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COUNTABLE FRAMES FOR BIMODAL LOGICS $S5 \otimes S5$ and $Grz.3 \otimes Grz.3$

Abstract

In this paper we consider bimodal logics $S5\otimes S5$ and $Grz.3\otimes Grz.3$. We construct and describe two single countable frames which characterize systems $S5\otimes S5$ and $Grz.3 \otimes Grz.3$, respectively.

Introduction

Multimodal logics are widely studied. They find substantial applications in computer science, in particular, to the knowledge representation (see e.g. [6]). For a modal logic L, a frame $\mathcal{F}$ is called an L-frame if all theorems of L are true in $\mathcal{F}$. Let S be a class of L-frames. A modal logic L is *characterized* (or *determined*) by S if S refutes all formulas which are not theorems of L. If $S = \{\mathcal{F}\}$ then we say that L is characterized by the frame $\mathcal{F}$. Or $\mathcal{F}$ is adequate for L. It is well-known that monomodal system $S5$ is characterized by the class of all finite frames whose relation is an equivalence relation, and also by the infinite countable cluster (see e.g. [1]). The system $Grz.3$ (also known as $S4.3Grz$, $Grz.3$ is equivalent to $S4.3Grz$ see e.g. [3]) is the smallest monomodal logic containing axiom K, Dummett's axiom $\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi)$ and Grzegorczyk's schema $\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$. This system is determined by the class of finite frames whose relation is a linear order, and also by one infinite frame $\langle\omega, \geq\rangle$ (see e.g. Goldblatt [7]). In monomodal logics, the completeness theorem is often formulated for a class of frames. For some modal logics, it is possible to replace the class with a single frame which can be countably infinite, as in the case of $S5$ and $Grz.3$. For bimodal logics, the problem of existence of one appropriate frame is more complicated. We will consider logics $S5 \otimes S5$ and $Grz.3 \otimes Grz.3$ which are fusions of $S5$ and $S5$ and $Grz.3$ and $Grz.3$, respectively (see [6]). According to transfer theorems from [8], canonicity is preserved under the formation of fusion. The system $S5 \otimes S5$ is canonical, so one could consider canonical frame as a single frame for $S5 \otimes S5$. But canonical frames are not easy to be described and applied. Moreover, usually they are uncountable. The system $Grz.3 \otimes Grz.3$ is not even canonical (see [4]). We construct and describe two countable frames which characterize systems $S5 \otimes S5$ and $Grz.3 \otimes Grz.3$, respectively. These results have practical applications. Both defined frames will allow us to find some of their finite subframes to reject non-theorems of $S5 \otimes S5$ or $Grz.3 \otimes Grz.3$, respectively.

Preliminaries

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional monomodal languages with modal operators $\Box_1$ and $\Box_2$, respectively. Let $\mathcal{L}_{1,2}$ be propositional bimodal language with both operators $\Box_1$ and $\Box_2$. Bimodal logic $L \subset \mathcal{L}_{1,2}$ is called a fusion of $L_1 \subset \mathcal{L}_1$ and $L_2 \subset \mathcal{L}_2$ if L is the smallest system containing $L_1 \cup L_2$. In this case, we write $L_1 \otimes L_2$ instead of L (see [6]). Let $\mathbb{N}$ be the set of positive integers $\{1, 2, \ldots\}$.

A Kripke frame for monomodal logic is a pair $\mathcal{F} = \langle W, R \rangle$ where W is a nonempty set and R is a binary relation on W (R is an accessibility relation).

A Kripke frame for a bimodal logic is a triple $\mathcal{F} = \langle W, R_1, R_2 \rangle$ where W is a nonempty set and R_1, R_2 are accessibility relations. $\mathcal{F} = \langle W, R_1, R_2 \rangle$ is *connected* if for every $x, y \in W$ and $x \neq y$ there exists a sequence $(x_1, ..., x_{k-1})$ of elements from W such that xS_1x_1, $x_1S_2x_2, \ldots$, $x_{k-2}S_{k-1}x_{k-1}$, $x_{k-1}S_ky$, where $S_j \in \{R_1, R_2, R_1^{-1}, R_2^{-1}\}$ for $j \in \{1, \ldots, k\}$. Let $\mathcal{F}$ be a frame and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ parewise disjoint connected parts of $\mathcal{F}$. If a formula φ is refuted in $\mathcal{F}$, then φ is refuted in $\mathcal{F}_i$ for some $i \in \{1, \ldots, n\}$. The other parts $\mathcal{F}_1, \ldots, \mathcal{F}_{i-1}, \mathcal{F}_{i+1}, \ldots, \mathcal{F}_n$ do not interfere with refutation of φ in $\mathcal{F}_i$. Hence it is enough to consider connected frames only. The relation $\models$ is defined in a standard way (see [6]).

Below we list some axioms and corresponding to them conditions on relations in frames (see for example [1] and [2]).

K_i	$\Box_i(\varphi\to\psi)\to(\Box_i\varphi\to\Box_i\psi)$	no condition
T_i	$\Box_i\varphi\to\varphi$	$\forall_x xR_ix$ (reflexivity)
4_i	$\Box_i\varphi\to\Box_i\Box_i\varphi$	$\forall_x\forall_y\forall_z((xR_iy\wedge yR_iz)\Rightarrow xR_iz)$ (transitivity)
B_i	$\Diamond_i\Box_i\varphi\to\varphi$	$\forall_x\forall_y(xR_iy\Rightarrow yR_ix)$ (symmetry)
$D1_i$	$\Box_i(\Box_i\varphi\to\psi)\vee\Box_i(\Box_i\psi\to\varphi)$	$\forall_x\forall_y\forall_z((xR_iy\wedge xR_iz)\Rightarrow(yR_iz\vee zR_iy))$
Grz_i	$\Box_i(\Box_i(\varphi\to\Box_i\varphi)\to\varphi)\to\varphi$	There is no infinite chain $x_1, x_2, \ldots$ with $x_jR_ix_{j+1}$ and $x_j\neq x_{j+1}$, for all j.

Given two frames $\mathfrak{F} = \langle W, R_1, R_2\rangle$ and $\mathfrak{V} = \langle V, S_1, S_2\rangle$, a map f from W onto V is called a *p-morphism from $\mathfrak{F}$ to $\mathfrak{V}$* if, for all $x, y \in W$ and $z \in V$, it satisfies the following conditions:

(i) if xR_iy, then $f(x)S_if(y)$

(ii) if $f(x)S_iz$, then there is $y \in W$ such that xR_iy and $f(y) = z$

for $i = 1, 2$.

