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QUANTIFIED INTUITIONISTIC PROPOSITIONAL LOGIC AND CANTOR SPACE

Abstract

We consider propositional quantification in intuitionistic logic. We prove that, under topological interpretation over Cantor space, it enjoys surprising and interesting properties such as maximum property and a kind of distribution of existential quantifier over conjunction. Moreover, by pointing to the appropriate examples, we show that the set of quantified formulas valid in Cantor space strictly contains the set of formulas provable in the minimal system of intuitionistic logic with propositional quantification.

The well-known topological interpretation of intuitionistic propositional connectives, introduced by A. Tarski in [6], can be naturally extended to propositional quantifiers. This gives us a large variety of models for propositional quantification, since, of course, different topological properties of the spaces in question give rise to different sets of validated formulas.

From among these models, some seem to draw our special attention. Cantor space is one of the examples and there are several reasons for this choice. First, as it is well-known, Cantor space is a universal space for Heyting calculus, IPC, which means that a formula F is IPC-provable if and only if it is valid in Cantor space. Thus, in the context of intuitionistic logic, the topological semantics over Cantor space seems to be very natural. Second, let us consider the minimal system of intuitionistic logic with propositional quantification, IPC^2, which results in adding to the usual axiomatization of Heyting calculus the basic axioms and rules for propositional quantification, like the axioms $\forall p F(p) \to F(q)$ and $F(q) \to \exists p F(p)$, and the rules

$F(p) \to G / \exists p F(p) \to G$ and $G \to F(p) / G \to \forall p F(p)$, where p is not free in G. Any acceptable interpretation of propositional quantification must validate these axioms and rules of inference, but the point is that the minimal system is intuitively incomplete: there are formulas which, although intuitively acceptable on the intuitionistic ground, are not derivable in it. The formula $\neg\forall p(p \vee \neg p)$ is probably the most natural example. It can be shown that this formula is not generally valid in topological models. On the other hand, as we will see, in Cantor space, the formula in question is valid. Moreover, we will show that many other formulas of this kind are valid in Cantor space. Finally, using the well-known result of M. O. Rabin, cf. [5], we can prove that the set of all formulas valid in Cantor space is decidable. This contrasts with the case of the set of formulas valid in the class of all topological spaces, cf. [1].

In the sequel, we extend the notion of a formula (in the language of propositional variables $p, q, \ldots$ and propositional connectives $\neg, \vee, \wedge, \to$) by allowing quantification over propositional variables. We start with a general property of topological interpretation of propositional quantifiers. Given a topological space $\mathcal{T}$, and an assignment $P, \vec{Q}$ of open subset of $\mathcal{T}$ to the propositional variables $p, \vec{q}$, each formula $F(\vec{q})$ is interpreted as an open subset $F[\vec{Q}]$ of the space. Our definition of $F[\vec{Q}]$ is standard in case that F is formed by using propositional connectives only; the quantifiers are interpreted as follows:

$$
\begin{aligned}
(\exists p F)[\vec{Q}] &= \bigcup \{F[P, \vec{Q}] \ : \ P - \text{open}\} \\
(\forall p F)[\vec{Q}] &= \operatorname{int} \bigcap \{F[P, \vec{Q}] \ : \ P - \text{open}\}.
\end{aligned}
$$

The basic fact that will be frequently used can be stated as follows:

THEOREM 1. *For any topological space $\mathcal{T}$, any formula $F(p, \vec{q})$, and any open sets $P, \vec{Q}, S$ of $\mathcal{T}$:*

$$F[P, \vec{Q}] \cap S = F[P \cap S, \vec{Q}].$$

From Theorem 1 it follows that for every $x \in \mathcal{T}$ and every neighborhood S of x we have

$$x \in F[P_1, \ldots, P_n] \quad \text{iff} \quad x \in F[P_1 \cap S, \ldots, P_n \cap S].$$

So, accordingly, the topological interpretation of formulas has local character. To illustrate this fact let us consider some operators defined by monadic formulas of the language of IPC. Consider a formula $F(p)$ such that $\mathsf{IPC} \vdash (p \vee \neg p) \to F$. Notice that $F[P]$ is always a dense subset of $\mathcal{T}$, thus $x \notin F[P]$ iff x is an element of some nowhere dense subset of the space. In particular, if $F = p \vee \neg p$ then $x \notin F[P]$ iff x belongs to the boundary of P, and if $F = \neg\neg p \to p$ then $x \notin F[P]$ iff in any neighborhood of x there are points of intclP not belonging to P etc.

