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Linearization Technique and its Application to Numerical Solution of Bidimensional Nonlinear Convection Diffusion Equation

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Abstract

In this work, a numerical scheme using Crank-Nicolson scheme and the high-order Finite Difference Method were used, respectively, for the temporal and spatial discretization of the nonlinear two-dimensional convection-diffusion equation. Numerical applications are implemented and valuated making use of the L_2 error norm and the numerical results were compared with the exact solution obtained via literature review.

Keywords: Convection-diffusion equation, linearization technique, high-order finite difference method

1 Introduction

The two-dimensional parabolic differential equations appeared in many scientific fields of engineering and sciences such as neutron diffusion, heat transfer and fluid flow problems [1]. An attractive problem is convection-diffusion equation for which various numerical methods have been suggested by a number of researchers to solve them. Application of this equation may be seen in computational fluid dynamics, hydrodynamic turbulence, shockwave theory, wave processes in thermoelastic medium, transport and dispersion of pollutants in rivers and sediment transport for modeling convection-diffusion of quantities such as mass, energy, vorticity, heat, among others [2,3]. In recent years, the solution of two-dimensional nonlinear convection-diffusion equation has attracted a lot of attention and many authors have used various numerical techniques for the solution of this equation.

2 Numerical Formulation

In this work, we propose a solution, by the high-order finite difference method for the nonlinear two-dimensional convection-diffusion equation which is given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

where $u(x,y,t)$ is the velocity field in the x,y-directions and υ is the kinematic viscosity.

Here, due to efficiency and simple implementation, to carry out the time discretization of the equation (1) will use the Crank-Nicolson method [4] as follows:

$$\left(\frac{u^{n+1}-u^n}{\Delta t}\right)=0{,}5\left(\upsilon\frac{\partial^2 u^{n+1}}{\partial x^2}+\upsilon\frac{\partial^2 u^{n+1}}{\partial y^2}-u^{n+1}\frac{\partial u^{n+1}}{\partial x}-u^{n+1}\frac{\partial u^{n+1}}{\partial y}\right)$$
$$+0{,}5\left(\upsilon\frac{\partial^2 u^{n}}{\partial x^2}+\upsilon\frac{\partial^2 u^{n}}{\partial y^2}-u^{n}\frac{\partial u^{n}}{\partial x}-u^{n}\frac{\partial u^{n}}{\partial y}\right) \tag{2}$$

The main purpose of this work is to use the Newton's method for the linearization of the terms $u\frac{\partial u}{\partial x}$ and $u\frac{\partial u}{\partial y}$ in the following manner

$$u^{n+1}\frac{\partial u^{n+1}}{\partial x}\approx u^n\frac{\partial u^{n+1}}{\partial x}+u^{n+1}\frac{\partial u^n}{\partial x}-u^n\frac{\partial u^n}{\partial x} \tag{3a}$$

$$u^{n+1}\frac{\partial u^{n+1}}{\partial y}\approx u^n\frac{\partial u^{n+1}}{\partial y}+u^{n+1}\frac{\partial u^n}{\partial y}-u^n\frac{\partial u^n}{\partial y} \tag{3b}$$

Note that this technique does not require an iterative linearization at each time step, making quicker the computation of u.

Substituing the Eqs. (3a-b) in the Eq. (2), we have:

$$\left(\frac{u^{n+1}-u^{n}}{\Delta t}\right)=0{,}5\left(\upsilon\frac{\partial^2 u^{n+1}}{\partial x^2}+\upsilon\frac{\partial^2 u^{n+1}}{\partial y^2}-u^{n}\frac{\partial u^{n+1}}{\partial x}-u^{n+1}\frac{\partial u^{n}}{\partial x}+u^{n}\frac{\partial u^{n}}{\partial x}-u^{n}\frac{\partial u^{n+1}}{\partial y}\right.$$

$$\left.-u^{n+1}\frac{\partial u^{n}}{\partial y}+u^{n}\frac{\partial u^{n}}{\partial y}\right)+0{,}5\left(\upsilon\frac{\partial^2 u^{n}}{\partial x^2}+\upsilon\frac{\partial^2 u^{n}}{\partial y^2}-u^{n}\frac{\partial u^{n}}{\partial x}-u^{n}\frac{\partial u^{n}}{\partial y}\right)$$