Let f be a p-morphism from $\mathfrak{F}$ to $\mathfrak{V}$. Then f is called *a p-morphism from a model $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$ to a model $\mathfrak{N} = \langle\mathfrak{V}, \mathfrak{U}\rangle$* if $x \in \mathfrak{V}(p)$ iff $f(x) \in \mathfrak{U}(p)$ for every propositional variable p and $x \in W$.

It is well-known that for all $\mathcal{L}_{1,2}$-formulas φ and all $x \in W$, $(\mathfrak{M}, x) \models \varphi$ iff $(\mathfrak{N}, f(x)) \models \varphi$ (see e.g. [6]). It follows that if $\mathfrak{V}$ is a p-morphic image of $\mathfrak{F}$ and $\mathfrak{F} \models \varphi$ then $\mathfrak{V} \models \varphi$ for every $\mathcal{L}_{1,2}$-formula φ. These definitions and properties also have monomodal counterpart.

Consider two classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of frames that are closed under disjoint unions and isomorphic copies. The set $\mathcal{C}_1 \otimes \mathcal{C}_2 = \{\langle W, R, S\rangle; \langle W, R\rangle \in \mathcal{C}_1, \langle W, S\rangle \in \mathcal{C}_2\}$ will be called a *fusion of $\mathcal{C}_1$ and $\mathcal{C}_2$*.

It is possible to transfer some theorems from monomodal to bimodal case. The monomodal system $S5$ is characterized by the class of finite frames whose relation is an equivalence relation. We will need the following theorem (see Theorem 4.1 from [6]):

THEOREM 1. *If the modal logics L_1 and L_2 are characterized by classes of frames C_1 and C_2, respectively, and both classes are closed under the formation of disjoint unions and isomorphic copies, then the fusion $L_1 \otimes L_2$ is characterized by $C_1 \otimes C_2$.*

Moreover, working with finite frames allows us to consider finite disjoint unions of frames only.

$S5 \otimes S5$ is the smallest bimodal system which contains the axioms $K_i, T_i, 4_i, B_i$ and is closed under the rule of Modus Ponens (MP) $\frac{\varphi\to\psi,\varphi}{\psi}$

and the rules of Necessitation (RN_i) $\frac{\varphi}{\Box_i \varphi}$, for $i = 1, 2$. $Grz.3 \otimes Grz.3$ is the smallest bimodal system which contains the axioms $K_i, Grz_i, D1_i$ and is closed under the rule of Modus Ponens and the rules of Necessitation, for $i = 1, 2$. Monomodal logic $Grz.3$ is characterized by the class of finite frames whose relation is a linear order. If we close this class under the formation of finite disjoint unions it will still characterize $Grz.3$. From the previous theorem, it immediately follows:

COROLLARY 2.
(i) The system $S5 \otimes S5$ *is characterized by the class of finite frames* $\mathfrak{V} = \langle V, S_1, S_2 \rangle$ *whose relations are equivalence relations.*
(ii) The system $Grz.3 \otimes Grz.3$ *is characterized by the class of finite frames* $\mathfrak{V} = \langle V, S_1, S_2 \rangle$ *whose relations are linear orders, on every* S_1 *(or* S_2*) – connected component, which is connected with respect to* S_1 *(or* S_2*).*

As mentioned before, if a formula φ is refuted in some frame, then φ is refuted in some connected part of this frame. Both classes from the previous corollary are closed under getting connected subframes. Hence, if φ is refuted in some frame from our class, then φ is refuted in some connected frame from our class. Therefore, both classes in the corollary above can be replaced by their subclasses consisting only of connected frames.

Countable frame adequate for $S5 \otimes S5$

Now we will describe the countable frame $\mathfrak{F} = \langle U, R, B \rangle$ which characterizes the system $S5 \otimes S5$. Set $U = \{(a_1, \ldots, a_n) \in \mathbb{N}^n; n \in \{2, 3, \ldots\}\}$, R and B are binary relations on U defined as follow:

$(a_1, \ldots, a_n) R (b_1, \ldots, b_m)$ iff

- $n = m = 2$ or
- $2 < m = n$ is even and $a_1 = b_1, \ldots, a_{n-2} = b_{n-2}$ or
- $m = n$ is odd and $a_1 = b_1, \ldots, a_{n-1} = b_{n-1}$ or
- $k = \min\{n, m\}$ is odd, $|n - m| = 1$ and $a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$.

$(a_1, \ldots, a_n) B (b_1, \ldots, b_m)$ iff

- $m = n$ is even and $a_1 = b_1, \ldots, a_{n-1} = b_{n-1}$ or
- $m = n$ is odd and $a_1 = b_1, \ldots, a_{n-2} = b_{n-2}$ or
- $k = \min\{n, m\}$ is even, $|n - m| = 1$ and $a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$.

One can check that both R and B are equivalence relations and that $\mathfrak{F}$ is connected. By R^0 we denote the R-cluster which consists of all sequences of length 2. By B^{n_1} we denote the B-cluster which consists of all sequences of length 2 and 3 beginning with n_1. R^{n_1,n_2} denotes the R-cluster which consists of all sequences of length 3 and 4 beginning with n_1, n_2. Generally, for all $l \in \{1, 3, 5, \ldots\}$, $B^{n_1,\ldots,n_l}$ denotes the B-cluster containing all sequences of length $l+1$ and $l+2$ beginning with $n_1, \ldots, n_l$. Analogously, for all $l \in \{2, 4, 6 \ldots\}$, $R^{n_1,\ldots,n_l}$ denotes the R-cluster containing all sequences of length $l+1$ and $l+2$ beginning with $n_1, \ldots, n_l$. Let us note that both sets $R^0 \cap B^{n_1}$ and $R^0 \setminus B^{n_1}$ have infinitely many elements for each $n_1 \in \mathbb{N}$. All sets $R^{n_1,\ldots,n_k} \cap B^{n_1,\ldots,n_{k-1}}$, $R^{n_1,\ldots,n_k} \setminus B^{n_1,\ldots,n_{k-1}}$, $B^{n_1,\ldots,n_{k+1}} \cap R^{n_1,\ldots,n_k}$ and $B^{n_1,\ldots,n_{k+1}} \setminus R^{n_1,\ldots,n_k}$ are infinite for each $k \in \{2, 4, 6, ...\}$. In other cases, the defined sets have empty intersection.