Now let us turn to specific properties of the interpretation of propositional quantification in Cantor space. By Cantor space we mean the set 2^ω with the usual product topology. As a basis of 2^ω we can take the sets V_n of all sequences α with a fixed initial segment whose code is n. Notice that the sets V_n are both closed and open. The family of all open subsets of 2^ω will be denoted by $\mathcal{Q}(2^\omega)$.

First, we show that, for any formula $F(p, \vec{q})$ and any open sets $\vec{Q}$, the intersection of all $F[P, \vec{Q}]$, with P ranging over all open subsets of 2^ω, is always an open set. In particular, we can simplify the interpretation of the universal quantifier in the following way.

THEOREM 2. *For every formula $F(p, \vec{q})$ and any $\vec{Q} \in \mathcal{Q}(2^\omega)$ we have*

$$(\forall p F)[\vec{Q}] = \bigcap_{P \in \mathcal{Q}(2^\omega)} F[P, \vec{Q}].$$

PROOF. Assume that $\alpha \notin \operatorname{int} \bigcap_{P \in \mathcal{Q}(2^\omega)} F[P, \vec{Q}]$. Hence, by the properties of Cantor space, we can find a sequence $(\alpha_n)_{n\in\omega}$ with $\alpha_n \in 2^\omega \setminus \{\alpha\}$ and sets $P_n \in \mathcal{Q}(2^\omega)$ such that $\alpha_n \notin F[P_n, \vec{Q}]$ for $n \in \omega$, and such that α is the unique accumulation point of $(\alpha_n)_{n\in\omega}$, i.e. α is the unique point of 2^ω such that in every neighborhood of α there are some elements of the sequence $(\alpha_n)_{n\in\omega}$. Let, for every $n \in \omega$, U_n be a basic set such that $\alpha_n \in U_n$; since α is the only point of accumulation of the sequence $(\alpha_n)_{n\in\omega}$, we can additionally assume that the sets U_n are pairwise disjoint. Now put $P = \bigcup_{n\in\omega}(U_n \cap P_n)$. We shall show that $\alpha \notin F[P, \vec{Q}]$. Observe that, since the sets U_n are closed and open and pairwise disjoint, by Theorem 1 we have $U_n \cap -F[P_n, \vec{Q}] = U_n \cap F[P, \vec{Q}]$. Hence, for every $n \in \omega$, we get $\alpha_n \in U_n \cap -F[P_n, \vec{Q}] \subseteq -F[P, \vec{Q}]$ which implies $\alpha \in \mathrm{cl} - \mathrm{F}[\mathrm{P}, \vec{\mathrm{Q}}] = -\mathrm{F}[\mathrm{P}, \vec{\mathrm{Q}}]$, since α is the accumulation point of $(\alpha_n)_{n\in\omega}$. □

Our next goal is to prove that for any propositional formula F which is not intuitionistically valid, the negation of the universal closure of F is true in 2^ω.

We begin with a technical fact. Let $\Phi_k : 2^\omega \to V_k$ be the mapping defined by $\Phi_k(\alpha) = k*\alpha$ where $k*\alpha$ is the concatenation of a finite sequence of the code k and an (infinite) one denoted by α. Notice that each Φ_k is an homeomorphism from 2^ω onto V_k. By induction on the complexity of the formula F, the following can be proved:

LEMMA 3. *Let $F(p_1, \dots, p_n)$ and $P_1, \dots, P_n \in \mathcal{Q}(2^\omega)$ be arbitrary. Then, for every k,*

$$\Phi_k\big(F[P_1, \dots, P_n]\big) = F[\Phi_k(P_1), \dots, \Phi_k(P_n)] \cap V_k.$$

Using Lemma 3 we prove the announced property of the interpretation over Cantor space.