$$\Rightarrow 0{,}5\left(-\upsilon\frac{\partial^2 u^{n+1}}{\partial x^2}-\upsilon\frac{\partial^2 u^{n+1}}{\partial y^2}+u^{n}\frac{\partial u^{n+1}}{\partial x}+u^{n}\frac{\partial u^{n+1}}{\partial y}+u^{n+1}\frac{\partial u^{n}}{\partial x}+u^{n+1}\frac{\partial u^{n}}{\partial y}\right)+\frac{u^{n+1}}{\Delta t}=F$$

where $F=\dfrac{u^{n}}{\Delta t}+0{,}5\left(\upsilon\dfrac{\partial^2 u^{n}}{\partial x^2}+\upsilon\dfrac{\partial^2 u^{n}}{\partial y^2}\right)$.

In the applications 1 and 2 below, the initial value of the exact solution is taken as the initial condition, and boundary conditions are also specified by the above equation and change with time. However, for the application 3, the conditions will be specified in the same case.

Considering nodes with Δx or Δy distance from the boundary using the Central Difference Method with $O(\Delta x^2)$, we have:

$$-0{,}5\upsilon\left(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{\Delta x^2}\right)-0{,}5\upsilon\left(\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{\Delta y^2}\right)+0{,}5u_{i,j}^{n}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2\Delta x}\right)$$

$$+0{,}5u_{i,j}^{n}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2\Delta y}\right)+\left(\frac{1}{\Delta t}+0{,}5\frac{\partial u_{i,j}^{n}}{\partial x}\right)u_{i,j}^{n+1}=F_{1_i}$$

$$\Rightarrow\left(-\frac{0{,}5\upsilon}{\Delta y^2}-\frac{0{,}5u_{i,j}^{n}}{2\Delta y}\right)u_{i,j-1}^{n+1}+\left(-\frac{0{,}5\upsilon}{\Delta x^2}-\frac{0{,}5u_{i,j}^{n}}{2\Delta x}\right)u_{i-1,j}^{n+1}+\left(\frac{\upsilon}{\Delta x^2}+\frac{\upsilon}{\Delta y^2}+\frac{1}{\Delta t}+0{,}5\frac{\partial u_{i,j}^{n}}{\partial x}\right)u_{i,j}^{n+1}$$

$$+\left(-\frac{0{,}5\upsilon}{\Delta x^2}+\frac{0{,}5u_{i,j}^{n}}{2\Delta x}\right)u_{i+1,j}^{n+1}+\left(-\frac{0{,}5\upsilon}{\Delta y^2}+\frac{0{,}5u_{i,j}^{n}}{2\Delta y}\right)u_{i,j+1}^{n+1}=F_{1_i}$$

where $F_{1_i}=\dfrac{u_{i,j}^{n}}{\Delta t}+0{,}5\left(\upsilon\dfrac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{\Delta x^2}+\upsilon\dfrac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{\Delta y^2}\right)$.

Now, considering the internal nodes using the Central Difference Method with $O(\Delta x^4)$, we have:

$$-0{,}5\upsilon\left(\frac{-u_{i+2,j}^{n+1}+16u_{i+1,j}^{n+1}-30u_{i,j}^{n+1}+16u_{i-1,j}^{n+1}-u_{i-2,j}^{n+1}}{12\Delta x^2}+\right.$$

$$\left.\frac{-u_{i,j+1}^{n+1}+16u_{i,j+1}^{n+1}-30u_{i,j}^{n+1}+16u_{i,j-1}^{n+1}-u_{i,j-1}^{n+1}}{12\Delta y^2}\right)+0{,}5u_{i,j}^{n}\left(\frac{-u_{i+2,j}^{n+1}+8u_{i+1,j}^{n+1}-8u_{i-1,j}^{n+1}+u_{i-2,j}^{n+1}}{12\Delta x}\right.$$

$$\left.+\frac{-u_{i,j+2}^{n+1}+8u_{i,j+1}^{n+1}-8u_{i,j-1}^{n+1}+u_{i,j-2}^{n+1}}{12\Delta y}\right)+\left(\frac{1}{\Delta t}+0{,}5\frac{\partial u_{i,j}^{n}}{\partial x}+\frac{0{,}5u_{i,j}^{n}}{\partial y}\right)u_{i,j}^{n+1}=F_{2_i}$$

