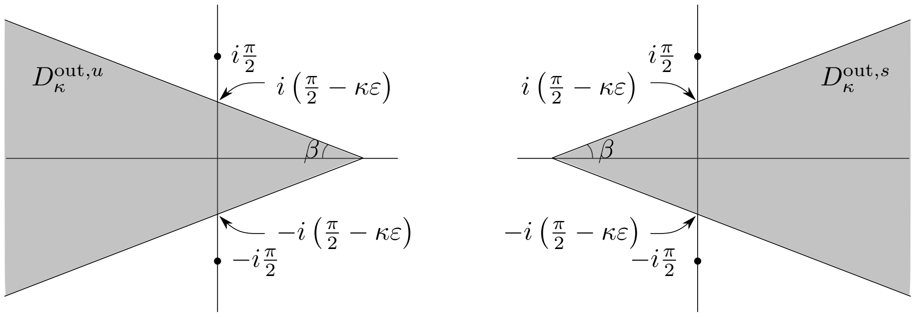


Figure 8.2: Outer domains $D_\kappa^{\mathrm{out},u}$ and $D_\kappa^{\mathrm{out},s}$.

for $0<\varepsilon<\varepsilon_0$ *and* $\tau\in\mathbb{T}$.

3. *If* $C_{\mathrm{in}}\neq 0$, *then the invariant manifolds* $W^u(0)$ *and* $W^s(0)$ *do not intersect the first time that they reach* Σ.

We highlight that Theorem U concerns with the breakdown of breathers which crosses the transversal section Σ only one time. Nevertheless, (8.1.12) may admit breathers which cross Σ at n distinct points, $n\geq 2$.

8.3 Description of the Proof

In this section we give an overall description of the steps to prove Theorem U. First, notice that the homoclinic orbit $(v_1^h(y)\sin(\tau),\partial_y v_1^h(y)\sin(\tau))$ of the singular limit (8.1.15) is transverse to the section Σ, and $v_1^h(y)$ has poles at $y=\pm i(\pi/2+k\pi)$, $k\in\mathbb{Z}$.

In order to compute the distance between the perturbed $W^s(0)$ and $W^u(0)$ in Σ, we obtain complex parameterizations of them in the **outer domains**

$$\begin{aligned} D_\kappa^{\mathrm{out},u}&=\left\{y\in\mathbb{C};|\mathrm{Im}(y)|\leq-\tan\beta\,\mathrm{Re}(y)+\frac{\pi}{2}-\kappa\varepsilon\right\},\\ D_\kappa^{\mathrm{out},s}&=\left\{y\in\mathbb{C};-y\in D_\kappa^{\mathrm{out},u}\right\},\end{aligned}\tag{8.3.1}$$

where $0<\beta<\pi/4$ is a fixed parameter independent of ε and $\kappa\geq 1$ (see Figure 8.2).

Also, notice that in the complex domains, the invariant manifolds $W^u(0)$ and $W^s(0)$ are characterized as solutions v^u and v^s of (8.1.12) such that

$$\lim_{\mathrm{Re}(y)\to+\infty}v^s(y,\tau)=\lim_{\mathrm{Re}(y)\to-\infty}v^u(y,\tau)=0,\ \forall\tau\in\mathbb{T},\tag{8.3.2}$$

respectively.

Theorem 8.3.1 (Outer). *Consider the equation* (8.1.9). *There exist* $\kappa_0\geq 1$ *and* $\varepsilon_0>0$, *such that, for each* $\varepsilon\leq\varepsilon_0$ *and* $\kappa\geq\kappa_0$, *the invariant manifold* $W^\star(0)$ *of* (8.1.9), $\star=u,s$, *is parameterized by*

$$v^\star(y,\tau)=v_1^h(y)\sin(\tau)+\xi^\star(y,\tau),\ y\in D_\kappa^{\mathrm{out},\star},\ \tau\in\mathbb{T},$$

where v_1^h *is given by* (8.1.17) *and* $\xi^\star:D_\kappa^{\mathrm{out},\star}\times\mathbb{T}\to\mathbb{C}$ *is a real-analytic function in the variable* y *satisfying* $\partial_y\Pi_1[\xi^\star](0)=0$ *and the asymptotic condition* (8.3.2). *Moreover,* $\Pi_{2l}[\xi^\star](y)\equiv 0$, *for every* $l\in\mathbb{N}$, *and there exists a constant* $M_1>0$ *independent of* ε *and* κ, *such that*

1. $\|\xi^\star\|_{\ell_1}(y), \|\partial_\tau\xi^\star\|_{\ell_1}(y), \|\partial_\tau^2\xi^\star\|_{\ell_1}(y) \leq \dfrac{M_1\varepsilon^2}{|y^2+\pi^2/4|^3}$ *and* $\|\partial_y\xi^\star\|_{\ell_1}(y) \leq \dfrac{M_1\varepsilon^2}{|y^2+\pi^2/4|^4}$, *for every* $y \in D_\kappa^{\text{out},\star} \cap \{|\text{Re}(y)| \leq 1\}$;

2. $\|\xi^\star\|_{\ell_1}(y), \|\partial_y\xi^\star\|_{\ell_1}(y), \|\partial_\tau\xi^\star\|_{\ell_1}(y), \|\partial_\tau^2\xi^\star\|_{\ell_1}(y) \leq \dfrac{M_1\varepsilon^2}{|\cosh(y)|}$, *for every* $y \in D_\kappa^{\text{out},\star} \cap \{|\text{Re}(y)| > 1\}$.

Notice that, the parameterization $v^\star(y,\tau)$ of $W^\star(0)$, $\star = u, s$, given by Theorem 8.3.1 has the homoclinic orbit $v_1^h(y)\sin(\tau)$ as a first order, for $y \in \mathbb{R}$. Nevertheless, at distance $\mathcal{O}(\varepsilon)$ of the poles $y = \pm i\pi/2$ of v_1^h, we have that v_1^h has the same size of the error $\xi^\star$.

In light of this, we need to analyze the first order of the invariant manifolds at distance $\mathcal{O}(\varepsilon)$ of the poles of the unperturbed homoclinic to compute a correct asymptotic formula for the distance between the invariant manifolds $W^{u,s}(0)$.

We focus on the singularity $i\pi/2$. Nevertheless similar results can be proved near the singularity $-i\pi/2$ in an analogous way.

Consider the **inner variable**

$$z = \varepsilon^{-1}\left(y - i\frac{\pi}{2}\right), \tag{8.3.3}$$

and the scaling

$$\phi(z,\tau) = \varepsilon v\left(i\frac{\pi}{2} + \varepsilon z, \tau\right). \tag{8.3.4}$$

Writing equation (8.1.9) for ϕ in the inner variable, we obtain

$$\partial_z^2\phi - \omega^2\partial_\tau^2\phi - \phi + \frac{1}{3}\phi^3 + f(\phi) = 0,\ \omega = \sqrt{1-\varepsilon^2}. \tag{8.3.5}$$

The first order of (8.3.5) corresponds to the limit case $\varepsilon = 0$, which gives the so-called **inner equation**

$$\partial_z^2\phi^0 - \partial_\tau^2\phi^0 - \phi^0 + \frac{1}{3}(\phi^0)^3 + f(\phi^0) = 0. \tag{8.3.6}$$

Now, we present the results concerning the existence of two solutions $\phi^{0,\star}$ of (8.3.5), $\star = u, s$, which will give a good approximation for $W^\star(0)$ for y near the pole $i\pi/2$. Moreover, we provide an asymptotic expression for the difference $\phi^{0,u} - \phi^{0,s}$ which will be crucial to compute the first order of the difference $v^u - v^s$.

Consider the **inner domains**

$$\begin{aligned} D_{\theta,\kappa}^{u,\text{in}} &= \{z \in \mathbb{C};\, |\text{Im}(z)| > \tan\theta\,\text{Re}(z) + \kappa\}, \\ D_{\theta,\kappa}^{s,\text{in}} &= \{z \in \mathbb{C};\, -z \in D_{\theta,\kappa}^{u,\text{in}}\}, \end{aligned} \tag{8.3.7}$$

where $0 < \theta < \pi/2$ and $\kappa > 0$ (see Figure 8.3).

Theorem 8.3.2 (Inner). *Let* $\theta > 0$ *be fixed. There exists* $\kappa_0 \geq 1$ *such that, for each* $\kappa \geq \kappa_0$,

1. *equation* (8.3.6) *has two solutions* $\phi^{0,\star} : D_{\theta,\kappa}^{\star,\text{in}} \times \mathbb{T} \to \mathbb{C} \times \ell_1(\mathbb{C})$, $\star = u, s$, *given by*

$$\phi^{0,\star}(z,\tau) = -\frac{2\sqrt{2}i}{z}\sin(\tau) + \psi^\star(z,\tau), \tag{8.3.8}$$

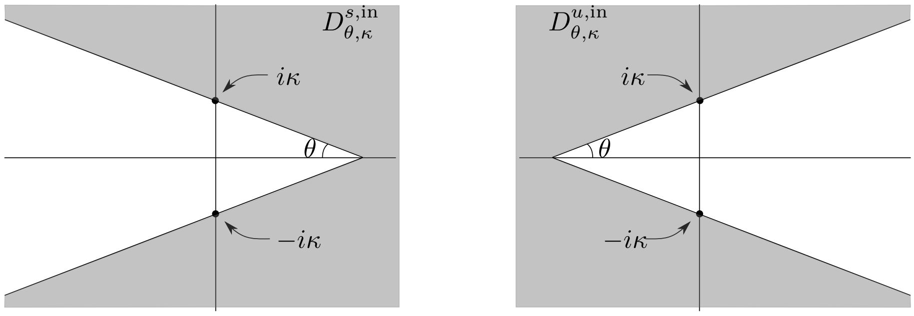


Figure 8.3: Inner domains $D^{s,\text{in}}_{\theta,\kappa}$ and $D^{u,\text{in}}_{\theta,\kappa}$.

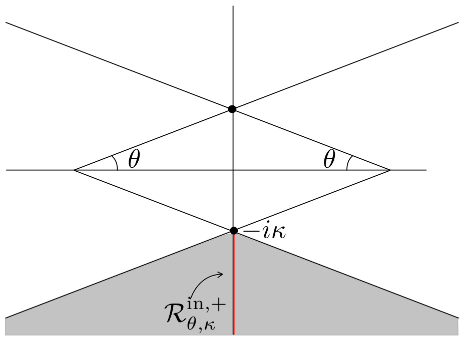


Figure 8.4: Domain $\mathcal{R}^{\text{in},+}_{\theta,\kappa}$.

which are analytic in the variable z. Moreover, $\Pi_{2l}\left[\phi^{0,\star}\right] \equiv 0$, for every $l \in \mathbb{N}$, and there exists a constant $M_2 > 0$ independent of κ such that, for every $z \in D^{\star,\text{in}}_{\theta,\kappa}$

$$\left\|\psi^\star\right\|_{\ell_1}(z), \left\|\partial_\tau \psi^\star\right\|_{\ell_1}(z) \leq \frac{M_2}{|z|^3}.$$

2. *the difference $\Delta\phi_0(z,\tau) = \phi^{0,u}(z,\tau) - \phi^{0,s}(z,\tau)$ is given by*

$$\Delta\phi_0(z,\tau) = e^{-i\lambda_{0,3} z}\left(C_{\text{in}} \sin(3\tau) + \chi(z,\tau)\right), \tag{8.3.9}$$

for each $z \in \mathcal{R}^{\text{in},+}_{\theta,\kappa} = D^{u,\text{in}}_{\theta,\kappa} \cap D^{s,\text{in}}_{\theta,\kappa} \cap \{z;\ z \in i\mathbb{R} \text{ and } \operatorname{Im}(z) < 0\}$ (see Figure 8.4), where $\lambda_{0,3} = 2\sqrt{2}$, and χ is an analytic function in the variable z such that

$$\|\chi\|_{\ell_1}(z), \|\partial_\tau \chi\|_{\ell_1}(z) \leq \frac{M_2}{|z|} \quad \text{and} \quad \|\partial_z \chi\|_{\ell_1}(z) \leq \frac{M_2}{|z|^2},\ \forall z \in \mathcal{R}^{\text{in},+}_{\theta,\kappa}.$$

Notice that, for $\star = u, s$, the outer solution $v^\star(y)$ given by Theorem 8.3.1 provides a good approximation for $W^\star(0)$ when y is $\mathcal{O}(1)-$distant from the poles $\pm i\pi/2$, but it does not give us an accurate approximation near the poles. On the other hand, the inner solution $\varepsilon^{-1}\phi^{0,\star}(\varepsilon^{-1}(y - i\pi/2), \tau)$ obtained in Theorem (8.3.2) approximates $W^\star(0)$ near the pole $y = i\pi/2$ with good bounds, but it is not a good approximation for real values of y.

In light of this, we perform the complex matching near the pole $y = i\pi/2$ between the outer solution $v^\star(y,\tau)$ and the inner solution $\varepsilon^{-1}\phi^{0,\star}(\varepsilon^{-1}(y - i\pi/2), \tau)$, in order to take advantage of the good properties of these solutions.

Take $0 < \beta_1 < \beta < \beta_2 < \pi/4$ constants independents of ε and κ, and define $y_j \in \mathbb{C}$, $j = 1, 2$ as the two points satisfying

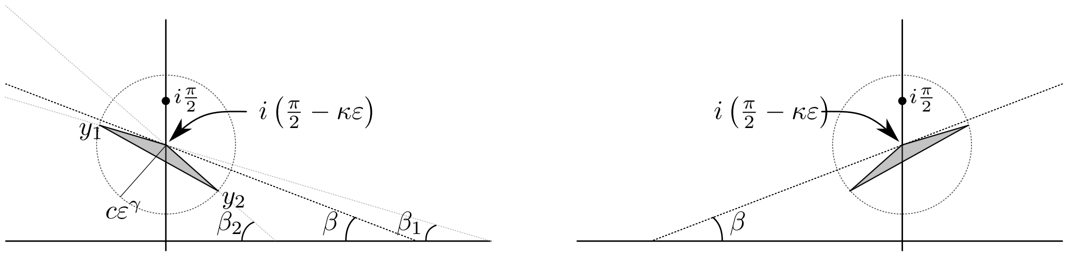


Figure 8.5: Matching domains $D^{\mathrm{mch},u}_{+,\kappa}$ (on the left) and $D^{\mathrm{mch},s}_{+,\kappa}$ (on the right).

1. $\mathrm{Im}(y_j) = -\tan\beta_j \,\mathrm{Re}(y_j) + \pi/2 - \kappa\varepsilon$;
2. $|y_j - i(\pi/2-\kappa\varepsilon)| = c\varepsilon^\gamma$, where $\gamma \in (0,1)$ and $c > 0$ are constants independent of ε and κ;
3. $\mathrm{Re}(y_1) < 0$ and $\mathrm{Re}(y_2) > 0$;
4. $\mathrm{Im}(z_1) \neq \mathrm{Im}(z_2)$;
5. $\|y_1\| \neq \|y_2\|$.

Consider the following **matching domains** for $\gamma \in (0,1)$ (see Figure 8.5),

$$\begin{aligned} D^{\mathrm{mch},u}_{+,\kappa} &= \{y\in\mathbb{C};\ \mathrm{Im}(y) \le -\tan\beta_1 \mathrm{Re}(y) + \pi/2 - \kappa\varepsilon,\ \mathrm{Im}(y) \le -\tan\beta_2 \mathrm{Re}(y) + \pi/2-\kappa\varepsilon, \\ &\qquad \left.\mathrm{Im}(y) \ge \mathrm{Im}(y_1) - \tan\left(\frac{\beta_1+\beta_2}{2}\right)(\mathrm{Re}(y) - \mathrm{Re}(y_1))\right\}, \\ D^{\mathrm{mch},s}_{+,\kappa} &= \left\{y \in \mathbb{C}; -\overline{y} \in D^{\mathrm{mch},u}_{+,\kappa}\right\}. \end{aligned} \tag{8.3.10}$$

Notice that there exist constants $M_1, M_2 > 0$ independent of ε and κ such that

$$M_1\varepsilon^\gamma \le |y_j - i\pi/2| \le M_2\varepsilon^\gamma,\ j=1,2,$$

and for $y \in D^{\mathrm{mch},u}_{+,\kappa}$,

$$M_1\kappa\varepsilon \le |y - i\pi/2| \le M_2\varepsilon^\gamma.$$

In terms of the inner variable z (see (8.3.3)), we obtain that

$$M_1\kappa \le |z| \le M_2\varepsilon^{\gamma-1},\ |z| \le M_2|z_j|,\ j=1,2,\ \forall z \in D^{\mathrm{mch},u}_{+,\kappa},$$

where z_1 and z_2 are the vertices of the inner domain y_1 and y_2, respectively, expressed in the inner variable.

Theorem 8.3.3 (Matching). *Fix $\gamma \in (0,1)$. Let $\phi^\star(z,\tau) = \varepsilon v^\star(i\pi/2 + \varepsilon z, \tau)$, $\star = u, s$, where $v^\star$ is the parameterization obtained in Theorem 8.3.1. Then, there exist $\varepsilon_0 > 0$ and $\kappa_0 \ge 1$ such that, for each $\varepsilon \le \varepsilon_0$, $\kappa \ge \kappa_0$, and $z \in D^{\mathrm{mch},+,\star}_{\kappa,c}$,*

$$\phi^\star(z,\tau) = \phi^{0,\star}(z,\tau) + \varphi^\star(z,\tau),$$

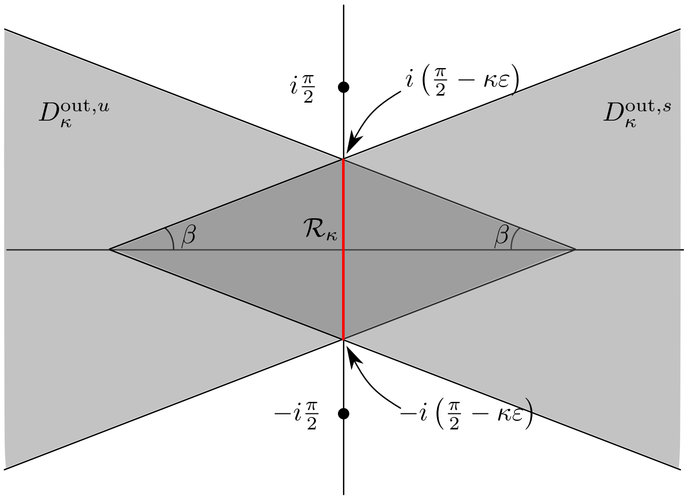


Figure 8.6: Domain $\mathcal{R}_\kappa$.

where $\phi^{0,\star}$ *is the solution of the inner equation* (8.3.6) *obtained in Theorem 8.3.2, and there exists a constant* $M_3 > 0$ *independent of* ε *and* κ *such that*

$$\|\varphi^\star\|_{\ell_1}(z), \|\partial_\tau \varphi^\star\|_{\ell_1}(z) \leq \frac{M_3(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1})}{|z|^2} \quad \text{and} \quad \|\partial_z \varphi^\star\|_{\ell_1}(z) \leq \frac{M_3(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1})}{\kappa|z|^2},$$

for every $z \in D_{\kappa,c}^{\mathrm{mch},+,\star}$.

Finally, we study the difference $\Delta v(y,\tau) = v^u(y,\tau) - v^s(y,\tau)$, between the solutions obtained in Theorem 8.3.1 in the domain

$$\mathcal{R}_\kappa = D_\kappa^{\mathrm{out},u} \cap D_\kappa^{\mathrm{out},s} \cap i\mathbb{R},$$

illustrated in Figure 8.6.

From Theorem 8.3.3, we have that, near the poles, the solutions v^u and v^s are well approximated by the solutions $\phi^{0,u}$ and $\phi^{0,s}$ of the inner equation 8.3.6 given by Theorem 8.3.2, respectively. Therefore, the asymptotic formula for the difference $\Delta\phi_0$ given by Theorem 8.3.2, provides a first order for the total difference Δv near the poles. In Section 8.7, we use functional analysis to show that the knowledge of the asymptotic behavior of the difference near the poles induces a first order for the total difference. This will conclude the proof of Theorem U.

8.4 Proof of Theorem 8.3.1

We prove Theorem 8.3.1 by setting a fixed point argument. In order to do this, we replace $v(y,\tau) = \sum_{n\geq 1} v_n(y)\sin(n\tau)$ into (8.1.9), and write the equation in sine Fourier expansion as

$$\begin{cases} \ddot{v}_1 = v_1 - \dfrac{v_1^3}{4} + \left(-\dfrac{1}{\varepsilon^3}\Pi_1\left[g(\varepsilon v)\right] + \dfrac{v_1^3}{4}\right), \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\varepsilon^2} v_n - \dfrac{1}{\varepsilon^3}\Pi_n\left[g(\varepsilon v)\right], \ n \geq 2, \end{cases} \tag{8.4.1}$$

where Π_n is the Fourier projection given by (8.1.7) and g is given by (8.1.6).

We study the invariant manifolds $W^{u,s}(0)$ of (8.1.9) as a perturbation of the homoclinic orbit $v_1^h(y)\sin(\tau)$ given by (8.1.17). Thus, we set

$$\xi(y,\tau) = v(y,\tau) - v_1^h(y)\sin(\tau) = \sum_{n\geq 1} \xi_n(y)\sin(n\tau),$$

and thus, system (8.4.1) is brought into

$$\begin{cases} \ddot{\xi}_1 = \xi_1 - \dfrac{3(v_1^h)^2\xi_1}{4} - \dfrac{3v_1^h\xi_1^2}{4} - \dfrac{\xi_1^3}{4} + \left(-\dfrac{1}{\varepsilon^3}\pi_1(g(\varepsilon(\xi + v_1^h\sin(\tau)))) + \dfrac{(\xi + v_1^h)^3}{4}\right), \\ \ddot{\xi}_n = -\dfrac{\lambda_n^2}{\varepsilon^2}\xi_n - \dfrac{1}{\varepsilon^3}\pi_n(g(\varepsilon(\xi + v_1^h\sin(\tau)))),\ n\geq 2. \end{cases}$$

Now, define the operators

$$\mathcal{L}(\xi) = \left(\ddot{\xi}_1 - \xi_1 + \frac{3(v_1^h)^2\xi_1}{4}\right)\sin(\tau) + \sum_{n\geq 2}\left(\ddot{\xi}_n + \frac{\lambda_n^2}{\varepsilon^2}\xi_n\right)\sin(n\tau), \tag{8.4.2}$$

and

$$\mathcal{F}(\xi) = -\frac{1}{\varepsilon^3}g(\varepsilon(\xi + v_1^h\sin(\tau))) + \left(\frac{(\xi_1 + v_1^h)^3}{4} - \frac{3v_1^h\xi_1^2}{4} - \frac{\xi_1^3}{4}\right)\sin(\tau), \tag{8.4.3}$$

and notice that, for $\star = u, s$, to find a solution $v^\star$ of (8.1.9) satisfying (8.1.13) is equivalent to find a fixed point $\xi^\star$ of the functional equation

$$\mathcal{L}(\xi) = \mathcal{F}(\xi), \tag{8.4.4}$$

which satisfies

$$\lim_{y\to-\infty}\xi^u(y,\tau) = \lim_{y\to\infty}\xi^s(y,\tau) = 0,\ \forall\tau\in\mathbb{T}. \tag{8.4.5}$$

In the remainder of this section, we find a fixed point of (8.4.4) in some appropriate Banach space. We consider only the unstable case, since the stable one is completely analogous.

8.4.1 Banach Spaces and Linear Operators

First, we set the Banach spaces we will work with to invert the operator $\mathcal{L}$ given in (8.4.2).

Given $\kappa \geq 1$ and a real-analytic function $h : D_\kappa^{\text{out},u} \to \mathbb{C}$, we define

$$\|h\|_{\kappa,m,\alpha} = \sup_{y\in D_\kappa^{\text{out},u}\cap\{\text{Re}(y)\leq -1\}} |\cosh(y)^m h(y)| + \sup_{y\in D_\kappa^{\text{out},u}\cap\{\text{Re}(y)\geq -1\}} |(y^2+\pi^2/4)^\alpha h(y)|, \tag{8.4.6}$$

and given a function $\xi : D_\kappa^{\text{out},u}\times\mathbb{T}\to\mathbb{C}$ which is 2π-periodic in $\tau\in\mathbb{T}$ and real analytic in $y\in D_\kappa^{\text{out},u}$, we define

$$\|\xi\|_{\ell_1,\kappa,m,\alpha} = \sum_{n\geq 1}\|\Pi_n[\xi]\|_{\kappa,m,\alpha},$$

where Π_n is given by (8.1.7).

Consider the Banach spaces

$$\mathcal{E}_{\kappa,m,\alpha} = \{\xi : D_\kappa^{\text{out},u}\to\mathbb{C};\ \xi \text{ is real-analytic in } y, \text{ and } \|\xi\|_{\kappa,m,\alpha} < \infty\}.$$

and

$$\mathcal{E}_{\ell_1,\kappa,m,\alpha} = \{\xi : D_\kappa^{\mathrm{out},u}\times\mathbb{T}\to\mathbb{C};\ \xi(y,\tau) \text{ is real-analytic in } y, 2\pi\text{−periodic in } \tau, \text{ and } \|\xi\|_{\ell_1,\kappa,m,\alpha}<\infty\}.$$

If there is no misunderstanding about the domain $D_\kappa^{\mathrm{out},u}$ and the norm $\|\cdot\|_{\ell_1,\kappa,m,\alpha}$, we will not write the dependence on κ, and therefore

$$\|\cdot\|_{m,\alpha}= \|\cdot\|_{\kappa,m,\alpha},\ \|\cdot\|_{\ell^1,m,\alpha}= \|\cdot\|_{\ell^1,\kappa,m,\alpha},\ \mathcal{E}_{m,\alpha} = \mathcal{E}_{\kappa,m,\alpha},\ \text{and } \mathcal{E}_{\ell^1,m,\alpha} = \mathcal{E}_{\ell^1,\kappa,m,\alpha}.$$

Proposition 8.4.1. *Given an analytic function $f : B(R_0)\to\mathbb{C}$, and $g,h : D_\kappa^{\mathrm{out},u}\times\mathbb{T}_\sigma\to\mathbb{C}$, where $B(R_0)\subset\mathbb{C}$ is a ball with center at the origin and radius R_0, the following statements hold.*

1. *If $\alpha_2\geq\alpha_1\geq 0$, then*

$$\|h\|_{\ell_1,m,\alpha_2}\leq M\|h\|_{\ell_1,m,\alpha_1} \quad \text{and} \quad \|h\|_{\ell_1,m,\alpha_1}\leq \frac{M}{(\kappa\varepsilon)^{\alpha_2-\alpha_1}}\|h\|_{\ell_1,m,\alpha_2}.$$

2. *If $\alpha_1,\alpha_2\geq 0$, and $\|g\|_{\ell_1,m,\alpha_1}, \|h\|_{\ell_1,m,\alpha_2}<\infty$, then*

$$\|gh\|_{\ell_1,m,\alpha_1+\alpha_2}\leq \|g\|_{\ell_1,m,\alpha_1}\|h\|_{\ell_1,m,\alpha_2}.$$

3. *If $\|g\|_{\ell_1,m,\alpha}$, $\|h\|_{\ell_1,m,\alpha}\leq R_0/4$, then*

$$\|f(g)-f(h)\|_{\ell_1,m,\alpha}\leq M\|g-h\|_{\ell_1,m,\alpha}.$$

4. *Given $n\geq 1$, if $f^{(k)}(0)=0$, for every $1\leq k\leq n-1$, and $\|g\|_{\ell_1,\alpha}\leq R_0/4$, where R_0 is the convergence ratio of $f^{(n)}$ at 0, then*

$$\|f(g)\|_{\ell_1,m,n\alpha}\leq M(\|g\|_{\ell_1,m,\alpha})^n.$$

Consider

$$\zeta_1(y) = -2\sqrt{2}\frac{\sinh(y)}{\cosh^2(y)} \quad \text{and} \quad \zeta_2(y) = -\frac{\sqrt{2}}{16}\frac{\sinh(y)}{\cosh^2(y)}(6y-4\coth(y)+\sinh(2y)), \tag{8.4.7}$$

and define the operator $\mathcal{G}(\xi)$ acting on the Fourier coefficients of ξ as

$$\mathcal{G}(\xi) = \sum_{n\geq 1}\mathcal{G}_n(\xi_n)\sin(n\tau),$$

where

$$\mathcal{G}_1(\xi_1) = -\zeta_1(y)\int_0^y \zeta_2(s)\xi_1(s)ds + \zeta_2(y)\int_{-\infty}^y \zeta_1(s)\xi_1(s)ds, \tag{8.4.8}$$

and

$$\mathcal{G}_n(\xi_n) = -\frac{i\varepsilon}{2\lambda_n}e^{i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{-i\frac{\lambda_n}{\varepsilon}s}\xi_n(s)ds + \frac{i\varepsilon}{2\lambda_n}e^{-i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{i\frac{\lambda_n}{\varepsilon}s}\xi_n(s)ds,\ n\geq 2. \tag{8.4.9}$$

Proposition 8.4.2. *Consider $\kappa\geq 1$. Given $\alpha\geq 5$ and $m>1$, the operator*

$$\mathcal{G} : \mathcal{E}_{\ell_1,m,\alpha}\to\mathcal{E}_{\ell_1,1,\alpha-2}$$

is well defined and the following statements hold.

1. *$\partial_y \pi_1(\mathcal{G}(\xi))(0) = 0$.*

2. *$\mathcal{G} \circ \mathcal{L}(\xi) = \mathcal{L} \circ \mathcal{G}(\xi) = \xi$.*

3. *There exists a constant $M > 0$ independent of ε and κ such that, for every $\xi \in \mathcal{E}_{\ell_1,\alpha}$,*

$$\|\mathcal{G}(\xi)\|_{\ell_1,1,\alpha-2} \leq M\|\xi\|_{\ell_1,m,\alpha}. \tag{8.4.10}$$

Moreover, given $l, \beta \geq 0$ and denoting $\mathcal{E}^1_{\ell_1,l,\beta} = \{\xi \in \mathcal{E}_{\ell_1,l,\beta};\ \pi_1(\xi) = 0\}$, then

$$\mathcal{G} : \mathcal{E}^1_{\ell_1,l,\beta} \to \mathcal{E}^1_{\ell_1,l,\beta}$$

is well defined and, for every $\xi \in \mathcal{E}^1_{\ell_1,l,\beta}$,

$$\|\mathcal{G}(\xi)\|_{\ell_1,l,\beta} \leq M\varepsilon^2\|\xi\|_{\ell_1,l,\beta}. \tag{8.4.11}$$

4. *The operators $\partial_\tau \circ \mathcal{G}$ and $\partial^2_\tau \circ \mathcal{G}$ are well defined and satisfy* (8.4.10)*.*

5. *The operator $\partial_y \circ \mathcal{G} : \mathcal{E}_{\ell_1,m,\alpha} \to \mathcal{E}_{\ell_1,1,\alpha-1}$ is well defined and*

$$\|\partial_y \circ \mathcal{G}(\xi)\|_{\ell_1,1,\alpha-1} \leq M\|\xi\|_{\ell_1,m,\alpha}.$$

Proof. Despite when it is said, M will denote any constant independent of κ and ε.

First, we must construct an inverse for the operator $\mathcal{L}$ given in (8.4.2). Since $\mathcal{L}$ acts on the Fourier coefficients of ξ, it is sufficient to construct inverses of the operators $\Pi_n \circ \mathcal{L}$, which will be denoted by $\mathcal{L}_n$, to obtain an inverse for $\mathcal{L}$.

From (8.4.5), we have that $\xi(y,\tau)$ satisfies $\xi(y,\tau) \to 0$ as $\mathrm{Re}(y) \to -\infty$ for every τ. Thus, we must look for $\zeta(y)$ such that $\mathcal{L}_n(\zeta) = h$ and

$$\lim_{\mathrm{Re}(y)\to-\infty} \zeta(y) = 0, \tag{8.4.12}$$

where $h : D^{\mathrm{out},u}_\kappa \to \mathbb{C}$ is a real-analytic function.

First, consider $n \geq 2$. Notice that the homogeneous equation $\mathcal{L}_n(\zeta) = 0$ has $\zeta_{1,n}(y) = e^{i\frac{\lambda_n}{\varepsilon}y}$ and $\zeta_{2,n}(y) = e^{-i\frac{\lambda_n}{\varepsilon}y}$ as fundamental solutions. Thus, using the variation of constants formula, we obtain that the solutions of $\mathcal{L}_n(\zeta) = h$ are given by

$$\zeta(y) = -\frac{i\varepsilon}{2\lambda_n}e^{i\frac{\lambda_n}{\varepsilon}y}\left(\int_{y_0}^{y} e^{-i\frac{\lambda_n}{\varepsilon}s}h(s)ds - C_0\right) + \frac{i\varepsilon}{2\lambda_n}e^{-i\frac{\lambda_n}{\varepsilon}y}\left(\int_{y_1}^{y} e^{i\frac{\lambda_n}{\varepsilon}s}h(s)ds - C_1\right),$$

where C_0, C_1, y_0, y_1 are constants.