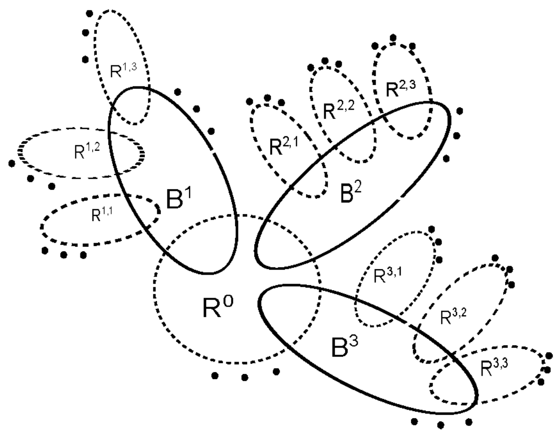
$R^0 \cap B^{n_1} = \{(n_1, n_2); n_2 \in \mathbb{N}\}$ for $n_1 \in \mathbb{N}$

if l is odd:

$B^{n_1,\ldots,n_l} \cap R^{n_1,\ldots,n_l,n_{l+1}} = \{(n_1, \ldots, n_{l+2}); n_{l+2} \in \mathbb{N}\}$ for $n_1, \ldots, n_{l+1} \in \mathbb{N}$

if l is even:

$R^{n_1,\ldots,n_l} \cap B^{n_1,\ldots,n_l,n_{l+1}} = \{(n_1, \ldots, n_{l+2}); n_{l+2} \in \mathbb{N}\}$ for $n_1, \ldots, n_{l+1} \in \mathbb{N}$



LEMMA 3. *$\mathfrak{F}$ is an $S5 \otimes S5$-frame.*

PROOF. Since R (correspond to $\Box_1$) and B (correspond to $\Box_2$) are equivalence relations we have $\mathfrak{F} \models T_i, 4_i, B_i$, for $i = 1, 2$. $\Box$

LEMMA 4. *Every finite connected $S5 \otimes S5$-frame is a p-morphic image of $\mathfrak{F}$.*

PROOF. (Sketch) Let $\mathfrak{V} = \langle V, S_1, S_2 \rangle$ be a finite connected $S5 \otimes S5$-frame. We choose an S_1-cluster and denote it by S_1^0. Let $S_2^1, S_2^2, \ldots, S_2^{m_0}$ denote all pairwise different S_2-clusters having nonempty intersections with S_1^0. m_0 is the number of all those clusters. Next, let $S_1^{k_1,1}, S_1^{k_1,2}, \ldots S_1^{k_1,m_{k_1}}$ denote all pairwise different S_1-clusters having nonempty intersections with $S_2^{k_1}$ for $k_1 \in \{1, 2, \ldots, m_0\}$ where m_{k_1} is the number of all those clusters. Suppose that $S_2^{k_1,\ldots,k_l}$ is already defined for some $l \in \{1, 3, 5, \ldots\}$ and $m_{k_1,\ldots,k_l}$ is the number of all pairwise different S_1-clusters having nonempty intersections with $S_2^{k_1,\ldots,k_l}$. We denote those clusters by $S_1^{k_1,\ldots,k_l,1}$, $S_1^{k_1,\ldots,k_l,2}, \ldots$, $S_1^{k_1,\ldots,k_l,m_{k_1,\ldots,k_l}}$. Analogously, if we have $S_1^{k_1,\ldots,k_l}$ for some $l \in \{2, 4, 6, \ldots\}$, by $S_2^{k_1,\ldots,k_l,1}$, $S_2^{k_1,\ldots,k_l,2}$, $\ldots$, $S_2^{k_1,\ldots,k_l,m_{k_1,\ldots,k_l}}$ we denote all pairwise different S_2-clusters having nonempty intersections with $S_1^{k_1,\ldots,k_l}$. This procedure will not stop in finitely many steps. Hence, every cluster will be named infinitely many times. Now let us denote all elements from V as follows:
$S_1^0 \cap S_2^{k_1} = \{x_1^{k_1}, x_2^{k_1}, \ldots, x_{t_{k_1}}^{k_1}\}$ for $k_1 \in \{1, \ldots, m_0\}$

Generally, if l is odd, for $k_1 \in \{1, \ldots, m_0\}$ and $k_i \in \{1, \ldots, m_{k_1,\ldots,k_{i-1}}\}$ we put:

$S_2^{k_1,\ldots,k_l} \cap S_1^{k_1,\ldots,k_l,k_{l+1}} = \{x_1^{k_1,\ldots,k_{l+1}}, x_2^{k_1,\ldots,k_{l+1}}, \ldots, x_{t_{k_1,\ldots,k_{l+1}}}^{k_1,\ldots,k_{l+1}}\}$

if l is even, for $k_1 \in \{1, \ldots, m_0\}$ and $k_i \in \{1, \ldots, m_{k_1,\ldots,k_{i-1}}\}$ we put:

$S_1^{k_1,\ldots,k_l} \cap S_2^{k_1,\ldots,k_l,k_{l+1}} = \{x_1^{k_1,\ldots,k_{l+1}}, x_2^{k_1,\ldots,k_{l+1}}, \ldots, x_{t_{k_1,\ldots,k_{l+1}}}^{k_1,\ldots,k_{l+1}}\}$

where $t_{k_1,\ldots,k_{l+1}}$ is the number of elements in the intersections above. Every point will be named infinitely many times.

Now let us define a mapping $f : U \to V$:

$f((n_1, \ldots, n_l)) = x_z^{k_1,\ldots,k_{l-1}}$ where $k_1 = \min\{n_1, m_0\}$ and
$k_i = \min\{n_i, m_{k_1,\ldots,k_{i-1}}\}$ for $i \in \{2, \ldots, l-1\}$, and $n_l = z(\mod t_{k_1,\ldots,k_{l-1}})$.

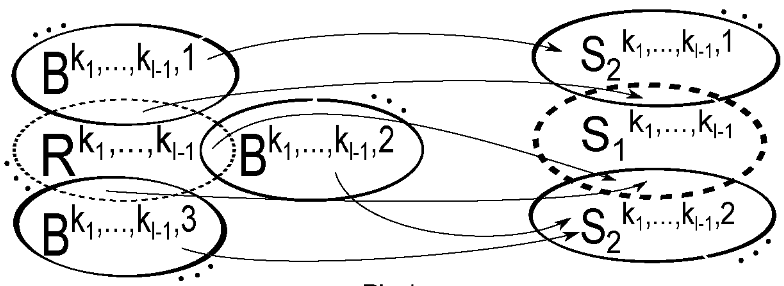
For our proof, it is enough to show that f is a p-morphism. Every point from $\mathfrak{V}$ belongs to intersection of two clusters and is named as $x_z^{k_1,\ldots,k_{l-1}}$.