$$\Rightarrow\left(\frac{\upsilon}{24\Delta y^2}+\frac{u_{i,j}^n}{24\Delta y}\right)u_{i,j-2}^{n+1}+\left(\frac{\upsilon}{24\Delta x^2}+\frac{u_{i,j}^n}{24\Delta x}\right)u_{i-2,j}^{n+1}+\left(-\frac{2\upsilon}{3\Delta y^2}-\frac{u_{i,j}^n}{3\Delta y}\right)u_{i,j-1}^{n+1}+\left(-\frac{2\upsilon}{3\Delta x^2}-\frac{u_{i,j}^n}{3\Delta x}\right)u_{i-1,j}^{n+1}+$$

$$+\left(\frac{1{,}25\upsilon}{\Delta x^2}+\frac{1{,}25\upsilon}{\Delta y^2}+\frac{1}{\Delta t}+0{,}5\frac{\partial u_{i,j}^n}{\partial x}+0{,}5\frac{\partial u_{i,j}^n}{\partial y}\right)u_{i,j}^{n+1}+\left(-\frac{2\upsilon}{3\Delta y^2}-\frac{u_{i,j}^n}{3\Delta y}\right)u_{i,j+1}^{n+1}$$

$$+\left(-\frac{2\upsilon}{3\Delta x^2}+\frac{u_{i,j}^n}{3\Delta x}\right)u_{i+1,j}^{n+1}+\left(\frac{\upsilon}{24\Delta x^2}-\frac{u_{i,j}^n}{24\Delta x}\right)u_{i+2,j}^{n+1}\left(\frac{\upsilon}{24\Delta y^2}-\frac{u_{i,j}^n}{24\Delta y}\right)u_{i,j+2}^{n+1}=F_{2_i}$$

where

$$F_{2_i}=\frac{u_{i,j}^n}{\Delta t}+0{,}5\upsilon\left(\frac{-u_{i+2,j}^n+16u_{i+1,j}^n-30u_{i,j}^n+16u_{i-1,j}^n-u_{i-2,j}^n}{12\Delta x^2}+\frac{-u_{i,j-2}^n+16u_{i,j-1}^n-u_{i,j+2}^n-30u_{i,j}^n+16u_{i,j+1}^n}{12\Delta y^2}\right).$$

3 Numerical Applications

In the following applications, the numerical solution to nonlinear two-dimensional convection-diffusion equation on a computational domain $0 \le x \le L_x$, $0 \le y \le L_y$ with $t > 0$. In the applications 1 and 2, the numerical solution is compared to the exact solution found in the literature using the L_2 norm defined as: $\|\varepsilon\|_2=\left[\left(\sum_{i=1}^{Nnost}e_i^2\right)/Nnost\right]^{1/2}$ where *Nnost* is the total number of nodes in the mesh and $e_i=\left|u_{(num)_i}-u_{(an)_i}\right|$, where $u_{(num)}$ is the result from the numerical solution and $u_{(an)}$ is the result from the numerical solution and the result from the analytical solution. Now, since the application 3 has no analytical solution used a coarse mesh and a more refined to present the numerical oscillations in the solution in each case, it is possible to note that the oscillations decreased as the refinement.

Application 1: In this application, the exact solution of the Eq. (1) is specified by the following equation $u(x,y,t)=\dfrac{1}{1+\exp\left(\dfrac{x+y-t}{2\nu}\right)}$ [5] and compared with the numerical solution. The values of x, y, t e ν were chosen so that the velocity profile assume values between 0 and 1. Thus, in the by Fig. (1) on the right, for example, were taken, for $\nu = 0.1$ and $t = 0.1$, $0 \le x,y \le 0.1$.