Using (8.4.12), we have that

$$C_0 = -\int_{y_0}^{-\infty} e^{-i\frac{\lambda_n}{\varepsilon}s}h(s)ds \quad \textit{and} \quad C_1 = -\int_{y_1}^{-\infty} e^{i\frac{\lambda_n}{\varepsilon}s}h(s)ds,$$

and notice that both integrals in the definition of C_0 and C_1 are convergent since $h \in \mathcal{E}_{\ell_1,m,\alpha}$, with $m > 1$.

Thus, $\zeta = \mathcal{G}_n(h)$, where $\mathcal{G}_n$ is given by (8.4.9), $n \geq 2$. Now, we construct an inverse for $\mathcal{L}_1$. In this case, notice that the homogeneous equation $\mathcal{L}_1(\zeta) = 0$ is the variational equation of the solution v^h_1, thus it follows that $\zeta_1 = \dot{v}^h_1$ is a solution of this equation, which is given in (8.4.7).

Now, applying the reduction of order technique, we obtain another solution ζ_2 which is linearly independent from ζ_1 and since we are dealing with a second order differential equation, $\{\zeta_1, \zeta_2\}$ is a fundamental set of solutions of the homogeneous problem. In this case

$$\zeta_2(y) = \zeta_1(y)\left(\int_{y_0}^{y} \frac{1}{\zeta_1^2(s)} ds + A\right),$$

where A, y_0 are constants.Choosing A, y_0 such that $\dot{\zeta}_2(0) = 0$, we obtain that ζ_2 is given in (8.4.7).

Notice that these solutions satisfy the asymptotic conditions

$$\lim_{\mathrm{Re}(y)\to-\infty} \zeta_1(y) = 0 \text{ and } \lim_{\mathrm{Re}(y)\to-\infty} \zeta_2(y) = -\infty,$$

and their Wronskian is given by $W(\zeta_1, \zeta_2) = 1$. Again, using the variation of constants formula, we obtain that the solutions of $\mathcal{L}_1(\zeta) = h$ are given by

$$\zeta(y) = -\zeta_1(y)\left(\int_{y_0}^{y} \zeta_2(s)h(s)ds - C_0\right) + \zeta_2(y)\left(\int_{y_1}^{y} \zeta_1(s)h(s)ds - C_1\right),$$

where C_0, C_1, y_0, y_1 are constants.

Now, using (8.4.12), it follows that

$$C_1 = -\int_{y_1}^{-\infty} \zeta_1(s)h(s)ds.$$

Also, we choose

$$C_0 = -\int_{y_0}^{0} \zeta_1(s)h(s)ds,$$

in order to have $\dot{\zeta}(0) = 0$.

Thus $\zeta = \mathcal{G}_1(h)$, where $\mathcal{G}_1$ is given by (8.4.8) and notice that $\mathcal{G}_1$ is an inverse of $\mathcal{L}_1$ such that $\partial_y \mathcal{G}_1(h)(0) = 0$ and $\mathcal{G}_1(h)(y)$ decays at infinity. This proves items (1) and (2) of this proposition.

Now, to prove item (3), let $h \in \mathcal{E}_{m,\alpha}$, and consider the twisted path $w = e^{\mp i\beta}\eta$. Thus, if $y \in D_\kappa^{out,u}$, then

$$\begin{aligned}
\int_{-\infty}^{y} e^{\pm i\frac{\lambda_n}{\varepsilon}(s-y)} h(s)ds &= \int_{-\infty}^{0} e^{\pm i\frac{\lambda_n}{\varepsilon}w} h(w+y)dw \\
&= \int_{-\infty}^{0} e^{\pm i\frac{\lambda_n}{\varepsilon}e^{\mp i\beta}\eta} h(e^{\mp i\beta}\eta + y)e^{\mp i\beta}d\eta.
\end{aligned}$$

If $\mathrm{Re}(y) \leq -1$, then

$$\begin{aligned}
\left|\cosh^m(y)\int_{-\infty}^{y} e^{\pm i\frac{\lambda_n}{\varepsilon}(s-y)} h(s)ds\right| &\leq \|h\|_{m,\alpha}\int_{-\infty}^{0} e^{\frac{\lambda_n}{\varepsilon}\sin(\beta)\eta}\left|\frac{\cosh(y)}{\cosh(e^{\mp i\beta}\eta + y)}\right|^m d\eta \\
&\leq M\|h\|_{m,\alpha}\int_{-\infty}^{0} e^{\frac{\lambda_n}{\varepsilon}\sin(\beta)\eta}d\eta \\
&\leq M\frac{\varepsilon}{\lambda_n}\|h\|_{m,\alpha}.
\end{aligned}$$

If $\mathrm{Re}(y) \geq -1$, then let $\rho^*(y) \leq 0$ be such that

$$\mathrm{Re}(e^{\mp i\beta}\rho^*(y) + y) = -1.$$

In this case, we have that

$$\begin{aligned}
\left|(y^2+\pi^2/4)^\alpha \int_{-\infty}^{y} e^{\pm i\frac{\lambda_n}{\varepsilon}(s-y)}h(s)ds\right| &\leq \left|(y^2+\pi^2/4)^\alpha \int_{-\infty}^{\rho^*(y)} e^{\pm i\frac{\lambda_n}{\varepsilon}e^{\mp i\beta}\eta}h(e^{\mp i\beta}\eta+y)e^{\mp i\beta}d\eta\right| \\
&+ \left|(y^2+\pi^2/4)^\alpha \int_{\rho^*(y)}^{0} e^{\pm i\frac{\lambda_n}{\varepsilon}e^{\mp i\beta}\eta}h(e^{\mp i\beta}\eta+y)e^{\mp i\beta}d\eta\right| \\
&\leq \|h\|_{m,\alpha}\int_{-\infty}^{\rho^*(y)} e^{\frac{\lambda_n}{\varepsilon}\sin(\beta)\eta}\left|\frac{(y^2+\pi^2/4)^\alpha}{\cosh^m(e^{\mp i\beta}\eta+y)}\right| d\eta \\
&+ \|h\|_{m,\alpha}\int_{\rho^*(y)}^{0} e^{\frac{\lambda_n}{\varepsilon}\sin(\beta)\eta}\left|\frac{y^2+\pi^2/4}{(e^{\mp i\beta}\eta+y)^2+\pi^2/4}\right|^\alpha d\eta \\
&\leq M\|h\|_{m,\alpha}\int_{-\infty}^{0} e^{\frac{\lambda_n}{\varepsilon}\sin(\beta)\eta}d\eta \\
&\leq M\frac{\varepsilon}{\lambda_n}\|h\|_{m,\alpha}.
\end{aligned}$$

Notice that $M>0$ is a constant independent of ε, κ and n, and thus it follows that

$$\|\mathcal{G}_n(h)\|_{m,\alpha}\leq \frac{M\varepsilon^2}{\lambda_n^2}\|h\|_{m,\alpha},\ n\geq 2. \tag{8.4.13}$$

As we have seen before, the operator $\mathcal{G}_1$ has to be considered separately since it demands a special attention. Roughly speaking, we have no exponentials in the integrals of (8.4.8), and thus the technique used to show the estimates above can not be applied to this case.

First, we bound $\mathcal{G}_1(h)(y)$ for values of y in $D_\kappa^{\text{out},u}\cap\{\text{Re}(y)\leq -1\}$. Notice that the functions $\zeta_1(y),\zeta_2(y)$ given in (8.4.7) satisfy

$$|\zeta_1(y)|\leq \frac{M}{|\cosh(y)|} \text{ and } |\zeta_2(y)|\leq M|\cosh(y)|, \tag{8.4.14}$$

for every $y\in D_\kappa^{\text{out},u}\cap\{y\in\mathbb{C};\ |\text{Im}(y)|\leq -K\,\text{Re}(y)\}$, where

$$K=\left(\tan(\beta)+\frac{\pi}{2}-\kappa\varepsilon\right).$$

The second integral in the following expression

$$\mathcal{G}_1(h)(y)=-\zeta_1(y)\int_0^y \zeta_2(s)h(s)ds+\zeta_2(y)\int_{-\infty}^{y}\zeta_1(s)h(s)ds.$$

satisfies

$$\begin{aligned}
\left|\int_{-\infty}^{y}\zeta_1(s)h(s)ds\right| &\leq \|h\|_{m,\alpha}\int_{-\infty}^{0}\frac{1}{|\cosh^{m+1}(s+y)|}ds \\
&\leq M\frac{\|h\|_{m,\alpha}}{|\cosh^m(y)|}\int_{-\infty}^{0}\frac{1}{|\cosh(s+y)|}ds \\
&\leq M\frac{\|h\|_{m,\alpha}}{|\cosh^m(y)|}\int_{-\infty}^{0}e^{s+y}ds \\
&\leq M\frac{\|h\|_{m,\alpha}}{|\cosh^{m+1}(y)|},
\end{aligned}$$

for every $y \in D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \leq -1\}$.

From (8.4.14), we conclude that

$$\left|\zeta_2(y)\int_{-\infty}^{y}\zeta_1(s)h(s)ds\right| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh^m(y)|}, \tag{8.4.15}$$

for every $y \in D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \leq -1\}$.

Now, let y^* be the unique point in the segment of line between 0 and y such that $\mathrm{Re}(y^*) = -1$. Hence, it follows from (8.4.14) that

1. if s is in the line between 0 and y^*, then

$$\begin{aligned}|\zeta_2(s)h(s)| &\leq \frac{M\|h\|_{m,\alpha}|\cosh(s)|}{|s^2+\pi^2/4|^\alpha}\\ &\leq M\|h\|_{m,\alpha}\sup_{s\in U}\frac{|\cosh(s)|}{|s^2+\pi^2/4|^\alpha}\\ &\leq M\|h\|_{m,\alpha}\end{aligned}$$

where $U = \{y \in \mathbb{C};\ |\mathrm{Im}(y)| \leq -K\,\mathrm{Re}(y) \text{ and } \mathrm{Re}(y) \geq -1\}$;

2. if s is in the line between y^* and y, then

$$|\zeta_2(s)h(s)| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh^{m-1}(s)|}.$$

Notice that $\mathrm{Re}(y^*) = -1$ and $|\mathrm{Im}(y^*)| \leq \left|\tan(\beta) + \frac{\pi}{2} - \kappa\varepsilon\right|$, thus since $m > 1$, we have that

$$\begin{aligned}\left|\int_0^y \zeta_2(s)h(s)ds\right| &\leq \left|\int_{y^*}^0 \zeta_2(s)h(s)ds\right| + \left|\int_y^{y^*}\zeta_2(s)h(s)ds\right|\\ &\leq M\|h\|_{m,\alpha}|y^*| + M\|h\|_{m,\alpha}\int_y^{y^*}\frac{1}{|\cosh^{m-1}(s)|}ds\\ &\leq M\|h\|_{m,\alpha} + M\|h\|_{m,\alpha}\int_{-\infty}^{y^*}\frac{1}{|\cosh^{m-1}(s)|}ds\\ &\leq M\|h\|_{m,\alpha}.\end{aligned}$$

Hence, it follows from (8.4.14) that

$$\left|\zeta_1(y)\int_0^y \zeta_2(s)h(s)ds\right| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh(y)|}, \tag{8.4.16}$$

for every $y \in D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \leq -1\}$.

Now, from (8.4.8), (8.4.15) and (8.4.16), we obtain that

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\leq -1\}} |\cosh(y)\mathcal{G}_1(h)(y)| \leq M\|h\|_{m,\alpha}. \tag{8.4.17}$$

The set of fundamental solutions $\{\zeta_1, \zeta_2\}$ given in (8.4.7) of the equation $\mathcal{L}_1(\zeta) = 0$ provides the operator $\mathcal{G}_1$ defined in (8.4.8) which is quite useful to bound $\mathcal{G}_1(h)(y)$ for values of y in $D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \leq -1\}$, as we have seen in (8.4.17).

Nevertheless, both solutions $\zeta_1(y)$, $\zeta_2(y)$ have poles of order 2 at $\pm i\pi/2$, and this might prevent us to see cancellations of poles in the domain $D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \geq -1\}$. In order to avoid this kind of problem, we consider a new set of fundamental solutions $\{\zeta_+, \zeta_-\}$ of $\mathcal{L}_1(\zeta) = 0$ which has good properties at $\pm i\pi/2$.

The main idea is to rewrite the solutions $\zeta_1(y)$ and $\zeta_2(y)$ as linear combinations of $\zeta_+(y)$ and $\zeta_-(y)$ and use them to obtain a new expression of the operator $\mathcal{G}_1$, which will be useful to bound $\mathcal{G}_1(h)(y)$ for values of y in $D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \geq -1\}$. We emphasize that the operator $\mathcal{G}_1$ is already defined. We will only express it in a different manner.

Since $\zeta_1(y)$ is a solution of $\mathcal{L}_1(\zeta) = 0$, it follows from the method of reduction of order that

$$\zeta_\pm(y) = \zeta_1(y) \int_{\pm i\frac{\pi}{2}}^{y} \frac{1}{\zeta_1^2(s)} ds, \tag{8.4.18}$$

are also solutions of the homogeneous equation. Notice that

$$\zeta_\pm(y) = -\frac{\sqrt{2}}{4} \frac{\sinh(y)}{\cosh^2(y)} \int_{\pm i\frac{\pi}{2}}^{y} \frac{\cosh^4(s)}{\sinh^2(s)} ds. \tag{8.4.19}$$

Claim: The solutions $\zeta_\pm : D_\kappa^{out,u} \to \mathbb{C}$ of $\mathcal{L}_1(\zeta) = 0$ given in (8.4.18) are well defined and they are given by

$$\zeta_\pm(y) = -\frac{\sqrt{2}}{4} \frac{1}{\cosh^2(y)} \left(\frac{3y \sinh(y)}{2} - \cosh(y) + \frac{1}{4} \sinh(y) \sinh(2y) \mp i \frac{3\pi}{4} \sinh(y) \right), \tag{8.4.20}$$

for each $y \in D_\kappa^{\mathrm{out},u}$. Furthermore,

- $\zeta_\pm$ are linearly independent and

$$W(\zeta_+, \zeta_-) = \zeta_+ \dot{\zeta}_- - \zeta_- \dot{\zeta}_+ = -i\frac{3\pi}{16}. \tag{8.4.21}$$

- There exist uniformly bounded (with respect to ε and κ) analytic functions $\xi_\pm : D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \geq -1\} \to \mathbb{C}$ such that

$$\zeta_\pm(y) = \frac{(y \mp i\pi/2)^3}{(y \pm i\pi/2)^2} \xi_\pm(y), \tag{8.4.22}$$

  for each $y \in D_\kappa^{\mathrm{out},u} \cap \{\mathrm{Re}(y) \geq -1\}$.

Proof: The integrand of (8.4.19) is analytic in $D_\kappa^{out,u} \setminus \{0\}$, and thus, $\zeta_\pm(y)$ is well defined for each $y \in D_\kappa^{out,u} \setminus \{0\}$.

Now, $F(s) = \dfrac{3s}{2} - \coth(s) + \dfrac{1}{4}\sinh(2s)$ is a primitive of $f(s) = \dfrac{\cosh^4(s)}{\sinh^2(s)}$, for every $s \in \mathbb{C} \setminus (\{2k\pi;\ k \in \mathbb{Z}\} \cup \{i\pi + 2k\pi;\ k \in \mathbb{Z}\})$.

Thus, for each $y \in D_\kappa^{out,u} \setminus \{0\}$, we have that

$$\int_{\pm i\frac{\pi}{2}}^{y} \frac{\cosh^4(s)}{\sinh^2(s)} ds = \frac{3y}{2} - \coth(y) + \frac{1}{4}\sinh(2y) \mp i\frac{3\pi}{4}.$$

It follows from (8.4.19) that (8.4.20) holds for each $y \in D_\kappa^{out,u} \setminus \{0\}$. From the expression in (8.4.20), both $\zeta_\pm(y)$ can be analytically extended to $D_\kappa^{out,u}$ by defining $\zeta_\pm(0) = \dfrac{\sqrt{2}}{4}$.

A straightforward computation shows (8.4.21) and thus $\zeta_\pm$ are linearly independent. A simple analysis of (8.4.20) allows us to conclude (8.4.22). ■

Since $\{\zeta_+, \zeta_-\}$ is a fundamental set of solutions of $\mathcal{L}_1(\zeta) = 0$, we are able to rewrite the operator $\mathcal{G}_1$ given in (8.4.8) in terms of ζ_+ and ζ_-.
Claim: Given $\kappa \geq 1$, consider the operator $\mathcal{G}_1$ given by (8.4.8). Let $h : D_\kappa^{\mathrm{out,u}} \to \mathbb{C}$ be a real-analytic function, then

$$\mathcal{G}_1(h) = i\frac{16}{3\pi}\left(-\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds, \tag{8.4.23}$$

where ζ_1, ζ_2 and $\zeta_\pm$ are given in (8.4.7) and (8.4.20), respectively.
<u>Proof:</u> In fact, using the expressions of ζ_1, ζ_2 and $\zeta_\pm$ in (8.4.7) and (8.4.20), we can see that

$$\zeta_1(y) = i\frac{16}{3\pi}(\zeta_+(y) - \zeta_-(y)) \text{ and } \zeta_2(y) = \frac{\zeta_+(y) + \zeta_-(y)}{2}. \tag{8.4.24}$$

From definition of $\mathcal{G}_1$ in (8.4.8) and (8.4.24), we have

$$\begin{aligned}
\mathcal{G}_1(h) &= -i\frac{8}{3\pi}\zeta_+(y)\int_0^y \zeta_+(s)h(s)ds - i\frac{8}{3\pi}\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds \\
&\quad +i\frac{8}{3\pi}\zeta_-(y)\int_0^y \zeta_+(s)h(s)ds + i\frac{8}{3\pi}\zeta_-(y)\int_0^y \zeta_-(s)h(s)ds \\
&\quad +i\frac{8}{3\pi}\zeta_+(y)\int_{-\infty}^y \zeta_+(s)h(s)ds - i\frac{8}{3\pi}\zeta_+(y)\int_{-\infty}^y \zeta_-(s)h(s)ds \\
&\quad +i\frac{8}{3\pi}\zeta_-(y)\int_{-\infty}^y \zeta_+(s)h(s)ds - i\frac{2}{3\pi}\zeta_-(y)\int_{-\infty}^y \zeta_-(s)h(s)ds \\
&= -i\frac{8}{3\pi}\zeta_+(y)\left(\int_0^y \zeta_-(s)h(s)ds + \int_{-\infty}^y \zeta_-(s)h(s)ds\right) \\
&\quad +i\frac{8}{3\pi}\zeta_-(y)\left(\int_0^y \zeta_+(s)h(s)ds + \int_{-\infty}^y \zeta_+(s)h(s)ds\right) \\
&\quad +i\frac{8}{3\pi}\zeta_+(y)\int_{-\infty}^0 \zeta_+(s)h(s)ds - i\frac{8}{3\pi}\zeta_-(y)\int_{-\infty}^0 \zeta_-(s)h(s)ds \\
&= -i\frac{8}{3\pi}\zeta_+(y)\left(2\int_0^y \zeta_-(s)h(s)ds + \int_{-\infty}^0 \zeta_-(s)h(s)ds\right) \\
&\quad +i\frac{8}{3\pi}\zeta_-(y)\left(2\int_0^y \zeta_+(s)h(s)ds + \int_{-\infty}^0 \zeta_+(s)h(s)ds\right) \\
&\quad +i\frac{8}{3\pi}\zeta_+(y)\int_{-\infty}^0 \zeta_+(s)h(s)ds - i\frac{2}{3\pi}\zeta_-(y)\int_{-\infty}^0 \zeta_-(s)h(s)ds \\
&= i\frac{16}{3\pi}\left(-\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right) \\
&\quad +\frac{\zeta_+(y) + \zeta_-(y)}{2}\int_{-\infty}^0 i\frac{16}{3\pi}(\zeta_+(s) - \zeta_-(s))h(s)ds \\
&= i\frac{16}{3\pi}\left(-\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds.
\end{aligned}$$

■

Now, let $y \in D_\kappa^{\mathrm{out},u}$ satisfying $\mathrm{Re}(y) \geq -\rho$, and use (8.4.23) to write $\mathcal{G}_1(h)$ as

$$\mathcal{G}_1(h)(y) = i\frac{16}{3\pi}\left(-\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds.$$

First, notice that we can use (8.4.14) to see that

$$\begin{aligned}\left|\int_{-\infty}^{0}\zeta_1(s)h(s)ds\right| &\leq M\|h\|_{m,\alpha}\left(\int_{-\infty}^{-1}\frac{1}{|\cosh^{m+1}(s)|}ds+\int_{-1}^{0}\frac{1}{|\cosh(s)(s^2+\pi^2/4)^{\alpha}|}ds\right)\\ &\leq M\|h\|_{m,\alpha}.\end{aligned}$$

From the expression of $\zeta_2(y)$ given in (8.4.7), we have that $\zeta_2(y)$ has poles only at $\pm i\pi/2+i2k\pi$, $k\in\mathbb{Z}$, and they have order 2. Since $\alpha\geq 5$, it follows that

$$\sup_{y\in D_{\kappa}^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_2(y)\int_{-\infty}^{0}\zeta_1(s)h(s)ds\right|\leq M\|h\|_{m,\alpha}. \tag{8.4.25}$$

Now, we use that $\alpha\geq 5$ and equation (8.4.22) to see that

$$\begin{aligned}\left|\zeta_+(y)\int_0^y\zeta_-(s)h(s)ds\right| &\leq M\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\int_0^y\frac{|s+i\pi/2|^3}{|s-i\pi/2|^2}|h(s)|ds\\ &\leq M\|h\|_{m,\alpha}\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\int_0^y\frac{1}{|s+i\pi/2|^{\alpha-3}|s-i\pi/2|^{\alpha+2}}ds\\ &\leq M\|h\|_{m,\alpha}\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\left(\int_0^y\frac{1}{|s+i\pi/2|^{\alpha-3}}ds+\int_0^y\frac{1}{|s-i\pi/2|^{\alpha+2}}ds\right)\\ &\leq M\|h\|_{m,\alpha}\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\left(\frac{1}{|y+i\pi/2|^{\alpha-5}}\int_0^y\frac{1}{|s+i\pi/2|^2}ds\right.\\ &\quad\left.+\frac{1}{|y-i\pi/2|^{\alpha}}\int_0^y\frac{1}{|s-i\pi/2|^2}ds\right)\\ &\leq M\|h\|_{m,\alpha}\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\frac{1}{|y+i\pi/2|^{\alpha-4}|y-i\pi/2|^{\alpha+1}}\\ &\leq \frac{M\|h\|_{m,\alpha}}{|y^2+\pi^2/4|^{\alpha-2}}.\end{aligned}$$

We conclude that

$$\sup_{y\in D_{\kappa}^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_+(y)\int_0^y\zeta_-(s)h(s)ds\right|\leq M\|h\|_{m,\alpha}. \tag{8.4.26}$$

In a similar way, we can prove that

$$\sup_{y\in D_{\kappa}^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_-(y)\int_0^y\zeta_+(s)h(s)ds\right|\leq M\|h\|_{m,\alpha}. \tag{8.4.27}$$

It follows from (8.4.23), (8.4.25), (8.4.26) and (8.4.27) that

$$\sup_{y\in D_{\kappa}^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\mathcal{G}_1(h)(y)\right|\leq M\|h\|_{m,\alpha}. \tag{8.4.28}$$

Hence, using (8.4.6), (8.4.17) and (8.4.28) we obtain

$$\|\mathcal{G}_1(h)\|_{1,\alpha-2}\leq M\|h\|_{m,\alpha}.$$

Thus, item (3) is proved. To prove item (4) it is sufficient to use (8.4.13) and remark that

$$\Pi_n[\partial_\tau\circ\mathcal{G}(h)]=n\Pi_n[\mathcal{G}(h)]\quad \mathit{and}\quad \Pi_n[\partial_\tau^2\circ\mathcal{G}(h)]=n^2\Pi_n[\mathcal{G}(h)].$$

Finally, notice that

$$\partial_y \circ \mathcal{G}_n(h) = \frac{1}{2} e^{i\frac{\lambda_n}{\varepsilon} y} \int_{-\infty}^{y} e^{-i\frac{\lambda_n}{\varepsilon} s} h(s) ds + \frac{1}{2} e^{-i\frac{\lambda_n}{\varepsilon} y} \int_{-\infty}^{y} e^{i\frac{\lambda_n}{\varepsilon} s} h(s) ds, \ n \geq 2,$$

and thus, following the same ideas previously used in this proof, we obtain

$$\|\partial_y \circ \mathcal{G}_n(h)\|_{m,\alpha} \leq \frac{M\varepsilon}{\lambda_n} \|h\|_{m,\alpha}, \ n \geq 2.$$

Also,

$$\partial_y \circ \mathcal{G}_1(h) = i\frac{16}{3\pi}\left(-\zeta_+'(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-'(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2'(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds,$$

and

$$\zeta_\pm'(y) = \frac{(y \mp i\pi/2)^2}{(y \pm i\pi/2)^3}\widehat{\xi_\pm}(y),$$

where $\widehat{\xi_\pm}$ are analytic functions. Hence, since ξ_2' has a pole of order 1 at $\pm i\pi/2 + i2k\pi, \ k \in \mathbb{Z}$, we follow the same ideas above to obtain

$$\|\partial_y \circ \mathcal{G}_1(h)\|_{1,\alpha-1} \leq M\|h\|_{m,\alpha}.$$

This proves item (5). □

8.4.2 Fixed Point Argument

Now, we use Proposition 8.4.2 to rewrite (8.4.4) as

$$\xi = \mathcal{G} \circ \mathcal{F}(\xi),$$

and in the following proposition we study some properties of the operator

$$\widetilde{\mathcal{F}} = \mathcal{G} \circ \mathcal{F}. \tag{8.4.29}$$

Proposition 8.4.3. *Consider $\kappa \geq 1$. The following statements hold.*

1. *There exists a constant $M_1 > 0$ independent of ε and κ such that, for ε sufficiently small,*
$$\|\widetilde{\mathcal{F}}(0)\|_{\ell^1,1,3} \leq M_1\varepsilon^2.$$

2. *Given $R > 0$, there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$, the operator*
$$\widetilde{\mathcal{F}} : \mathcal{B}_0(R\varepsilon^2) \subset \mathcal{E}_{\ell^1,1,3} \to \mathcal{E}_{\ell^1,1,3}$$
is well defined, and there exists a constant $M_2 > 0$ independent of ε and κ such that, for every $\xi, \xi' \in \mathcal{B}_0(R\varepsilon^2) \subset \mathcal{E}_{\ell^1,1,3}$ and $0 < \varepsilon \leq \varepsilon_0$,
$$\|\widetilde{\mathcal{F}}(\xi) - \widetilde{\mathcal{F}}(\xi')\|_{\ell_1,1,3} \leq M_2\left(\left(\varepsilon^2 + \frac{1}{\kappa^2}\right)\|\xi - \xi'\|_{\ell^1,1,3} + \|\widehat{\Pi}[\xi] - \widehat{\Pi}[\xi']\|_{\ell_1,1,3}\right).$$
Furthermore,
$$\|\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi)] - \widehat{\Pi}[\widetilde{\mathcal{F}}(\xi')]\|_{\ell_1,1,3} \leq \frac{M_2}{\kappa^2}\|\xi - \xi'\|_{\ell_1,1,3}.$$

Proof. First, we need to rewrite the operator $\mathcal{F}$ given in (8.4.3), in order to make explicit some cancellations between its terms. Recall that g is given by (8.1.6).

$$\begin{aligned}
\mathcal{F}(\xi) &= -\frac{1}{\varepsilon^3} g(\varepsilon(\xi + v_1^h \sin(\tau))) + \left(\frac{(\xi_1 + v_1^h)^3}{4} - \frac{3 v_1^h \xi_1^2}{4} - \frac{\xi_1^3}{4} \right) \sin(\tau) \\
&= -\frac{1}{\varepsilon^3} \widehat{\Pi} \left[g(\varepsilon(\xi + v_1^h \sin(\tau))) \right] + \left(-\frac{1}{3} \Pi_1 \left[\left((\xi_1 + v_1^h) \sin(\tau) + \widehat{\Pi}(\xi) \right)^3 \right] \right. \\
&\quad \left. -\frac{1}{\varepsilon^3} \Pi_1 \left[f(\varepsilon(\xi + v_1^h \sin(\tau))) \right] + \frac{(\xi_1 + v_1^h)^3}{4} - \frac{3 v_1^h \xi_1^2}{4} - \frac{\xi_1^3}{4} \right) \sin(\tau) \\
&= -\frac{1}{\varepsilon^3} \widehat{\Pi} \left[g(\varepsilon(\xi + v_1^h \sin(\tau))) \right] + \left(-\frac{1}{3} \Pi_1 \left[(\xi_1 + v_1^h)^3 \sin^3(\tau) + 3(\xi_1 + v_1^h)^2 \sin^2(\tau) \widehat{\Pi}[\xi] \right. \right. \\
&\quad \left. + 3(\xi_1 + v_1^h) \sin(\tau) (\widehat{\Pi}[\xi])^2 + (\widehat{\Pi}[\xi])^3 \right] - \frac{1}{\varepsilon^3} \Pi_1 \left[f(\varepsilon(\xi + v_1^h \sin(\tau))) \right] \\
&\quad \left. + \frac{(\xi_1 + v_1^h)^3}{4} - \frac{3 v_1^h \xi_1^2}{4} - \frac{\xi_1^3}{4} \right) \sin(\tau).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{F}(\xi) &= -\frac{1}{\varepsilon^3} \widehat{\Pi}[g(\varepsilon(\xi + v_1^h \sin(\tau)))] + \left(\Pi_1 \left[-(\xi_1 + v_1^h)^2 \sin^2(\tau) \widehat{\Pi}[\xi] - (\xi_1 + v_1^h) \sin(\tau) (\widehat{\Pi}[\xi])^2 \right. \right. \\
&\quad \left. \left. -\frac{1}{3} (\widehat{\Pi}[\xi])^3 \right] - \frac{1}{\varepsilon^3} \Pi_1 \left[f(\varepsilon(\xi + v_1^h \sin(\tau))) \right] - \frac{3 v_1^h \xi_1^2}{4} - \frac{\xi_1^3}{4} \right) \sin(\tau).
\end{aligned} \tag{8.4.30}$$

Now,

$$\mathcal{F}(0) = -\frac{1}{\varepsilon^3} \widehat{\Pi} \left[g(\varepsilon(v_1^h \sin(\tau))) \right] - \frac{1}{\varepsilon^3} \Pi_1 \left[f(\varepsilon(v_1^h \sin(\tau))) \right] \sin(\tau).$$

Since g, and therefore, f are analytic functions such that $g(z) = \mathcal{O}(z^3)$ and $f(z) = \mathcal{O}(z^5)$ and $v_1^h \sin(\tau) \in \mathcal{E}_{\ell_1,1,1}$, and using Proposition 8.4.2 (with $m = \alpha = 5$, $l = 1$ for f, and $\beta = 3$, $l = 1$ for g) it follows that

$$\begin{aligned}
\|\widetilde{\mathcal{F}}(0)\|_{\ell_1,1,3} &\leq M\varepsilon^2 \left\| \frac{1}{\varepsilon^3} \widehat{\Pi} \left[g(\varepsilon(v_1^h \sin(\tau))) \right] \right\|_{\ell_1,1,3} + M \left\| \frac{1}{\varepsilon^3} \Pi_1 \left[f(\varepsilon(v_1^h \sin(\tau))) \right] \right\|_{\ell_1,5,5} \\
&\leq M\varepsilon^2 \left\| v_1^h \sin(\tau) \right\|_{\ell_1,1,1}^3 + M\varepsilon^2 \left\| v_1^h \sin(\tau) \right\|_{\ell_1,1,1}^5 \\
&\leq M\varepsilon^2.
\end{aligned}$$

Now, let $\xi, \xi' \in \mathcal{B}_0(R\varepsilon^2)$, thus we have that

$$\begin{aligned}\mathcal{F}(\xi)-\mathcal{F}(\xi') &= -\frac{1}{\varepsilon^3}\widehat{\Pi}\left[g(\varepsilon(\xi+v_1^h\sin(\tau)))-g(\varepsilon(\xi'+v_1^h\sin(\tau)))\right]\\ &\quad -\Pi_1\left[(\xi_1+v_1^h)^2\sin^2(\tau)(\widehat{\Pi}[\xi]-\widehat{\Pi}[\xi'])\right]\sin(\tau)\\ &\quad +\Big(-\Pi_1\left[\widehat{\Pi}[\xi']\sin^2(\tau)\left((\xi_1+v_1^h)^2-(\xi_1'+v_1^h)^2\right)\right]\\ &\quad -\Pi_1\left[(\xi_1+v_1^h)\sin(\tau)((\widehat{\Pi}[\xi])^2-(\widehat{\Pi}[\xi'])^2)\right]\Big)\sin(\tau)\\ &\quad +\left(-\Pi_1\left[(\widehat{\Pi}[\xi'])^2\sin(\tau)(\xi_1-\xi_1')\right]-\frac{1}{3}\Pi_1\left[(\widehat{\Pi}[\xi])^3-(\widehat{\Pi}[\xi'])^3\right]\right)\sin(\tau)\\ &\quad +\Bigg(-\frac{1}{\varepsilon^3}\Pi_1\left[f(\varepsilon(\xi+v_1^h\sin(\tau)))-f(\varepsilon(\xi'+v_1^h\sin(\tau)))\right]\\ &\quad -\frac{3v_1^h(\xi_1^2-(\xi_1')^2)}{4}-\frac{\xi_1^3-(\xi_1')^3}{4}\Bigg)\sin(\tau)\end{aligned}$$