It is easy to see that $x_z^{k_1,\dots,k_{l-1}} = f((k_1,\dots,k_{l-1},z))$. The complete proof is in Appendix I. Let us consider the following three typical cases in the range of f (Pic.1-3):

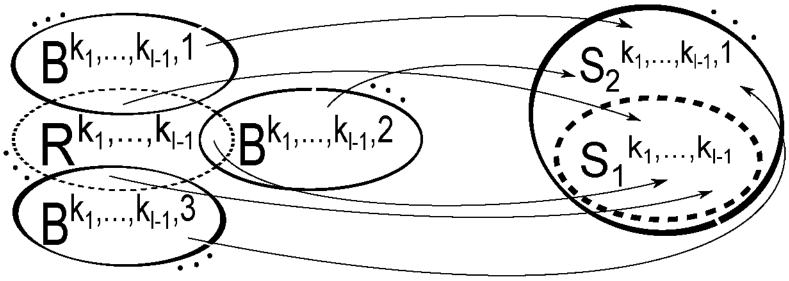
- Case I (Pic.1) $S_1^{k_1,\dots,k_{l-1}} \cap S_2^{k_1,\dots,k_{l-1},i} \neq \emptyset \neq S_2^{k_1,\dots,k_{l-1},i} \setminus S_1^{k_1,\dots,k_{l-1}}$ for $i \in \{1,2\}$ and there is no more S_2-clusters which have nonempty intersection with $S_1^{k_1,\dots,k_{l-1}}$. Then:
  $f(R^{k_1,\dots,k_{l-1}} \cap B^{k_1,\dots,k_{l-1},1}) = S_1^{k_1,\dots,k_{l-1}} \cap S_2^{k_1,\dots,k_{l-1},1}$, and
  $f(R^{k_1,\dots,k_{l-1}} \cap B^{k_1,\dots,k_{l-1},j}) = S_1^{k_1,\dots,k_{l-1}} \cap S_2^{k_1,\dots,k_{l-1},2}$ for $j \geq 2$.

- Case II (Pic.2) $S_1^{k_1,\dots,k_{l-1}} \subsetneq S_2^{k_1,\dots,k_{l-1},1}$. Then:
  $f(R^{k_1,\dots,k_{l-1}} \cap B^{k_1,\dots,k_{l-1},j}) = S_1^{k_1,\dots,k_{l-1}}$ for $j \in \mathbb{N}$.

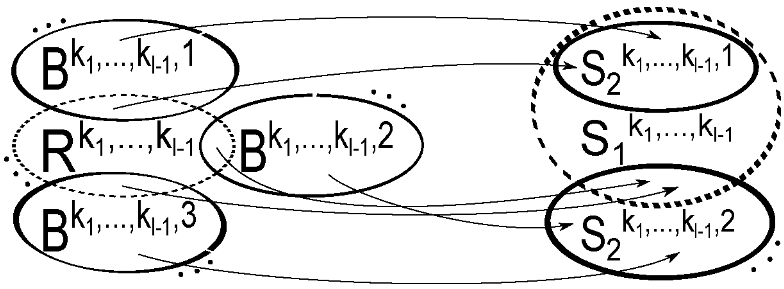
- Case III (Pic.3) $S_2^{k_1,\dots,k_{l-1},1} \subsetneq S_1^{k_1,\dots,k_{l-1}}$, $S_1^{k_1,\dots,k_{l-1}} \cap S_2^{k_1,\dots,k_{l-1},2} \neq \emptyset$ and there are no more S_2-clusters which have nonempty intersection with $S_1^{k_1,\dots,k_{l-1}}$. Then:
  $f(R^{k_1,\dots,k_{l-1}} \cap B^{k_1,\dots,k_{l-1},1}) = S_2^{k_1,\dots,k_{l-1},1}$, and
  $f(R^{k_1,\dots,k_{l-1}} \cap B^{k_1,\dots,k_{l-1},j}) = S_1^{k_1,\dots,k_{l-1}} \cap S_2^{k_1,\dots,k_{l-1},2}$ for $j \geq 2$.



Pic.1



Pic.2



Pic.3

$\square$

From Corollary 2, Lemma 3 and Lemma 4 it follows that:

THEOREM 5. *$S5 \otimes S5$ is characterized by the frame $\mathfrak{F}$.*

REMARK. In order to refute a formula in $\mathfrak{F}$, it is enough to consider finite subframes of $\mathfrak{F}$. This is shown by the following example.

EXAMPLE. To show that $\Box_2(\Box_1(p \to \Box_2 p) \to p) \to (\Diamond_1\Box_2 p \to p)$ is not theorem of $S5 \otimes S5$, it is sufficient to find a falsifying valuation v. This will be done at the point $(1,1)$. To falsify $\Diamond_1\Box_2 p \to p$, we need $\not\Vdash_{\langle \mathfrak{F},v,(1,1)\rangle} p$ and $\forall_{x\in B^3} \Vdash_{\langle \mathfrak{F},v,x\rangle} p$. Putting $\forall_{x\in B^1\setminus\{(1,1)\}} \Vdash_{\langle \mathfrak{F},v,x\rangle} p$ and $\exists_{x\in B^2} \not\Vdash_{\mathfrak{F}} p$, we validate $\Box_2(\Box_1(p \to \Box_2 p) \to p$. Let $\mathfrak{F}' = \langle U', R', B'\rangle$ where $U' = \{(1,1),(2,1),(3,1),(1,1,1),(2,1,1),(3,1,1)\}$, R' and B' are restriction of R and B, respectively. Then $\mathfrak{F}'$ with valuation $v' = v|_{U'}$ is the desired model.

