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PARTIALLY-ELEMENTARY EXTENSION KRIPKE MODELS AND BURR'S HIERARCHY

Abstract

We investigate Kripke models of subtheories $i\Phi_n$ of Heyting Arithmetic. The theories $i\Phi_n$, defined by W. Burr, can be regarded as the natural intuitionistic counterparts of subtheories $I\Pi_n$ of Peano Arithmetic. In the paper we consider n-elementary extension Kripke models which are models whose worlds are ordered by the elementary extension relation with respect to Σ_n formulae instead of merely the (weak) submodel relation. We prove that every $I\Pi_n$-normal, n-elementary extension model is a model of $i\Phi_n$. This suggests a method of constructing non-trivial Kripke models of $i\Phi_n$. We also show that every $(n+1)$-elementary extension model of $i\Phi_n$ is $I\Pi_n$-normal.

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Let $\mathcal{L}$ be the language of arithmetic containing, as usual, the symbols $\leq, 0, S, \cdot, +$. In the sequel we will consider the standard axiomatization of the first-order arithmetic in the language $\mathcal{L}$, which consists of the defining axioms for the symbols of $\mathcal{L}$ and the induction schema $A(0) \wedge \forall x(A(x) \rightarrow A(Sx)) \rightarrow \forall x A(x)$. This collection of axioms, augmented by the axioms and rules of intuitionistic predicate logic with equality (for the language $\mathcal{L}$), gives rise to the so-called Heyting Arithmetic, HA. Peano Arithmetic, PA, is then obtained by adding to the axiomatization of HA the principle of excluded middle. We will consider subtheories of HA and PA resulting in restricting induction schema to a class Γ of formulae of the language $\mathcal{L}$. These theories will be denoted by $i\Gamma$ and $I\Gamma$ respectively.

In the classical context we will, as usual, consider the hierarchy of prenex formulae of the language $\mathcal{L}$. Recall that the class Δ_0 consists of all formulae of $\mathcal{L}$ which can be obtained from atomic ones by applying propositional connectives and bounded quantification; the classes Σ_n and Π_n result in prefixing to Δ_0 formulae n alternating (blocks of) quantifiers beginning with existential and universal quantifier, respectively.

The well-known subtheories of PA are the theories $I\Pi_n$ which form a strict hierarchy within PA. It is known that the intuitionistic counterparts of $I\Pi_n$, i.e. the theories $i\Pi_n$, do not form a corresponding hierarchy in HA: the union of all theories $i\Pi_n$ does not exhaust HA. However, W. Burr recently discovered a class of subtheories of HA with the properties analogous to the class of the subtheories $I\Pi_n$ of PA (cf. [2]). Before we specify the theories in question, we define the classes Φ_n of formulae of the language $\mathcal{L}$:

- $\Phi_0 := \Delta_0$
- $\Phi_1 := \Sigma_1$
- Suppose Φ_{n-2}, Φ_{n-1} are defined for $n > 1$. Then Φ_n consists of all formulae $\forall x(B \to \exists y C)$, where $B \in \Phi_{n-1}$ and $C \in \Phi_{n-2}$.

It turns out that every formula of the language $\mathcal{L}$ is equivalent, over $i\Delta_0$, to some formula in Φ_n, for some n. So, the classes Φ_n provide, modulo equivalence in $i\Delta_0$, normal forms for formulae of the language $\mathcal{L}$. It is also worth noting that, over the classical theory $I\Delta_0$, the classes Φ_n and Π_n coincide. The classes of formulae Φ_n give rise in the natural way to the theories $i\Phi_n$. W. Burr proved that, for every n, $I\Pi_n$ is Π_2 conservative over $i\Phi_n$, so the theories $i\Phi_n$ can be regarded as the appropriate counterparts of the classical subtheories $I\Pi_n$ of PA.

In the general case, a *Kripke model* $\mathcal{K}$ for the language $\mathcal{L}$ can be viewed as a tuple $\langle K, \preceq, \{\mathrm{M}_\alpha : \alpha \in K\}\rangle$ where $\langle K, \preceq\rangle$ is a poset and $\{\mathrm{M}_\alpha : \alpha \in K\}$ is a family of classical structures for the language $\mathcal{L}$ satisfying the condition: if $\alpha \preceq \beta$ then M_α is a weak substructure of M_β. The forcing relation $\Vdash$ in the model $\mathcal{K}$ is defined in the standard way (cf. [5] for details). With each world M_α we relate the language $\mathcal{L}_\alpha$ resulting in expanding $\mathcal{L}$ with a constant symbol c for every element c in the domain of M_α.