Now, using the Mean Value Theorem, we obtain that

$$g(\varepsilon(\xi+v_1^h\sin(\tau)))-g(\varepsilon(\xi'+v_1^h\sin(\tau)))=\varepsilon(\xi-\xi')\int_0^1 g'(\varepsilon(s(\xi+v_1^h\sin(\tau))+(1-s)(\xi'+v_1^h\sin(\tau))))ds,$$

thus, since $g'(x)=\mathcal{O}(x^2)$ and $\xi+v_1^h\sin(\tau)\in\mathcal{E}_{\ell_1,1,1}$, it follows that

$$\|g(\varepsilon(\xi+v_1^h\sin(\tau)))-g(\varepsilon(\xi'+v_1^h\sin(\tau)))\|_{\ell_1,3,3}\leq\frac{M\varepsilon}{\kappa^2}\|\xi-\xi'\|_{\ell_1,1,3}.$$

Analogously, we obtain

- $\left\|\Pi_1\left[(\xi_1+v_1^h)^2\sin^2(\tau)(\widehat{\Pi}[\xi]-\widehat{\Pi}[\xi'])\right]\sin(\tau)\right\|_{3,5}\leq M\left\|\widehat{\Pi}[\xi]-\widehat{\Pi}[\xi']\right\|_{\ell_1,1,3};$
- $\left\|\Pi_1\left[\widehat{\Pi}[\xi']\sin^2(\tau)\left((\xi_1+v_1^h)^2-(\xi_1'+v_1^h)^2\right)\right]\sin(\tau)\right\|_{3,5}\leq\frac{M}{\kappa^2}\|\xi_1-\xi_1'\|_{1,3};$
- $\left\|\Pi_1\left[(\xi_1+v_1^h)\sin(\tau)((\widehat{\Pi}[\xi])^2-(\widehat{\Pi}[\xi'])^2)\right]\sin(\tau)\right\|_{3,5}\leq\frac{M}{\kappa^2}\left\|\widehat{\Pi}[\xi]-\widehat{\Pi}[\xi']\right\|_{\ell_1,1,3};$
- $\left\|\Pi_1\left[(\widehat{\Pi}[\xi'])^2\sin(\tau)(\xi_1-\xi_1')\right]\sin(\tau)\right\|_{3,5}\leq\frac{M}{\kappa^4}\|\xi_1-\xi_1'\|_{1,3};$
- $\left\|\Pi_1\left[(\widehat{\Pi}(\xi))^3-(\widehat{\Pi}(\xi'))^3\right]\sin(\tau)\right\|_{3,5}\leq\frac{M}{\kappa^4}\left\|\widehat{\Pi}[\xi]-\widehat{\Pi}[\xi']\right\|_{\ell_1,1,3};$
- $\left\|\Pi_1\left[f(\varepsilon(\xi+v_1^h\sin(\tau)))-f(\varepsilon(\xi'+v_1^h\sin(\tau)))\right]\sin(\tau)\right\|_{3,5}\leq\frac{M\varepsilon^3}{\kappa^2}\|\xi-\xi'\|_{\ell_1,1,3};$
- $\left\|v_1^h(\xi_1^2-(\xi_1')^2\sin(\tau)\right\|_{3,5}\leq\frac{M}{\kappa^2}\|\xi_1-\xi_1'\|_{1,3};$
- $\|(\xi_1^3-(\xi_1')^3)\sin(\tau)\|_{3,5}\leq\frac{M}{\kappa^4}\|\xi_1-\xi_1'\|_{1,3}.$

Therefore, applying (8.4.10) to bound $\mathcal{G}$, we have the second statement of this proposition. Finally, using (8.4.11), notice that

$$\begin{aligned}\left\|\widehat{\Pi}\left[\widetilde{\mathcal{F}}(\xi))\right]-\widehat{\Pi}\left[\widetilde{\mathcal{F}}(\xi')\right]\right\|_{\ell_1,1,3} &= \|\mathcal{G}(\widehat{\Pi}\circ\mathcal{F}(\xi)-\widehat{\Pi}\circ\mathcal{F}(\xi'))\|_{\ell_1,1,3}\\ &= \left\|\mathcal{G}\left(-\frac{1}{\varepsilon^3}\widehat{\Pi}\left[g(\varepsilon(\xi+v_1^h\sin(\tau)))-g(\varepsilon(\xi'+v_1^h\sin(\tau)))\right]\right)\right\|_{\ell_1,1,3}\\ &\leq \frac{M}{\varepsilon}\left\|\widehat{\Pi}\left[g(\varepsilon(\xi+v_1^h\sin(\tau)))-g(\varepsilon(\xi'+v_1^h\sin(\tau)))\right]\right\|_{\ell_1,1,3}\\ &\leq \frac{M}{\kappa^2}\|\xi-\xi'\|_{\ell_1,1,3}.\end{aligned}$$

The proof is complete.

□

We want to find a small fixed point of the operator $\widetilde{\mathcal{F}}:\mathcal{E}_{\ell^1,1,3}\to\mathcal{E}_{\ell^1,1,3}$ given in (8.4.29). Now, in order to use a Gauss-Seidel type argument, we set the operator

$$\widetilde{\mathcal{F}}_{GS}(\xi)=\Pi_1\left[\widetilde{\mathcal{F}}(\xi_1\sin(\tau)+\widehat{\Pi}\left[\widetilde{\mathcal{F}}(\xi)\right)\right]\sin(\tau)+\widehat{\Pi}\left[\widetilde{\mathcal{F}}(\xi)\right] \tag{8.4.31}$$

If $\widehat{\Pi}[\xi^*]=\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi^*)]$, then

$$\Pi_1\left[\widetilde{\mathcal{F}}(\xi^*)\right]=\Pi_1\left[\widetilde{\mathcal{F}}(\xi_1\sin(\tau)+\widehat{\Pi}(\widetilde{\mathcal{F}}(\xi)))\right] \tag{8.4.32}$$

which implies that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}_{GS}$ have the same fixed points.

Proposition 8.4.4. *Consider $\kappa\geq 1$. The following statements hold.*

1. *There exists a constant $M_1>0$ independent of ε and κ such that, for ε sufficiently small,*
$$\|\widetilde{\mathcal{F}}_{GS}(0)\|_{\ell^1,1,3}\leq M_1\varepsilon^2.$$

2. *Given $R>0$, there exists $\varepsilon_0>0$ such that, for every $0<\varepsilon\leq\varepsilon_0$, the operator*
$$\widetilde{\mathcal{F}}_{GS}:\mathcal{B}_0(R\varepsilon^2)\subset\mathcal{E}_{\ell^1,1,3}\to\mathcal{E}_{\ell^1,1,3}$$
is well defined, and there exists a constant $M_2>0$ independent of ε and κ such that, for every $\xi,\xi'\in\mathcal{B}_0(R\varepsilon^2)\subset\mathcal{E}_{\ell^1,1,3}$ and $0<\varepsilon\leq\varepsilon_0$,
$$\|\widetilde{\mathcal{F}}_{GS}(\xi)-\widetilde{\mathcal{F}}_{GS}(\xi')\|_{\ell_1,1,3}\leq M_2\left(\varepsilon^2+\frac{1}{\kappa^2}\right)\|\xi-\xi'\|_{\ell^1,1,3}.$$

Proof. Notice that

$$\widetilde{\mathcal{F}}_{GS}(0)=\Pi_1\left[\widetilde{\mathcal{F}}(\widehat{\Pi}(\widetilde{\mathcal{F}}(0)))\right]\sin(\tau)+\widehat{\Pi}(\widetilde{\mathcal{F}}(0)).$$

Therefore, using Proposition 8.4.3, we obtain

$$\begin{aligned}\|\widetilde{\mathcal{F}}_{GS}(0)\|_{\ell_1,1,3} &\leq \|\widetilde{\mathcal{F}}(\widehat{\Pi}(\widetilde{\mathcal{F}}(0)))\|_{\ell_1,1,3}+\|\widetilde{\mathcal{F}}(0)\|_{\ell_1,1,3}\\ &\leq \|\widetilde{\mathcal{F}}(\widehat{\Pi}(\widetilde{\mathcal{F}}(0)))-\widetilde{\mathcal{F}}(0)\|_{\ell_1,1,3}+2\|\widetilde{\mathcal{F}}(0)\|_{\ell_1,1,3}\\ &\leq M\varepsilon^2,\end{aligned}$$

Finally, item (2) follows from follows and the following estimate

$$\begin{aligned}
&\left\|\Pi_1\left[\widetilde{\mathcal{F}}(\xi_1\sin(\tau)+\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi)])\right]-\Pi_1\left[\widetilde{\mathcal{F}}(\xi_1'\sin(\tau)+\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi')])\right]\right\|_{1,3}\\
&\leq\left\|\widetilde{\mathcal{F}}(\xi_1\sin(\tau)+\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi)])-\widetilde{\mathcal{F}}(\xi_1'\sin(\tau)+\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi')])))\right\|_{\ell_1,1,3}\\
&\leq M\left(\varepsilon^2+\frac{1}{\kappa^2}\right)\left\|(\xi_1-\xi_1')\sin(\tau)+\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi)]-\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi')]\right\|_{\ell_1,1,3}\\
&+M\left\|\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi)]-\widehat{\Pi}[\widetilde{\mathcal{F}}(\xi')]\right\|_{\ell_1,1,3}\\
&\leq M\left(\varepsilon^2+\frac{1}{\kappa^2}\right)\|\xi-\xi'\|_{\ell_1,1,3}.
\end{aligned}$$

□

Recall that $W^u(0)$ is parameterized by $v^u(y,\tau)=v_1^h(y)\sin(\tau)+\xi^u(y,\tau)$, where $\xi^u(y,\tau)$ is a fixed point of

$$\xi=\widetilde{\mathcal{F}}_{GS}(\xi).$$

In what follows, we prove Theorem 8.3.1 by showing that $\widetilde{\mathcal{F}}_{GS}$ has a fixed point in $\mathcal{E}_{\ell_1,1,3}$.

Proposition 8.4.5. *There exist $\varepsilon_0>0$ and $\kappa_0\geq 1$ such that the operator $\widetilde{\mathcal{F}}_{GS}:\mathcal{E}_{\ell^1,1,3}\to\mathcal{E}_{\ell^1,1,3}$ has a fixed point $\xi^u\in\mathcal{E}_{\ell^1,1,3}$, for every $\varepsilon\leq\varepsilon_0$. Furthermore, there exists $M>0$ independent of ε such that*

$$\|\xi^u\|_{\ell^1,1,3},\ \|\partial_\tau\xi^u\|_{\ell_1,1,3},\ \|\partial_\tau^2\xi^u\|_{\ell_1,1,3}\leq M\varepsilon^2. \tag{8.4.33}$$

Proof. From Proposition 8.4.4, it follows that there exists a constant $b_1>0$, such that

$$\|\widetilde{\mathcal{F}}_{GS}(0)\|_{\ell^1,1,3}\leq\frac{b_1}{2}\varepsilon^2.$$

Also, given $\xi,\xi'\in\overline{\mathcal{B}}_0(b_1\varepsilon^2)$, it follows that

$$\|\widetilde{\mathcal{F}}_{GS}(\xi)-\widetilde{\mathcal{F}}_{GS}(\xi')\|_{\ell^1,1,3}\leq M\left(\varepsilon^2+\frac{1}{\kappa^2}\right)\|\xi-\xi'\|_{\ell^1,1,3}.$$

Thus, choosing $\varepsilon_0>0$ sufficiently small and $\kappa_0\geq 1$ sufficiently big such that $\mathrm{Lip}(\widetilde{\mathcal{F}}_{GS})\leq 1/2$, it follows that $\widetilde{\mathcal{F}}_{GS}$ sends $\overline{\mathcal{B}}_0(b_1\varepsilon^2)$ into itself and it is a contraction. Thus, it follows from Banach's Fixed Point Theorem that $\widetilde{\mathcal{F}}_{GS}$ admits a unique fixed point ξ^u in $\overline{\mathcal{B}}_0(b_1\varepsilon^2)$.

Now, since $\xi^u=\mathcal{G}\circ\mathcal{F}(\xi^u)$, then it follows from Proposition 8.4.2 that

$$\begin{aligned}
\|\partial_\tau\xi^u\|_{\ell_1,1,3}&\leq\|\partial_\tau\circ\mathcal{G}\circ\mathcal{F}(\xi^u)-\partial_\tau\circ\mathcal{G}\circ\mathcal{F}(0)\|_{\ell_1,1,3}+\|\partial_\tau\circ\mathcal{G}\circ\mathcal{F}(0)\|_{\ell_1,1,3}\\
&\leq M\varepsilon^2.
\end{aligned}$$

Analogously, we prove that $\|\partial_\tau^2\xi^u\|_{\ell_1,1,3}\leq M\varepsilon^2$ and $\|\partial_y\xi^u\|_{\ell_1,1,4}\leq M\varepsilon^2$. □

Since g given in (8.1.6) is an odd function, and the product $s=s_1\cdots s_{2k+1}$ of $2k+1$ terms of type $s_l=\sin(2k_l+1)$, with $k_l\geq 0$, $1\leq l\leq 2k+1$, is written as

$$s=\sum_{m\geq 0}a_m\sin((2m+1)\tau),$$

where $a_m\in\mathbb{R}$, it follows that the operator $\mathcal{F}$ given in (8.4.3) leaves invariant the subspace of functions $\xi:D_\kappa^{out,u}\times\mathbb{T}\to\to\mathbb{C}$ such that $\Pi_{2l}[\xi]=0$, $\forall l\geq 0$. The same remark holds for the operator $\widetilde{\mathcal{F}}_{GS}$.

Now, since ξ^* is a fixed point of $\widetilde{\mathcal{F}}_{GS}$, we have that $\Pi_{2l}[\xi]=0$, $\forall l\geq 0$. Thus the proof of Theorem (8.3.1) is complete.

8.5 Proof of Theorem 8.3.2

In this section we use a fixed point argument to prove the existence of certain solutions of the inner equation (8.3.6). Since, we look solutions odd in τ of (8.3.6), we write

$$\phi^0 = \sum_{n\geq 1} \phi_n^0 \sin(n\tau). \tag{8.5.1}$$

Replacing (8.5.1) in (8.3.6), we obtain that

$$(\partial_z^2 + (n^2-1))\phi_n^0 + \Pi_n\left[\frac{1}{3}(\phi^0)^3 + f(\phi^0)\right] = 0, n \geq 1. \tag{8.5.2}$$

Since we are interested in matching the solutions of (8.3.6) with the outer solutions $v^{u,s}$ given in Theorem 8.3.1, we must look for solutions $\phi^{0,u,s}$ of (8.3.6) which have the same expansion of

$$\varepsilon v^{u,s}\left(i\left(\frac{\pi}{2}+\varepsilon z\right),\tau\right) = \frac{-2\sqrt{2}i}{z}\sin(\tau) + h.o.t.,$$

and satisfy the asymptotic conditions

$$\lim_{z\to-\infty}\phi^{0,u}(z,\tau) = \lim_{z\to+\infty}\phi^{0,s}(z,\tau) = 0,\ \forall \tau\in\mathbb{T}, \quad \textit{and} \quad \mathrm{Im}(z)<0. \tag{8.5.3}$$

Near the pole $y = i\pi/2$, we have

$$v^{u,s}(y,\tau) = \frac{-2\sqrt{2}i}{y-i\pi/2}\sin(\tau) + \mathcal{O}(y-i\pi/2) + \mathcal{O}\left(\frac{\varepsilon^2}{(y-i\pi/2)^3}\right),$$

which corresponds to

$$\phi^{u,s}(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin(\tau) + \mathcal{O}(\varepsilon^2 z) + \mathcal{O}\left(z^{-3}\right),$$

in the inner variables (8.3.3) and (8.3.4).

In the limit case $\varepsilon = 0$, we have that

$$\phi^{u,s}(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin(\tau) + \mathcal{O}\left(z^{-3}\right).$$

It means that we must look solutions of the inner equation (8.3.6) of the form

$$\phi^0(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi(z,\tau),$$

where $\psi = \mathcal{O}(z^{-3})$.

Since ϕ^0 has to satisfy (8.5.2), it follows that $\psi(y,\tau) = \sum_{n\geq 1}\psi_n(y)\sin(n\tau)$ must satisfy

$$\begin{cases} \psi_1'' - \dfrac{6}{z^2}\psi_1 = -\Pi_1\left[-\dfrac{8}{z^2}\sin^2(\tau)\widehat{\Pi}\left[\psi\right] - \dfrac{2\sqrt{2}i}{z}\sin(\tau)\psi^2 + \dfrac{1}{3}\psi^3 + f\left(\dfrac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)\right], \\ \psi_n'' + \lambda_{0,n}^2\psi_n = -\Pi_n\left[\dfrac{1}{3}\left(\dfrac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)^3 + f\left(\dfrac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)\right],\ n\geq 2, \end{cases} \tag{8.5.4}$$

where $' = d/dz$, and $\lambda_{0,n} = \sqrt{n^2-1}$.

Now, define the operators

$$\mathcal{I}(\psi) = \left(\psi_1'' - \frac{6}{z^2}\psi_1\right)\sin(\tau) + \sum_{n\geq 2}\left(\psi_n'' + \lambda_{0,n}^2\psi_n\right)\sin(n\tau), \tag{8.5.5}$$

and

$$\begin{aligned}\mathcal{W}(\psi) &= -\Pi_1\left[-\frac{8}{z^2}\sin^2(\tau)\widehat{\Pi}\left[\psi\right] - \frac{2\sqrt{2}i}{z}\sin(\tau)\psi^2 + \frac{1}{3}\psi^3\right]\sin(\tau)\\ &- \widehat{\Pi}\left[\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)^3\right] - f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)\end{aligned} \tag{8.5.6}$$

and notice that, for $\star = u, s$, to find a solution $\phi^{0,\star}$ of (8.3.6) satisfying (8.5.3) is equivalent to find a fixed point $\psi^\star$ of the functional equation

$$\mathcal{I}(\psi) = \mathcal{W}(\psi), \tag{8.5.7}$$

which satisfies

$$\lim_{z\to-\infty}\psi^u(z,\tau) = \lim_{z\to+\infty}\psi^s(z,\tau) = 0,\ \forall\tau\in\mathbb{T}. = 0,\ \forall\tau\in\mathbb{T}\quad \textit{and}\quad \operatorname{Im}(z)<0. \tag{8.5.8}$$

In the remainder of this section, we find a fixed point of (8.5.7) and (8.5.8) in some appropriate Banach space. As before, we consider only the unstable case, since the stable one is completely analogous.

8.5.1 Banach Spaces and Linear Operators

Given $\alpha \geq 0$ and an analytic function $f: D^{u,\text{in}}_{\theta,\kappa} \to \mathbb{C}$, where $D^{u,\text{in}}_{\theta,\kappa}$ is given in (8.3.7), consider the norm

$$\|f\|_\alpha = \sup_{z\in D^{u,\text{in}}_{\theta,\kappa}} |z^\alpha f(z)|,$$

and the Banach space

$$\mathcal{X}_\alpha = \{f: D^{u,\text{in}}_{\theta,\kappa} \to \mathbb{C};\ f \text{ is an analytic function and } \|f\|_\alpha < \infty\}.$$

Also, if $f: D^{u,\text{in}}_{\theta,\kappa}\times\mathbb{T} \to \mathbb{C}$ is an analytic function in the variable z, we define

$$\|f\|_{\ell_1,\alpha} = \sum_{n\geq 1}\|f_n\|_\alpha,$$

and the Banach space

$$\mathcal{X}_{\ell_1,\alpha} = \left\{f: D^{u,\text{in}}_{\theta,\kappa}\times\mathbb{T}\to\mathbb{C};\ f \text{ is an analytic function in the variable } z \text{ and } \|f\|_{\ell_1,\alpha}<\infty\right\}.$$

Proposition 8.5.1. *Given an analytic function $f: B(R_0)\to\mathbb{C}$, and $g, h: D^{u,\text{in}}_{\theta,\kappa}\times\mathbb{T}\to\mathbb{C}$, where $B(R_0)\subset\mathbb{C}$ is a ball with center at the origin and radius R_0, the following statements hold*

1. *If $\alpha\geq\beta\geq 0$, then*

$$\|h\|_{\ell_1,\alpha-\beta} \leq \frac{M}{\kappa^\beta}\|h\|_{\ell_1,\alpha}.$$

2. *If $\alpha, \beta \geq 0$, and $\|g\|_{\ell_1,\alpha}, \|h\|_{\ell_1,\beta} < \infty$, then*

$$\|gh\|_{\ell_1,\alpha+\beta} \leq \|g\|_{\ell_1,\alpha}\|h\|_{\ell_1,\beta}.$$

3. *If $\|g\|_{\ell_1,\alpha}$, $\|h\|_{\ell_1,\alpha} \leq R_0/4$, then*

$$\|f(g) - f(h)\|_{\ell_1,\alpha} \leq M\|g - h\|_{\ell_1,\alpha}.$$

4. *Given $n \geq 1$, if $f^{(k)}(0) = 0$, for every $1 \leq k \leq n-1$, and $\|g\|_{\ell_1,\alpha} \leq R_0/4$, where R_0 is the convergence ratio of $f^{(n)}$ at 0, then*

$$\|f(g)\|_{\ell_1,n\alpha} \leq M(\|g\|_{\ell_1,\alpha})^n.$$

5. *If $h \in \mathcal{X}_{\ell_1,\alpha}$ (with respect to the inner domain $D^{u,\text{in}}_{\theta,\kappa}$), then $\partial_z h \in \mathcal{X}_{\ell_1,\alpha+1}$ (with respect to the inner domain $D^{u,\text{in}}_{2\theta,4\kappa}$), and*

$$\|\partial_z h\|_{\ell_1,\alpha+1} \leq M\|h\|_{\ell_1,\alpha}.$$

The proposition above is proved in [7, 53].

Now, define the linear operator acting on the Fourier coefficients of ψ

$$\mathcal{J}(\psi) = \sum_{n\geq 1} \mathcal{J}_n(\psi_n)\sin(n\tau),$$

where

$$\mathcal{J}_1(\psi_1)(z) = \frac{z^3}{5}\int_{-\infty}^{z} \frac{\psi_1(s)}{s^2}ds - \frac{1}{5z^2}\int_{-\infty}^{z} s^3\psi_1(s)ds,$$

and

$$\mathcal{J}_n(\psi_n)(z) = \frac{1}{2i\lambda_{0,n}}\int_{-\infty}^{z} e^{-i\lambda_{0,n}(s-z)}\psi_n(s)ds - \frac{1}{2i\lambda_{0,n}}\int_{-\infty}^{z} e^{i\lambda_{0,n}(s-z)}\psi_n(s)ds, \; n \geq 2.$$

Proposition 8.5.2. *Consider $\kappa \geq 1$. Given $\alpha \geq 3$, the operator*

$$\mathcal{J} : \mathcal{X}_{\ell_1,\alpha+2} \to \mathcal{X}_{\ell_1,\alpha}$$

is well defined and the following statements hold.

1. *$\mathcal{J} \circ \mathcal{I}(\psi) = \mathcal{I} \circ \mathcal{J}(\psi) = \psi$.*

2. *There exists a constant $M > 0$ independent of κ such that, for every $\psi \in \mathcal{X}_{\ell_1,\alpha+2}$,*

$$\|\mathcal{J}(\psi)\|_{\ell_1,\alpha} \leq M\|\psi\|_{\ell_1,\alpha+2}. \tag{8.5.9}$$

Moreover, given $\beta \geq 0$ and denoting $\mathcal{X}^1_{\ell_1,\beta} = \{\psi \in \mathcal{X}_{\ell_1,\beta};\ \pi_1(\psi) = 0\}$, then

$$\mathcal{J} : \mathcal{X}^1_{\ell_1,\beta} \to \mathcal{X}^1_{\ell_1,\beta}$$

is well defined and, for every $\psi \in \mathcal{X}^1_{\ell_1,\beta}$,

$$\|\mathcal{J}(\psi)\|_{\ell_1,\beta} \leq M\|\psi\|_{\ell_1,\beta}.$$

3. The operators $\partial_\tau \circ \mathcal{J}$ and $\partial_\tau^2 \circ \mathcal{J}$ are well defined and satisfy (8.5.9).

Proof. Consider the equation $\mathcal{I}(\psi)(z,\tau) = h(z,\tau) = \sum_{n\geq 1} h_n(z)\sin(n\tau)$ and denote $\mathcal{I}_n = \Pi_n \circ \mathcal{I}$.

First, we consider $n=1$. Using that $\eta_1^1(z) = z^3$ and $\eta_2^1(z) = -(5z^2)^{-1}$ are fundamental solutions of the homogeneous equation $\mathcal{I}_1(\psi_1) = 0$, we obtain from the method of variation of constants that

$$\psi_1(z) = \frac{z^3}{5}\left(\int_{z_0^1}^{z} \frac{h_1(s)}{s^2}ds + C_0^1\right) - \frac{1}{5z^2}\left(\int_{z_1^1}^{z} s^3 h_1(s)ds + C_1^1\right),$$

where z_0^1, z_1^1, C_0^1 and C_1^1 are constants.

Now, for $n \geq 2$, the fundamental solutions of the homogeneous equation $\mathcal{I}_n(\psi_n) = 0$ are given by $\eta_1^n(z) = e^{i\lambda_{0,n}z}$ and $\eta_2^n(z) = e^{-i\lambda_{0,n}z}$. Again, it follows from the method of variation of constants that the equation $\mathcal{I}_n(\psi_n) = h_n$ implies that

$$\psi_n(z) = \frac{e^{i\lambda_{0,n}z}}{2i\lambda_{0,n}}\left(\int_{z_0^n}^{z} e^{-i\lambda_{0,n}s}h_n(s)ds + C_0^n\right) - \frac{e^{-i\lambda_{0,n}z}}{2i\lambda_{0,n}}\left(\int_{z_1^n}^{z} e^{i\lambda_{0,n}s}h_n(s)ds + C_1^n\right),$$

where z_0^n, z_1^n, C_0^n and C_1^n are constants, for each $n \geq 2$.

Since we are looking for solutions of (8.5.4) satisfying (8.5.3) and such that $\|\psi\|_{\ell_1}(z) \sim z^{-3}$, we choose

$$C_0^1 = -\int_{z_0^1}^{\infty} \frac{h_1(s)}{s^2}ds \quad \textit{and} \quad C_1^1 = -\int_{z_1^1}^{-\infty} s^3 h_1(s)ds,$$

and

$$C_0^n = -\int_{z_0^n}^{-\infty} e^{-i\lambda_{0,n}s}h_n(s)ds \quad \textit{and} \quad C_1^n = -\int_{z_1^n}^{-\infty} e^{i\lambda_{0,n}s}h_n(s)ds, \ n \geq 2,$$

which proves item (1). Notice that,the integrals in the definition of the constants above converge for every $h \in \mathcal{X}_{\ell_1,\alpha+2}$.

Now, let $h_1 \in \mathcal{X}_{\alpha+2}$ and assume that $\alpha \geq 3$. Thus,

$$\begin{aligned} |z^\alpha \mathcal{J}_1(h_1)(z)| &= \left|\frac{z^{\alpha+3}}{5}\int_{-\infty}^{z} \frac{h_1(s)}{s^2}ds - \frac{z^{\alpha-2}}{5}\int_{-\infty}^{z} s^3 h_1(s)ds\right| \\ &\leq M\|h_1\|_{\alpha+2}\left(\int_{-\infty}^{z} \frac{|z|^{\alpha+3}}{|s|^{\alpha+4}}ds + \int_{-\infty}^{z} \frac{|z|^{\alpha-2}}{|s|^{\alpha-1}}ds\right) \\ &\leq M\|h_1\|_{\alpha+2}, \end{aligned}$$

for each $z \in D_{\theta,\kappa}^{u,\text{in}}$.

Also, if $\beta \geq 0$ and $h_n \in \mathcal{X}_\beta$, we have that, for each $n \geq 2$, changing the path of integration,

$$
\begin{aligned}
\left|z^\beta \mathcal{J}_n(h_n)(z)\right| &= \left|\frac{1}{2i\lambda_{0,n}}\int_{-\infty}^{z} e^{-i\lambda_{0,n}(s-z)} z^\beta h_n(s)ds - \frac{1}{2i\lambda_{0,n}}\int_{-\infty}^{z} e^{i\lambda_{0,n}(s-z)} z^\beta h_n(s)ds\right| \\
&= \left|\frac{1}{2i\lambda_{0,n}}\int_{-\infty}^{0} e^{-i\lambda_{0,n}e^{i\theta}r} z^\beta h_n(z+e^{i\theta}r)e^{i\theta}dr\right. \\
&\quad \left. -\frac{1}{2i\lambda_{0,n}}\int_{-\infty}^{0} e^{i\lambda_{0,n}e^{-i\theta}r} z^\beta h_n(z+e^{-i\theta}r)e^{-i\theta}dr\right| \\
&\leq \frac{M}{\lambda_{0,n}}\|h_n\|_\beta\left(\int_{-\infty}^{0} e^{\lambda_{0,n}\sin(\theta)r}\frac{|z|^\beta}{|z+e^{i\theta}r|^\beta}dr + \int_{-\infty}^{0} e^{\lambda_{0,n}\sin(\theta)r}\frac{|z|^\beta}{|z+e^{-i\theta}r|^\beta}dr\right) \\
&\leq \frac{M}{\lambda_{0,n}^2}\|h_n\|_\beta,
\end{aligned}
$$

for each $z \in D^{u,\text{in}}_{\theta,\kappa}$. Item (2) follows directly, and the proof of item (3) of this proposition is analogous to the proof of item (4) of Proposition 8.4.2. □

8.5.2 Fixed Point Argument

Now, we use Proposition 8.5.2 to rewrite (8.5.7) as

$$\psi = \mathcal{J} \circ \mathcal{W}(\psi),$$

where $\mathcal{W}$ is given by (8.5.6), and in the following proposition we study some properties of the operator

$$\widetilde{\mathcal{W}} = \mathcal{J} \circ \mathcal{W}.$$

Proposition 8.5.3. *Consider $\kappa \geq 1$. The following statements hold.*

1. *There exists a constant $M_1 > 0$ independent of κ such that,*

$$\|\widetilde{\mathcal{W}}(0)\|_{\ell^1,3} \leq M_1.$$

2. *Given $R > 0$, the operator*

$$\widetilde{\mathcal{W}} : \mathcal{B}_0(R) \subset \mathcal{X}_{\ell^1,3} \to \mathcal{X}_{\ell^1,3}$$

is well defined, and there exists a constant $M_2 > 0$ independent of κ such that, for every $\psi, \psi' \in \mathcal{B}_0(R) \subset \mathcal{X}_{\ell^1,3}$,

$$\left\|\widetilde{\mathcal{W}}(\psi) - \widetilde{\mathcal{W}}(\psi')\right\|_{\ell_1,3} \leq M_2\left(\frac{1}{\kappa^2}\|\psi-\psi'\|_{\ell^1,3} + \|\widehat{\Pi}[\psi] - \widehat{\Pi}[\psi]\|_{\ell_1,3}\right).$$

Furthermore,

$$\left\|\widehat{\Pi}[\widetilde{\mathcal{W}}(\psi)] - \widehat{\Pi}[\widetilde{\mathcal{W}}(\psi')]\right\|_{\ell_1,3} \leq \frac{M_2}{\kappa^2}\|\psi-\psi'\|_{\ell_1,3}.$$

Proof. In fact,

$$\mathcal{W}(0) = -\widehat{\Pi}\left[\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)\right)^3\right] - f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)\right),$$

and thus, since $f(z) = \mathcal{O}(z^5)$, it follows from Proposition 8.5.1 that

$$\|\Pi_1[\mathcal{W}(0)]\|_5 \leq M\left\|\frac{-2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}^5 \leq M,$$

and

$$\left\|\widehat{\Pi}[\mathcal{W}(0)]\right\|_{\ell_1,3} \leq M\left(\left\|\frac{-2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}^3 + \frac{1}{\kappa^2}\left\|\frac{-2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}^5\right) \leq M.$$

Hence, from Proposition 8.5.2, we have that

$$\begin{aligned}
\left\|\widetilde{W}(0)\right\|_{\ell_1,3} &= \|\mathcal{J}\left(\Pi_1[\mathcal{W}(0)]\sin(\tau)\right)\|_{\ell_1,3} + \left\|\mathcal{J}\left(\widehat{\Pi}[\mathcal{W}(0)]\right)\right\|_{\ell_1,3} \\
&\leq M\left(\|\Pi_1[\mathcal{W}(0)]\|_5 + \left\|\widehat{\Pi}[\mathcal{W}(0)]\right\|_{\ell_1,3}\right) \\
&\leq M.
\end{aligned}$$

Now, to prove item (2), assume that $\|\psi\|_{\ell_1,3}, \|\psi'\|_{\ell_1,3} \leq R$, and notice that

$$\begin{aligned}
\mathcal{W}(\psi) - \mathcal{W}(\psi') &= -\Pi_1\left[-\frac{8}{z^2}\sin^2(\tau)\left(\widehat{\Pi}[\psi] - \widehat{\Pi}[\psi']\right) - \frac{2\sqrt{2}i}{z}\sin(\tau)\left(\psi^2 - (\psi')^2\right)\right. \\
&\quad \left. + \frac{1}{3}\left(\psi^3 - (\psi')^3\right)\right]\sin(\tau) \\
&\quad -\widehat{\Pi}\left[\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi\right)^3 - \left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi'\right)^3\right] \\
&\quad -f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi\right) + f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi'\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\Pi_1[\mathcal{W}(\psi) - \mathcal{W}(\psi')]\|_5 &\leq \left\|\frac{8}{z^2}\sin^2(\tau)\right\|_{\ell_1,2}\left\|\widehat{\Pi}[\psi] - \widehat{\Pi}[\psi']\right\|_{\ell_1,3} \\
&\quad + \left\|\frac{2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}\|\psi + \psi'\|_{\ell_1,1}\|\psi - \psi'\|_{\ell_1,3} \\
&\quad + \|\psi^2 + \psi\psi' + (\psi')^2\|_{\ell_1,2}\|\psi - \psi'\|_{\ell_1,3} \\
&\quad + \left\|\int_0^1 f'\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + s\psi + (1-s)\psi'\right)ds\right\|_{\ell_1,2}\|\psi - \psi'\|_{\ell_1,3} \\
&\leq M\left(\left\|\widehat{\Pi}[\psi] - \widehat{\Pi}[\psi']\right\|_{\ell_1,3} + \frac{1}{\kappa^2}\|\psi - \psi'\|_{\ell_1,3}\right),
\end{aligned}$$

and, recalling that $g(z) = z^3 + f(z)$ is an analytic function such that $g(z) = \mathcal{O}(z^3)$, we have that

$$\begin{aligned}\left\|\widehat{\Pi}\left[\mathcal{W}(\psi)-\mathcal{W}(\psi')\right]\right\|_{\ell_1,3} &\leq \left\|\int_0^1 g'\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+s\psi+(1-s)\psi'\right)ds\right\|_{\ell_1,0}\|\psi-\psi'\|_{\ell_1,3}\\ &\leq \frac{M}{\kappa^2}\|\psi-\psi'\|_{\ell_1,3}.\end{aligned}$$

Item (2) follows from the estimates above and Proposition 8.5.2. □

As in Section 8.4.2, we also have to use a Gauss-Seidel argument to obtain a contractive operator. Therefore, consider the operator

$$\widetilde{\mathcal{W}}_{GS}(\psi) = \Pi_1\left[\widetilde{\mathcal{W}}(\psi_1\sin(\tau)+\widehat{\Pi}\left[\widetilde{\mathcal{W}}(\psi)\right)\right]\sin(\tau)+\widehat{\Pi}\left[\widetilde{\mathcal{W}}(\psi)\right],$$

which has the same fixed points of $\widetilde{\mathcal{W}}$.