Countable frame adequate for $Grz.3 \otimes Grz.3$

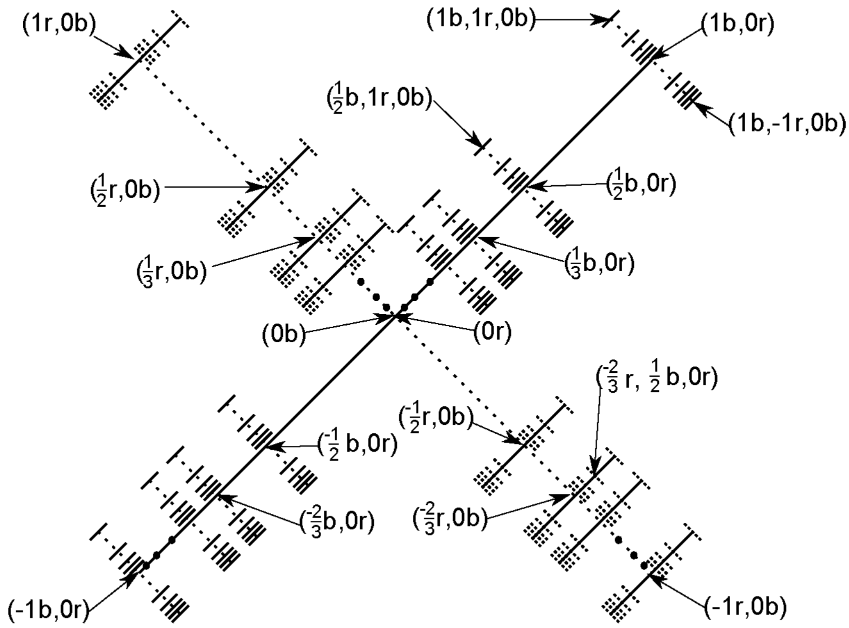
Let $\mathfrak{D} = \langle U, R, B\rangle$ be a frame built of r, b (two distinct constants) and some rational numbers, i.e. $U = \{(p_1c_1, \ldots, p_{n-1}c_{n-1}, 0c_n); n \in \mathbb{N}, c_k \in \{r, b\}, c_k \neq c_{k+1}, p_k \in \{-\frac{n}{n+1}; n \in \mathbb{N}\} \cup \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{-1\}\}$. $(0r)$ and $(0b)$ are the same element. Both R and B are binary relations on U(see the picture below):

$(p_1c_1^1, \ldots, p_{n-1}c_{n-1}^1, 0c_n^1)R(q_1c_1^2, \ldots, q_{m-1}c_{m-1}^2, 0c_m^2)$ iff

- $n = m$, $c_1^1 = c_1^2$, $p_s = q_s$ for $s \leq n-2$, $c_{n-1}^1 = r$, $p_{n-1} \leq q_{m-1}$ or
- $n = m-1$, $c_1^1 = c_1^2$, $p_s = q_s$ for $s \leq n-1$, $c_n^1 = r$, $0 < q_{m-1}$ or
- $n-1 = m$, $c_1^1 = c_1^2$, $p_s = q_s$ for $s \leq n-2$, $c_m^2 = r$, $p_{n-1} < 0$.

$(p_1c_1^1, \ldots, p_{n-1}c_{n-1}^1, 0c_n^1)B(q_1c_1^2, \ldots, q_{m-1}c_{m-1}^2, 0c_m^2)$ iff

- $n = m$, $c_1^1 = c_1^2$, $p_s = q_s$ for $s \leq n-2$, $c_{n-1}^1 = b$, $p_{n-1} \leq q_{m-1}$ or
- $n = m-1$, $c_1^1 = c_1^2$, $p_s = q_s$ for $s \leq n-1$, $c_n^1 = b$, $0 < q_{m-1}$ or
- $n-1 = m$, $c_1^1 = c_1^2$, $p_s = q_s$ for $s \leq n-2$, $c_m^2 = b$, $p_{n-1} < 0$.



It is easy to check that every R (or B) $-$ *connected* part of the frame $\mathfrak{D}$ is isomorphic to a frame $\mathfrak{K} = \langle U', \leq \rangle$ where $U' = \{-\frac{n}{n+1}; n \in \mathbb{N}\} \cup \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{-1, 0\}$. The frame $\mathfrak{K}$ is reflexive, transitive and converse weakly well-founded (for every nonempty set $X \subset U'$, there is an maximal element of X). According to [3], $\mathfrak{K} \models Grz$ and so $\mathfrak{D} \models Grz_i$ for $i = 1, 2$. U' is linearly ordered so $\mathfrak{K} \models D1$ and therefore $\mathfrak{D} \models D1_i$ for $i = 1, 2$. Hence:

LEMMA 6. *$\mathfrak{D}$ is an $Grz.3 \otimes Grz.3$-frame.*

LEMMA 7. *Every finite connected $Grz.3 \otimes Grz.3$-frame is a p-morphic image of $\mathfrak{D}$.*

PROOF. Before giving a formal proof we present its main idea. Let $\mathfrak{B} = \langle V, R', B' \rangle$ be a finite connected $Grz.3 \otimes Grz.3$-frame and x an element from V. We name it x_0. Let z be an element in V such that $x_0 R' z$ and $z R' t$ for no $t \in V$ other than z. We name it $x_{0,1r'}$. Let $x_{0,nr'}$ be the name of an element from $V \setminus \{x_{0,n-1r'}\}$ which is directly before $x_{0,n-1r'}$ (with respect to R') and $x_0 R' x_{0,nr'}$ (it can be x_0). $x_{0,1b'}$ and $x_{0,nb'}$ are defined in the same way, with respect to B'. Let $x_{0,-1r'}$ be another name for element x_0 and $x_{0,-nr'}$ be the name for the element in $V \setminus \{x_{0,-(n-1)r'}\}$ which is the closest to $x_{0,-(n-1)r'}$ (with respect to R') and $x_{0,-nr'} R' x_{0,-(n-1)r'}$. Analogously for $x_{0,-1b'}$ and $x_{0,-nb'}$. Now suppose that we have already defined $x_{0,\pm n_1 c'_1,\ldots,\pm n_k c'_k} (c'_i \in \{r', b'\}$ and $c'_i \neq c'_{i+1})$. $x_{0,\pm n_1 c'_1,\ldots,\pm(n_k+1)c'_k}$, $x_{0,\pm n_1 c'_1,\ldots,\pm n_k c'_k, 1c'_{k+1}}$ and $x_{0,\pm n_1 c'_1,\ldots,\pm n_k c'_k, -1c'_{k+1}}$ are defined analogously $(c'_{k+1} \neq c'_k)$. For every element x_0 from V we define sets of successors and predecessors with respect to R' and B':

$$R'^{+}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k b'}} = \{z \in V; x_{0,\pm n_1 c'_1,\ldots,\pm n_k b'} R' z\},$$

$$R'^{-}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k b'}} = \{z \in V; z R' x_{0,\pm n_1 c'_1,\ldots,\pm n_k b'}\},$$

$$B'^{+}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k r'}} = \{z \in V; x_{0,\pm n_1 c'_1,\ldots,\pm n_k r'} B' z\} \text{ and}$$

$$B'^{-}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k r'}} = \{z \in V; z B' x_{0,\pm n_1 c'_1,\ldots,\pm n_k r'}\}.$$

By $m^{R'+}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k b'}}$, $m^{R'-}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k b'}}$, $m^{B'+}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k r'}}$ and $m^{B'-}_{x_{0,\pm n_1 c'_1,\ldots,\pm n_k r'}}$ we denote the number of elements of the sets defined above, respectively.