THEOREM 4. *Let $F(p_1, \dots, p_n)$ be a quantifier-free formula which is not provable in* IPC. *Then*

$$2^\omega \models \neg\forall p_1 \dots \forall p_n F(p_1, \dots, p_n).$$

PROOF. It is well-known that every IPC-non-provable formula can be falsified in 2^ω. Thus we have

$$F[P_1^*, \dots, P_n^*] \neq 2^\omega \quad \text{for some } P_1^*, \dots, P_n^* \in \mathcal{Q}(2^\omega). \tag{1}$$

Suppose, for a proof by contradiction, that $\bigcap_{P_1,\dots,P_n \in \mathcal{Q}(2^\omega)} F \neq \emptyset$. Then we find a basic set V_k such that $V_k \subseteq F[P_1, \dots, P_n]$ for every open sets $P_1, \dots, P_n$. On the other hand, by (1) and Lemma 3, we get

$$V_k \setminus F[\Phi(P_1^*), \dots, \Phi(P_n^*)] \neq \emptyset,$$

a contradiction. □

Notice that, in general, in 2^ω it is impossible to reduce an (infinite) intersection $\bigcap_{P \in \mathcal{Q}(2^\omega)} F[P]$ to a finite one. To see this consider $\forall p F(p)$ for a formula $F(p)$ of the language of IPC which is a classical tautology but is not provable in IPC. Then in 2^ω we have $\bigcap_{P \in \mathcal{Q}(2^\omega)} F[P] = \emptyset$, i.e.,

$$\bigcup_{P \in \mathcal{Q}(2^\omega)} -F[P] = 2^\omega, \tag{2}$$

(cf. [3]). Notice that the sets $F[P]$ are all open and dense and hence their complements are nowhere-dense. Now, if the union (2) were reducible to some finite union, then 2^ω would be a finite union of its nowhere-dense subsets, quod non.

This property shows an asymmetry in topological interpretation of the propositional quantifiers because, as we shall show, the interpretation over Cantor space bears a kind of maximum property.

THEOREM 5. *For every formula $F(p, \vec{q})$ and every $\vec{Q} \in \mathcal{Q}(2^\omega)$ there is an open set P_F such that*

$$(\exists p F)[\vec{Q}] = F[P_F, \vec{Q}].$$

PROOF. Let $F(p, \vec{q})$ be a formula and let $\vec{Q} \in \mathcal{Q}(2^\omega)$. We claim that, for any basic set V of 2^ω,

> if $V \subseteq \bigcup_{P \in \mathcal{Q}(2^\omega)} F[P, \vec{Q}]$, then there exists $P_V \in \mathcal{Q}(2^\omega)$ such that $V \subseteq F[P_V, \vec{Q}]$.

For the proof of the claim, notice that for every $\alpha \in V$ we can find $P^\alpha \in \mathcal{Q}(2^\omega)$ and a basic set V^α with $\alpha \in V^\alpha \subseteq F[P^\alpha, \vec{Q}]$. Consider the family $\mathcal{V} = \{V^\alpha : \alpha \in V\}$. Obviously, $\mathcal{V}$ is a covering of V. By compactness of V, we find $V^{\alpha_1}, \dots, V^{\alpha_k} \in \mathcal{V}$ such that $V \subseteq V^{\alpha_1} \cup \ldots \cup V^{\alpha_k}$; we can additionally assume that the sets $V^{\alpha_1}, \dots, V^{\alpha_k}$ are pairwise disjoint. Put

$$P_V = \bigcup_{1 \le j \le k} (P^{\alpha_j} \cap V^{\alpha_j}).$$

Notice that $P_V \cap V^{\alpha_j} = P^{\alpha_j}$ for $1 \le j \le k$. Moreover, by Theorem 1,

$$V^{\alpha_j} \cap F[P_V, \vec{Q}] = V^{\alpha_j} \cap F[P^{\alpha_j}, \vec{Q}]. \tag{3}$$

Now, let $\alpha \in V$. Then for some i we have $\alpha \in V^{\alpha_i} \subseteq F[P^{\alpha_i}, \vec{Q}]$. Hence, by (3), we get $\alpha \in F[P_V, \vec{Q}]$. Thus $V \subseteq F[P_V, \vec{Q}]$, which proves the claim.