Analogously to Proposition 8.4.4, we obtain the following result.

Proposition 8.5.4. *Consider $\kappa \geq 1$. The following statements hold.*

1. *There exists a constant $M_1 > 0$ independent of κ such that,*

$$\|\widetilde{\mathcal{W}}_{GS}(0)\|_{\ell^1,3} \leq M_1.$$

2. *Given $R > 0$, the operator*

$$\widetilde{\mathcal{W}}_{GS} : \mathcal{B}_0(R) \subset \mathcal{X}_{\ell^1,3} \to \mathcal{X}_{\ell^1,3}$$

is well defined, and there exists a constant $M_2 > 0$ independent of κ such that, for every $\psi, \psi' \in \mathcal{B}_0(R) \subset \mathcal{X}_{\ell^1,3}$

$$\|\widetilde{\mathcal{W}}_{GS}(\psi) - \widetilde{\mathcal{W}}_{GS}(\psi')\|_{\ell_1,3} \leq \frac{M_2}{\kappa^2}\|\psi-\psi'\|_{\ell^1,3}.$$

Therefore, item (1) of Theorem 8.3.2 is proved by the next proposition which shows that $\widetilde{\mathcal{W}}_{GS}$ has a fixed point in $\mathcal{X}_{\ell_1,3}$.

Proposition 8.5.5. *There exist $\kappa_0 \geq 1$ such that the operator $\widetilde{\mathcal{W}}_{GS} : \mathcal{E}_{\ell^1,1,3} \to \mathcal{E}_{\ell^1,1,3}$ has a fixed point $\psi^u \in \mathcal{X}_{\ell^1,3}$, for every $\kappa \geq \kappa_0$. Furthermore, there exists $M > 0$ independent of κ such that*

$$\|\psi^u\|_{\ell^1,3},\ \|\partial_\tau\psi^u\|_{\ell_1,3}, \|\partial_\tau^2\psi^u\|_{\ell_1,3} \leq M.$$

We omit the proof of Proposition 8.5.5 due to its similarity with the proof of Proposition 8.4.5. Finally, we remark that using the same arguments presented in the end of Section 8.4.2, we conclude $\Pi_{2l}[\psi] \equiv 0$, $\forall l \geq 0$.

8.5.3 The Difference between the Solutions of the Inner Equation

This section is devoted to prove the second statement of Theorem 8.3.2. In Proposition 8.5.5, we have proven the existence of two solutions $\phi_0^{u,s}$ of the inner equation (8.3.6) which are given by (8.3.8). Now, we study the difference

$$\Delta\psi(z,\tau) = \phi_0^u(z,\tau) - \phi_0^s(z,\tau) = \psi^u(z,\tau) - \psi^s(z,\tau), \tag{8.5.10}$$

for $z \in \mathcal{R}_{\theta,\kappa}^{\mathrm{in},+} = D_{\theta,\kappa}^{u,\mathrm{in}} \cap D_{\theta,\kappa}^{s,\mathrm{in}} \cap \{z;\ z \in i\mathbb{R} \text{ and } \operatorname{Im}(z) < 0\}$ and $\tau \in \mathbb{T}$.

Remark 8.5.6. *We are interested in the behavior of the difference in the connected component $\mathcal{R}_{\theta,\kappa}^{\mathrm{in},+}$ of $D_{\theta,\kappa}^{u,\mathrm{in}} \cap D_{\theta,\kappa}^{s,\mathrm{in}} \cap i\mathbb{R}$ because the change $z = \varepsilon^{-1}(y - i\pi/2)$ brings the origin $y = 0$ into $z = -i\varepsilon^{-1}\pi/2 \in \mathcal{R}_{\theta,\kappa}^{\mathrm{in},+}$.*

Proposition 8.5.7. *The function $\Delta\psi(z,\tau)$ given in (8.5.10) satisfies the following differential equation*

$$\mathcal{I}(\Delta\psi) = \mathcal{B}(\Delta\psi), \tag{8.5.11}$$

where $\mathcal{I}$ is given in (8.5.5) and

$$\mathcal{B} : \mathcal{X}_{\ell_1,0} \to \mathcal{X}_{\ell_1,2} \tag{8.5.12}$$

is a linear operator. Moreover, there exists a constant $M > 0$ independent of κ such that

$$\|\mathcal{B}(\Delta\psi)\|_{\ell_1,2} \leq M \|\Delta\psi\|_{\ell_1,0}.$$

Proof. Since $\psi^{u,s}$ satisfy (8.5.7), subtracting the solution and using the Mean Value Theorem, we obtain

$$\begin{aligned}
\mathcal{I}(\Delta\psi) \;=\; & \Pi_1\left[\frac{8}{z^2}\sin(\tau)\widehat{\Pi}\left[\Delta\psi\right] + \frac{2\sqrt{2}i}{z}\sin(\tau)(\psi^u + \psi^s)\Delta\psi\right.\\
& \left. -\frac{1}{3}\left((\psi^u)^2 + \psi^u\psi^s + (\psi^s)^2\right)\Delta\psi\right]\sin(\tau)\\
& -\widehat{\Pi}\left[\int_0^1 3\left(-\frac{2\sqrt{2}i}{z}\sin(\tau) + r\psi^u + (1-r)\psi^s\right)^2 dr\Delta\psi\right]\\
& -\int_0^1 f'\left(-\frac{2\sqrt{2}i}{z}\sin(\tau) + r\psi^u + (1-r)\psi^s\right) dr\Delta\psi.
\end{aligned}$$

The proof follows by taking $\mathcal{B}$ as the righthand side of the equation above and recalling that ψ^u, ψ^s are known functions such that $\|\psi^{u,s}\|_{\ell_1,3} \leq M$ and $f(z) = \mathcal{O}(z^5)$.

□

Now, given an analytic function $f : \mathcal{R}_{\theta,\kappa}^{\mathrm{in},+} \to \mathbb{C}$, we define the norm

$$\|f\|_{\alpha,\mathrm{in}} = \sup_{z\in\mathcal{R}_{\theta,\kappa}^{\mathrm{in},+}} |z^\alpha e^{i\lambda_{0,3}z} f(z)|,$$

and if $f : \mathcal{R}_{\theta,\kappa}^{\mathrm{in},+} \times \mathbb{T} \to \mathbb{C}$ is an analytic function in the variable z, then we define the norm

$$\|f\|_{\ell_1,\alpha,\mathrm{in}} = \sum_{k\geq 0} \|\Pi_n[f]\|_{\alpha,\mathrm{in}},$$

and the Banach space

$$\begin{aligned}\mathcal{Z}_{\ell_1,\alpha,\mathrm{in}} &= \Big\{f : \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C};\ f \text{ is analytic in the variable } z,\ \Pi_{2l}[f] = 0, \forall l \geq 0,\\ &\quad \text{and } \|f\|_{\ell_1,\alpha,\mathrm{in}} < \infty\Big\}.\end{aligned}$$

In particular, we denote

$$\|\cdot\|_{\mathcal{Z}} = \|\cdot\|_{\ell_1,0,\mathrm{in}} \quad \textit{and} \quad \mathcal{Z} = \mathcal{Z}_{\ell_1,0,\mathrm{in}}.$$

Remark 8.5.8. *One can see that, for $f \in \mathcal{Z}_{\ell_1,\alpha}$ and $g \in \mathcal{Z}$, the norm $\|\cdot\|_{\ell_1,\alpha,\mathrm{in}}$ satisfies the property $\|fg\|_{\ell_1,\alpha,\mathrm{in}} \leq \|f\|_{\ell_1,\alpha}\|g\|_{\mathcal{Z}}$, Hence, we can adapt the proof of Proposition 8.5.7 to see that the linear operator$\mathcal{B}$ satisfies*

$$\|\mathcal{B}(\Delta\psi)\|_{\ell_1,2,\mathrm{in}} \leq M\,\|\Delta\psi\|_{\mathcal{Z}}.$$

We write equation (8.5.11) as an integral equation. To this end, for a function $h : \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, we define the following linear operator

$$\mathcal{A}(h) = \sum_{k\geq 0} \mathcal{A}_{2k+1}(h), \tag{8.5.13}$$

where

$$\mathcal{A}_1(h) = \frac{z^3}{5}\int_{-i\infty}^{z} \frac{\Pi_1[h](s)}{s^2}ds - \frac{1}{5z^2}\int_{-i\infty}^{z} s^3\Pi_1[h](s)ds,$$

and

$$\mathcal{A}_{2k+1}(h) = \int_{-i\infty}^{z} \frac{e^{-i\lambda_{0,2k+1}(s-z)}\Pi_{2k+1}[h](s)}{2i\lambda_{0,2k+1}}ds - \int_{-i\kappa}^{z} \frac{e^{i\lambda_{0,2k+1}(s-z)}\Pi_{2k+1}[h](s)}{2i\lambda_{0,2k+1}}ds, k \geq 1. \tag{8.5.14}$$

Lemma 8.5.9. *The operator $\widetilde{\mathcal{B}} : \mathcal{Z} \to \mathcal{Z}$ given by*

$$\widetilde{\mathcal{B}} = \mathcal{A} \circ \mathcal{B} \tag{8.5.15}$$

is well defined, where $\mathcal{B}$ and $\mathcal{A}$ are given by (8.5.12) *and* (8.5.13), *respectively. Moreover, there exists a constant $M > 0$ independent of κ such that, for each $\kappa \geq 1$,*

1. *the operators $\widetilde{\mathcal{B}}, \partial_\tau \circ \widetilde{\mathcal{B}} : \mathcal{Z} \to \mathcal{Z}$ satisfies*

$$\|\widetilde{\mathcal{B}}\|_{\mathcal{Z}}, \|\partial_\tau \circ \widetilde{\mathcal{B}}\|_{\mathcal{Z}} \leq \frac{M}{\kappa}; \tag{8.5.16}$$

2. *for each $h \in \mathcal{Z}$,*

$$\left\|\widetilde{\mathcal{B}}(h) - K(\kappa, h)e^{-i\lambda_{0,3}z}\sin(3\tau)\right\|_{\ell_1,1,\mathrm{in}} \leq M\|h\|_{\mathcal{Z}},$$

where $K(\kappa, h) \in \mathbb{C}$ is given by

$$K(\kappa, h) = -\int_{-i\kappa}^{-i\infty} \frac{e^{i\lambda_{0,3}s}\Pi_3[\mathcal{B}(h)](s)}{2i\lambda_{0,3}}ds. \tag{8.5.17}$$

Proof. First, we prove that the operator $\mathcal{A}:\mathcal{Z}_{\ell_1,2,\mathrm{in}}\to\mathcal{Z}$ is well defined and satisfies

$$\|\mathcal{A}(h)\|_{\mathcal{Z}},\|\partial_\tau\circ\mathcal{A}(h)\|_{\mathcal{Z}}\leq\frac{M}{\kappa}\|h\|_{\ell_1,2,\mathrm{in}}. \tag{8.5.18}$$

Let $h(z,\tau)=\sum_{k\geq0}h_{2k+1}(z)\sin((2k+1)\tau)\in\mathcal{Z}_{\ell_1,2,\mathrm{in}}$. We have that, for each $z\in\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$,

$$\begin{aligned}|\mathcal{A}_1(h)(z)e^{i\lambda_{0,3}z}| &\leq \frac{1}{5}\left(|z|^3\int_{-i\infty}^{z}\left|\frac{h_1(s)}{s^2}\right|e^{-\lambda_{0,3}\operatorname{Im}(z)}ds+\frac{1}{|z|^2}\int_{-i\infty}^{z}\left|s^3h_1(s)\right|e^{-\lambda_{0,3}\operatorname{Im}(z)}ds\right)\\ &\leq M\|h_1\|_{2,\mathrm{in}}\left(|z|^3\int_{-i\infty}^{z}\frac{e^{\lambda_{0,3}\operatorname{Im}(s-z)}}{|s|^4}ds+\frac{1}{|z|^2}\int_{-i\infty}^{z}|s|e^{\lambda_{0,3}\operatorname{Im}(s-z)}ds\right)\end{aligned}$$

Considering the integration path $s=i\operatorname{Im}(z)l$, $l\in(-\infty,1]$, and integrating by parts, we obtain

$$\begin{aligned}|\mathcal{A}_1(h)(z)e^{i\lambda_{0,3}z}| &\leq M\|h_1\|_{2,\mathrm{in}}\left(\frac{1}{|z|}\int_{-i\infty}^{z}e^{\lambda_{0,3}\operatorname{Im}(s-z)}ds+\frac{|\operatorname{Im}(z)|^2}{|z|^2}\int_{\infty}^{1}le^{\lambda_{0,3}\operatorname{Im}(z)(l-1)}dl\right)\\ &\leq M\|h_1\|_{2,\mathrm{in}}\left(\frac{1}{\lambda_{0,3}|z|}+\frac{M|\operatorname{Im}(z)|}{\lambda_{0,3}|z|^2}\right)\\ &\leq \frac{M}{\lambda_{0,3}|z|}\|h_1\|_{\mathcal{Z},\ell_1},\end{aligned}$$

and

$$\begin{aligned}|\mathcal{A}_3(h)(z)e^{i\lambda_{0,3}z}| &\leq \int_{-i\infty}^{z}\left|e^{-2i\lambda_{0,3}(s-z)}\frac{h_3(s)e^{i\lambda_{0,3}s}}{2i\lambda_{0,3}}\right|ds+\int_{-i\kappa}^{z}\left|\frac{e^{i\lambda_{0,3}s}h_3(s)}{2i\lambda_{0,3}}\right|ds\\ &\leq \frac{M\|h_3\|_{2,\mathrm{in}}}{\lambda_{0,3}}\left(\frac{1}{\kappa^2}\int_{-i\infty}^{z}e^{2\lambda_{0,3}\operatorname{Im}(s-z)}ds+\int_{-i\kappa}^{z}\frac{1}{|s|^2}ds\right)\\ &\leq \frac{M\|h_3\|_{2,\mathrm{in}}}{\lambda_{0,3}\kappa}.\end{aligned}$$

Now, for each $k\geq 2$, we have that $\lambda_{0,3}\leq\lambda_{0,2k+1}$ and thus,

$$\begin{aligned}&\left|\mathcal{A}_{2k+1}(h)(z)e^{i\lambda_{0,3}z}\right|\\ &\leq \int_{-i\infty}^{z}\left|e^{-i(\lambda_{0,2k+1}+\lambda_{0,3})(s-z)}\frac{h_{2k+1}(s)e^{i\lambda_{0,3}s}}{2i\lambda_{0,2k+1}}\right|ds+\int_{-i\kappa}^{z}\left|e^{i(\lambda_{0,2k+1}-\lambda_{0,3})(s-z)}\frac{h_{2k+1}(s)e^{i\lambda_{0,3}s}}{2i\lambda_{0,2k+1}}\right|ds\\ &\leq \frac{M\|h_{2k+1}\|_{2,\mathrm{in}}}{\lambda_{0,2k+1}}\left(\int_{-i\infty}^{z}\frac{e^{(\lambda_{0,2k+1}+\lambda_{0,3})\operatorname{Im}(s-z)}}{|s|^2}ds+\int_{-i\kappa}^{z}\frac{e^{-(\lambda_{0,2k+1}-\lambda_{0,3})\operatorname{Im}(s-z)}}{|s|^2}ds\right)\\ &\leq \frac{M\|h_{2k+1}\|_{2,\mathrm{in}}}{\lambda_{0,2k+1}}\left(\frac{1}{|z|^2}\int_{-i\infty}^{z}e^{(\lambda_{0,2k+1}+\lambda_{0,3})\operatorname{Im}(s-z)}ds+\int_{-i\kappa}^{z}\frac{e^{-(\lambda_{0,2k+1}-\lambda_{0,3})\operatorname{Im}(s-z)}}{|s|^2}ds\right).\end{aligned}$$

In the expression above, we compute the first integral and bound the second one by the maximum of the function and the length of the interval of integration. Thus, we get

$$\left|\mathcal{A}_{2k+1}(h)(z)e^{i\lambda_{0,3}z}\right|\leq\frac{M\|h_{2k+1}\|_{2,\mathrm{in}}}{\lambda_{0,2k+1}|z|},$$

for every $k \geq 2$. Consequently,

$$\|\mathcal{A}_{2k+1}(h)\|_{0,\text{in}} \leq \frac{M}{\lambda_{0,2k+1}\kappa}\|h_{2k+1}\|_{2,\text{in}},$$

for every $k \geq 1$, and (8.5.18) follows directly.

From Proposition 8.5.7 (see Remark 8.5.8), it follows that $\mathcal{B} : \mathcal{Z} \to \mathcal{Z}_{\ell_1,2,\text{in}}$ is well defined and

$$\|\mathcal{B}(\Delta\psi)\|_{\ell_1,2,\text{in}} \leq M\|\Delta\psi\|_{\mathcal{Z}}. \tag{8.5.19}$$

Hence, from the definition of $\widetilde{\mathcal{B}}$ given in (8.5.15), we have that $\widetilde{\mathcal{B}}, \partial_\tau \circ \widetilde{\mathcal{B}} : \mathcal{Z} \to \mathcal{Z}$ are well defined and (8.5.16) follows from (8.5.18) and (8.5.19). This proves item (1).

To prove item (2), let $h \in \mathcal{Z}$, and notice that, using similar arguments, we obtain

$$\begin{aligned} |\mathcal{A}_3(\mathcal{B}(h))e^{i\lambda_{0,3}z} - K(\kappa, h)\sin(3\tau)| &= \left| e^{i\lambda_{0,3}z}\int_{-i\infty}^{z} \frac{e^{-i\lambda_{0,3}(s-z)}\Pi_3[\mathcal{B}(h)](s)}{2i\lambda_{0,3}}ds \right. \\ &\quad \left. - \int_{-i\infty}^{z} \frac{e^{i\lambda_{0,3}s}\Pi_3[\mathcal{B}(h)](s)}{2i\lambda_{0,3}}ds \right| \\ &\leq \frac{M\|\Pi_3(\mathcal{B}(h))\|_{2,\text{in}}}{\lambda_{0,3}|z|}. \end{aligned}$$

It proves item (2). □

Corollary 8.5.10. *There exists $\kappa_0 \geq 1$ such that the linear operator* $\text{Id} - \widetilde{\mathcal{B}} : \mathcal{Z} \to \mathcal{Z}$ *is invertible, where $\widetilde{\mathcal{B}}$ is given by* (8.5.15)*.*

Proof. From item (1) of Lemma 8.5.9, there exists $\kappa_0 \geq 1$ sufficiently big such that, $\|\widetilde{\mathcal{B}}\|_{\mathcal{Z}} \leq 1/2$. Therefore, $\text{Id} - \widetilde{\mathcal{B}} : \mathcal{Z} \to \mathcal{Z}$ is invertible and $\|(\text{Id} - \widetilde{\mathcal{B}})^{-1}\|_{\mathcal{Z}} \leq 2$. □

Given a sequence $a = (a_{2k+1})_{k\geq 1}$, we define the function

$$\mathcal{C}_{\text{in}}(a)(z, \tau) = \sum_{k\geq 1} a_{2k+1}e^{-i\lambda_{0,2k+1}z}\sin((2k+1)\tau). \tag{8.5.20}$$

Proposition 8.5.11. *Let $\Delta\psi(z, \tau)$ be the function given in* (8.5.10) *and κ_0 given in Corollary 8.5.10. There exists a unique sequence of constants $b = (b_{2k+1})_{k\geq 1}$ such that $\mathcal{C}_{\text{in}}(b) \in \mathcal{Z}$ and $\Delta\psi$ satisfies*

$$\Delta\psi(z, \tau) = \mathcal{C}_{\text{in}}(b)(z, \tau) + \widetilde{\mathcal{B}}(\Delta\psi)(z, \tau), \tag{8.5.21}$$

for every $z \in \mathcal{R}^{\text{in},+}_{\theta,\kappa_0}$ and $\tau \in \mathbb{T}$, where $\widetilde{\mathcal{B}}$ and $\mathcal{C}_{\text{in}}$ are given by (8.5.15)*(with $\kappa = \kappa_0$) and* (8.5.20)*, respectively. Furthermore, $\Delta\psi \in \mathcal{Z}$.*

Proof. Recall that $\Pi_{2k}[\Delta\psi] = 0$, $\forall k \geq 0$ and denote $\Pi_{2k+1}[\Delta\psi] = \Delta\psi_{2k+1}$. Since $\Delta\psi$ satisfies (8.5.11), we use the Method of Variation of Constants as before to obtain

$$\Delta\psi_1 = \frac{z^3}{5}\left(\int_{z_0^1}^{z} \frac{\Pi_1\left[\mathcal{B}(\Delta\psi)\right](s)}{s^2}ds + C_0^1\right) - \frac{1}{5z^2}\left(\int_{z_1^1}^{z} s^3\Pi_1\left[\mathcal{B}(\Delta\psi)\right](s)ds + C_1^1\right),$$

and

$$\begin{aligned} \Delta\psi_{2k+1} &= e^{i\lambda_{0,2k+1}z}\left(\int_{z_0^{2k+1}}^{z} \frac{e^{-i\lambda_{0,2k+1}s}\Pi_{2k+1}\left[\mathcal{B}(\Delta\psi)\right](s)}{2i\lambda_{0,2k+1}}ds + C_0^{2k+1}\right) \\ &\quad -e^{-i\lambda_{0,2k+1}z}\left(\int_{z_0^{2k+1}}^{z} \frac{e^{i\lambda_{0,2k+1}s}\Pi_{2k+1}\left[\mathcal{B}(\Delta\psi)\right](s)}{2i\lambda_{0,2k+1}}ds + C_1^{2k+1}\right), \end{aligned}$$

where $z_0^{2k+1}, z_1^{2k+1}, C_0^{2k+1}$ and C_1^{2k+1} are constants, for each $k \geq 0$.

Recalling that $\|\Delta\psi\|_{\ell_1,3} \leq M$ (see Proposition 8.5.5), and taking $z \to -i\infty$, we obtain that the equations above are satisfied, if and only if

$$C_0^1 = -\int_{z_0^1}^{-i\infty} \frac{\Pi_1 \left[\mathcal{B}(\Delta\psi)\right](s)}{s^2} ds, \; C_1^1 = -\int_{z_1^1}^{-i\infty} s^3 \Pi_1 \left[\mathcal{B}(\Delta\psi)\right](s) ds$$

and

$$C_0^{2k+1} = -\int_{z_0^{2k+1}}^{-i\infty} \frac{e^{-i\lambda_{0,2k+1}s} \Pi_{2k+1} \left[\mathcal{B}(\Delta\psi)\right](s)}{2i\lambda_{0,2k+1}} ds.$$

Hence, choosing $z_1^{2k+1} = -i\kappa_0$, for every $k \geq 1$, we have that (8.5.21) is satisfied with constants $b_{2k+1} = C_1^{2k+1}$, $k \geq 1$. Using the expression of $\mathcal{A}_{2k+1}$ given in (8.5.14) (with $\kappa = \kappa_0$), we have that, for each $k \geq 1$,

$$|\mathcal{A}_{2k+1}(\mathcal{B}(\Delta\psi))(-i\kappa_0)| = \left| \int_{-i\infty}^{-i\kappa_0} \frac{e^{-i\lambda_{0,2k+1}(s+i\kappa_0)} \Pi_{2k+1} \left[\mathcal{B}(\Delta\psi)\right](s)}{2i\lambda_{0,2k+1}} ds \right| \leq \frac{M}{\kappa_0} \|\Pi_{2k+1} \left[\Delta\psi\right] \|_{\ell_1,0},$$

and thus, it follows from (8.5.15) that

$$\|\Delta\psi - \widetilde{\mathcal{B}}(\Delta\psi)\|_{\ell_1}(-i\kappa_0) \leq M.$$

Hence, $\|\mathcal{C}_{\text{in}}(b)\|_{\ell_1}(-i\kappa_0) \leq M$ and

$$\begin{aligned} \|\mathcal{C}_{\text{in}}(b)\|_{\mathcal{Z}} &= \sum_{k \geq 1} |b_{2k+1}| \|e^{-i\lambda_{0,2k+1}z}\|_{0,\text{in}} \\ &= \sum_{k \geq 0} |b_{2k+1}| e^{-(\lambda_{0,2k+1} - \lambda_{0,3})\kappa_0} \\ &= e^{\lambda_{0,3}\kappa} \|\mathcal{C}_{\text{in}}(b)\|_{\ell_1}(-i\kappa_0) \\ &\leq M e^{\lambda_{0,3}\kappa_0}, \end{aligned}$$

which proves that $\mathcal{C}_{\text{in}}(b) \in \mathcal{Z}$. Finally, it follows from Corollary 8.5.10, that

$$\Delta\psi = (\text{Id} - \widetilde{\mathcal{B}})^{-1}(\mathcal{C}_{\text{in}}(b)) \in \mathcal{Z}.$$

□

Finally, we prove the second statement of Theorem 8.3.2. Let

$$K_0 = K(\kappa_0, \Delta\psi), \tag{8.5.22}$$

where K, κ_0 and $\Delta\psi$ are given by (8.5.17), Corollary 8.5.10 and (8.5.10), respectively. Take

$$C_{\text{in}} = b_3 + K_0,$$

where b_3 is the first term of the sequence b given in 8.5.11 and K_0 is given by (8.5.22). Also, notice that $\widetilde{\mathcal{C}_{\text{in}}(b)} = \mathcal{C}_{\text{in}}(b) - b_3 e^{-i\lambda_{0,3}z} \sin(3\tau) \in \mathcal{Z}_{\ell_1,1,\text{in}}$.

Thus, using the second item of Lemma 8.5.9 (with $\kappa = \kappa_0$ and $h = \Delta\psi$) and Proposition 8.5.11, we have that

$$\begin{aligned}\|\Delta\psi - C_{\rm in}e^{-i\lambda_{0,3}z}\sin(3\tau)\|_{\ell_1,1,{\rm in}} &= \|\Delta\psi - \mathcal{C}_{\rm in}(b) - K_0 e^{-i\lambda_{0,3}z} + \widetilde{\mathcal{C}_{\rm in}(b)}]\|_{\ell_1,1,{\rm in}}\\ &\leq \|\widetilde{\mathcal{B}}(\Delta\psi) - K_0 e^{-i\lambda_{0,3}z}\|_{\ell_1,1,{\rm in}} + \|\widetilde{\mathcal{C}_{\rm in}(b)}\|_{\ell_1,1,{\rm in}}\\ &\leq M(1+\|\Delta\psi\|_{\ell_1,1,{\rm in}})\\ &\leq M_0.\end{aligned}$$

Taking

$$\chi(z,\tau) = e^{i\lambda_{0,3}z}\left(\Delta\psi - C_{\rm in}e^{-i\lambda_{0,3}z}\sin(3\tau)\right),$$

it follows that

$$\|\chi\|_{\ell_1,1}, \|\partial_\tau\chi\|_{\ell_1,1} \leq M,$$

and formula (8.3.9) holds for every $\kappa \geq \kappa_0$, since $\mathcal{R}^{{\rm in},+}_{\theta,\kappa} \subset \mathcal{R}^{{\rm in},+}_{\theta,\kappa_0}$ provided that $\kappa \geq \kappa_0$.

Finally, it follows from item (5) of Proposition 8.5.1 that $\partial_z\chi \in \mathcal{X}_{\ell_1,\alpha+1}$ (with respect to the domain $\mathcal{R}^{{\rm in},+}_{2\theta,4\kappa_0}$) and

$$\|\partial_z\chi\|_{\ell_1,2} \leq M.$$

The proof of Theorem 8.3.2 follows by reducing the initial domain $\mathcal{R}^{{\rm in},+}_{\theta,\kappa_0}$ to $\mathcal{R}^{{\rm in},+}_{2\theta,4\kappa_0}$. In order to simplify the notation, we make no distinction between $\mathcal{R}^{{\rm in},+}_{\theta,\kappa_0}$ and $\mathcal{R}^{{\rm in},+}_{2\theta,4\kappa_0}$.

8.6 Proof of Theorem 8.3.3

As usual, we consider only the unstable case, and in order to simplify the notation, we omit the superscript u of the solutions. Also, throughout this section, we change the domain $D^{u,{\rm in}}_{\theta,\kappa}$ by $D^{{\rm mch},u}_{+,\kappa}$ (see (8.3.7) and (8.3.10))in the definition of the norms and Banach spaces introduced in Section 8.5.1, but we keep the same notation.