Now let us define a mapping $f : U \to V$ such that $f((0c_1)) = x_0$ and $f((p_1c_1, \ldots, p_{k+1}c_{k+1}, 0c_{k+2})) = x_{0,o_1c'_1,\ldots,o_kc'_k,o_{k+1}c'_{k+1}}$ where $f((p_1c_1, \ldots, p_kc_k, 0c_{k+1})) = x_{0,o_1c'_1,\ldots,o_kc'_k}$ and:

a) $o_{k+1} = n_{k+1}$ if $p_{k+1} = \frac{1}{n_{k+1}}$ and $n_{k+1} \leq m^{C'_{k-1}+}_{x_{0,o_1c'_1,\ldots,o_kc'_k}}$

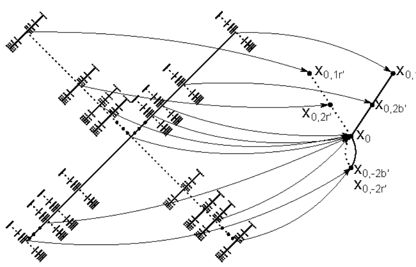
b) $o_{k+1} = m^{C'_{k-1}+}_{x_{0,o_1c'_1,\ldots,o_kc'_k}}$ if $p_{k+1} = \frac{1}{n_{k+1}}$ and $n_{k+1} > m^{C'_{k-1}+}_{x_{0,n_1c'_1,\ldots,o_kc'_k}}$

c) $o_{k+1} = -n_{k+1}$ if $p_{k+1} = \frac{-n_{k+1}}{n_{k+1}+1}$ and $n_{k+1} \leq m^{C'_{k-1}-}_{x_{0,o_1c'_1,\ldots,o_kc'_k}}$

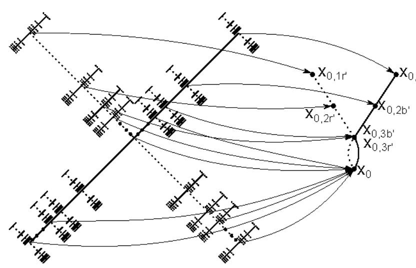
d) $o_{k+1} = -m^{C'_{k-1}-}_{x_{0,o_1c'_1,\ldots,o_kc'_k}}$ if $p_{k+1} = \frac{-n_{k+1}}{n_{k+1}+1}$ and $n_{k+1} > m^{C'_{k-1}-}_{x_{0,o_1c'_1,\ldots,o_kc'_k}}$

e) $o_{k+1} = -m^{C'_{k-1}-}_{x_{0,o_1c'_1,\ldots,o_kc'_k}}$ if $p_{k+1} = -1$

We will show that f is a p-morphism. Let $x_{0,o_1c'_1,\ldots,o_kc'_k}$ be an element from V. $x_{0,o_1c'_1,\ldots,o_kc'_k} = f((p_1c_1, \ldots, p_kc_k, 0c_{k+1}))$ where $p_i = \frac{1}{n_i}$ if $o_i = n_i$ or $p_i = \frac{-n_i}{n_i+1}$ if $o_i = -n_i$. Every R (or B) – *connected* part of the frame $\mathfrak{D}$ is mapped onto some R' (or B') – *connected* part of the frame $\mathfrak{B}$ with preserving order. Below we consider two examples that show how to map initial elements of $\mathfrak{D}$. In Pic.1, $f((0r))$ is not the first element of $\mathfrak{B}$. In Pic.2, $f((0r))$ is the first element of $\mathfrak{B}$.



Pic.1



Pic.2

Of course each element has infinitely many names. In the first example (Pic.1) x_0 is named $x_{0,-1b'}$, $x_{0,-1r'}$, $x_{0,3b'}$, $x_{0,3r'}$ among others. In the second example (Pic.2) x_0 is named $x_{0,-1b'}$, $x_{0,-1r'}$, $x_{0,4b'}$, $x_{0,4r'}$ among others. For details check Appendix II. □

From Corollary 2, Lemma 6 and Lemma 7 it follows that:

THEOREM 8. *$Grz.3 \otimes Grz.3$ is characterized by the frame $\mathfrak{D}$.*

REMARK. Let us mention that in order to refute a formula in $\mathfrak{D}$ it is enough to consider finite subframes of $\mathfrak{D}$.

EXAMPLE. To show that $\Box_1(\Box_2(p \to \Box_1 p) \to p) \to p$ is not a theorem of $Grz.3 \otimes Grz.3$ it is sufficient to find a falsifying valuation v. This will be done at the point $(0r)$. We need to falsify p in the point $(0r)$ ($\nVdash_{\langle \mathfrak{D},v,(0r)\rangle} p$). Putting $\forall_{n\in\mathbb{N}} \Vdash_{\langle \mathfrak{D},v,(\frac{1}{n}r,0b)\rangle} p$, $\Vdash_{\langle \mathfrak{D},v,(\frac{1}{n_0}b,0r)\rangle} p$ and $\nVdash_{\langle \mathfrak{D},v,(\frac{1}{n_0}b,\frac{1}{m_0}r,0b)\rangle} p$ for some $n_0, m_0 \in \mathbb{N}$ (let $n_0 = m_0 = 1$), we validate $\Box_1(\Box_2(p \to \Box_1 p) \to p)$. Let $\mathfrak{D}' = \langle U', R', B' \rangle$ where $U' = \{(0r), (1b, 0r), (1b, 1r, 0b), (1r, 0b)\}$, R' and B' are restriction of R and B, respectively. Then $\mathfrak{D}'$ with valuation $v' = v|_{U'}$ is the desired model.

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Appendix I (Complete proof of Lemma 4)

Our aim is to prove that mapping:

$f((n_1, \ldots, n_l)) = x_z^{k_1,\ldots,k_{l-1}}$ where $k_1 = \min\{n_1, m_0\}$ and
$k_i = \min\{n_i, m_{k_1,\ldots,k_{i-1}}\}$ for $i \in \{2, \ldots, l-1\}$, and $n_l = z(\mod t_{k_1,\ldots,k_{l-1}})$.
is a p-morphism.

Let $x_z^{k_1,\ldots,k_{l-1}}$ be an element from V. Then $x_z^{k_1,\ldots,k_{l-1}} = f((k_1, \ldots, k_{l-1}, z))$. To show that the first condition is fulfilled, let $(n_1, \ldots, n_p)$, $(h_1, \ldots, h_q) \in U$ such that $(n_1, \ldots, n_p)R(h_1, \ldots, h_q)$.