Now we can show that $(\exists p F)[\vec{Q}] \subseteq F[P_F, \vec{Q}]$. Assume that

$$(\exists pF)[\vec{Q}] = \bigcup_{i \in I} V_i$$

for some family $\{V_i : i \in I\}$ of pairwise disjoint basic sets. By the Claim, for every $i \in I$ we can find $P_i \in \mathcal{Q}(2^\omega)$ such that $V_i \subseteq F[P_i, \vec{Q}]$. Put

$$P_F = \bigcup_{i \in I} P_i \cap V_i.$$

It is easy to see that $V_i \subseteq F[P_F, \vec{Q}]$ for every $i \in I$. Hence

$$\bigcup_{P \in \mathcal{Q}(2^\omega)} F[P, \vec{Q}] = \bigcup_{i \in I} V_i \subseteq F[P_F, \vec{Q}].$$

□

It is worth noting that the set P_F of Theorem 5 is not, in general, definable by a propositional formula. Indeed, recall that, according to [2], there exists an interpretation of propositional quantification within intuitionistic propositional logic IPC. In this interpretation, $\exists pF$ is interpreted as the weakest upper interpolant of F not containing the variable p, denoted by $\mathsf{E}pF$. Notice that the formula $\exists pF \to \mathsf{E}pF$ is valid in any model of IPC^2. Now, if P_F were defined over Cantor space by means of a propositional formula then, for F quantifier-free, the formula $\exists pF(p, \vec{q})$ would be equivalent to $\mathsf{E}p\ (F)$. On the other hand, it is known that for formulas involving at least two variables, the two interpretations of propositional quantification in question differ from each other (cf. [4]).

Theorem 5 motivates the introduction of the following definition.

Definition. For every formula $F(p, \vec{q})$ and every $\vec{Q} \in \mathcal{Q}(2^\omega)$, we put

$$\mathcal{E}_{\vec{Q}}(F) = \{R : (\exists pF)[\vec{Q}] = F[R, \vec{Q}]\}.$$

Notice that, by Theorem 5, for every formula $F(p, \vec{q})$ and every $\vec{Q} \in \mathcal{Q}(2^\omega)$ the family $\mathcal{E}_{\vec{Q}}(F)$ is non-empty, moreover it is easy to see that $\mathcal{E}_{\vec{Q}}(F)$ can be infinite. However, given a formula $F(p, \vec{q})$ and $\vec{Q} \in \mathcal{Q}(2^\omega)$, the proof of Theorem 5 does not provide us with a method of constructing any of the sets of $\mathcal{E}_{\vec{Q}}(F)$, and in general, it is not easy to find its elements. However, in some cases, given a member of $\mathcal{E}_{\vec{Q}}(F)$, we can find its elements by other means. This is the essence of the next theorem.

THEOREM 6. *For every formula $F(p,\vec{q})$, every open sets $P,\vec{Q}$ of Cantor space, and every basic set $V \subseteq F[P,\vec{Q}]$, there is a set $R \in \mathcal{E}_{\vec{Q}}(F)$ such that $V \cap P = V \cap R$.*

PROOF. Assume that V is a basic set and $V \subseteq F[P,\vec{Q}]$. Moreover, let

$$\bigcup_{P\in\mathcal{Q}(2^\omega)} F[P,\vec{Q}] = \bigcup_{i\in I} V_{n_i}$$

for some I and pairwise disjoint basic sets V_{n_i}. Then, of course, $V \subseteq V_{n_j}$ for some $j \in I$. First, assume that $V = V_{n_j}$. Recall that the set $\mathcal{E}_{\vec{Q}}(F)$ is non-empty, so we can choose $S \in \mathcal{E}_{\vec{Q}}(F)$. Put

$$R = \big(S \cap \bigcup_{i\neq j} V_{n_i}\big) \cup \big(P \cap V\big).$$

It is easy to see that $V_{n_i} \cap R = V_{n_i} \cap S$ for $i \neq j$ and $V \cap R = V \cap S$. We shall show that $R \in \mathcal{E}_{\vec{Q}}(F)$. Let $i \neq j$; we have $V_{n_i} = V_{n_i} \cap F[S,\vec{Q}]$. Hence, by Theorem 1, we get $V_{n_i} = V_{n_i} \cap F[R,\vec{Q}]$, i.e., $V_{n_i} \subseteq F[R,\vec{Q}]$ for $i \neq j$. Consequently,

$$V = V_{n_j} = V_{n_j} \cap F[S,\vec{Q}] = V_{n_j} \cap F[R,\vec{Q}],$$

whence

$$\bigcup_{P\in\mathcal{Q}(2^\omega)} F[P,\vec{Q}] = \bigcup_{i\in I} \subseteq F[R,\vec{Q}],$$

i.e., $R \in \mathcal{E}_{\vec{Q}}(F)$. In the case when $V \subset V_{n_j}$ for some $j \in I$, we represent V_{n_j} as the union $V \cup V_{m_1} \cup \ldots \cup V_{m_k}$ of pairwise disjoint basic sets and so reduce this case to the previous one. □

We apply Theorem 6 to prove another property of propositional quantification.