We begin by studying the equation satisfied by the difference

$$\varphi(z,\tau) = \phi(z,\tau) - \phi^0(z,\tau). \tag{8.6.1}$$

Proposition 8.6.1. *The function $\varphi : D^{{\rm mch},u}_{+,\kappa} \times \mathbb{T} \to \mathbb{C}$ given by* (8.6.1) *satisfies the following differential equation*

$$\mathcal{I}(\varphi)(z,\tau) = \mathcal{C}_{\rm mch}(z,\tau) + \left(L(\varphi)(z) + \widehat{L}(\widehat{\Pi}[\varphi])(z)\right)\sin(\tau) + K(\varphi)(z,\tau),$$

where $\mathcal{I}$ is the operator given by (8.5.5)*, $L : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\alpha+4}$, $\widehat{L} : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\alpha+2}$, and $K : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+2}$ are linear operators and $\mathcal{C}_{\rm mch} : D^{{\rm mch},u}_{+,\kappa} \times \mathbb{T} \to \mathbb{C}$ is an analytic function in the variable z given by*

$$\mathcal{C}_{\rm mch}(z,\tau) = -\frac{2\sqrt{2}i\varepsilon^2}{z}\sin(\tau) + d_1(z)\sin(\tau) + d_2(z,\tau). \tag{8.6.2}$$

Moreover, $\Pi_1 \circ K \equiv 0$, and there exists a constant $M > 0$ independent of ε and κ such that

1. $\|L(\varphi)\|_{\alpha+4} \leq M\|\varphi\|_{\ell_1,\alpha}$;
2. $\|\widehat{L}(\varphi)\|_{\alpha+2} \leq M\|\varphi\|_{\ell_1,\alpha}$;

3. $\|K(\varphi)\|_{\ell_1,\alpha+2} \leq M\|\varphi\|_{\ell_1,\alpha}$*;*

4. $\|d_2\|_{\ell_1,3} \leq M\varepsilon^2$ *and* $|d_1(z)| \leq M\varepsilon^4|z|$*, for every* $z \in D^{\mathrm{mch},u}_{+,\kappa}$*.*

Proof. Since ϕ and ϕ^0 satisfy (8.3.5) and (8.3.6), respectively, we have that $\varphi(z,\tau)$ satisfies

$$\partial_z^2\varphi - \partial_\tau^2\varphi - \varphi = -\varepsilon^2\partial_\tau^2\phi - \frac{1}{3}(\phi^3 - (\phi^0)^3) - f(\phi) + f(\phi^0). \tag{8.6.3}$$

Now, recall that $\phi(z,\tau) = \varepsilon v(i\pi/2 + \varepsilon z, \tau)$, where $v(y,\tau) = v_1^h(y)\sin(\tau) + \xi(y,\tau)$, v_1^h is given by (8.1.17) and ξ is given by Theorem 8.3.1. An easy computation shows that

$$\varepsilon v_1^h(i\pi/2 + \varepsilon z) = -\frac{2\sqrt{2}i}{z} + l_1(z,\varepsilon),$$

where l_1 is an analytic function such that $|l_1(z,\varepsilon)| \leq M\varepsilon^2|z|$, for each $z \in D^{\mathrm{mch},u}_{+,\kappa}$. For the sake of simplicity, we omit the dependence of ł$_1$ on ε. Thus,

$$\phi(z,\tau) = -\frac{2\sqrt{2}i}{z}\sin(\tau) + l_1(z)\sin(\tau) + l_2(z,\tau), \tag{8.6.4}$$

where $l_2(z,\tau) = \varepsilon\xi(i\pi/2 + \varepsilon z, \tau)$.

Notice that, since $y = i\pi/2 + \varepsilon z$, we have

$$\left|(y - i\pi/2)^3\Pi_n\left[\partial_\tau^2\xi(y,\tau)\right]\right| = \varepsilon^3\left|z^3\Pi_n\left[\partial_\tau^2\xi(i\pi/2 + \varepsilon z, \tau)\right]\right|, \forall\ n \geq 1,$$

and thus, from Proposition 8.4.5,

$$\|l_2\|_{\ell_1,3} = \|\varepsilon\partial_\tau^2\xi(i\pi/2 + \varepsilon z, \tau)\|_{\ell_1,3} \leq \frac{1}{\varepsilon^2}\|\partial_\tau^2\xi(y,\tau)\|_{\ell_1,1,3} \leq M,$$

where $\|\cdot\|_{\ell_1,1,3}$ is the norm introduced in Section 8.4.1.

Differentiating ϕ with respect to τ twice, we obtain

$$\partial_\tau^2\phi(z,\tau) = \frac{2\sqrt{2}i}{z}\sin(\tau) - l_1(z)\sin(\tau) + \partial_\tau^2 l_2(z,\tau). \tag{8.6.5}$$

Since $M\kappa \leq |z| \leq M\varepsilon^{\gamma-1}$, for every $z \in D^{\mathrm{mch},u}_{+,\kappa}$, and

$$\left\|\phi^0(z,\tau) + \frac{2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,3} \leq M, \tag{8.6.6}$$

we obtain that

$$\begin{aligned} -\frac{1}{3}\left(\phi^3 - (\phi^0)^3\right) &= -\frac{1}{3}\left(\phi^2 + \phi\phi^0 + (\phi^0)^2\right)\varphi \\ &= \frac{6}{z^2}\Pi_1[\varphi]\sin(\tau) - \frac{2}{z^2}\Pi_1[\varphi]\sin(3\tau) + l_3(\widehat{\Pi}\,[\varphi]) + l_4(\varphi) + l_5(z,\tau), \end{aligned} \tag{8.6.7}$$

where $l_3 : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+2}$, $l_4 : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+4}$, are linear operators such that

$$\|l_3(\widehat{\Pi}\,[\varphi])\|_{\ell_1,\alpha+2} \leq M\|\widehat{\Pi}\,[\varphi]\,\|_{\ell_1,\alpha} \quad \text{and} \quad \|l_4(\varphi)\|_{\ell_1,\alpha+4} \leq M\|\varphi\|_{\ell_1,\alpha},$$

and $l_5 : D^{\mathrm{mch},u}_{+,\kappa} \times \mathbb{T} \to \mathbb{C}$ is an analytic function in the variable z such that

$$\|l_5\|_{\ell_1,3} \le M\varepsilon^2.$$

Also, using that $\|\phi\|_{\ell_1,1}, \|\phi^0\|_{\ell_1,1} \le M$, $f(z) = \mathcal{O}(z^5)$, and the Mean Value Theorem, we have that

$$-f(\phi) + f(\phi^0) = -\varphi \int_0^1 f'(s\phi + (1-s)\phi^0)ds = l_6(\varphi), \tag{8.6.8}$$

where $l_6 : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+4}$, is a linear operator such that

$$\|l_6(\varphi)\|_{\ell_1,\alpha+4} \le M\|\varphi\|_{\ell_1,\alpha}.$$

Taking,

- $\widehat{L}(\widehat{\Pi}\,[\varphi]) = \Pi_1 \left[l_3(\widehat{\Pi}\,[\varphi])\right]$,
- $L(\varphi) = \Pi_1 \left[l_4(\varphi) + l_6(\varphi)\right]$,
- $K(\varphi) = \widehat{\Pi}\left[l_3(\widehat{\Pi}\,[\varphi]) - \dfrac{2}{z^2}\Pi_1[\varphi]\sin(3\tau) + l_4(\varphi) + l_6(\varphi)\right]$
- $d_1(z) = \varepsilon^2 l_1(z)$, and $d_2(z,\tau) = -\varepsilon^2\partial_\tau^2 l_2(z,\tau) + l_5(z,\tau)$,

the proof follows from (8.6.3), (8.6.5), (8.6.7) and (8.6.8).

□

Let $z_j = \varepsilon^{-1}(y_j - i\pi/2)$, $j = 1, 2$, where y_1 and y_2 are the vertices of the matching domain $D^{\mathrm{mch},u}_{+,\kappa}$ given by (8.3.10). Consider the following linear operator acting on the Fourier coefficients of $h = \sum_{k\ge 0} h_{2k+1}(z)\sin((2k+1)\tau)$.

$$\mathcal{T}(\varphi) = \sum_{k\ge 0} \mathcal{T}_{2k+1}(h_{2k+1})\sin((2k+1)\tau), \tag{8.6.9}$$

where $\mathcal{T}_{2k+1}$ is given by

$$\begin{aligned} \mathcal{T}_1(h_1) \;&=\; \frac{z^3}{5}\int_{z_1}^{z}\frac{h_1(s)}{s^2}ds - \frac{1}{5z^2}\int_{z_2}^{z} h_1(s)s^3 ds \\ &\quad -\frac{1}{5(z_2^5 - z_1^5)}\left(\left(z^3 - \frac{z_2^5}{z^2}\right)\int_{z_2}^{z_1} h_1(s)s^3 ds + \left(z^3 z_2^5 - \frac{(z_1 z_2)^5}{z^2}\right)\int_{z_1}^{z_2}\frac{h_1(s)}{s^2}ds\right), \end{aligned} \tag{8.6.10}$$

and

$$\begin{aligned} \mathcal{T}_{2k+1}(h_{2k+1}) \;&=\; \int_{z_2}^{z}\frac{h_{2k+1}(s)e^{-i\lambda_{0,2k+1}(s-z)}}{2i\lambda_{0,2k+1}}ds - \int_{z_1}^{z}\frac{h_{2k+1}(s)e^{i\lambda_{0,2k+1}(s-z)}}{2i\lambda_{0,2k+1}}ds, \\ &\quad + \frac{\sin(\lambda_{0,2k+1}(z_2 - z))}{\sin(\lambda_{0,2k+1}(z_1 - z_2))}\int_{z_2}^{z_1}\frac{h_{2k+1}(s)e^{-i\lambda_{0,2k+1}(s-z_1)}}{2i\lambda_{0,2k+1}}ds \\ &\quad + \frac{\sin(\lambda_{0,2k+1}(z_1 - z))}{\sin(\lambda_{0,2k+1}(z_1 - z_2))}\int_{z_1}^{z_2}\frac{h_{2k+1}(s)e^{-i\lambda_{0,2k+1}(s-z_2)}}{2i\lambda_{0,2k+1}}ds, \text{ for every } k \ge 1. \end{aligned} \tag{8.6.11}$$

Also, consider the following function $\mathcal{Q} : D^{\mathrm{mch},u}_{+,\kappa} \times \mathbb{T} \to \mathbb{C}$, which is analytic in the variable z, given by

$$\mathcal{Q}(z,\tau;\varphi) = \sum_{k\geq 0} \mathcal{Q}_{2k+1}(z_1,z_2;\varphi_{2k+1})(z)\sin((2k+1)\tau), \tag{8.6.12}$$

where

$$\mathcal{Q}_1(z_1,z_2,\varphi_1)(z) = \frac{1}{z_2^5 - z_1^5}\left(z^3(z_2^2\varphi_1(z_2) - z_1^2\varphi_1(z_1)) - \frac{1}{z^2}\left(z_1^5 z_2^2\varphi_1(z_2) - z_1^2 z_2^5\varphi_1(z_1)\right)\right), \tag{8.6.13}$$

for $k = 0$,

$$\mathcal{Q}_{2k+1}(z_1,z_2,\varphi_{2k+1})(z) = \frac{\sin(\lambda_{0,2k+1}(z-z_2))}{\sin(\lambda_{0,2k+1}(z_1-z_2))}\varphi_{2k+1}(z_1) - \frac{\sin(\lambda_{0,2k+1}(z-z_1))}{\sin(\lambda_{0,2k+1}(z_1-z_2))}\varphi_{2k+1}(z_2), \tag{8.6.14}$$

for $k \geq 1$, and φ is given by (8.6.1).

Remark 8.6.2. *Notice that the functions $\mathcal{Q}_{2k+1}$, $k \geq 0$, are chosen in such way that $\Pi_{2k+1}[\mathcal{Q} + \mathcal{T}(h)](z_j) = \varphi_{2k+1}(z_j)$, $j = 1,2$ and $k \geq 0$.*

Proposition 8.6.3. *Consider the operator $\mathcal{I}$ given by* (8.5.5)*. Let $h, \widehat{\varphi} : D^{\mathrm{mch},u}_{+,\kappa} \times \mathbb{C} \to \mathbb{C}$ be functions which are analytic in the variable z and assume that*

$$\mathcal{I}(\widehat{\varphi}) = h,$$

and $\widehat{\varphi}(z_j) = \varphi(z_j)$, $j = 1,2$, where φ is given in (8.6.1)*, and z_1, z_2 are the vertices of the matching domain $D^{\mathrm{mch},u}_{+,\kappa}$ given by* (8.3.10)*. Therefore, there exist angles β_0 and β_2 of* (8.3.10)*, such that*

$$\widehat{\varphi}(z,\tau) = \mathcal{Q}(z,\tau;\varphi) + \mathcal{T}(h)(z,\tau),$$

where $\mathcal{T}$ and $\mathcal{Q}$ are given by (8.6.9) *and* (8.6.12)*, respectively.*

Proof. Denote $\Pi_n[h] = h_n$, $\Pi_n \circ \mathcal{I} = \mathcal{I}_n$, $\Pi_1[\varphi]$ and $\Pi_n[\widehat{\varphi}] = \widehat{\varphi}_n$. First, we consider the operator $\mathcal{I}_1$ (see (8.5.5)) and we solve the equation $\mathcal{I}_1(\widehat{\varphi}_1) = h_1$. Considering the solutions $\zeta_1(z) = z^3$ and $\zeta_2(z) = -z^2/5$ of the homogeneous equation $\mathcal{I}_1(\widehat{\varphi}_1) = 0$, and applying the method of variation of constants, we obtain

$$\widehat{\varphi}_1(z) = \frac{z^3}{5}\left(\int_{z_1}^{z}\frac{h_1(s)}{s^2}ds + C_1^1\right) - \frac{1}{5z^2}\left(\int_{z_2}^{z} h_1(s)s^3 ds + C_2^1\right),$$

where z_1 and z_2 are the vertices of the matching domain $D^{\mathrm{in},+,u}_{\kappa,c}$.

Recall that the function φ_1 is already known, therefore the points $\varphi_1(z_1)$ and $\varphi_1(z_2)$ are already given, and using the initial conditions $\widehat{\varphi}_1(z_1) = \varphi_1(z_1)$ and $\widehat{\varphi}_1(z_2) = \varphi_1(z_2)$, we determine the constants C_1^1 and C_2^1. In fact,

$$\begin{pmatrix} 5^{-1}z_1^3 & -(5z_1^2)^{-1} \\ 5^{-1}z_2^3 & -(5z_2^2)^{-1}\end{pmatrix}\begin{pmatrix} C_1^1 \\ C_2^1\end{pmatrix} = \begin{pmatrix} \varphi_1(z_1) + \dfrac{1}{5z_1^2}\displaystyle\int_{z_2}^{z_1} h_1(s)s^3 ds \\ \varphi_1(z_2) - \dfrac{z_2^3}{5}\displaystyle\int_{z_1}^{z_2}\frac{h_1(s)}{s^2}ds\end{pmatrix},$$

and since $\|z_1\| \neq \|z_2\|$ (see (8.3.10)), we have that the matrix on the right side of the last equation is invertible, and thus, we obtain the values of C_1^1 and C_2^1. Therefore,

$$\widehat{\varphi}_1(z) = \mathcal{Q}_1(z_1, z_2, \varphi_1)(z) + \mathcal{T}_1(h_1)(z),$$

where $\mathcal{T}_1$ is the linear operator given in (8.6.10), and $\mathcal{Q}_1$ is the independent term given in (8.6.13).

Proceeding in the same way for the higher modes, we obtain that the homogeneous equation $\mathcal{I}_{2k+1}(\widehat{\varphi}_{2k+1}) = 0$ has independent solutions $\zeta^{\mp}_{2k+1} = e^{\pm i\lambda_{0,2k+1}z}$, where $\lambda_{0,2k+1} = \sqrt{(2k+1)^2 - 1}$, and thus applying the method of variation of constants to the equation $\mathcal{I}_{2k+1}(\widehat{\varphi}_{2k+1}) = h_{2k+1}$, we obtain

$$\begin{aligned} \widehat{\varphi}_{2k+1}(z) &= -\frac{e^{-i\lambda_{0,2k+1}z}}{2i\lambda_{0,2k+1}} \left(\int_{z_1}^{z} h_{2k+1}(s) e^{i\lambda_{0,2k+1}s} ds + C_1^{2k+1} \right) \\ &\quad + \frac{e^{i\lambda_{0,2k+1}z}}{2i\lambda_{0,2k+1}} \left(\int_{z_2}^{z} h_{2k+1}(s) e^{-i\lambda_{0,2k+1}s} ds + C_2^{2k+1} \right). \end{aligned}$$

Again, using that the function $\varphi_{2k+1}(z)$ is already known and imposing the initial conditions $\widehat{\varphi}_{2k+1}(z_1) = \varphi_{2k+1}(z_1)$ and $\widehat{\varphi}_{2k+1}(z_2) = \varphi_{2k+1}(z_2)$, we determine the constants C_1^{2k+1} and C_2^{2k+1} through the following system

$$\begin{pmatrix} -e^{-i\lambda_{0,2k+1}z_1} & e^{i\lambda_{0,2k+1}z_1} \\ -e^{-i\lambda_{0,2k+1}z_2} & e^{i\lambda_{0,2k+1}z_2} \end{pmatrix} \begin{pmatrix} C_1^{2k+1} \\ C_2^{2k+1} \end{pmatrix} =$$
$$\begin{pmatrix} 2i\lambda_{0,2k+1}\varphi_{2k+1}(z_1) - e^{i\lambda_{0,2k+1}z_1} \displaystyle\int_{z_2}^{z_1} h_{2k+1}(s) e^{-i\lambda_{0,2k+1}s} ds \\ 2i\lambda_{0,2k+1}\varphi_{2k+1}(z_2) + e^{-i\lambda_{0,2k+1}z_2} \displaystyle\int_{z_1}^{z_2} h_{2k+1}(s) e^{i\lambda_{0,2k+1}s} ds \end{pmatrix}.$$

Since $\mathrm{Im}(z_1) \neq \mathrm{Im}(z_2)$ (see (8.3.10)), we have that $e^{i\lambda_{0,2k+1}(z_1 - z_2)} - e^{-i\lambda_{0,2k+1}(z_1 - z_2)} \neq 0$, for every $k \geq 1$. Thus, the matrix on the right of the equation above is invertible and consequently, we obtain the values of C_1^{2k+1} and C_2^{2k+1}. Hence,

$$\widehat{\varphi}_{2k+1}(z) = \mathcal{Q}_{2k+1}(z_1, z_2, \varphi_{2k+1})(z) + \mathcal{T}_{2k+1}(h_{2k+1})(z),$$

where $\mathcal{T}_{2k+1}$ is the linear operator given in (8.6.11), and $\mathcal{Q}_{2k+1}$ is the independent term given in (8.6.14), for $k \geq 1$. The proof is complete. □

Now, we study the operator $\mathcal{T}$ given in (8.6.9) in some appropriate Banach spaces.

Proposition 8.6.4. *The following statements hold.*

1. *Given $\alpha \geq 4$, the linear operator $\mathcal{T} : \mathcal{X}_\alpha \to \mathcal{X}_{\alpha-2}$ is well defined and there exists a constant M independent of ε and κ such that*

$$\|\mathcal{T}(h)\|_{\ell_1,\alpha-2} \leq M \|h\|_{\ell_1,\alpha} \quad \text{and} \quad \|\partial_\tau \circ \mathcal{T}(h)\|_{\ell_1,\alpha-2} \leq M \|h\|_{\ell_1,\alpha}.$$

2. *Given $\alpha \geq 0$, the linear operator $\mathcal{T} : \mathcal{X}^1_\alpha \to \mathcal{X}^1_\alpha$ is well defined, where $\mathcal{X}^1_\alpha$ is the Banach space*

$$\mathcal{X}^1_\alpha = \{h \in \mathcal{X}_\alpha;\ \Pi_1[h] = 0\},$$

and there exists a constant M independent of ε and κ such that

$$\|\mathcal{T}(h)\|_{\ell_1,\alpha} \leq M \|h\|_{\ell_1,\alpha} \quad \text{and} \quad \|\partial_\tau \circ \mathcal{T}(h)\|_{\ell_1,\alpha} \leq M \|h\|_{\ell_1,\alpha}.$$

3. Given $2 \leq \alpha \leq 3$*, there exists a constant* $M > 0$ *independent of* ε *and* κ *such that*

$$\|\mathcal{Q}(z_1, z_2, \varphi)\|_{\ell_1,\alpha} \leq M(\varepsilon^{(\alpha-3)(\gamma-1)} + \varepsilon^{2+(\alpha+1)(\gamma-1)}),$$

where φ *is given in* (8.6.1).

Proof. Notice that, for every $z \in D^{\mathrm{in},+,u}_{\kappa,c}$, we have $|z_i| \geq M|z|$, $i = 1, 2$, for some constant $M > 0$ independent of ε and k. Also, $|z_2|^5 - |z_1|^5 \geq M|z_2|^5 \geq M|z|^5$, for every $z \in D^{\mathrm{in},+,u}_{\kappa,c}$. Now, using these properties, we show the following bounds for $\alpha \geq 4$.

$$\begin{aligned}\left|\frac{z^3}{5}\int_{z_1}^{z}\frac{h_1(s)}{s^2}ds\right| &\leq M\|h_1\|_\alpha|z|^3\int_{z_1}^{z}\frac{1}{|s|^{2+\alpha}}ds\\ &\leq M\|h_1\|_\alpha|z|^3\left(\frac{1}{|z|^{1+\alpha}}+\frac{1}{|z_1|^{1+\alpha}}\right)\\ &\leq M\|h_1\|_\alpha|z|^{2-\alpha},\end{aligned}$$

$$\begin{aligned}\left|\frac{1}{5z^2}\int_{z_2}^{z}h_1(s)s^3ds\right| &\leq \frac{M\|h_1\|_\alpha}{|z|^2}\int_{z_2}^{z}|s|^{3-\alpha}ds\\ &\leq \frac{M\|h_1\|_\alpha}{|z|^2}|z_2|^{3-\alpha}|z-z_2|\\ &\leq \frac{M\|h_1\|_\alpha}{|z|^{\alpha-2}}\left(|z_2|^{3-\alpha}|z|^{\alpha-3}+|z_2|^{4-\alpha}|z|^{\alpha-4}\right)\\ &\leq M\|h_1\|_\alpha|z|^{2-\alpha},\end{aligned}$$

$$\begin{aligned}\left|\frac{1}{5(z_2^5-z_1^5)}\left(z^3-\frac{z_2^5}{z^2}\right)\int_{z_2}^{z_1}h_1(s)s^3ds\right| &\leq \frac{M\|h\|_\alpha}{|z|^2}\int_{z_2}^{z_1}|s|^{3-\alpha}ds\\ &\leq \frac{M\|h_1\|_\alpha}{|z|^2}|z_2|^{3-\alpha}|z_1-z_2|\\ &\leq \frac{M\|h_1\|_\alpha}{|z|^{\alpha-2}}|z|^{\alpha-4}|z_2|^{4-\alpha}\\ &\leq M\|h_1\|_\alpha|z|^{2-\alpha},\end{aligned}$$

$$\begin{aligned}\left|\frac{1}{5(z_2^5-z_1^5)}\left(z^3z_2^5-\frac{(z_1z_2)^5}{z^2}\right)\int_{z_1}^{z_2}\frac{h_1(s)}{s^2}ds\right| &\leq M\|h_1\|_\alpha\left(|z|^3+\frac{|z_2|^5}{|z|^2}\right)\int_{z_1}^{z_2}\frac{1}{|s|^{2+\alpha}}ds\\ &\leq M\|h_1\|_\alpha\left(|z|^3+\frac{|z_2|^5}{|z|^2}\right)\frac{1}{|z_2|^{1+\alpha}}\\ &\leq \frac{M\|h_1\|_\alpha}{|z|^{\alpha-2}}\left(|z|^{1+\alpha}|z_2|^{-1-\alpha}+|z|^{\alpha-4}|z_2|^{4-\alpha}\right)\\ &\leq M\|h_1\|_\alpha|z|^{2-\alpha}.\end{aligned}$$

Hence, we can see that

$$\|\mathcal{T}_1(h_1)\|_{\alpha-2} \leq M\|h_1\|_\alpha, \ \alpha \geq 4. \tag{8.6.15}$$

Now, to deal with the higher modes, we will see that

$$\left|\frac{\sin(\lambda_{0,2k+1}(z_j-z))}{\sin(\lambda_{0,2k+1}(z_1-z_2))}\right| \leq M, \ j = 1, 2, \ \forall\, z \in D^{\mathrm{mch},u}_{+,\kappa}, \ \forall\, k \geq 1, \tag{8.6.16}$$

where $M > 0$ is independent of ε and k.

In fact, recalling that $|\sin^2(z)|= \frac{1}{2}(\cosh(2\operatorname{Im}(z)) - \cos(2\operatorname{Re}(z)))$, we have

$$\begin{aligned}\left|\frac{\sin(\lambda_{0,2k+1}(z_j - z))}{\sin(\lambda_{0,2k+1}(z_1 - z_2))}\right|^2 &= \frac{\cosh(2\lambda_{0,2k+1}\operatorname{Im}(z_j - z)) - \cos(2\lambda_{0,2k+1}\operatorname{Re}(z_j - z))}{\cosh(2\lambda_{0,2k+1}\operatorname{Im}(z_1 - z_2)) - \cos(2\lambda_{0,2k+1}\operatorname{Re}(z_1 - z_2))}\\ &\leq \frac{\cosh(2\lambda_{0,2k+1}\operatorname{Im}(z_j - z)) + 1}{\cosh(2\lambda_{0,2k+1}\operatorname{Im}(z_1 - z_2)) - 1},\end{aligned}$$

and since $\operatorname{Im}(z_1 - z_2) = K\varepsilon^{\gamma-1}$ and $|\operatorname{Im}(z_j - z)|\leq |\operatorname{Im}(z_1 - z_2)|$, we obtain that

$$\left|\frac{\sin(\lambda_{0,2k+1}(z_j - z))}{\sin(\lambda_{0,2k+1}(z_1 - z_2))}\right|^2 \leq M\frac{\cosh(2\lambda_{0,2k+1}\operatorname{Im}(z_j - z)) + 1}{\cosh(2\lambda_{0,2k+1}\operatorname{Im}(z_1 - z_2))} \leq 2M.$$

Now, assume that $\alpha \geq 0$. For each $z \in D^{\mathrm{mch},u}_{+,\kappa}$, there exist β_1^*, β_2^* (depending on z) between β_0 and β_2 and $t_2^*, t_1^* > 0$ (depending on z) such that $z_2 = z + e^{-i\beta_2^*}t_2^*$ and $z_1 = z + e^{i\beta_1^*}t_1^*$. Thus, we have that

$$\begin{aligned}\left|\int_{z_2}^{z} h_{2k+1}(s)e^{-i\lambda_{0,2k+1}(s-z)}ds\right| &= \left|\int_{t_2^*}^{0} h_{2k+1}\left(z + e^{-i\beta_2^*}t\right) e^{-i\lambda_{0,2k+1}te^{-i\beta_2^*}} e^{-i\beta_2^*}dt\right|\\ &\leq \int_0^{t_2^*} \left|h_{2k+1}\left(z + e^{-i\beta_2^*}t\right)\right| e^{-\lambda_{0,2k+1}\sin(\beta_2^*)t}dt\\ &\leq \|h_{2k+1}\|_\alpha \int_0^{t_2^*} \frac{e^{-\lambda_{0,2k+1}\sin(\beta_2^*)t}}{|z + e^{-i\beta_2^*}t|^\alpha}dt\\ &\leq \frac{\|h_{2k+1}\|_\alpha}{|z|^\alpha}\int_0^{\infty} e^{-\lambda_{0,2k+1}\sin(\beta_2^*)t}dt\\ &\leq \frac{\|h_{2k+1}\|_\alpha}{\lambda_{0,2k+1}\sin(\beta_2^*)|z|^\alpha}\\ &\leq \frac{M\|h_{2k+1}\|_\alpha}{\lambda_{0,2k+1}|z|^\alpha}\end{aligned}$$

Analogously, we prove that

$$\left|\int_{z_1}^{z} h_{2k+1}(s)e^{i\lambda_{0,2k+1}(s-z)}ds\right| \leq \frac{M\|h_{2k+1}\|_\alpha}{\lambda_{0,2k+1}|z|^\alpha},$$

and in particular,

$$\left|\int_{z_2}^{z_1} h_{2k+1}(s)e^{-i\lambda_{0,2k+1}(s-z_1)}ds\right| \leq \frac{M\|h_{2k+1}\|_\alpha}{\lambda_{0,2k+1}|z_1|^\alpha} \leq \frac{M\|h_{2k+1}\|_\alpha}{\lambda_{0,2k+1}|z|^\alpha},$$

and

$$\left|\int_{z_1}^{z_2} h_{2k+1}(s)e^{i\lambda_{0,2k+1}(s-z_2)}ds\right| \leq \frac{M\|h_{2k+1}\|_\alpha}{\lambda_{0,2k+1}|z_2|^\alpha} \leq \frac{M\|h_{2k+1}\|_\alpha}{\lambda_{0,2k+1}|z|^\alpha}.$$

Hence,

$$\|\mathcal{T}_{2k+1}(h_{2k+1})\|_\alpha\leq \frac{M}{\lambda_{0,2k+1}}\|h_{2k+1}\|_\alpha,\ k \geq 1,\ \alpha \geq 0. \tag{8.6.17}$$

Items (1) and (2) follows from (8.6.15) and (8.6.17).

Now, using (8.6.4) and (8.6.6), we obtain that

$$\varphi(z,\tau) = l_1(z)\sin(\tau) + b(z,\tau),$$

where

$$\|b\|_{\ell_1,3} \leq M$$

and $|l_1(z)| \leq M\varepsilon^2|z|$, for each $z \in D^{\mathrm{mch},u}_{+,\kappa}$. Notice that $b(z,\tau) = l_2(z,\tau) - (\phi_0(z,\tau) + 2\sqrt{2}i/z\sin(\tau))$, where l_2 is the function given in (8.6.4).