- Case I ($p = q = 2$)
  $f((n_1, n_2)) \in S_1^0$ and $f((h_1, h_2)) \in S_1^0$. Then $f((n_1, n_2))S_1f((h_1, h_2))$.
- Case II ($2 < p = q$ is even and $n_1 = h_1, \ldots, n_{p-2} = h_{p-2}$)
  $f((n_1, \ldots, n_{p-2}, n_{p-1}, n_p)) = x_z^{k_1,\ldots,k_{p-2},k_{p-1}}$ where $k_1 = \min\{n_1, m_0\}$ and $k_i = \min\{n_i, m_{k_1,\ldots,k_{i-1}}\}$ for $i \in \{2, \ldots, p-1\}$ and
  $n_p = z(\mod t_{k_1,\ldots,k_{p-2},k_{p-1}})$

  $f((n_1, \ldots, n_{p-2}, h_{p-1}, h_p)) = x_{z'}^{k_1,\ldots,k_{p-2},k'_{p-1}}$ where
  $k'_{p-1} = \min\{h_{p-1}, m_{k_1,\ldots,k_{p-2}}\}$ and $h_p = z'(\mod t_{k_1,\ldots,k_{p-2},k'_{p-1}})$.

  Both are elements of $S_1^{k_1,\ldots,k_{p-2}}$, so
  $f((n_1, \ldots, n_{p-2}, n_{p-1}, n_p))S_1f((n_1, \ldots, n_{p-2}, h_{p-1}, h_p))$.
- Case III ($p = q$ is odd and $n_1 = h_1, \ldots, n_{p-1} = h_{p-1}$)
  $f((n_1, \ldots, n_{p-1}, n_p)) = x_z^{k_1,\ldots,k_{p-1}}$ where $k_1 = \min\{n_1, m_0\}$ and $k_i = \min\{n_i, m_{k_1,\ldots,k_{i-1}}\}$ for $i \in \{2, \ldots, p-1\}$ and
  $n_p = z(\mod t_{k_1,\ldots,k_{p-1}})$

  $f((n_1, \ldots, n_{p-1}, h_p)) = x_{z'}^{k_1,\ldots,k_{p-1}}$ where $h_p = z'(\mod t_{k_1,\ldots,k_{p-1}})$.

  Both are elements of $S_1^{k_1,\ldots,k_{p-1}}$, so
  $f((n_1, \ldots, n_{p-2}, n_{p-1}, n_p))S_1f((n_1, \ldots, n_{p-2}, n_{p-1}, h_p))$.
- Case IV ($k = \min\{p, q\}$ is odd, $|p - q| = 1$ and $n_1 = h_1, \ldots, n_{k-1} = h_{k-1}$)
  Suppose that $k = q$.

$f((n_1,\ldots,n_{p-1},n_p)) = x_z^{k_1,\ldots,k_{p-1}}$ where $k_1 = \min\{n_1, m_0\}$ and $k_i = \min\{n_i, m_{k_1,\ldots,k_{i-1}}\}$ for $i \in \{2,\ldots,p-1\}$ and $n_p = z(\mod t_{k_1,\ldots,k_{p-1}})$

$f((n_1,\ldots,n_{p-2},h_{p-1})) = x_{z'}^{k_1,\ldots,k_{p-2}}$ where $h_{p-1} = z'(\mod t_{k_1,\ldots,k_{p-2}})$.

Both are elements of $S_1^{k_1,\ldots,k_{p-2}}$, so
$f((n_1,\ldots,n_{p-2},n_{p-1},n_p))S_1 f((n_1,\ldots,n_{p-2},h_{p-1}))$.

Now suppose $f((n_1,\ldots,n_p))S_1 x_{z_1}^{h_1,\ldots,h_l}$ and let $f((n_1,\ldots,n_p)) = x_{z_2}^{k_1,\ldots,k_{p-1}}$ ($k_1 = \min\{n_1,m_0\}$, $k_i = \min\{n_i, m_{k_1,\ldots,k_{i-1}}\}$ and $n_p = z_2(\mod t_{k_1,\ldots,k_{p-1}})$).

- Case I (p is odd)
$x_{z_2}^{k_1,\ldots,k_{p-1}} \in S_1^{k_1,\ldots,k_{p-1}}$ and naturally $x_{z_1}^{h_1,\ldots,h_l} \in S_1^{k_1,\ldots,k_{p-1}}$. Therefore $x_{z_1}^{h_1,\ldots,h_l}$ has another designation which defines its membership to $S_1^{k_1,\ldots,k_{p-1}}$. It can be $x_{z_3}^{k_1,\ldots,k_{p-1}}$ or $x_{z_4}^{k_1,\ldots,k_{p-1},k_p}$ for some z_3, z_4 and k_p. $x_{z_3}^{k_1,\ldots,k_{p-1}} = f((n_1,\ldots,n_{p-1},z_3))$ and of course $(n_1,\ldots,n_p)R(n_1,\ldots,n_{p-1},z_3)$. $x_{z_4}^{k_1,\ldots,k_{p-1},k_p} = f((n_1,\ldots,n_{p-1},k_p,z_4))$ and of course $(n_1,\ldots,n_p)R(n_1,\ldots,n_{p-1},k_p,z_4)$.
- Case II (p is even)
$x_{z_2}^{k_1,\ldots,k_{p-1}} \in S_1^{k_1,\ldots,k_{p-2}}$ and $x_{z_1}^{h_1,\ldots,h_l} \in S_1^{k_1,\ldots,k_{p-2}}$. Again we use another name for $x_{z_1}^{h_1,\ldots,h_l}$ which defines its membership to $S_1^{k_1,\ldots,k_{p-2}}$. It can be $x_{z_3}^{k_1,\ldots,k_{p-2}}$ or $x_{z_4}^{k_1,\ldots,k_{p-1}}$ for some z_3, z_4. $x_{z_3}^{k_1,\ldots,k_{p-2}} = f((n_1,\ldots,n_{p-2},z_3))$ and $(n_1,\ldots,n_p)R(n_1,\ldots,n_{p-2},z_3)$. $x_{z_4}^{k_1,\ldots,k_{p-1}} = f((n_1,\ldots,n_{p-1},z_4))$ and $(n_1,\ldots,n_p)R(n_1,\ldots,n_{p-1},z_4)$.