THEOREM 7. *For any formulas $F(p,\vec{q})$ and $G(p,\vec{q})$ and any $\vec{Q} \in \mathcal{Q}(2^\omega)$, we have*

$$\big(\exists p(F \wedge G)\big)[\vec{Q}] = \bigcup_{P\in\mathcal{E}_{\vec{Q}}(G)} F[P,\vec{Q}] \cup \bigcup_{P\in\mathcal{E}_{\vec{Q}}(F)} G[P,\vec{Q}].$$

PROOF. Let V be a basic set with $V \subseteq \exists p(F \wedge G)[\vec{Q}]$. Then, by Theorem 5, for some $R \in \mathcal{Q}(2^\omega)$, we have

$$V \subseteq F[R, \vec{Q}] \cap F[R, \vec{Q}].$$

Hence, according to Theorem 6, there are $R_F \in \mathcal{E}_{\vec{Q}}(F)$ and $R_G \in \mathcal{E}_{\vec{Q}}(G)$ such that $V \cap R_F = V \cap R_G = V \cap R$. Thus,

$$V \subseteq F[R_G, \vec{Q}] \cap G[R_F, \vec{Q}].$$

For the converse, assume that $R_F \in \mathcal{E}_{\vec{Q}}(F)$ and $R_G \in \mathcal{E}_{\vec{Q}}(G)$. Observe that

$$\begin{aligned} F[R_G, \vec{Q}] \cap G[R_F, \vec{Q}] &\subseteq (F \wedge G)[R_F, \vec{Q}] \cup (F \wedge G)[R_G, \vec{Q}] \\ &\subseteq \big(\exists p(F \wedge G)\big)[\vec{Q}], \end{aligned}$$

since, of course, $F[R_G, \vec{Q}] \cap G[R_F, \vec{Q}] \subseteq F[R_F, \vec{Q}] \cap G[R_G, \vec{Q}]$. □

Note that the unions $\bigcup_{P \in \mathcal{E}_{\vec{Q}}(G)} F[P, \vec{Q}]$ and $\bigcup_{P \in \mathcal{E}_{\vec{Q}}(F)} G[P, \vec{Q}]$ which occur in Theorem 7 can be viewed as bounded quantifiers. So, in a sense Theorem 7 expresses a kind of distributivity of existential quantification over conjunction.

Using Theorem 5 we prove some further properties of existential quantification under the interpretation over Cantor space. For example, the following fact is easy to verify.

THEOREM 8. *The following formulas are valid in* 2^ω*:*

1. $\neg\neg\exists pF \leftrightarrow \exists p\neg\neg F$,
2. $(G \to \exists pF) \leftrightarrow \exists p(G \to F)$, *where* p *is not free in* G.

Let us conclude with some remarks.

Theorem 8 shows that, as far as we consider the topological interpretation of propositional quantification in Cantor space, we can shift the existential quantifier in front of the implication. Note that Cantor space does not validate the following, intuitively not acceptable on intuitionistic ground, shift of the universal quantifier from the antecedent on an implication: $(\forall pF \to G) \to \exists p(F \to G)$, with p not free in G. Indeed, consider any formula quantifier-free formula $F(p)$ with one variable p such that F is a tautology of classical propositional logic but it is not provable in IPC.

Then, by Theorem 4, in Cantor space the formula $\forall pF \to G$ is valid. On the other hand, it is shown in [3] that (in particular) in Cantor space the formula $\exists p\big(F \to G\big)$ is equivalent to $\neg\neg G$.

As we mentioned, the set of all formulas valid in Cantor space is decidable and, as we have shown, strictly contains the minimal system of intuitionistic logic with propositional quantification. We think that it would be interesting to find a natural axiomatization of quantified propositional intuitionistic logic of Cantor space.

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