Thus, we can see that

$$\begin{aligned}
|\mathcal{Q}_1(z_1,z_2,\varphi_1)(z)| &= \left|\frac{1}{z_2^5 - z_1^5}\left(z^3(z_2^2\varphi_1(z_2) - z_1^2\varphi_1(z_1)) - \frac{1}{z^2}\left(z_1^5 z_2^2\varphi_1(z_2) - z_1^2 z_2^5\varphi_1(z_1)\right)\right)\right| \\
&\leq M\left(|\varphi_1(z_1)| + |\varphi_1(z_2)| + \frac{|z_1^2|}{|z|^2}|\varphi_1(z_1)| + \frac{|z_2^2|}{|z|^2}|\varphi_1(z_2)|\right) \\
&\leq M\left(\frac{1}{|z_2||z|^2} + \varepsilon^2|z_2| + \frac{\varepsilon^2|z_2|^3}{|z|^2}\right).
\end{aligned}$$

Now, recalling that $|z_2| = M\varepsilon^{\gamma-1}$ and $M\kappa \leq |z| \leq M\varepsilon^{\gamma-1}$, for every $z \in D^{\mathrm{in},+,u}_{\kappa,c}$, we obtain that, for $\alpha \geq 2$,

$$\|\mathcal{Q}_1(z_1,z_2,\varphi_1)\|_\alpha \leq M\left(\varepsilon^{(\alpha-3)(\gamma-1)} + \varepsilon^{2+(\alpha+1)(\gamma-1)}\right).$$

Finally, from (8.6.16) and (8.6.12), we can see that, for $0 \leq \alpha \leq 3$ and $k \geq 1$,

$$\begin{aligned}
|z^\alpha \mathcal{Q}_{2k+1}(z_1,z_2,\varphi_{2k+1})(z)| &= \left|\frac{\sin(\lambda_{0,2k+1}(z-z_2))}{\sin(\lambda_{0,2k+1}(z_1-z_2))}z^\alpha\varphi_{2k+1}(z_1)\right. \\
&\qquad \left. -\frac{\sin(\lambda_{0,2k+1}(z-z_1))}{\sin(\lambda_{0,2k+1}(z_1-z_2))}z^\alpha\varphi_{2k+1}(z_2)\right| \\
&\leq M\|\Pi_{2k+1}[b]\|_3\frac{|z|^\alpha}{|z_2|^3} \\
&\leq M\|\Pi_{2k+1}[b]\|_3\frac{1}{|z_2|^{3-\alpha}} \\
&\leq M\|\Pi_{2k+1}[b]\|_3\varepsilon^{(\alpha-3)(\gamma-1)},
\end{aligned}$$

and thus

$$\|\mathcal{Q}_{2k+1}(z_1,z_2,\varphi_{2k+1})\|_\alpha \leq M\|\Pi_{2k+1}[b]\|_3\varepsilon^{(\alpha-3)(\gamma-1)}, \ \alpha \leq 3, \ k \geq 1.$$

Now, for $2 \leq \alpha \leq 3$, we have that

$$\begin{aligned}
\|\mathcal{Q}(z_1,z_2,\varphi)\|_{\ell_1,\alpha} &= \sum_{k\geq 0}\|\mathcal{Q}_{2k+1}(z_1,z_2,\varphi_{2k+1})\|_\alpha \\
&\leq M\left(\varepsilon^{(\alpha-3)(\gamma-1)} + \varepsilon^{2+(\alpha+1)(\gamma-1)}\right) + M\varepsilon^{(\alpha-3)(\gamma-1)}\sum_{k\geq 1}\|\Pi_{2k+1}[b]\|_3 \\
&\leq M\left(\varepsilon^{(\alpha-3)(\gamma-1)} + \varepsilon^{2+(\alpha+1)(\gamma-1)}\right) + M\varepsilon^{(\alpha-3)(\gamma-1)}\|b\|_{\ell_1,3} \\
&\leq M\left(\varepsilon^{(\alpha-3)(\gamma-1)} + \varepsilon^{2+(\alpha+1)(\gamma-1)}\right),
\end{aligned}$$

which proves item (3). □

Finally, for $z \in D^{\text{mch},u}_{+,\kappa}$, from Propositions 8.6.1 and 8.6.3, we have that φ given in (8.6.1) is written as

$$\varphi(z,\tau) = \mathcal{Q}(z_1,z_2,\varphi)(z,\tau) + \mathcal{T}\left(\mathcal{C}_{\text{mch}}(z,\tau) + \left(L(\varphi)(z) + \widehat{L}(\widehat{\Pi}[\varphi])(z)\right)\sin(\tau) + K(\varphi)(z,\tau)\right), \tag{8.6.18}$$

and thus, we prove Theorem (8.3.3) through the following proposition

Proposition 8.6.5. *Consider the function $\varphi(z,\tau)$ given in* (8.6.1). *There exist $\kappa_0 > 0$ and a constant $M > 0$ independent of ε, such that, for every $\kappa \geq \kappa_0$ and $\gamma \in (1/3,1)$ (in the definition of $D^{\text{mch},u}_{+,\kappa}$ given in* (8.3.10)*), we have*

$$\|\varphi\|_{\ell_1,2}, \|\partial_\tau\varphi\|_{\ell_1,2}, \|\partial_z\varphi\|_{\ell_1,3} \leq M(\varepsilon^{1-\gamma} + \varepsilon^{3\gamma-1}).$$

Proof. From (8.6.18), and Propositions 8.6.1 and 8.6.4, we have that

$$\begin{aligned}
\|\varphi_1\|_2 &= \left\|\mathcal{Q}_1(z_1,z_2,\varphi_1) + \mathcal{T}_1\left(\Pi_1\left[\mathcal{C}_{\text{mch}}\right] + L(\varphi) + \widehat{L}(\widehat{\Pi}[\varphi])\right)\right\|_2 \\
&\leq \|\mathcal{Q}_1(z_1,z_2,\varphi_1)\|_2 + M\left(\|\Pi_1\left[\mathcal{C}_{\text{mch}}\right]\|_4 + \|L(\varphi)\|_4 + \left\|\widehat{L}(\widehat{\Pi}[\varphi])\right\|_4\right) \\
&\leq M(\varepsilon^{1-\gamma} + \varepsilon^{2+3(\gamma-1)}) + M\left(\varepsilon^{3\gamma-1} + \varepsilon^{\gamma+1} + \varepsilon^{5\gamma-1} + \|\varphi\|_{\ell_1,0} + \left\|\widehat{\Pi}[\varphi]\right\|_{\ell_1,2}\right) \\
&\leq M\left(\varepsilon^{1-\gamma} + \varepsilon^{3\gamma-1} + \frac{1}{\kappa^2}\|\varphi\|_{\ell_1,2} + \left\|\widehat{\Pi}[\varphi]\right\|_{\ell_1,2}\right).
\end{aligned}$$

Also, since $\Pi_1 \circ K \equiv 0$, we have that

$$\begin{aligned}
\left\|\widehat{\Pi}[\varphi]\right\|_{\ell_1,2} &= \left\|\widehat{\Pi}\circ\mathcal{Q}(z_1,z_2,\varphi) + \mathcal{T}\left(\widehat{\Pi}\left[\mathcal{C}_{\text{mch}}\right] + K(\varphi)\right)\right\|_{\ell_1,2} \\
&\leq \left\|\widehat{\Pi}\circ\mathcal{Q}(z_1,z_2,\varphi)\right\|_{\ell_1,2} + M\left(\left\|\widehat{\Pi}\left[\mathcal{C}_{\text{mch}}\right]\right\|_{\ell_1,2} + \|K(\varphi)\|_{\ell_1,2}\right) \\
&\leq M(\varepsilon^{1-\gamma} + \varepsilon^{2+3(\gamma-1)}) + M\left(\frac{\varepsilon^2}{\kappa} + \|\varphi\|_{\ell_1,0}\right) \\
&\leq M\left(\varepsilon^{1-\gamma} + \varepsilon^{3\gamma-1} + \frac{1}{\kappa^2}\|\varphi\|_{\ell_1,2}\right)
\end{aligned}$$

It follows that

$$\|\varphi\|_{\ell_1,2} \leq M\left(\varepsilon^{1-\gamma} + \varepsilon^{3\gamma-1} + \frac{1}{\kappa^2}\|\varphi\|_{\ell_1,2}\right).$$

Now, choosing κ_0 sufficiently big, we have that, for every $\kappa \geq \kappa_0$

$$\|\varphi\|_{\ell_1,2} \leq M(\varepsilon^{1-\gamma} + \varepsilon^{3\gamma-1}).$$

Also, it follows from Proposition 8.6.4 that

$$\|\partial_\tau\varphi\|_{\ell_1,2} \leq M(\varepsilon^{1-\gamma} + \varepsilon^{3\gamma-1}).$$

Finally, from Lemma 8.1 of [8], reducing the domain $D^{\text{mch},u}_{+,\kappa}$ (see (8.3.10)), with vertices y_1 and y_2 such that $|y_j - i(\pi/2 - \kappa\varepsilon)| = c\varepsilon^\gamma$, $j = 1,2$, to $D^{\text{mch},u}_{+,2\kappa} \subset D^{\text{mch},u}_{+,\kappa}$ having vertices $\widetilde{y}_1$ and $\widetilde{y}_2$ such that $|\widetilde{y}_j - i(\pi/2 - 2\kappa\varepsilon)| = \widetilde{c}\varepsilon^\gamma$, $j = 1,2$, and $0 < \widetilde{c} < c$, we obtain that

$$\|\partial_z\varphi\|_{\ell_1,2} \leq \frac{M}{\kappa}(\varepsilon^{1-\gamma} + \varepsilon^{3\gamma-1}).$$

It completes the proof of this proposition. In order to simplify the notation, we make no distinction between $D^{\text{mch},u}_{+,\kappa}$ and $D^{\text{mch},u}_{+,2\kappa}$.

□

Remark 8.6.6. *Notice that $\gamma = 1/2$ minimizes the size of $\|\varphi\|_{\ell_1,2}$ in Theorem 8.6.5. In this case,*

$$\|\varphi\|_{\ell_1,2} \leq M\varepsilon^{1/2}.$$

8.7 Proof of Theorem U

In Theorem 8.3.1, we have obtained the existence of the solutions $v^\star : D_\kappa^{\text{out},\star} \times \mathbb{T} \to \mathbb{C}$ of (8.1.9), $\star = u, s$ which parameterize the invariant manifolds $W^u(0)$ and $W^s(0)$. Notice that both solutions are defined in the domain

$$\mathcal{R}_\kappa = D_\kappa^{\text{out},u} \cap D_\kappa^{\text{out},s} \cap i\mathbb{R},$$

where $D_\kappa^{\text{out},\star}$ is given in (8.3.1).

Our aim is to obtain an asymptotic formula for the function

$$d(\tau;\varepsilon) = v^u(0,\tau) - v^s(0,\tau). \tag{8.7.1}$$

In order to do this, we write the equations satisfied by the difference

$$\Delta v(y,\tau) = v^u(y,\tau) - v^s(y,\tau), \tag{8.7.2}$$

and recall that $\Pi_{2l}[\Delta v] = 0$, for every $l \geq 0$.

From (8.4.4) and Theorem 8.3.1, we have that $\Delta v(y,\tau) = \xi^u(y,\tau) - \xi^s(y,\tau)$ and

$$\mathcal{L}(\Delta v) = \mathcal{F}(\xi^u) - \mathcal{F}(\xi^s),$$

where $\mathcal{L}$ and $\mathcal{F}$ are the operators given in (8.4.2) and (8.4.3), respectively.

Proposition 8.7.1. *The function $\Delta v(y,\tau)$ given by (8.7.2) satisfies the equation*

$$\mathcal{L}(\Delta v) = \Pi_1\left[\eta_1(y,\tau)\Pi_1[\Delta v]\sin(\tau) + \eta_2(y,\tau)\widehat{\Pi}[\Delta v]\right]\sin(\tau) + \widehat{\Pi}[\eta_3(y,\tau)\Delta v],$$

where $\mathcal{L}$ is the operator given in (8.4.2) and $\eta_j : \mathcal{R}_\kappa \times \mathbb{T} \to \mathbb{C}$, $j = 1,2,3$ are real analytic functions in the variable y. Moreover, there exists a constant $M > 0$ independent of κ and ε such that

$$\|\eta_1\|_{\ell_1,4} \leq M\varepsilon^2, \quad \text{and} \quad \|\eta_2\|_{\ell_1,2}, \|\eta_3\|_{\ell_1,2} \leq M.$$

Proof. Using the expression of $\mathcal{F}$ given in (8.4.3) (see also (8.4.30)), we obtain that

$$\begin{aligned}
\mathcal{F}(\xi^u) - \mathcal{F}(\xi^s) \quad = \quad & -\frac{1}{\varepsilon^3}\widehat{\Pi}\left[g(\varepsilon(\xi^u + v_1^h\sin(\tau))) - g(\varepsilon(\xi^s + v_1^h\sin(\tau)))\right] \\
& -\Pi_1\left[(\xi_1^u + v_1^h)^2\sin^2(\tau)\widehat{\Pi}(\xi^u) - (\xi_1^s + v_1^h)^2\sin^2(\tau)\widehat{\Pi}(\xi^s)\right]\sin(\tau) \\
& -\Pi_1\left[(\xi_1^u + v_1^h)\sin(\tau)(\widehat{\Pi}[\xi^u])^2 - (\xi_1^s + v_1^h)\sin(\tau)(\widehat{\Pi}[\xi^s])^2\right. \\
& \left. +\frac{1}{3}\left((\widehat{\Pi}[\xi^u])^3 - (\widehat{\Pi}[\xi^s])^3\right)\right]\sin(\tau) \\
& +\left(-\frac{1}{\varepsilon^3}\Pi_1\left[f(\varepsilon(\xi^u + v_1^h\sin(\tau))) - f(\varepsilon(\xi^s + v_1^h\sin(\tau)))\right]\right. \\
& \left. -\frac{3v_1^h\left((\xi_1^u)^2 - (\xi_1^s)^2\right)}{4} - \frac{(\xi_1^u)^3 - (\xi_1^s)^3}{4}\right)\sin(\tau).
\end{aligned}$$

The proof follows from the Mean Value Theorem, the estimates $\|v_1^h\|_1 \leq M$, $\|\xi^{u,s}\|_{\ell_1,3} \leq M\varepsilon^2$ obtained in Theorem 8.3.1, and the fact that $g(z) = \mathcal{O}(z^3)$ and $f(z) = \mathcal{O}(z^5)$. $\square$

As an abuse of notation, we consider the spaces

$$\mathcal{E}_\alpha = \{f : \mathcal{R}_\kappa \to \mathbb{C};\ f \text{ is real-analytic and } \|f\|_\alpha < \infty\}$$

and

$$\mathcal{E}_{\ell_1,\alpha} = \{f : \mathcal{R}_\kappa \times \to \mathbb{C};\ f \text{ is real-analytic in the variable } y \text{ and } \|f\|_{\ell_1,\alpha} < \infty\},$$

where

$$\|f\|_\alpha = \sup_{y\in\mathcal{R}_\kappa} |(y^2+\pi^2/4)^\alpha f(y)| \quad \textit{and} \quad \|f\|_{\ell_1,\alpha} = \sum_{n\geq 1} \|\Pi_n[f]\|_\alpha.$$

Recall that equation (8.1.9) is a Hamiltonian Partial Differential Equation. In fact, if we write it as the system

$$\begin{cases} \partial_y v = w, \\ \partial_y w = \dfrac{\omega^2}{\varepsilon^2}\partial_\tau^2 v + \dfrac{1}{\varepsilon^2} v - \dfrac{1}{3} v^3 - \dfrac{1}{\varepsilon^3} f(\varepsilon v), \end{cases} \tag{8.7.3}$$

we obtain that (8.7.3) is a Hamiltonian system with respect to

$$\mathcal{H}(v,w) = \frac{1}{\pi}\int_{\mathbb{T}} \left(\frac{w^2}{2} + \frac{(\omega\partial_\tau v)^2}{2\varepsilon^2} - \frac{v^2}{2\varepsilon^2} + \frac{v^4}{12} + \frac{F(\varepsilon v)}{\varepsilon^4} \right) d\tau,$$

where F is an analytic function such that $F(z) = \mathcal{O}(z^6)$ and $F'(z) = f(z)$.

Notice that the solutions $v^\star(y,\tau)$ of (8.1.9), $\star = u, s$, obtained in Theorem 8.3.1 are contained in the same energy level of $\mathcal{H}$. We use the Hamiltonian $\mathcal{H}$ to obtain the variable $\Pi_1[\Delta v]$ in terms of the variables $\Pi_1[\Delta w]$, $\widehat{\Pi}[\Delta w]$ and $\widehat{\Pi}[\Delta v]$, where $\Delta w = \partial_y \Delta v$.

Proposition 8.7.2. *There exist two linear operators $A : \mathcal{E}_{\ell_1,0} \to \mathcal{E}_1$ and $B : \mathcal{E}_{\ell_1,0} \to \mathcal{E}_0$, such that*

$$\Pi_1[\Delta v](y) = \frac{\dot{v}_1^h(y)}{\ddot{v}_1^h(y)} \Pi_1[\Delta w](y) + A(\Delta w)(y) + B(\widehat{\Pi}[\Delta v])(y), \tag{8.7.4}$$

where

1. $|A(\Delta w)(y)| \leq \dfrac{M\varepsilon^2}{|y^2+\pi^2/4|} \|\Delta w\|_{\ell_1}(y)$, *for every $y \in \mathcal{R}_\kappa$;*

2. $|B(\widehat{\Pi}[\Delta v])(y)| \leq M \|\widehat{\Pi}[\Delta v]\|_{\ell_1}(y)$, *for every $y \in \mathcal{R}_\kappa$;*

Proof. First, recall that the projections Π_1 and $\widehat{\Pi}$ given in (8.1.7) are orthogonal. Therefore, we obtain that $\mathcal{H}$ is given by

$$\mathcal{H}(v,w) = \frac{(\Pi_1[w])^2}{2} - \frac{(\Pi_1[v])^2}{2} + \frac{1}{\pi}\int_{\mathbb{T}} \left(\frac{(\widehat{\Pi}[w])^2}{2} + \frac{(\omega\partial_\tau \widehat{\Pi}[v])^2}{2\varepsilon^2} - \frac{(\widehat{\Pi}[v])^2}{2\varepsilon^2} + \frac{v^4}{12} + \frac{F(\varepsilon v)}{\varepsilon^4} \right) d\tau.$$

Using that $\mathcal{H}(v^\star, w^\star) = 0$, $\star = u, s$, integrability by parts of the ∂_τ terms and the Mean Value Theorem, we have that

$$
\begin{aligned}
0 &= \mathcal{H}(v^u, w^u) - \mathcal{H}(v^u, w^u) \\
&= \frac{\Pi_1[w^u] + \Pi_1[w^s]}{2}\Pi_1[\Delta w] - \frac{\Pi_1[v^u] + \Pi_1[v^s]}{2}\Pi_1[\Delta v] \\
&\quad + \frac{1}{\pi}\int_{\mathbb{T}} \left[\frac{\widehat{\Pi}[w^u] + \widehat{\Pi}[w^s]}{2}\widehat{\Pi}[\Delta w] - \frac{\omega^2}{\varepsilon^2}\frac{\partial_\tau^2\widehat{\Pi}[v^u] + \partial_\tau^2\widehat{\Pi}[v^s]}{2}\widehat{\Pi}[\Delta v] - \frac{\widehat{\Pi}[v^u] + \widehat{\Pi}[v^s]}{2\varepsilon^2}\widehat{\Pi}[\Delta v] \right] d\tau \\
&\quad + \frac{1}{\pi}\int_{\mathbb{T}} \left[\frac{(v^u)^3 + (v^u)^2(v^s) + (v^u)(v^s)^2 + (v^s)^3}{12}\Delta v \right. \\
&\quad \left. + \left(\frac{1}{\varepsilon^3}\int_0^1 f(\varepsilon(\sigma v^u(y,\tau) + (1-\sigma)\sigma v^s(y,\tau)))d\sigma \right) \Delta v \right] d\tau.
\end{aligned}
$$

Finally, we use that $v^\star = v_1^h \sin(\tau) + \xi^\star(y,\tau)$, $\ddot{v}_1^h = v_1^h - (v_1^h)^3/4$ and

$$
\|\xi^\star\|_{\ell_1,3},\ \|\partial_\tau^2\xi^\star\|_{\ell_1,3},\ \|\partial_y\xi^\star\|_{\ell_1,4} \leq M\varepsilon^2,\ \star = u, s
$$

we conclude that

$$
0 = -(\ddot{v}_1^h + \tilde{a}(y))\Pi_1[\Delta v] + \dot{v}_1^h\Pi_1[\Delta w] + \widetilde{A}(\Delta w) + \widetilde{B}(\widehat{\Pi}[\Delta v]),
$$

where

- $\|\tilde{a}\|_5 \leq M\varepsilon^2$;
- $|\widetilde{A}(\Delta w)(y)| \leq \dfrac{M\varepsilon^2}{|y^2 + \pi^2/4|^4}\|\Delta w\|_{\ell_1}(y)$, for every $y \in \mathcal{R}_\kappa$;
- $|\widetilde{B}(\widehat{\Pi}[\Delta v])(y)| \leq \dfrac{M}{|y^2 + \pi^2/4|^3}\|\widehat{\Pi}[\Delta v]\|_{\ell_1}(y)$, for every $y \in \mathcal{R}_\kappa$;

Now, observe that $\ddot{v}_1^h(y) = \sqrt{2}(\cosh(2y) - 3)\operatorname{sech}^3(y)$ is strictly negative, for every $y = i\tilde{y}$ with $\tilde{y} \in (-\pi/2, \pi/2)$. Also, since $\ddot{v}_1^h(y)$ has a third order pole at the points $y = \pm i\pi/2$, we obtain that $\left\|\tilde{a}/\ddot{v}_1^h\right\|_2 \leq M\varepsilon^2$, which means that

$$
\left|\frac{\tilde{a}}{\ddot{v}_1^h}\right| \leq \frac{M}{\kappa^2},
$$

for every $y \in \mathcal{R}_\kappa$.

Hence, taking κ sufficiently big, we have that the function

$$
D(y) = \ddot{v}_1^h(y)\left(1 + \frac{\tilde{a}(y)}{\ddot{v}_1^h(y)}\right),
$$

is non-zero for every $y \in \mathcal{R}_\kappa$. Moreover, the function $D(y)^{-1}$ has a third order zero at the points $y = \pm i\pi/2$, and

$$
D(y)^{-1} = (\ddot{v}_1^h(y))^{-1}(1 + a(y)),
$$

where $a : \mathcal{R}_\kappa \to \mathbb{C}$ is a real-analytic function such that $\|a\|_2 \leq M\varepsilon^2$.

Hence, it follows that

$$\begin{aligned} \Pi_1[\Delta v] &= \frac{\dot v_1^h \Pi_1[\Delta w] + \widetilde{A}(\Delta w) + \widetilde{B}(\widehat{\Pi}[\Delta v])}{D} \\ &= \left(\frac{\dot v_1^h \Pi_1[\Delta w] + \widetilde{A}(\Delta w) + \widetilde{B}(\widehat{\Pi}[\Delta v])}{\ddot v_1^h}\right)(1+a) \\ &= \frac{\dot v_1^h}{\ddot v_1^h}\Pi_1[\Delta w] + A(\Delta w) + B(\widehat{\Pi}[\Delta v]), \end{aligned}$$

where A and B are the linear operators

$$A(\Delta w)(y) = \frac{1+a(y)}{\ddot v_1^h(y)}\widetilde{A}(\Delta w)(y) + \frac{\dot v_1^h(y)}{\ddot v_1^h(y)} a(y)\Pi_1[\Delta w](y)$$

and

$$B(\widehat{\Pi}[\Delta v]) = \frac{1+a(y)}{\ddot v_1^h(y)}\widetilde{B}(\widehat{\Pi}[\Delta v]).$$

The proof of the proposition follows directly from the estimates of $\widetilde{A}(\Delta w)$, $\widetilde{B}(\widehat{\Pi}[\Delta v])$, a and the fact that $\ddot v_1^h$ and $\dot v_1^h$ have a third and second order pole at the points $y = \pm i\pi/2$, respectively.

□

Remark 8.7.3. *Notice that* $\Delta w = \Pi_1[\Delta w]\sin(\tau) + \widehat{\Pi}[\Delta w]$, *and thus the operator* A *in* (8.7.4) *also acts in the first harmonic.*

Now, denote $\Delta v_{2k+1} = \Pi_{2k+1}[\Delta v]$ and $\Delta w_{2k+1} = \Pi_{2k+1}[\Delta w]$, for every $k \geq 0$, and consider the following change of variables

$$\begin{cases} \Gamma_{2k+1} = \lambda_{2k+1}\Delta v_{2k+1} + i\varepsilon\Delta w_{2k+1}, \\ \Theta_{2k+1} = \lambda_{2k+1}\Delta v_{2k+1} - i\varepsilon\Delta w_{2k+1}, \end{cases} \tag{8.7.5}$$

for every $k \geq 1$.

Consider

$$\Gamma = \sum_{k\geq 1}\Gamma_{2k+1}(y)\sin((2k+1)\tau) \quad \textit{and} \quad \Theta = \sum_{k\geq 1}\Theta_{2k+1}(y)\sin((2k+1)\tau),$$

and define the operator

$$\begin{aligned} \mathcal{N}(\Delta w_1, \Gamma, \Theta) \;=\; &\Bigg(\dot{\Delta w}_1 - \frac{\dddot v_1^h}{\ddot v_1^h}\Delta w_1, \sum_{k\geq 1}\left(\dot\Gamma_{2k+1} + i\frac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1}\right)\sin((2k+1)\tau), \\ &\sum_{k\geq 1}\left(\dot\Theta_{2k+1} - i\frac{\lambda_{2k+1}}{\varepsilon}\Theta_{2k+1}\right)\sin((2k+1)\tau)\Bigg) \end{aligned} \tag{8.7.6}$$

Notice that, from Theorem 8.3.1, Δv satisfies

$$\sum_{k\geq 1}\lambda_{2k+1}^2\|\Delta v_{2k+1}\|_{\ell_1,3} \leq M\varepsilon^2,$$

and thus

$$\sum_{k\geq 1}\lambda_{2k+1}\|\Gamma_{2k+1}\|_3 \leq M\varepsilon^2 \quad \textit{and} \quad \sum_{k\geq 1}\lambda_{2k+1}\|\Theta_{2k+1}\|_3 \leq M\varepsilon^2.$$

It follows that operator $\mathcal{N}$ is well defined.

Consider the Banach space given by

$$\begin{aligned}\Upsilon_{\ell_1,\alpha} &= \{f : \mathcal{R}_\kappa \times \mathbb{T} \to \mathbb{C};\ f \text{ is an analytic function in the variable } y \text{ such that} \\ &\quad \Pi_1[f] = \Pi_{2l}[f] = 0, \forall l \geq 0 \text{ and } \|f\|_{\ell_1,\alpha} < \infty\}.\end{aligned} \tag{8.7.7}$$

Proposition 8.7.4. *Let $\Delta v(y,\tau)$ be the function given in* (8.7.2)*, and consider $\Delta w = \partial_y \Delta v$. Therefore, $\Delta v_1(y)$ is given by* (8.7.4)*, and $(\Delta w_1(y), \Gamma(y,\tau), \Theta(y,\tau))$ (see* (8.7.5)*) satisfies the following equation*

$$\mathcal{N}(\Delta w_1, \Gamma, \Theta) = \mathcal{M}(\Delta w_1, \Gamma, \Theta), \tag{8.7.8}$$

where $\mathcal{N}$ is given in (8.7.6) *and $\mathcal{Y}$ is a linear operator which can be written as*

$$\mathcal{M}(\Delta w_1, \Gamma, \Theta) = \begin{pmatrix} m_W(y)\Delta w_1 + \mathcal{M}_W(\Gamma,\Theta) \\ m_{\text{osc}}(y,\tau)\Delta w_1 + \mathcal{M}_{\text{osc}}(\Gamma,\Theta) \\ -m_{\text{osc}}(y,\tau)\Delta w_1 - \mathcal{M}_{\text{osc}}(\Gamma,\Theta) \end{pmatrix}, \tag{8.7.9}$$

where, $m_W : \mathcal{R}_\kappa \to \mathbb{C}$, $m_{\text{osc}} : \mathcal{R}_\kappa \times \mathbb{T} \to \mathbb{C}$ are real-analytic functions in the variable y, and $\mathcal{M}_W : \Upsilon_{\ell_1,0} \times \Upsilon_{\ell_1,0} \to \Upsilon_2$, $\mathcal{M}_{\text{osc}} : \Upsilon_{\ell_1,0} \times \Upsilon_{\ell_1,0} \to \Upsilon_{\ell_1,2}$ are linear operators, where $\Upsilon_{\ell_1,\alpha}$ is given by (8.7.7)*. Moreover, there exists a constant $M > 0$ independent of ε and κ such that*

1. *$\|m_W\|_3 \leq M\varepsilon^2$ and $\|m_{\text{osc}}\|_{\ell_1,1} \leq M\varepsilon$;*
2. *$|\mathcal{M}_W(\Gamma,\Theta)(y)| \leq \dfrac{M}{|y^2+\pi^2/4|^2}\left(\|\Gamma\|_{\ell_1}(y) + \|\Theta\|_{\ell_1}(y)\right)$, for every $y \in \mathcal{R}_\kappa$;*
3. *$\|\mathcal{M}_{\text{osc}}(\Gamma,\Theta)\|_{\ell_1}(y) \leq \dfrac{M\varepsilon}{|y^2+\pi^2/4|^2}\left(\|\Gamma\|_{\ell_1}(y) + \|\Theta\|_{\ell_1}(y)\right)$, for every $y \in \mathcal{R}_\kappa$.*

Proof. In fact, from (8.7.5) and Proposition 8.7.1, we have that, for each $k \geq 1$,

$$\begin{aligned}\dot{\Gamma}_{2k+1} &= \lambda_{2k+1}\Delta w_{2k+1} + i\varepsilon \Delta\ddot{v}_{2k+1} \\ &= \lambda_{2k+1}\Delta w_{2k+1} + i\varepsilon\left(-\frac{\lambda_{2k+1}^2}{\varepsilon^2}\Delta v_{2k+1} + \Pi_{2k+1}\left[\eta_3(y,\tau)\Delta v\right]\right) \\ &= -i\frac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1} + i\varepsilon\Pi_{2k+1}\left[\eta_3(y,\tau)\Delta v\right].\end{aligned} \tag{8.7.10}$$

Analogously, for each $k \geq 1$,

$$\dot{\Theta}_{2k+1} - i\frac{\lambda_{2k+1}}{\varepsilon}\Theta_n = -i\varepsilon\Pi_{2k+1}\left[\eta_3(y,\tau)\Delta v\right]. \tag{8.7.11}$$

Also, for the variable Δw_1, we have that

$$\Delta\dot{w}_1 = \left(1 - \frac{3(v_1^h)^2}{4}\right)\Delta v_1 + \Pi_1\left[\eta_1(y,\tau)\Delta v_1\sin(\tau) + \eta_2(y,\tau)\widehat{\Pi}[\Delta v]\right]. \tag{8.7.12}$$

Therefore, by the definition of $\mathcal{N}$ in (8.7.6), equations (8.7.10), (8.7.11) and (8.7.12) are equivalent to

$$\mathcal{N}(\Delta w_1,\Gamma,\Theta)=\begin{pmatrix} -\frac{\dddot{v}_1^h}{\ddot{v}_1^h}\Delta w_1+\left(1-\frac{3(v_1^h)^2}{4}\right)\Delta v_1+\Pi_1\left[\eta_1(y,\tau)\Delta v_1\sin(\tau)+\eta_2(y,\tau)\widehat{\Pi}[\Delta v]\right] \\ i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\Delta v\right] \\ -i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\Delta v\right] \end{pmatrix}, \tag{8.7.13}$$

where η_j, $j=1,2,3$ are given by Proposition 8.7.1. Using formula (8.7.4) for Δv_1, we get

$$\left(1-\frac{3(v_1^h)^2}{4}\right)\dot{v}_1^h=\dddot{v}_1^h,$$

we have

$$\begin{aligned}\dot{\Delta w_1}-\frac{\dddot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 &= \left(1-\frac{3(v_1^h)^2}{4}\right)\Big(A(\Delta w)+B(\widehat{\Pi}[\Delta v])\Big)\\ &\quad+\Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1+A(\Delta w)+B(\widehat{\Pi}[\Delta v])\right)\sin(\tau)+\eta_2(y,\tau)\widehat{\Pi}[\Delta v]\right]\\ &= \left(1-\frac{3(v_1^h)^2}{4}\right)A(\Delta w_1\sin(\tau))\\ &\quad+\Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1+A(\Delta w_1\sin(\tau))\right)\sin(\tau)\right]\\ &\quad+\left(1-\frac{3(v_1^h)^2}{4}\right)\Big(A(\widehat{\Pi}[\Delta w])+B(\widehat{\Pi}[\Delta v])\Big)\\ &\quad+\Pi_1\left[\eta_1(y,\tau)\Big(A(\widehat{\Pi}[\Delta w])+B(\widehat{\Pi}[\Delta v])\Big)\sin(\tau)+\eta_2(y,\tau)\widehat{\Pi}[\Delta v]\right].\end{aligned}$$

Using (8.7.5), we have

$$\begin{aligned}\dot{\Delta w_1}-\frac{\dddot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 &= \left(1-\frac{3(v_1^h)^2}{4}\right)A(\Delta w_1\sin(\tau))\\ &\quad+\Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1+A(\Delta w_1\sin(\tau))\right)\sin(\tau)\right]\\ &\quad+\left(1-\frac{3(v_1^h)^2}{4}\right)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta)+B\left(\sum_{n\geq 2}\frac{\Gamma_n+\Theta_n}{2\lambda_n}\sin(n\tau)\right)\right)\\ &\quad+\Pi_1\left[\eta_1(y,\tau)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta)\right)\sin(\tau)\right.\\ &\quad+\eta_1(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin(\tau)\\ &\quad\left.+\eta_2(y,\tau)\left(\sum_{n\geq 2}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right]\end{aligned}$$

and,

$$
\begin{aligned}
i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\Delta v\right] &= i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\left(\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 + A(\Delta w) + B(\widehat{\Pi}[\Delta v])\right)\sin(\tau) + \widehat{\Pi}[\Delta v]\right)\right] \\
&= i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 + A(\Delta w_1\sin(\tau))\right)\sin(\tau)\right] \\
&\quad + i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\left(A(\widehat{\Pi}[\Delta w])\sin(\tau) + B(\widehat{\Pi}[\Delta v])\sin(\tau) + \widehat{\Pi}[\Delta v]\right)\right] \\
&= i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 + A(\Delta w_1\sin(\tau))\right)\sin(\tau)\right] \\
&\quad + i\varepsilon\widehat{\Pi}\left[\frac{\eta_3(y,\tau)}{2i\varepsilon}A(\Gamma-\Theta)\sin(\tau)\right. \\
&\quad + \eta_3(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin(\tau) \\
&\quad \left. + \eta_3(y,\tau)\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right]
\end{aligned}
$$

Now, the proof is concluded by taking

$$
\begin{aligned}
m_W(y)\Delta w_1 = {} & \left(1 - \frac{3(v_1^h)^2}{4}\right)A(\Delta w_1\sin(\tau)) \\
& + \Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 + A(\Delta w_1\sin(\tau))\right)\sin(\tau)\right]; \\
\mathcal{M}_W(\Gamma,\Theta) = {} & \left(1 - \frac{3(v_1^h)^2}{4}\right)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta) + B\left(\sum_{n\geq 2}\frac{\Gamma_n+\Theta_n}{2\lambda_n}\sin(n\tau)\right)\right) \\
& + \Pi_1\left[\frac{\eta_1(y,\tau)}{2i\varepsilon}A(\Gamma-\Theta)\sin(\tau)\right. \\
& + \eta_1(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin(\tau) \\
& \left. + \eta_2(y,\tau)\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right]; \\
m_{\text{osc}}(y,\tau)\Delta w_1 = {} & i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 + A(\Delta w_1\sin(\tau))\right)\sin(\tau)\right]; \\
\mathcal{M}_{\text{osc}}(\Gamma,\Theta) = {} & i\varepsilon\widehat{\Pi}\left[\eta_3(y,\tau)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta)\sin(\tau)\right.\right. \\
& + B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin(\tau) \\
& \left.\left. + \sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right],
\end{aligned}
\tag{8.7.14}
$$

and using the bounds for the functions η_j, $j = 1, 2, 3$ and the operators A and B provided in Propositions 8.7.1 and 8.7.2. $\square$

8.7.1 Banach Space and Operators

In this section, we rewrite (8.7.8) as a fixed point of certain functional operator in some appropriate Banach space.