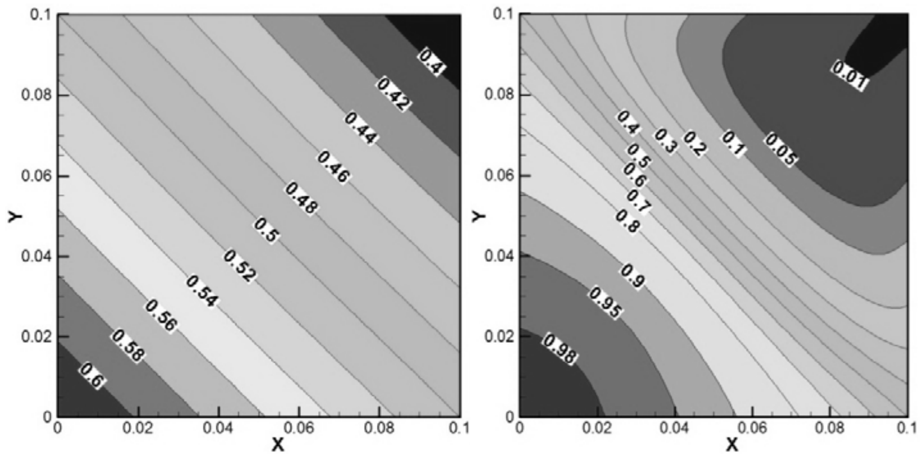

Figure 1: Mesh with $\Delta x = \Delta y = \Delta t = 0.1/100$ considering, on the left, $\nu = 0.1$ with $\|\varepsilon\|_2 = 7.78\text{E-}04$ and, on the right, $\nu = 0.01$ with $\|\varepsilon\|_2 = 6.12\text{E-}02$ both in $t = 0.1$.

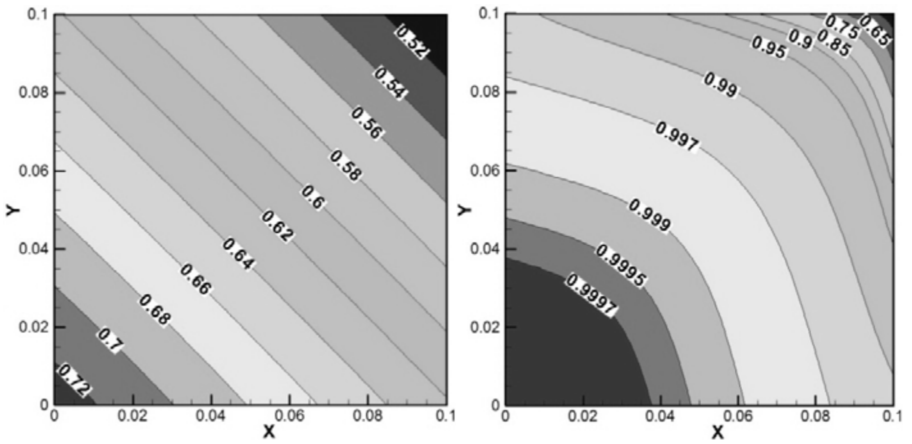

Figure 2: Mesh with $\Delta x = \Delta y = 0.1/100$ and $\Delta t = 0.2/100$ considering, on the left, $\nu = 0.1$ with $\|\varepsilon\|_2 = 4.56\text{E-}04$ and, on the right, $\nu = 0.01$ with $\|\varepsilon\|_2 = 2.61\text{E-}02$ both in $t = 0.2$.

In Figures 1 and 2 is noted that the numerical results for some refined meshes was obtained a precision of at least two decimal places, which is considered suitable for engineering.

Application 2: For the second example, we consider the following analytical solution $u(x,y,t) = \dfrac{1}{2} - \tanh\left(\dfrac{x+y-t}{2\upsilon}\right)$ [6]. The initial value of the exact solution is taken as the initial condition, and boundary conditions are also specified by the above equation and change with time.

It may be noted, the analysis of Figs. (1)-(4), for large values for ν, the forces dominate diffusive profile and the solution evolves toward a flat surface. As the ν decreases, the convective forces take over and the solution develops into a viscous shock that travels to the right.