Appendix II (Complete proof of Lemma 6)

Let us define the mapping $f : U \to V$ such that $f((0c_1)) = x_0$ and $f((p_1c_1,\ldots,p_{k+1}c_{k+1},0c_{k+2})) = x_{0,o_1c'_1,\ldots,o_kc'_k,o_{k+1}c'_{k+1}}$ where $f((p_1c_1,\ldots,p_kc_k,0c_{k+1})) = x_{0,o_1c'_1,\ldots,o_kc'_k}$ and:

a) $o_{k+1} = n_{k+1}$ if $p_{k+1} = \frac{1}{n_{k+1}}$ and $n_{k+1} \leq m^{C'_{k-1}+}_{x_{0,o_1 c'_1,\dots,o_k c'_k}}$

b) $o_{k+1} = m^{C'_{k-1}+}_{x_{0,o_1 c'_1,\dots,o_k c'_k}}$ if $p_{k+1} = \frac{1}{n_{k+1}}$ and $n_{k+1} > m^{C'_{k-1}+}_{x_{0,n_1 c'_1,\dots,o_k c'_k}}$

c) $o_{k+1} = -n_{k+1}$ if $p_{k+1} = \frac{-n_{k+1}}{n_{k+1}+1}$ and $n_{k+1} \leq m^{C'_{k-1}-}_{x_{0,o_1 c'_1,\dots,o_k c'_k}}$

d) $o_{k+1} = -m^{C'_{k-1}-}_{x_{0,o_1 c'_1,\dots,o_k c'_k}}$ if $p_{k+1} = \frac{-n_{k+1}}{n_{k+1}+1}$ and $n_{k+1} > m^{C'_{k-1}-}_{x_{0,o_1 c'_1,\dots,o_k c'_k}}$

e) $o_{k+1} = -m^{C'_{k-1}-}_{x_{0,o_1 c'_1,\dots,o_k c'_k}}$ if $p_{k+1} = -1$

We will show that f is a p-morphism. Let $x_{0,o_1 c'_1,\dots,o_k c'_k}$ be an element from V. $x_{0,o_1 c'_1,\dots,o_k c'_k} = f((p_1 c_1, \dots, p_k c_k, 0c_{k+1}))$ where $p_i = \frac{1}{n_i}$ if $o_i = n_i$ or $p_i = \frac{-n_i}{n_i+1}$ if $o_i = -n_i$. Now let us take $(p_1 c^1_1, \dots, p_k c^1_k, 0c^1_{k+1})$, $(q_1 c^2_1, \dots, q_l c^2_l, 0c^2_{l+1}) \in U$ such that $(p_1 c^1_1, \dots, p_k c^1_k, 0c^1_{k+1}) R (q_1 c^2_1, \dots, q_l c^2_l, 0c^2_{l+1})$.

- Case I ($k = l$, $c^1_1 = c^2_1$ ($c^1_k = c^2_k$), $p_s = q_s$ for $s \in \{1,\dots,k-1\}$, $c^1_k = r$, $p_k \leq q_k$)
  $f((p_1 c^1_1, \dots, p_k r, 0b)) = x_{0,o_1 c^{1\prime}_1,\dots,o_k r'}$.

  $f((q_1 c^2_1, \dots, q_k r, 0b)) = f((p_1 c^1_1, \dots, p_{k-1} b, q_k r, 0b)) =$ $x_{0,o_1 c^{1\prime}_1,\dots,o_{k-1} b', o'_k r'}$. It is easy to see that $o_k \leq o'_k$ (because $p_k \leq q_k$).

  $f((p_1 c^1_1, \dots, p_k c^1_k, 0c^1_{k+1})) R' f((q_1 c^2_1, \dots, q_k c^2_k, 0c^2_{k+1}))$.
- Case II ($k = l - 1$, $c^1_1 = c^2_1$ ($c^1_k = c^2_k$), $p_s = q_s$ for $s \in \{1,\dots,k\}$, $c^1_{k+1} = r$, $0 < q_l$)
  $f((p_1 c^1_1, \dots, p_{l-1} b, 0r)) = x_{0,o_1 c^{1\prime}_1,\dots,o_{l-1} b'}$.

  $f((q_1 c^2_1, \dots, q_{l-1} b, q_l r, 0b)) = f((p_1 c^1_1, \dots, p_{l-1} b, q_l r, 0b)) =$ $x_{0,o_1 c^{1\prime}_1,\dots,o_{l-1} b', o'_l r'}$. It is easy to see that $0 < o'_l$ (because $0 < q_l$).

  $f((p_1 c^1_1, \dots, p_{l-1} b, 0r)) R' f((q_1 c^2_1, \dots, q_l r, 0b))$.
- Case III ($k-1 = l$, $c^1_1 = c^2_1$ ($c^1_k = c^2_k$), $p_s = q_s$ for $s \in \{1,\dots,k-1\}$, $c^1_k = r$, $p_k < 0$)
  $f((p_1 c^1_1, \dots, p_{k-1} b, p_k r, 0b)) = x_{0,o_1 c^{1\prime}_1,\dots,o_{k-1} b', o_k r'}$ and $o_k < 0$.

  $f((q_1 c^2_1, \dots, q_{k-1} b, 0r)) = f((p_1 c^1_1, \dots, p_{k-1} b, 0r)) = x_{0,o_1 c^{1\prime}_1,\dots,o_{k-1} b'}$.

  $f((p_1 c^1_1, \dots, p_{k-1} c^1_{k-1}, p_k c^1_k, 0c^1_{k+1})) R' f((q_1 c^2_1, \dots, q_{k-1} c^2_{k-1}, 0c^2_k))$.

Now suppose $f((p_1c_1,\dots,p_kc_k,0c_{k+1}))R'z$ for some $z\in V$ and let $f((p_1c_1,\dots,p_kc_k,0c_{k+1}))=x_{0,o_1c'_1,\dots,o_kc'_k}$.

- Case I ($c'_k=r'$)
  a) $z=x_{0,o_1c'_1,\dots,o_{k-1}b',o'_kr'}$ for some $o_k\le o'_k$.
  $z=f((p_1c_1,\dots,p_{k-1}b,p'_kr,0b))$ for some $p_k\le p'_k$.

  $(p_1c_1,\dots,p_kc_k,0c_{k+1})R(p_1c_1,\dots,p_{k-1}b,p'_kr,0b)$.

  b) $o_k<0$ ($p_k<0$) and $z=x_{0,o_1c'_1,\dots,o_{k-2}r',o_{k-1}b'}$.
  $z=f((p_1c_1,\dots,p_{k-2}r,p_{k-1}b,0r))$.

  $(p_1c_1,\dots,p_kc_k,0c_{k+1})R(p_1c_1,\dots,p_{k-2}r,p_{k-1}b,0r)$.
- Case II ($c'_k=b'$)
  $z=x_{0,o_1c'_1,\dots,o_kb',o_{k+1}r'}$ for some $o_{k+1}>0$.
  $z=f((p_1c_1,\dots,p_{k-1}r,p_kb,\frac{1}{o_{k+1}}r,0b))$.

  $(p_1c_1,\dots,p_kc_k,0c_{k+1})R(p_1c_1,\dots,p_{k-1}r,p_kb,\frac{1}{o_{k+1}}r,0b)$.