Given an analytic function $f : \mathcal{R}_\kappa \to \mathbb{C}$, we define the norm

$$\|f\|_{\alpha,\exp} = \sup_{y \in \mathcal{R}_\kappa} \left| (y^2 + \pi^2/4)^\alpha e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2} - |\mathrm{Im}(y)|\right)} f(y) \right|, \tag{8.7.15}$$

and the Banach space

$$\mathcal{X}_{\alpha,\exp} = \{f : \mathcal{R}_\kappa \to \mathbb{C};\ f \text{ is an analytic function such that } \|f\|_{\alpha,\exp} < \infty\}. \tag{8.7.16}$$

Also, given an analytic odd function $f : \mathcal{R}_\kappa \times \mathbb{T} \to \mathbb{C}$, we define the norm

$$\|f\|_{\ell_1,\alpha,\exp} = \sum_{k \geq 1} \|\Pi_{2k+1}[f]\|_{\alpha,\exp}, \tag{8.7.17}$$

and the Banach space

$$\begin{aligned} \mathcal{X}_{\ell_1,\alpha,\exp} \;=\; & \{f : \mathcal{R}_\kappa \times \mathbb{T} \to \mathbb{C};\ f \text{ is an analytic function in the variable } y \text{ such that} \\ & \Pi_1[f] = \Pi_{2l}[f] = 0, \forall l \geq 0 \text{ and } \|f\|_{\ell_1,\alpha,\exp} < \infty\}. \end{aligned} \tag{8.7.18}$$

Finally, we consider the product Banach space

$$\mathcal{Y}_{\ell_1,-1,\exp} = \mathcal{X}_{-1,\exp} \times \mathcal{X}_{\ell_1,0,\exp} \times \mathcal{X}_{\ell_1,0,\exp}, \tag{8.7.19}$$

endowed with the weight norm

$$[\![(f, g, h)]\!]_{\ell_1,-1,\exp} = \frac{1}{\varepsilon}\|f\|_{-1,\exp} + \|g\|_{\ell_1,\exp} + \|h\|_{\ell_1,\exp}. \tag{8.7.20}$$

Now, given a sequence $a = (a_{2k+1})_{k \geq 1}$, we define the functions

$$\mathcal{I}_\Gamma(a)(y, \tau) = \sum_{k \geq 1} a_{2k+1} e^{-i\frac{\lambda_{2k+1}}{\varepsilon} y} \sin((2k+1)\tau) \tag{8.7.21}$$

and

$$\mathcal{I}_\Theta(a)(y, \tau) = \sum_{k \geq 1} a_{2k+1} e^{i\frac{\lambda_{2k+1}}{\varepsilon} y} \sin((2k+1)\tau). \tag{8.7.22}$$

Also, considering

$$y_\pm = \pm i\left(\frac{\pi}{2} - \kappa\varepsilon\right),$$

we define the diagonal linear operator

$$\mathcal{P}(f, g, h) = \left(\mathcal{P}_W(f), \mathcal{P}^\Gamma(g), \mathcal{P}^\Theta(h)\right), \tag{8.7.23}$$

where

$$\mathcal{P}_W(f)(y) = \int_0^y \frac{f(s)}{\ddot{v}_1^h(s)} ds, \tag{8.7.24}$$

$$\mathcal{P}^\Gamma(g) = \sum_{k\geq 1}\mathcal{P}^\Gamma_{2k+1}(g)\sin((2k+1)\tau) \quad \textit{and} \quad \mathcal{P}^\Theta(h) = \sum_{k\geq 1}\mathcal{P}^\Theta_{2k+1}(h)\sin((2k+1)\tau), \tag{8.7.25}$$

with

$$\mathcal{P}^\Gamma_{2k+1}(g)(y) = \int_{y_+}^{y} e^{i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\Pi_{2k+1}[g](s)ds$$

and

$$\mathcal{P}^\Theta_{2k+1}(h)(y) = \int_{y_-}^{y} e^{-i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\Pi_{2k+1}[h](s)ds, \ k\geq 1.$$

Lemma 8.7.5. *For $\alpha = 2, 3$, the operators $\mathcal{P}^\Gamma, \mathcal{P}^\Theta : \mathcal{X}_{\ell_1,\alpha,\exp} \to \mathcal{X}_{\ell_1,0,\exp}$ given by (8.7.25) are well-defined. Moreover, there exists a constant $M > 0$ independent of ε and κ such that, for every $h \in \mathcal{X}_{\ell_1,\alpha,\exp}$,*

$$\|\mathcal{P}^\Gamma(h)\|_{\ell_1,0,\exp}, \|\mathcal{P}^\Theta(h)\|_{\ell_1,0,\exp} \leq \frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h\|_{\ell_1,\alpha,\exp}. \tag{8.7.26}$$

Proof. We prove the lemma only for the operator $\mathcal{P}^\Gamma$, since the result for $\mathcal{P}^\Theta$ follows analogously. Let $h(y,\tau) = \sum_{k\geq 1} h_{2k+1}(y)\sin((2k+1)\tau)$. First, we prove that

$$\left\|\mathcal{P}^\Gamma_3(h_3)\right\|_{0,\exp} \leq \frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h_3\|_{\alpha,\exp}$$

In fact,

$$\begin{aligned}
\left|\mathcal{P}^\Gamma_3(h_3)e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\right| &\leq \|h_3\|_{\alpha,\exp}\int_{y_+}^{y}\left|e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\frac{e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(s)|\right)}}{|s^2+\pi^2/4|^\alpha}e^{i\frac{\lambda_3}{\varepsilon}(s-y)}\right|ds \\
&\leq \|h_3\|_{\alpha,\exp}\int_{y_+}^{y}\frac{e^{\frac{\lambda_3}{\varepsilon}(|\mathrm{Im}(s)|-\mathrm{Im}(s)-(|\mathrm{Im}(y)|-\mathrm{Im}(y)))}}{|s^2+\pi^2/4|^\alpha}ds \\
&\leq \|h_3\|_{\alpha,\exp}\int_{\frac{\pi}{2}-\kappa\varepsilon}^{\mathrm{Im}(y)}\frac{e^{\frac{\lambda_3}{\varepsilon}(|\sigma|-\sigma-(|\mathrm{Im}(y)|-\mathrm{Im}(y)))}}{|\sigma^2-\pi^2/4|^\alpha}d\sigma
\end{aligned}$$

Now, if $\mathrm{Im}(y) \geq 0$, and recalling that $\alpha = 2, 3$, we obtain

$$\begin{aligned}
\left|\mathcal{P}^\Gamma_3(h_3)e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\right| &\leq \|h_3\|_{\alpha,\exp}\int_{\frac{\pi}{2}-\kappa\varepsilon}^{\mathrm{Im}(y)}\frac{1}{|\sigma^2-\pi^2/4|^\alpha}d\sigma \\
&\leq \frac{M}{|y^2+\pi^2/4|^{\alpha-1}}\|h_3\|_{\alpha,\exp} \\
&\leq \frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h_3\|_{\alpha,\exp},
\end{aligned}$$

and, if $\mathrm{Im}(y) < 0$, then

$$\begin{aligned}
\left|\mathcal{P}^\Gamma_3(h_3)e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\right| &\leq \|h_3\|_{\alpha,\exp}\left(\int_{\frac{\pi}{2}-\kappa\varepsilon}^{0}\frac{1}{|\sigma^2-\pi^2/4|^\alpha}d\sigma + \int_0^{\mathrm{Im}(y)}\frac{e^{-2\frac{\lambda_3}{\varepsilon}(\sigma-\mathrm{Im}(y)))}}{|\sigma^2-\pi^2/4|^\alpha}d\sigma\right) \\
&\leq \frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h_3\|_{\alpha,\exp}.
\end{aligned}$$

For $k \geq 2$, we have that

$$\left\| \mathcal{P}^{\Gamma}_{2k+1}(h_{2k+1}) \right\|_{0,\exp} \leq \frac{M}{\kappa^{\alpha}\varepsilon^{\alpha-1}} \|h_{2k+1}\|_{\alpha,\exp}.$$

In fact,

$$\begin{aligned} \left| \mathcal{P}^{\Gamma}_{2k+1}(h_{2k+1}) e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)} \right| &\leq \|h_{2k+1}\|_{\alpha,\exp} \int_{y_+}^{y} \left| e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)} \frac{e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(s)|\right)}}{|s^2+\pi^2/4|^{\alpha}} e^{i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)} \right| ds \\ &\leq \|h_{2k+1}\|_{\alpha,\exp} \int_{y_+}^{y} \frac{e^{\frac{1}{\varepsilon}(\lambda_3|\mathrm{Im}(s)|-\lambda_{2k+1}\mathrm{Im}(s)-(\lambda_3|\mathrm{Im}(y)|-\lambda_{2k+1}\mathrm{Im}(y)))}}{|s^2+\pi^2/4|^{\alpha}} ds \end{aligned}$$

If $\mathrm{Im}(y) \geq 0$, then

$$\begin{aligned} \left| \mathcal{P}^{\Gamma}_{2k+1}(h_{2k+1}) e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)} \right| &\leq \|h_{2k+1}\|_{\alpha,\exp} \int_{y_+}^{y} \frac{e^{-\frac{\lambda_{2k+1}-\lambda_3}{\varepsilon}(\mathrm{Im}(s)-\mathrm{Im}(y))}}{|s^2+\pi^2/4|^{\alpha}} ds \\ &\leq M\|h_{2k+1}\|_{\alpha,\exp} \frac{\varepsilon}{(\lambda_{2k+1}-\lambda_3)(\kappa\varepsilon)^{\alpha}} \left(1+e^{-\frac{\lambda_{2k+1}-\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\kappa\varepsilon-\mathrm{Im}(y)\right)}\right) \\ &\leq \frac{M}{\kappa^{\alpha}\varepsilon^{\alpha-1}} \|h_{2k+1}\|_{\alpha,\exp}, \end{aligned}$$

and, if $\mathrm{Im}(y) < 0$, then

$$\begin{aligned} &\left| \mathcal{P}^{\Gamma}_{2k+1}(h_{2k+1}) e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)} \right| \\ &\leq \|h_{2k+1}\|_{\alpha,\exp} \left(\int_{y_+}^{0} \frac{e^{-\frac{\lambda_{2k+1}-\lambda_3}{\varepsilon}(\mathrm{Im}(s)-\mathrm{Im}(y))}}{|s^2+\pi^2/4|^{\alpha}} ds + \int_{0}^{y} \frac{e^{-\frac{\lambda_{2k+1}+\lambda_3}{\varepsilon}(\mathrm{Im}(s)-\mathrm{Im}(y))}}{|s^2+\pi^2/4|^{\alpha}} ds \right) \\ &\leq \frac{M}{\kappa^{\alpha}\varepsilon^{\alpha-1}} \|h_{2k+1}\|_{\alpha,\exp} \left(\frac{1}{\lambda_{2k+1}-\lambda_3} \left(e^{\frac{\lambda_{2k+1}-\lambda_3}{\varepsilon}\mathrm{Im}(y)} + e^{-\frac{\lambda_{2k+1}-\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\kappa\varepsilon-\mathrm{Im}(y)\right)} \right) \right. \\ &\left. + \frac{1}{\lambda_{2k+1}+\lambda_3} \left(e^{\frac{\lambda_{2k+1}+\lambda_3}{\varepsilon}\mathrm{Im}(y)} + 1 \right) \right) \\ &\leq \frac{M}{\kappa^{\alpha}\varepsilon^{\alpha-1}} \|h_{2k+1}\|_{\alpha,\exp}. \end{aligned}$$

This shows (8.7.26) and concludes the proof. □

Proposition 8.7.6. *Let $\Delta v(y,\tau)$ be the function given in (8.7.2), and consider $\Delta w = \partial_y \Delta v$. Therefore, $\Delta v_1(y)$ is given by (8.7.4), and there exist two unique sequences of constants $c = (c_{2k+1})_{k\geq 1}$ and $d = (d_{2k+1})_{k\geq 1}$ such that $\Delta w_1 = \ddot{v}_1^h \widetilde{\Delta w_1}$ and $(\widetilde{\Delta w_1}(y), \Gamma(y,\tau), \Theta(y,\tau))$ (see (8.7.5)) satisfies the following equation*

$$(\widetilde{\Delta w_1}, \Gamma, \Theta) = (0, \mathcal{I}_{\Gamma}(c), \mathcal{I}_{\Theta}(d)) + \widetilde{\mathcal{M}}\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right), \tag{8.7.27}$$

where

$$\widetilde{\mathcal{M}}\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right) = \mathcal{P} \circ \mathcal{M}\left(\ddot{v}_1^h \widetilde{\Delta w_1}, \Gamma, \Theta\right),$$

$\mathcal{P}$ is given by (8.7.23) and $\mathcal{M}$ is given by (8.7.9). In addition, the following statements hold.

1. $(0, \mathcal{I}_\Gamma(c), \mathcal{I}_\Theta(d)) \in \mathcal{Y}_{\ell_1,-1,\exp}$ *and*

$$[\![(0, \mathcal{I}_\Gamma(c), \mathcal{I}_\Theta(d))]\!]_{\ell_1,-1,\exp} \le \frac{Me^{\lambda_3\kappa}}{\varepsilon\kappa^3}$$

2. The operator $\widetilde{\mathcal{M}} : \mathcal{Y}_{\ell_1,-1,\exp} \to \mathcal{Y}_{\ell_1,-1,\exp}$ *is well-defined and there exists a constant independent of* ε *and* κ *such that*

$$\left[\!\!\left[\widetilde{\mathcal{M}}\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right)\right]\!\!\right]_{\ell_1,-1,\exp} \le M\left[\!\!\left[\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right)\right]\!\!\right]_{\ell_1,-1,\exp},$$

and, denoting $\widetilde{\mathcal{M}} = (\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{M}}_3)$*, we have that*

$$\left\|\widetilde{\mathcal{M}}_1\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right)\right\|_{-1,\exp} \le \frac{M}{\kappa^3}\|\widetilde{\Delta w_1}\|_{-1\exp} + M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp} + \|\Theta\|_{\ell_1,0,\exp}\right) \quad (8.7.28)$$

and

$$\left\|\widetilde{\mathcal{M}}_j\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right)\right\|_{\ell_1,0,\exp} \le \frac{M}{\kappa}\left[\!\!\left[\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right)\right]\!\!\right]_{\ell_1,-1,\exp}, \; j = 2, 3. \quad (8.7.29)$$

Proof. From (8.7.8), we have that

$$\begin{cases} \dot{\Delta w_1} - \dfrac{\dddot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 &= m_W(y)\Delta w_1 + \mathcal{M}_W(\Gamma, \Theta), \\ \dot{\Gamma}_{2k+1} + i\dfrac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1} &= \Pi_{2k+1}\left[m_{\text{osc}}(y,\tau)\Delta w_1 + \mathcal{M}_{\text{osc}}(\Gamma, \Theta)\right], \; k \ge 1, \\ \dot{\Theta}_{2k+1} - i\dfrac{\lambda_{2k+1}}{\varepsilon}\Theta_{2k+1} &= -\Pi_{2k+1}\left[m_{\text{osc}}(y,\tau)\Delta w_1 + \mathcal{M}_{\text{osc}}(\Gamma, \Theta)\right], \; k \ge 1. \end{cases}$$

Using that $\ddot{v}_1^h$, $e^{-i\frac{\lambda_{2k+1}}{\varepsilon}y}$ and $e^{i\frac{\lambda_{2k+1}}{\varepsilon}y}$ are solutions of the homogeneous equations $\dot{\Delta w_1} - \frac{\dddot{v}_1^h}{\ddot{v}_1^h}\Delta w_1 = 0$, $\dot{\Gamma}_{2k+1} + i\frac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1} = 0$ and $\dot{\Theta}_{2k+1} - i\frac{\lambda_{2k+1}}{\varepsilon}\Theta_{2k+1} = 0$. It follows from the method of variation of constants, using that $\Delta w_1(0) = 0$, that there exists constants c_{2k+1} and d_{2k+1}, $k \ge 1$, such that

$$\begin{cases} \Delta w_1 &= \ddot{v}_1^h\mathcal{P}_W\left(m_W(y)\Delta w_1 + \mathcal{M}_W(\Gamma, \Theta)\right), \\ \Gamma_{2k+1} &= c_{2k+1}e^{-i\frac{\lambda_{2k+1}}{\varepsilon}y} + \mathcal{P}^\Gamma_{2k+1}\left(\Pi_{2k+1}\left[m_{\text{osc}}(y,\tau)\Delta w_1 + \mathcal{M}_{\text{osc}}(\Gamma, \Theta)\right]\right), \; k \ge 1, \\ \Theta_{2k+1} &= d_{2k+1}e^{i\frac{\lambda_{2k+1}}{\varepsilon}y} - \mathcal{P}^\Theta_{2k+1}\left(\Pi_{2k+1}\left[m_{\text{osc}}(y,\tau)\Delta w_1 + \mathcal{M}_{\text{osc}}(\Gamma, \Theta)\right]\right), \; k \ge 1. \end{cases}$$

Hence, writing $\Delta w_1 = \ddot{v}_1^h\widetilde{\Delta w_1}$ and the definitions of $\mathcal{P}$, $\mathcal{M}$ and $\mathcal{I}_\Gamma, \mathcal{I}_\Theta$ given in (8.7.23), (8.7.9), (8.7.21) and (8.7.22), we obtain (8.7.27).

Now, notice that $\mathcal{I}_\Gamma(c)(y_+) = \Gamma(y_+)$, and since $\|\Gamma\|_{\ell_1,3} \le M\varepsilon^2$, we obtain that

$$\sum_{k\ge1}\left|c_{2k+1}e^{\frac{\lambda_{2k+1}}{\varepsilon}\left(\frac{\pi}{2}-\kappa\varepsilon\right)}\right| = \|\Gamma\|_{\ell_1}(y_+) \le \frac{M}{\varepsilon\kappa^3},$$

which implies that

$$\sum_{k\geq 1}\left|c_{2k+1}e^{\frac{\lambda_{2k+1}-\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\kappa\varepsilon\right)}e^{\frac{\lambda_3\pi}{2\varepsilon}}\right| = \|\Gamma\|_{\ell_1}(y_+)e^{\lambda_3\kappa} \leq \frac{Me^{\lambda_3\kappa}}{\varepsilon\kappa^3}.$$

Now, notice that

$$\sum_{k\geq 1}\left\|c_{2k+1}e^{-i\frac{\lambda_{2k+1}}{\varepsilon}y}\right\|_{0,\exp} = \sum_{k\geq 1}\left|c_{2k+1}e^{\frac{\lambda_{2k+1}-\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\kappa\varepsilon\right)}e^{\frac{\lambda_3\pi}{2\varepsilon}}\right|,$$

and thus

$$\|\mathcal{I}_\Gamma(c)\|_{\ell_1,0,\exp}\leq \frac{Me^{\lambda_3\kappa}}{\varepsilon\kappa^3}.$$

Analogously, we prove that $\|\mathcal{I}_\Theta(d)\|_{\ell_1,0,\exp}\leq \dfrac{Me^{\lambda_3\kappa}}{\varepsilon\kappa^3}$, and hence, item (1) is proved.

Assume that $\left(\widetilde{\Delta w_1},\Gamma,\Theta\right)\in\mathcal{Y}_{\ell_1,-1,\exp}$. Notice that, for each $y\in\mathcal{R}_\kappa$

$$\begin{aligned}\left|(y^2+\pi^2/4)^{-1}e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\widetilde{\mathcal{M}}_1\left(\widetilde{\Delta w_1},\Gamma,\Theta\right)\right| \;\leq\; & \frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}}{|y^2+\pi^2/4|}\left(\int_0^y\left|m_W(s)\widetilde{\Delta w_1}(s)\right|ds\right.\\ & \left.+\int_0^y\left|\frac{\mathcal{M}_W(\Gamma,\Theta)(s)}{\ddot{v}_1^h(s)}\right|ds\right)\end{aligned}$$

Now, from item (1) of Proposition 8.7.4, we have that

$$\begin{aligned}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y\left|m_W(s)\widetilde{\Delta w_1}(s)\right|ds \;&\leq\; M\|\widetilde{\Delta w_1}\|_{-1,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y\frac{\varepsilon^2e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(s)|\right)}}{|s^2+\pi^2/4|^2}ds\\ &\leq \frac{M\|\widetilde{\Delta w_1}\|_{-1,\exp}}{\kappa^3\varepsilon}e^{-\frac{\lambda_3}{\varepsilon}|\mathrm{Im}(y)|}\int_0^y e^{\frac{\lambda_3}{\varepsilon}|\mathrm{Im}(s)|}ds\\ &\leq \frac{M\|\widetilde{\Delta w_1}\|_{-1,\exp}}{\kappa^3\varepsilon}e^{-\frac{\lambda_3}{\varepsilon}|\mathrm{Im}(y)|}\frac{\varepsilon}{\lambda_3}\left(e^{\frac{\lambda_3}{\varepsilon}|\mathrm{Im}(y)|}-1\right)\\ &\leq \frac{M\|\widetilde{\Delta w_1}\|_{-1,\exp}}{\kappa^3}.\end{aligned}$$

Using that $y\in i\mathbb{R}$ and

$$\|\Gamma\|_{\ell_1}(y)e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\leq\|\Gamma\|_{\ell_1,0,\exp}\quad \textit{and}\quad \|\Theta\|_{\ell_1}(y)e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\leq\|\Theta\|_{\ell_1,0,\exp},$$

we obtain that, using item (2) of Proposition 8.7.4 and that $\ddot{v}_1^h$ has a a pole of order 3, for $\operatorname{Im}(y) > 0$,

$$\begin{aligned}
&\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y\left|\frac{\mathcal{M}_W(\Gamma,\Theta)(s)}{\ddot{v}_1^h(s)}\right|ds\\
&\le M\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y|s^2+\pi^2/4|(\|\Gamma\|_{\ell_1}(s)+\|\Theta\|_{\ell_1}(s))\,ds\\
&\le M\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right)\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y|s^2+\pi^2/4|e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(s)|\right)}ds\\
&\le M\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right)\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\operatorname{Im}(y)\right)}}{|y-i\pi/2|}\int_0^{\operatorname{Im}(y)}|\sigma-\pi/2|e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\sigma\right)}d\sigma\\
&\le M\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right)\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\operatorname{Im}(y)\right)}}{|y-i\pi/2|}\int_{\frac{\pi}{2\varepsilon}}^{\frac{\frac{\pi}{2}-\operatorname{Im}(y)}{\varepsilon}}\varepsilon re^{-\lambda_3 r}\varepsilon dr\\
&\le M\varepsilon^2\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right)\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\operatorname{Im}(y)\right)}}{|y-i\pi/2|}\left(e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\operatorname{Im}(y)\right)}\left(1+\frac{\pi/2-\operatorname{Im}(y)}{\varepsilon}\right)\right.\\
&\left.+e^{-\frac{\pi\lambda_3}{2\varepsilon}}\left(1+\frac{\pi}{2\varepsilon}\right)\right)\\
&\le\frac{M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right)}{|y-i\pi/2|}\left(\varepsilon+\frac{\pi}{2}-\operatorname{Im}(y)+e^{-\frac{\lambda_3}{\varepsilon}\operatorname{Im}(y)}\left(\varepsilon+\frac{\pi}{2}\right)\right)\\
&\le M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right)\left(\frac{1}{\kappa}+1+\left(\varepsilon+\frac{\pi}{2}\right)\frac{e^{-\frac{\lambda_3}{\varepsilon}\operatorname{Im}(y)}}{|y-i\pi/2|}\right)\\
&\le M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right).
\end{aligned}$$

Analogously, we obtain the same estimate for $\operatorname{Im}(y) < 0$. The proof of (8.7.28) follows directly from these bounds.

Using the expressions of m_{osc} and $\mathcal{M}_{osc}$ given in (8.7.14) (such as the bounds of the operators A and B obtained in the proof of Proposition 8.7.2), and the property

$$\|h_1h_2\|_{\ell_1,\alpha_1+\alpha_2,\exp}\le\|h_1\|_{\ell_1,\alpha_1}\|h_2\|_{\ell_1,\alpha_2,\exp}$$

we can see that

$$\|m_{osc}\ddot{v}_1^h\widetilde{\Delta w_1}\|_{\ell_1,3,\exp}\le M\varepsilon\|\widetilde{\Delta w_1}\|_{-1,\exp} \tag{8.7.30}$$

and

$$\left\|\mathcal{M}_{\text{osc}}\left(\widetilde{\Delta w_1},\Gamma,\Theta\right)\right\|_{\ell_1,2,\exp}\le M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right). \tag{8.7.31}$$

Also, notice that

$$\widetilde{\mathcal{M}_2}\left(\widetilde{\Delta w_1},\Gamma,\Theta\right)=\mathcal{P}^\Gamma\left(m_{osc}\ddot{v}_1^h\widetilde{\Delta w_1}\right)+\mathcal{P}^\Gamma\left(\mathcal{M}_{\text{osc}}\left(\widetilde{\Delta w_1},\Gamma,\Theta\right)\right). \tag{8.7.32}$$

From Lemma 8.7.5, the operators $\mathcal{P}^\Gamma,\mathcal{P}^\Theta:\mathcal{X}_{\ell_1,\alpha,\exp}\to\mathcal{X}_{\ell_1,0,\exp}$ are well-defined for $\alpha=2,3$ and satisfy (8.7.26). Now, (8.7.29) (with $j=2$) follows directly from (8.7.30),(8.7.31) and (8.7.32). Analogously, we prove (8.7.29) for $j=3$.

The proof of item (2) follows directly from these estimates.

□

In order to obtain good estimates for the linear operator in (8.7.27), we use a Gauss-Seidel argument. Since $\Gamma = \mathcal{I}_\Gamma(c) + \widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta)$, $\Theta = \mathcal{I}_\Theta(d) + \widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)$, and $\widetilde{\mathcal{M}}_1$ is a linear operator, it follows that

$$\begin{aligned}\widetilde{\Delta w_1} &= \widetilde{\mathcal{M}}_1(\widetilde{\Delta w_1}, \mathcal{I}_\Gamma(c) + \widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta), \mathcal{I}_\Theta(d) + \widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)) \\ &= \widetilde{\mathcal{M}}_1(0, \mathcal{I}_\Gamma(c), \mathcal{I}_\Theta(d)) + \widetilde{\mathcal{M}}_1(\widetilde{\Delta w_1}, \widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta), \widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)).\end{aligned}$$

Thus, we rewrite (8.7.27) as

$$(\widetilde{\Delta w_1}, \Gamma, \Theta) = (\mathcal{I}_W, \mathcal{I}_\Gamma(c), \mathcal{I}_\Theta(d)) + \widetilde{\mathcal{M}}_{GS}\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right), \tag{8.7.33}$$

where

$$\mathcal{I}_W = \widetilde{\mathcal{M}}_1(0, \mathcal{I}_\Gamma(c), \mathcal{I}_\Theta(d)) = \mathcal{P}_W \circ \mathcal{M}_W(\mathcal{I}_\Gamma(c), \mathcal{I}_\Theta(d)), \tag{8.7.34}$$

and

$$\begin{aligned}\widetilde{\mathcal{M}}_{GS}\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right) &= \Big(\widetilde{\mathcal{M}}_1\left(\widetilde{\Delta w_1}, \widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta), \widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)\right), \\ &\qquad \widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta), \widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)\Big).\end{aligned}$$

Proposition 8.7.7. *The linear operator $\widetilde{\mathcal{M}}_{GS} : \mathcal{X}_{\ell_1,-1,\exp} \to \mathcal{X}_{\ell_1,-1,\exp}$ is well-defined, and there exist $\kappa_0 \geq 1$ and a constant $M > 0$ independent of ε and κ such that, for every $\kappa \geq \kappa_0$*

$$\left[\!\left[\widetilde{\mathcal{M}}_{GS}\right]\!\right]_{\ell_1,-1,\exp} \leq \frac{M}{\kappa}.$$

Proof. In fact, from (8.7.28) and (8.7.29), it follows that

$$\begin{aligned}&\left[\!\left[\widetilde{\mathcal{M}}_{GS}\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right)\right]\!\right]_{\ell_1,-1,\exp} \\ &= \frac{1}{\varepsilon}\left\|\widetilde{\mathcal{M}}_1\left(\widetilde{\Delta w_1}, \widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta), \widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)\right)\right\|_{-1,\exp} \\ &+ \left\|\widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta)\right\|_{\ell_1,0,\exp} + \left\|\widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)\right\|_{\ell_1,0,\exp} \\ &\leq \frac{M}{\kappa^3}\frac{\|\widetilde{\Delta w_1}\|_{-1\exp}}{\varepsilon} + M\left(\left\|\widetilde{\mathcal{M}}_2(\widetilde{\Delta w_1}, \Gamma, \Theta)\right\|_{\ell_1,0,\exp} + \left\|\widetilde{\mathcal{M}}_3(\widetilde{\Delta w_1}, \Gamma, \Theta)\right\|_{\ell_1,0,\exp}\right) \\ &\leq \frac{M}{\kappa^3}\frac{\|\widetilde{\Delta w_1}\|_{-1\exp}}{\varepsilon} + \frac{M}{\kappa}\left(\frac{\|\widetilde{\Delta w_1}\|_{-1,\exp}}{\varepsilon} + \|\Gamma\|_{\ell_1,0,\exp} + \|\Theta\|_{\ell_1,0,\exp}\right) \\ &\leq \frac{M}{\kappa}\left[\!\left[\left(\widetilde{\Delta w_1}, \Gamma, \Theta\right)\right]\!\right]_{\ell_1,-1,\exp},\end{aligned}$$

which proves the result.