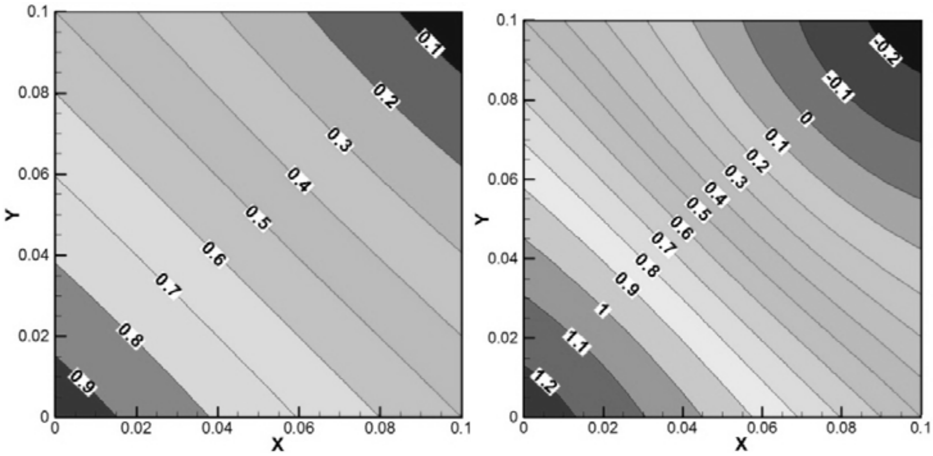

Figure 3: Mesh with $\Delta x = \Delta y = \Delta t = 0.1/100$ considering, on the left, $\nu = 0.05$ with $||\varepsilon||_2 = 3.74\text{E-}03$ and, on the right, $\nu = 0.01$ with $||\varepsilon||_2 = 3{,}07\text{E-}02$ both in $t = 0.1$.

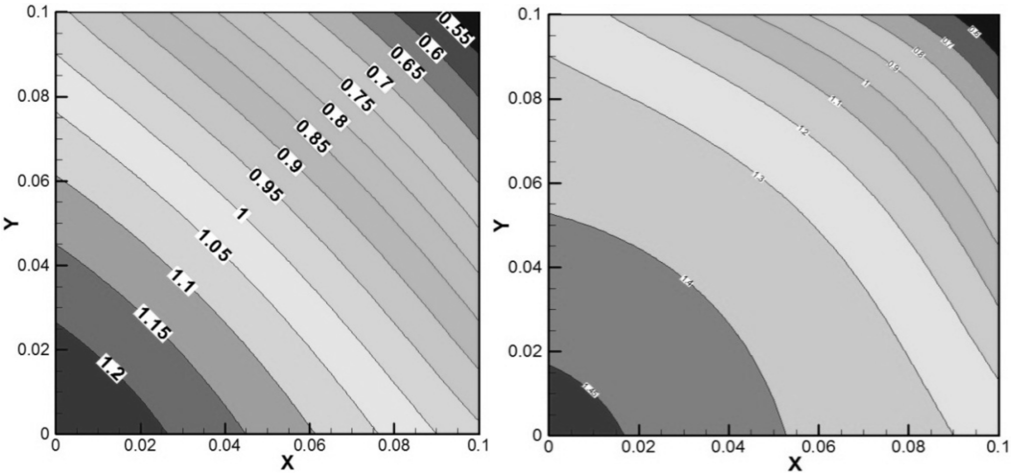

Figure 4: Mesh with $\Delta x = \Delta y = 0.1/100$ and $\Delta t = 0.2/100$ considering, on the left, $\nu = 0.1$ with $||\varepsilon||_2 = 1.27\text{E-}02$ and, on the right, $\nu = 0.05$ with $||\varepsilon||_2 = 1.27\text{E-}02$ both in $t = 0.2$.

Application 3: Here, was considered the computational domain $0 \leq x,y \leq 1$ and t > 0, and the following initial condition $u(x,y,0) = 0$, as well the boundary conditions $u(0,y,t) = 1$, $u(x,0,t) = u(x,0.1,t) = 0$ and $\frac{\partial u}{\partial x}(1.0, y, t) = 0$.

A qualitative analysis of the results of the two time instants for a mesh adopted and $\nu = 0.1$ (Fig. 5) shows that they have no numerical oscillations, and further, the numerical point are in the interval 0 to 1. However, for $\Delta t = 0.05$, obtained a maximum value for $u = 1.48537$, totally out of interval 0 to 1. In order to reduce numerical oscillations present in Figure 6 (left) was applied in step a refinement ($\Delta t = 0.001$) in step time.