□

8.7.2 Asymptotic Formula

Denote

$$\Delta(y, \tau) = (\widetilde{\Delta w_1}(y), \Gamma(y, \tau), \Theta(y, \tau)), \tag{8.7.35}$$

and define

$$\Delta_0(y,\tau)=\begin{pmatrix}\mathcal{P}_W\circ\mathcal{M}_W\left(\frac{2}{\varepsilon}C_{\text{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau),\frac{2}{\varepsilon}\overline{C_{\text{in}}}e^{i\frac{\lambda_3}{\varepsilon}(y+i\pi/2)}\sin(3\tau)\right)\\ \frac{2\lambda_{0,3}}{\varepsilon}C_{\text{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\\ \frac{2\lambda_{0,3}}{\varepsilon}\overline{C_{\text{in}}}e^{i\frac{\lambda_3}{\varepsilon}(y+i\pi/2)}\sin(3\tau)\end{pmatrix} \tag{8.7.36}$$

where C_{in} is given by Theorem 8.3.2. In order to prove Theorem U, we consider the following decomposition

$$\Delta=\Delta_0+\Delta_1. \tag{8.7.37}$$

Lemma 8.7.8. *Consider* $\kappa=\frac{1}{2\lambda_3}\log(\varepsilon^{-1})$ *and* $\mathcal{I}_{\text{tot}}=(\mathcal{I}_W,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d))$ *(see* (8.7.27) *and* (8.7.34)*). There exist* $\varepsilon_0>0$ *and a constant* $M>0$ *independent of* ε *such that, for each* $\varepsilon<\varepsilon_0$*,*

$$[\![\Delta_0-\mathcal{I}_{\text{tot}}]\!]_{\ell_1,-1,\exp}\leq\frac{M}{\varepsilon\log(\varepsilon^{-1})}.$$

Proof. First, notice that, from Theorems 8.3.2 and 8.3.3, we have that the function Δv given in (8.7.2) is written as

$$\begin{aligned}\Delta v(y,\tau)&=\frac{1}{\varepsilon}\phi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\phi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\&=\frac{1}{\varepsilon}\Delta\phi^0\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)+\frac{1}{\varepsilon}\varphi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\varphi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\&=\frac{1}{\varepsilon}e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}\left(C_{\text{in}}\sin(3\tau)+\chi\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\right)\\&\quad+\frac{1}{\varepsilon}\varphi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\varphi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\&=\frac{1}{\varepsilon}C_{\text{in}}e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}\sin(3\tau)+E_1^+(y,\tau)+E_2^+(y,\tau),\end{aligned}$$

for every $y\in\mathcal{R}^+_{\text{mch},\kappa}=D^{\text{mch},u}_{+,\kappa}\cap D^{\text{mch},s}_{+,\kappa}\cap i\mathbb{R}$ (with $\gamma=1/2$) and $\kappa\geq\kappa_0$ (κ_0 is given by Theorems 8.3.2 and 8.3.3), where $E_1^+,E_2^+:\mathcal{R}_{\text{mch},\kappa}\times\mathbb{T}\to\mathbb{C}$ are analytic functions in the variable y. It follows from Theorem 8.3.2 that

$$\|E_1^+\|_{\ell_1}(y),\|\partial_\tau E_1^+\|_{\ell_1}(y)\leq\frac{M|e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|}\quad\textit{and}\quad\|\partial_y E_1^+\|_{\ell_1}(y)\leq\frac{M|e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}|}{\varepsilon|y-i\pi/2|}, \tag{8.7.38}$$

and from Theorem 8.3.3, we obtain

$$\|E_2^+\|_{\ell_1}(y),\|\partial_\tau E_2^+\|_{\ell_1}(y)\leq\frac{M\varepsilon^{3/2}}{|y-i\pi/2|^2}\quad\textit{and}\quad\|\partial_y E_2^+\|_{\ell_1}(y)\leq\frac{M\varepsilon^{1/2}}{\kappa|y-i\pi/2|^2}. \tag{8.7.39}$$

Analogously, performing the same study for the pole $y=-i\pi/2$, we obtain that

$$\Delta v(y,\tau)=\frac{1}{\varepsilon}\overline{C_{\text{in}}}e^{i\lambda_{0,3}\frac{y+i\pi/2}{\varepsilon}}\sin(3\tau)+E_1^-(y,\tau)+E_2^-(y,\tau),$$

for every $y \in \mathcal{R}^-_{\text{mch},\kappa} = D^{\text{mch},-,u}_{\kappa,c} \cap D^{\text{mch},-,s}_{\kappa,c} \cap i\mathbb{R}$ (with $\gamma = 1/2$) and $\kappa \geq \kappa_0$ (κ_0 is given by Theorems 8.3.2 and 8.3.3), where $E_1^-, E_2^- : \mathcal{R}^-_{\text{mch},\kappa} \times \mathbb{T} \to \mathbb{C}$ are analytic functions in the variable y satisfying

$$\|E_1^-\|_{\ell_1}(y), \|\partial_\tau E_1^-\|_{\ell_1}(y) \leq \frac{M|e^{i\lambda_{0,3}\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|} \quad \textit{and} \quad \|\partial_y E_1^-\|_{\ell_1}(y) \leq \frac{M|e^{i\lambda_{0,3}\frac{y+i\pi/2}{\varepsilon}}|}{\varepsilon|y+i\pi/2|},$$

$$\|E_2^-\|_{\ell_1}(y), \|\partial_\tau E_2^-\|_{\ell_1}(y) \leq \frac{M\varepsilon^{3/2}}{|y+i\pi/2|^2} \quad \textit{and} \quad \|\partial_y E_2^-\|_{\ell_1}(y) \leq \frac{M\varepsilon^{1/2}}{\kappa|y+i\pi/2|^2}.$$

Now, using that $\lambda_3 = \lambda_{0,3} + \mathcal{O}(\varepsilon^2)$, we obtain that

$$\begin{aligned}
\Gamma(y,\tau) &= \sum_{k\geq 1}(\lambda_{2k+1}\Delta v_{2k+1}(y) + i\varepsilon\partial_y\Delta v_{2k+1}(y))\sin((2k+1)\tau) \\
&= \sum_{k\geq 1}\lambda_{2k+1}\Delta v_{2k+1}(y)\sin((2k+1)\tau) + i\varepsilon\widehat{\Pi}\left[\partial_y\Delta v\right](y,\tau) \\
&= \frac{2\lambda_{0,3}}{\varepsilon}C_{\text{in}}e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}(1+\mathcal{O}(\varepsilon^2))\sin(3\tau) \\
&\quad + \sum_{k\geq 1}\lambda_{2k+1}\Pi_{2k+1}\left[E_1^+ + E_2^+\right]\sin((2k+1)\tau) + i\varepsilon\widehat{\Pi}\left[\partial_y E_1^+ + \partial_y E_2^+\right](y,\tau),
\end{aligned}$$

for every $y \in \mathcal{R}^+_{\text{mch},\kappa}$ and $\tau \in \mathbb{T}$. Also, we have that

$$\begin{aligned}
\left\|\sum_{k\geq 1}\lambda_{2k+1}\Pi_{2k+1}\left[E_1^+ + E_2^+\right]\sin((2k+1)\tau)\right\|_{\ell_1}(y) &\leq M\left(\|\partial_\tau E_1^+\|_{\ell_1}(y) + \|\partial_\tau E_2^+\|_{\ell_1}(y)\right) \\
&\leq M\left(\frac{|e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|} + \frac{\varepsilon^{3/2}}{|y-i\pi/2|^2}\right),
\end{aligned}$$

and it follows from (8.7.38) and (8.7.39) that

$$\left\|i\varepsilon\widehat{\Pi}\left[\partial_y E_1^+ + \partial_y E_2^+\right]\right\|_{\ell_1}(y) \leq M\left(\frac{|e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|} + \frac{\varepsilon^{3/2}}{\kappa|y-i\pi/2|^2}\right),$$

which means that, near the pole $y = i\pi/2$, Γ satisfies

$$\Gamma(y,\tau) = \frac{2\lambda_{0,3}}{\varepsilon}C_{\text{in}}e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}\sin(3\tau) + E_\Gamma^+(y,\tau),$$

for every $y \in \mathcal{R}^+_{\text{mch},\kappa}$ and $\tau \in \mathbb{T}$, where $E_\Gamma^+ : \mathcal{R}^+_{\text{mch},\kappa} \times \mathbb{T} \to \mathbb{C}$ is an analytic function in the variable τ such that

$$\left\|E_\Gamma^+\right\|_{\ell_1}(y) \leq M\left(\frac{|e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|} + \frac{\varepsilon^{3/2}}{|y-i\pi/2|^2}\right)$$

Analogously, near the pole $y = -i\pi/2$, we can see that

$$\|\Gamma\|_{\ell_1}(y) \leq M\left(\frac{|e^{i\lambda_{0,3}\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|} + \frac{\varepsilon^{3/2}}{|y+i\pi/2|^2}\right),$$

for every $y \in \mathcal{R}^-_{\mathrm{mch},\kappa}$.

Proceeding in the same way for the function

$$\Theta(y,\tau) = \sum_{k\geq 0}(\lambda_{2k+1}\Delta v_{2k+1}(y) - i\varepsilon\partial_y \Delta v_{2k+1}(y))\sin((2k+1)\tau),$$

we conclude that there exists a function $E^-_\Theta : \mathcal{R}^-_{\mathrm{mch},\kappa} \times \mathbb{T} \to \mathbb{C}$ analytic in the variable y such that, near the pole $y = -i\pi/2$, Θ writes as

$$\Theta(y,\tau) = \frac{2\lambda_{0,3}}{\varepsilon}\overline{C_{\mathrm{in}}}e^{i\lambda_{0,3}\frac{y+i\pi/2}{\varepsilon}}\sin(3\tau) + E^-_\Theta(y,\tau),$$

for every $y \in \mathcal{R}^-_{\mathrm{mch},\kappa}$, and

$$\left\|E^-_\Theta\right\|_{\ell_1}(y) \leq M\left(\frac{|e^{i\lambda_{0,3}\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|} + \frac{\varepsilon^{3/2}}{|y+i\pi/2|^2}\right), \text{ for } y \in \mathcal{R}^-_{\mathrm{mch},\kappa},$$

and, near the pole $y = i\pi/2$,

$$\|\Theta\|_{\ell_1}(y) \leq M\left(\frac{|e^{-i\lambda_{0,3}\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|} + \frac{\varepsilon^{3/2}}{|y-i\pi/2|^2}\right),$$

for every $y \in \mathcal{R}^+_{\mathrm{mch},\kappa}$.

Now that we have good estimates for the function Γ and Θ near the poles $y = \pm i\pi/2$, we are able to bound the function $\mathcal{I}_{\mathrm{tot}}$.

In fact, recall that $\mathcal{I}_\Gamma(c)(y_+) = \Gamma(y_+)$. Therefore

$$\begin{aligned}\left\|\mathcal{I}_\Gamma(c) - \frac{2\lambda_{0,3}}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y_+) &= \left\|\Gamma - \frac{2\lambda_{0,3}}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y_+)\\ &= \left\|E^+_\Gamma\right\|_{\ell_1}(y_+)\\ &\leq M\left(\frac{|e^{-i\lambda_{0,3}\frac{y_+-i\pi/2}{\varepsilon}}|}{|y_+-i\pi/2|} + \frac{\varepsilon^{3/2}}{|y_+-i\pi/2|^2}\right)\\ &\leq M\left(\frac{e^{-\lambda_{0,3}\kappa}}{\kappa\varepsilon} + \frac{\varepsilon^{3/2}}{\kappa^2\varepsilon^2}\right),\end{aligned}$$

and notice that, from (8.7.21), we have that

$$\left\|\mathcal{I}_\Gamma(c) - \frac{2\lambda_{0,3}}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp} = e^{\lambda_3\kappa}\left\|\mathcal{I}_\Gamma(c) - \frac{2\lambda_{0,3}}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y_+),$$

and thus, taking $\kappa = \dfrac{1}{2\lambda_3}\log(\varepsilon^{-1})$, we have that

$$\begin{aligned}\left\|\mathcal{I}_\Gamma(c) - \frac{2\lambda_{0,3}}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp} &\leq M\left(\frac{e^{(\lambda_3-\lambda_{0,3})\kappa}}{\kappa\varepsilon} + \frac{\varepsilon^{3/2}e^{\lambda_3\kappa}}{\kappa^2\varepsilon^2}\right)\\ &\leq M\left(\frac{1}{\varepsilon\log(\varepsilon^{-1})} + \frac{\varepsilon^{3/2}\varepsilon^{-1/2}}{\log(\varepsilon^{-1})^2\varepsilon^2}\right)\\ &\leq \frac{M}{\varepsilon\log(\varepsilon^{-1})}.\end{aligned} \tag{8.7.40}$$

Analogously, we prove that

$$\left\|\mathcal{I}_\Theta(d)-\frac{2\lambda_{0,3}}{\varepsilon}\overline{C_{\rm in}}e^{i\frac{\lambda_3}{\varepsilon}(y+i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp}=\frac{M}{\varepsilon\log(\varepsilon^{-1})}. \tag{8.7.41}$$

Finally, recall that

$$\|\mathcal{P}_W\circ\mathcal{M}_W(\Gamma,\Theta)\|_{-1,\exp}\leq M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right),$$

and thus

$$\begin{aligned}&\frac{1}{\varepsilon}\left\|\mathcal{P}_W\circ\mathcal{M}_W\left(\mathcal{I}_\Gamma(c)-\frac{2}{\varepsilon}C_{\rm in}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau),\mathcal{I}_\Theta(d)-\frac{2}{\varepsilon}\overline{C_{\rm in}}e^{i\frac{\lambda_3}{\varepsilon}(y+i\pi/2)}\sin(3\tau)\right)\right\|_{-1,\exp}\\&\leq\frac{M}{\varepsilon\log(\varepsilon^{-1})}.\end{aligned} \tag{8.7.42}$$

The proof follows from (8.7.20), (8.7.34), (8.7.40), (8.7.41) and (8.7.42). □

Finally, the proof of Theorem U follows directly from the following result.

Proposition 8.7.9. *The function* Δ_1 *is completely determined by equation*

$$\Delta_1=\mathcal{I}_{\rm tot}-\Delta_0+\widetilde{\mathcal{M}}_{GS}(\Delta_0)+\widetilde{\mathcal{M}}_{GS}(\Delta_1), \tag{8.7.43}$$

and there exist $\varepsilon_0>0$ *and a constant* $M>0$ *independent of* ε *such that, for each* $\varepsilon<\varepsilon_0$ *and taking* $\kappa=\frac{1}{2\lambda_3}\log(\varepsilon^{-1})$,

$$[\![\Delta_1]\!]_{\ell_1,-1,\exp}\leq\frac{M}{\varepsilon\log(\varepsilon^{-1})}.$$

Proof. In fact, (8.7.43) follows directly from (8.7.33), (8.7.35), (8.7.37), and the linearity of $\widetilde{\mathcal{M}}_{GS}$. Now, taking ε_0 sufficiently small, we have from Proposition 8.7.7 that

$$\left[\!\!\left[\widetilde{\mathcal{M}}_{GS}\right]\!\!\right]_{\ell_1,-1,\exp}\leq\frac{M}{\log(\varepsilon^{-1})}\leq\frac{1}{2},$$

for every $\varepsilon<\varepsilon_0$. Thus the operator $\mathrm{Id}-\widetilde{\mathcal{M}}_{GS}:\mathcal{Y}_{\ell_1,-1,\exp}\to\mathcal{Y}_{\ell_1,-1,\exp}$ is invertible and

$$[\![(\mathrm{Id}-\widetilde{\mathcal{M}}_{GS})^{-1}]\!]\leq M.$$

Now,

$$\Delta_1=(\mathrm{Id}-\widetilde{\mathcal{M}}_{GS})^{-1}\left(\mathcal{I}_{\rm tot}-\Delta_0+\widetilde{\mathcal{M}}_{GS}(\Delta_0)\right),$$

and it is easy to check that

$$[\![\Delta_0]\!]_{\ell_1,-1,\exp}\leq\frac{M}{\varepsilon}.$$

Therefore

$$\begin{aligned}[\![\Delta_1]\!]_{\ell_1,-1,\exp}&\leq M\left([\![\mathcal{I}_{\rm tot}-\Delta_0]\!]_{\ell_1,-1,\exp}+\frac{1}{\log(\varepsilon^{-1})}[\![\Delta_0]\!]_{\ell_1,-1,\exp}\right)\\&\leq\frac{M}{\varepsilon\log(\varepsilon^{-1})}.\end{aligned}$$

□

Finally, from (8.7.5), we have that $d(\tau;\varepsilon)$ given by (8.7.1) writes as

$$\begin{aligned} d(\tau;\varepsilon) &= \begin{pmatrix} \widehat{\Pi}\left[\Delta v\right](0,\tau) \\ \widehat{\Pi}\left[\Delta w\right](0,\tau) \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle\sum_{k\geq 1} \frac{\Gamma_{2k+1}(0)+\Theta_{2k+1}(0)}{2\lambda_{2k+1}} \sin((2k+1)\tau) \\ \displaystyle\frac{\Gamma(0,\tau)-\Theta(0,\tau)}{2i\varepsilon} \end{pmatrix}, \end{aligned}$$

and thus, formula (8.2.1) in the second statement of Theorem U follows from (8.7.35), (8.7.36), (8.7.37) and Proposition 8.7.9. The proof of Theorem U is concluded by noticing that its third statement follows directly from the second one.

8.8 Conclusion and Further Directions

In this chapter we have associated breather solutions with period near 2π of reversible Klein-Gordon equations (8.1.3) with homoclinic solutions at the origin of a singularly perturbed Hamiltonian H_ε (where ε is the perturbation parameter). We have seen that, in the limit case $\varepsilon = 0$, H_0 has a homoclinic orbit and we have computed the distance between the invariant manifolds $W^u(0)$ and $W^s(0)$ of H_ε at the origin (in certain transversal section), for $\varepsilon > 0$, which happens to be exponentially small with respect to ε.

As a future work, one can prove that the constant C_{in} in Theorem U is generically non-vanishing. Moreover, based on numerical simulations and formal expansions, considering $f = 0$ in (8.1.3), we believe that $C_{\text{in}} \neq 0$, and thus the breather solution of the limit problem $\varepsilon = 0$ breaks down for $\varepsilon > 0$.

Bibliography

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *The inverse scattering transform-fourier analysis for nonlinear problems*, Studies in Applied Mathematics **53** (1974), no. 4, 249–315.

[2] K. da S. Andrade, *On degenerate cycles in discontinuous vector fields and the Dulac's problem*, Thesis (Unicamp), 2016.

[3] K. da S. Andrade, O. M. L. Gomide, and D. D. Novaes, *Qualitative analysis of polycycles in Filippov systems*, arXiv preprint, arXiv:1905.11950 (2019).

[4] K. da S. Andrade, M. R. Jeffrey, R. M. Martins, and M. A. Teixeira, *Homoclinic boundary-saddle bifurcations in nonsmooth vector fields*, arXiv preprint, arXiv:1701.05857 (2017), 1–39.

[5] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Theory of bifurcations of dynamic systems on a plane*, John Wiley and Sons, 1971.

[6] D. K. Arrowsmith and C. M. Place, *An introduction to dynamical systems*, Cambridge university press, 1990.

[7] I. Baldomá, *The inner equation for one and a half degrees of freedom rapidly forced Hamiltonian systems*, Nonlinearity **19** (2006), no. 6, 1415.

[8] I. Baldomá, E. Fontich, M. Guardia, and T. M. Seara, *Exponentially small splitting of separatrices beyond Melnikov analysis: rigorous results*, Journal of Differential Equations **253** (2012), no. 12, 3304–3439.

[9] I. Baldomá and T. M. Seara, *Breakdown of heteroclinic orbits for some analytic unfoldings of the hopf-zero singularity*, Journal of Nonlinear Science **16** (2006), no. 6, 543–582.

[10] A. D. Bazykin, *Nonlinear dynamics of interacting populations*, World Scientific, 1998.

[11] L. Benadero, E. Ponce, A. El Aroudi, and F. Torres, *Limit cycle bifurcations in resonant lc power inverters under zero current switching strategy*, Nonlinear Dynamics **91** (2018), no. 2, 1145–1161.

[12] ———, *Limit cycle bifurcations in resonant lc power inverters under zero current switching strategy*, Nonlinear Dynamics **91** (2018), no. 2, 1145–1161.

[13] W. Blaschke and G. Bol, *Geometrie der gewebe: Topologische fragen der differential geometrie*, Springer, 1938.

[14] C. Bonatti and M. A. Teixeira, *Topological equivalence of diffeomorphisms and curves*, Journal of differential equations **118** (1995), no. 2, 371–379.

[15] C. Bonet, T. M. Seara, E. Fossas, and M. R. Jeffrey, *A unified approach to explain contrary effects of hysteresis and smoothing in nonsmooth systems*, Communications in Nonlinear Science and Numerical Simulation **50** (2017), 142–168.

[16] C. Bonet-Reves, J. Larrosa, and T. M. Seara, *Regularization around a generic codimension one fold-fold singularity*, Journal of Differential Equations **265** (2018), no. 5, 1761–1838.

[17] C. Bonet Revés and T. M. Seara, *Regularization of sliding global bifurcations derived from the local fold singularity of Filippov systems*, Discrete and continuous dynamical systems. Series A **36** (2016), no. 7, 3545–3601.

[18] B. Brogliato, *Nonsmooth mechanics*, Springer, 1999.

[19] M. E Broucke, C. Pugh, and S. N. Simic, *Structural stability of piecewise smooth systems*, Computational and applied mathematics **20** (2001), no. 1-2, 51–89.

[20] C. A. Buzzi, T. Carvalho, and R. D. Euzébio, *Chaotic planar piecewise smooth vector fields with non-trivial minimal sets*, Ergodic Theory and Dynamical Systems **36** (2016), no. 2, 458–469.

[21] C. A. Buzzi, T. Carvalho, and M. A. Teixeira, *On three-parameter families of Filippov systems- the fold-saddle singularity*, International Journal of Bifurcation and Chaos **22** (2012), no. 12, 1250291.

[22] C. A. Buzzi, T. de Carvalho, and M. A. Teixeira, *On 3-parameter families of piecewise smooth vector fields in the plane*, SIAM Journal on Applied Dynamical Systems **11** (2012), no. 4, 1402–1424.

[23] C. A. Buzzi, J. C. R. Medrado, and M. A. Teixeira, *Generic bifurcation of refracted systems*, Advances in Mathematics **234** (2013), 653–666.

[24] A. Colombo and M. R. Jeffrey, *The two-fold singularity of discontinuous vector fields*, SIAM Journal on Applied Dynamical Systems **8** (2009), no. 2, 624–640.

[25] ———, *Nondeterministic chaos, and the two-fold singularity in piecewise smooth flows*, SIAM Journal on Applied Dynamical Systems **10** (2011), no. 2, 423–451.

[26] ———, *The two-fold singularity of nonsmooth flows: leading order dynamics in n-dimensions*, Physica D: Nonlinear Phenomena **263** (2013), 1–10.

[27] C. Conley and R. Easton, *Isolated invariant sets and isolating blocks*, Transactions of the American Mathematical Society **158** (1971), no. 1, 35–61.

[28] W. De Melo, *Moduli of stability of two-dimensional diffeomorphisms*, Topology **19** (1980), no. 1, 9–21.

[29] A. Delshams, R. de la Llave, and T. M. Seara, *A geometric mechanism for diffusion in hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model*, Mem. Amer. Math. Soc **179** (2003), no. 844, 1–141.

[30] A. Delshams, S. V. Gonchenko, V. S. Gonchenko, J. T. Lázaro, and O. Sten'Kin, *Abundance of attracting, repelling and elliptic periodic orbits in two-dimensional reversible maps*, Nonlinearity **26** (2012), no. 1, 1.

[31] J. Denzler, *Nonpersistence of breather families for the perturbed sine-Gordon equation*, Communications in mathematical physics **158** (1993), no. 2, 397–430.

[32] M. Di Bernardo, C. Budd, A. R. Champneys, and P. Kowalczyk, *Piecewise-smooth dynamical systems: theory and applications*, vol. 163, Springer Science & Business Media, 2008.

[33] M. Di Bernardo, M. I. Feigin, S. J. Hogan, and M. E. Homer, *Local analysis of C-bifurcations in n-dimensional piecewise-smooth dynamical systems*, Chaos, Solitons and Fractals: the interdisciplinary journal of Nonlinear Science, and Nonequilibrium and Complex Phenomena **11** (1999), no. 10, 1881–1908.

[34] I. Ekeland, *Discontinuités de champs hamiltoniens et existence de solutions optimales en calcul des variations*, Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques **47** (1977), no. 1, 5–32.

[35] E. G. Ekomasov, A. M. Gumerov, and R. V. Kudryavtsev, *On the possibility of the observation of the resonance interaction between kinks of the sine-Gordon equation and localized waves in real physical systems*, JETP letters **101** (2015), no. 12, 835–839.

[36] Z. Fei, Y. S. Kivshar, and L. Vázquez, *Resonant kink-impurity interactions in the sine-Gordon model*, Physical Review A **45** (1992), no. 8, 6019.

[37] M. I. Feigin, *On the structure of C-bifurcation boundaries of piecewise-continuous systems*, Journal of Applied mathematics and Mechanics **42** (1978), no. 5, 885–895.

[38] S. Fernández-García, D. A. García, G. O. Tost, M. di Bernardo, and M. R. Jeffrey, *Structural stability of the two-fold singularity*, SIAM Journal on Applied Dynamical Systems **11** (2012), no. 4, 1215–1230.

[39] A. F. Filippov, *Differential equations with discontinuous righthand sides: control systems*, vol. 18, Springer Science & Business Media, 1988.

[40] E. Freire, E. Ponce, and F. Torres, *On the critical crossing cycle bifurcation in planar Filippov systems*, Journal of Differential Equations **259** (2015), no. 12, 7086–7107.

[41] V. G. Gelfreich, *Melnikov method and exponentially small splitting of separatrices*, Physica D: Nonlinear Phenomena **101** (1997), no. 3-4, 227–248.

[42] S. W. Goatham, L. E. Mannering, R. Hann, and S. Krusch, *Dynamics of multi-kinks in the presence of wells and barriers*, Acta Physica Polonica B **42** (2011), no. 10.

[43] O. M. L. Gomide, M. Guardia, and T. M. Seara, *Critical velocity in kink-defect interaction models: rigorous results*, arXiv preprint, arXiv:1811.03699 (2018).

[44] O. M. L. Gomide and M. A. Teixeira, *Generic singularities of 3D piecewise smooth dynamical systems*, Advances in Mathematics and Applications (C. Lavor and Francisco A. M. Gomes, eds.), Springer, 2018, pp. 373–404.

[45] ______, *On structural stability of 3D Filippov systems*, Mathematische Zeitschrift (2019), 1–31.

[46] ______, *On typical homoclinic-like loops in 3d Filippov systems*, arXiv preprint, arXiv:1906.07814 (2019).

[47] R. H. Goodman and R. Haberman, *Interaction of sine-Gordon kinks with defects: the two-bounce resonance*, Physica D: Nonlinear Phenomena **195** (2004), no. 3-4, 303–323.

[48] ______, *Kink-antikink collisions in the phi 4 equation: The n-bounce resonance and the separatrix map*, SIAM Journal on Applied Dynamical Systems **4** (2005), no. 4, 1195–1228.

[49] ______, *Vector-soliton collision dynamics in nonlinear optical fibers*, Physical Review E **71** (2005), no. 5, 056605.

[50] R. H. Goodman, P. J. Holmes, and M. I. Weinstein, *Strong NLS soliton–defect interactions*, Physica D: Nonlinear Phenomena **192** (2004), no. 3-4, 215–248.

[51] R. H. Goodman, R. E. Slusher, and M. I. Weinstein, *Stopping light on a defect*, JOSA B **19** (2002), no. 7, 1635–1652.

[52] M. Guardia, C. Olivé, and T. M. Seara, *Exponentially small splitting for the pendulum: a classical problem revisited*, Journal of nonlinear science **20** (2010), no. 5, 595–685.

[53] ______, *Exponentially small splitting for the pendulum: a classical problem revisited*, Journal of nonlinear science **20** (2010), no. 5, 595–685.

[54] M. Guardia and T. M. Seara, *Exponentially and non-exponentially small splitting of separatrices for the pendulum with a fast meromorphic perturbation*, Nonlinearity **25** (2012), no. 5, 1367.

[55] M. Guardia, T. M. Seara, and M. A. Teixeira, *Generic bifurcations of low codimension of planar Filippov systems*, Journal of Differential Equations **250** (2011), no. 4, 1967–2023.

[56] J. Guckenheimer and P. J. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer-Verlag New York, 1983.

[57] P. Hartman, *On local homeomorphisms of Euclidean spaces*, Bol. Soc. Mat. Mexicana **5** (1960), no. 2, 220–241.

[58] J. Holmer, J. Marzuola, and M. Zworski, *Fast soliton scattering by delta impurities*, Communications in mathematical physics **274** (2007), no. 1, 187–216.

[59] ______, *Soliton splitting by external delta potentials*, Journal of Nonlinear Science **17** (2007), no. 4, 349–367.

[60] M. R. Jeffrey, *Hidden dynamics: The mathematics of switches, decisions and other discontinuous behaviour*, Springer, 2018.

[61] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, vol. 54, Cambridge university press, 1995.

[62] M. Kowalczyk, Y. Martel, and C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Letters in Mathematical Physics **107** (2017), no. 5, 921–931.

[63] V. S. Kozlova, *Roughness of a discontinuous system*, Vestnik Moskovskogo Universiteta Seriya 1 Matematika Mekhanika (1984), no. 5, 16–20.

[64] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, *Soliton stability in plasmas and hydrodynamics*, Physics Reports **142** (1986), no. 3, 103–165.

[65] Y. A. Kuznetsov, S. Rinaldi, and A. Gragnani, *One-parameter bifurcations in planar Filippov systems*, International Journal of Bifurcation and chaos **13** (2003), no. 08, 2157–2188.

[66] C. M. Lee, P. J. Collins, B. Krauskopf, and H. M. Osinga, *Tangency bifurcations of global poincaré maps*, SIAM Journal on applied dynamical systems **7** (2008), no. 3, 712–754.

[67] F. Liang and M. Han, *The stability of some kinds of generalized homoclinic loops in planar piecewise smooth systems*, International Journal of Bifurcation and Chaos **23** (2013), no. 02, 1350027.

[68] F. Liang and D. Wang, *Limit cycle bifurcations near a piecewise smooth generalized homoclinic loop with a saddle-fold point*, International Journal of Bifurcation and Chaos **27** (2017), no. 05, 1750071.

[69] X. Lin, *Using Melnikov's method to solve Silnikov's problems*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **116** (1990), no. 3-4, 295–325.

[70] Y. Liu and V. G. Romanovski, *Limit cycle bifurcations in a class of piecewise smooth systems with a double homoclinic loop*, Applied Mathematics and Computation **248** (2014), 235–245.

[71] N. Lu, *Small generalized breathers with exponentially small tails for Klein–Gordon equations*, Journal of Differential Equations **256** (2014), no. 2, 745–770.

[72] B. A. Malomed, *Variational methods in nonlinear fiber optics and related fields*, Progress in optics **43** (2002), 71–194.

[73] S. Mancini, M. Manoel, and M. A. Teixeira, *Divergent diagrams of folds and simultaneous conjugacy of involutions*, Discrete and Continuous Dynamical Systems **12** (2005), no. 4, 657–674.

[74] Y. Martel and F. Merle, *Description of two soliton collision for the quartic gKdV equation*, Annals of mathematics **174** (2011), no. 2, 757–857.

[75] V. K. Mel'nikov, *On the stability of a center for time-periodic perturbations*, Trudy moskovskogo matematicheskogo obshchestva **12** (1963), 3–52.

[76] K. Meyer, G. Hall, and D. Offin, *Introduction to hamiltonian dynamical systems and the n-body problem*, vol. 90, Springer Science & Business Media, 2008.

[77] D. D. Novaes, G. Ponce, and R. Varão, *Chaos induced by sliding phenomena in Filippov systems*, Journal of Dynamics and Differential Equations **29** (2017), no. 4, 1569–1583.

[78] D. D. Novaes and M. A. Teixeira, *Shilnikov problem in Filippov dynamical systems*, arXiv preprint, arXiv:1504.02425 (2015).

[79] D. D. Novaes, M. A. Teixeira, and I. O. Zeli, *The generic unfolding of a codimension-two connection to a two-fold singularity of planar Filippov systems*, Nonlinearity **31** (2018), no. 5, 2083.

[80] C. Olivé, D. Sauzin, and T. M. Seara, *Resurgence in a hamilton-jacobi equation*, Annales de l'institut Fourier, vol. 53, 2003, pp. 1185–1235.

[81] J. Palis, *A differentiable invariant of topological conjugacies and moduli of stability*, Asterisque **51** (1978), 335–346.

[82] J. Palis and W. De Melo, *Geometric theory of dynamical systems: an introduction*, Springer Science & Business Media, 1982.

[83] M. C. Peixoto and M. M. Peixoto, *Structural stability in the plane with enlarged boundary conditions*, An. Acad. Brasil. Ci **31** (1959), no. 2, 135–160.

[84] P. B. Percell, *Structural stability on manifolds with boundary*, Topology **12** (1973), no. 2, 123–144.

[85] J. V. Pereira and L. Pirio, *An introduction to web geometry*, IMPA Monographs, Springer, 2015.

[86] J. K. Perring and T. H. R. Skyrme, *A model unified field equation*, Nuclear Physics **31** (1962), 550–555.

[87] D. Pi, J. Yu, and X. Zhang, *On the sliding bifurcation of a class of planar Filippov systems*, International Journal of Bifurcation and Chaos **23** (2013), no. 03, 1350040.

[88] B. Piette and W. J. Zakrzewski, *Scattering of sine-Gordon kinks on potential wells*, Journal of Physics A: Mathematical and Theoretical **40** (2007), no. 22, 5995.

[89] B. M. A. G. Piette and W. J. Zakrzewski, *Dynamical properties of a soliton in a potential well*, Journal of Physics A: Mathematical and Theoretical **40** (2006), no. 2, 329.

[90] H. Poincaré, *Sur le problème des trois corps et les équations de la dynamique*, Acta mathematica **13** (1890), no. 1, A3–A270.

[91] C. Robinson, *Dynamical systems: Stability, symbolic dynamics, and chaos*, CRC Press LLC, Florida, 1998.

[92] T. M. Seara, *Estudi de fenomens homoclinics en sistemes propers a integrables*, 1991.

[93] Harvey Segur and Martin D Kruskal, *Nonexistence of small-amplitude breather solutions in phi 4 theory*, Physical review letters **58** (1987), no. 8, 747.

[94] I. M. Sigal, *Non-linear wave and Schrödinger equations*, Communications in mathematical physics **153** (1993), no. 2, 297–320.

[95] J. Sotomayor and M. A. Teixeira, *Vector fields near the boundary of a 3-manifold*, Dynamical Systems Valparaiso 1986, Springer, 1988, pp. 169–195.

[96] M. Tabor, *Chaos and integrability in nonlinear dynamics: An introduction*, 1989.

[97] M. A. Teixeira, *Generic bifurcation in manifolds with boundary*, Journal of Differential Equations **25** (1977), no. 1, 65–89.

[98] ______, *Generic singularities of discontinuous vector-fields*, Anais da Academia Brasileira de Ciencias **53** (1981), no. 2, 257–260.

[99] ______, *Local and simultaneous structural stability of certain diffeomorphisms*, Dynamical Systems and Turbulence, Warwick 1980, Springer, 1981, pp. 382–390.

[100] ______, *On topological stability of divergent diagrams of folds*, Mathematische Zeitschrift **180** (1982), no. 2, 361–371.

[101] ______, *Stability conditions for discontinuous vector fields*, Journal of Differential Equations **88** (1990), no. 1, 15–29.

[102] ______, *Generic bifurcation of sliding vector fields*, Journal of mathematical analysis and applications **176** (1993), 436–436.

[103] ______, *Perturbation theory for non-smooth systems*, Mathematics of complexity and dynamical systems. Vols. 1–3, Springer, New York, 2012, pp. 1325–1336.

[104] S. M. Vishik, *Vector fields near the boundary of a manifold*, Vestnik Moskovskogo Universiteta Matematika **27** (1972), no. 1, 21–28.

[105] D. Wang, J. Jung, and G. Biondini, *Detailed comparison of numerical methods for the perturbed sine-Gordon equation with impulsive forcing*, Journal of Engineering Mathematics **87** (2014), no. 1, 167–186.

[106] A. Wazwaz, *Partial differential equations and solitary waves theory*, Springer Science & Business Media, 2010.

[107] H. Weigel, *Kink-antikink scattering in phi 4 and phi 6 models*, Journal of Physics: Conference Series **482** (2014).

[108] S. Wiggins, *Global bifurcations and chaos: analytical methods*, vol. 73, Springer Science & Business Media, 2013.

[109] Y. Zhu and J. Yang, *Universal fractal structures in the weak interaction of solitary waves in generalized nonlinear Schrödinger equations*, Physical Review E **75** (2007), no. 3, 036605.