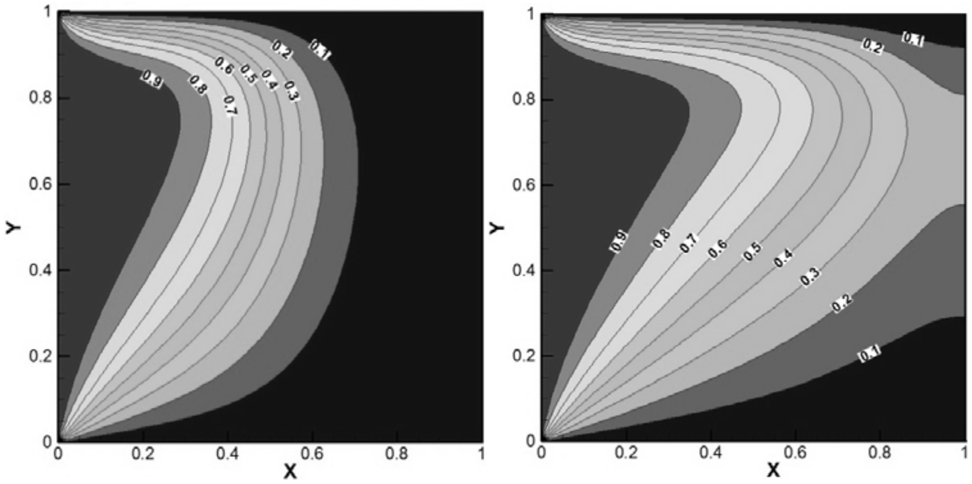

Figure 5: *u* profile for ν = 0.1 and *t* = 1, on the left, and for *t* = 2, on the right, with $\Delta x = \Delta y = 0.01$ and $\Delta t = 0.05$, in the application 3.

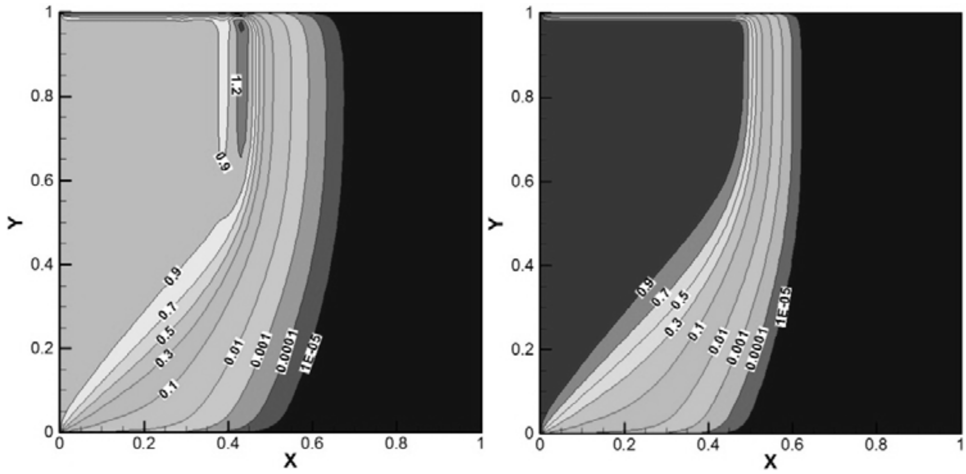

Figure 6: *v* profile for ν = 0.01 and t = 1 with $\Delta x = \Delta y = 0.01$ and $\Delta t = 0.05$ in the left and $\Delta t = 0.001$ in the right, in the application 3.

A qualitative analysis of the results of the two time instants for a mesh adopted and ν = 0.1 (Fig. 5) shows that they have no numerical oscillations, and further, the numerical point are in the interval 0 to 1. However, for $\Delta t = 0.05$, obtained a maximum value for $u = 1.48537$, totally out of interval 0 to 1. In order to reduce numerical oscillations present in Figure 6 (left) was applied in step a refinement ($\Delta t = 0.001$) in step time.

4 Conclusions

In this paper, we propose a high-order Finite Difference Method for nonlinear two-dimensional convection-diffusion equation. Numerical examples show that the method can be used to simulate the numerical solution of the equation. By observing the detailed comparison of numerical and analytical results, it is convinced that the proposed scheme is very simple, stable and accurate for the solutions of the equation. Especially regarding the linearization of nonlinear terms of the equation, this work was presented a very efficient technique, with large time intervals Δt, when compared with those used by other authors. Moreover, this technique does not require any type of iterative process within each time step, which represents a major saving on the computational time.

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