Following [3], we say that a model $\mathcal{K}$ is *T-normal*, for some (classical) theory T, if all its structures M_α are (classical) models of T. Recall that we call M_β a *n-elementary extension* of M_α, and write $\mathrm{M}_\alpha \prec_n \mathrm{M}_\beta$, if M_α is a substructure of M_β and for all $A(x_1, \ldots, x_n) \in \Sigma_n$ (or, equivalently, Π_n) and all $a_1, \ldots, a_n \in \mathrm{M}_\alpha$ we have $\mathrm{M}_\alpha \models A(a_1, \ldots, a_n)$ iff $\mathrm{M}_\beta \models A(a_1, \ldots, a_n)$. We say that M_β is an *elementary extension* of M_α if $\mathrm{M}_\alpha \prec_n \mathrm{M}_\beta$ for each n. For a Kripke model $\mathcal{K} = \langle \mathcal{K}, \preceq, \{\mathrm{M}_\alpha : \alpha \in K\}\rangle$, we say that $\mathcal{K}$ is a *n-elementary extension Kripke model* if for all $\alpha, \beta \in K$ with $\alpha \preceq \beta$ we have $\mathrm{M}_\alpha \prec_n \mathrm{M}_\beta$. Note that every Kripke model of a theory T in $\mathcal{L}$ in which all Δ_0 formulae are decidable (i.e. $T \vdash \forall \vec{x}(A \vee \neg A)$ for every Δ_0 formula A of $\mathcal{L}$) is in fact a 0-elementary extension model. Partially-elementary extension models were studied in the context of fragments of arithmetic in [4].

Since the natural context for considering the theories $i\Phi_n$ is $i\Delta_0$, we set $i\Delta_0$ our basic theory. Recall that in $i\Delta_0$ all Δ_0 formulae are decidable, so every Kripke model of $i\Phi_n$ is a 0-elementary extension model. As a direct consequence of this fact we can derive that for every Kripke model $\mathcal{K}$ of $i\Delta_0$, every $\alpha \in K$ and every $A \in \Sigma_1$ we have $\alpha \Vdash A$ iff $\mathrm{M}_\alpha \models A$. Notice that we can bound quantifiers in the usual axiomatization of $i\Delta_0$ to obtain a Π_1 axiomatization; in particular we can replace the induction schema with the schema

$$\forall y(A(0) \wedge (\forall x < y)(A(x) \rightarrow A(Sx)) \rightarrow A(y)).$$

This fact implies that a Kripke model $\mathcal{K}$ is a model of $i\Delta_0$ iff $\mathcal{K}$ is $I\Delta_0$ normal. It is known (cf. [3]) that, every $I\Sigma_1$-normal model is a model of $i\Sigma_1$. However, even a PA-normal model need not be a model of $i\Pi_1$ (cf. [3]). On the other hand, we can find an example of a Kripke model of $i(\mathit{open})$ which is not $I(\mathit{open})$-normal (cf. [1]). In this paper we prove that every $I\Pi_n$-normal, n-elementary extension model is a model of $i\Phi_n$ and that every $(n+1)$-elementary extension model of $i\Phi_n$ is $I\Pi_n$-normal. The former fact suggests a method of constructing (non-trivial) Kripke models of the theories $i\Phi_n$.

We begin with proving the following fact which will be used to prove the main results of this paper.

LEMMA 1. *Let $n \geq 0$. Let $\mathcal{K} \Vdash i\Delta_0$ be a n-elementary-extension Kripke model. Then, for every $\alpha \in K$ and every $A \in \Phi_n$ we have: $\alpha \Vdash A$ iff $M_\alpha \models A$.*

PROOF. Induction on n. The claim is obvious when $n < 2$. Assume that $n \geq 2$ and the lemma holds for all $k < n$. Consider a Φ_n formula $\forall x(B \rightarrow \exists y C)$; so, we assume that $B \in \Phi_{n-1}$ and $C \in \Phi_{n-2}$.

($\Rightarrow$) Assume that $\alpha \Vdash \forall x(B \rightarrow \exists y C)$. Using the induction hypothesis for the formulae B and C and the definition of forcing, one can easily show that $\mathrm{M}_\alpha \models \forall x(B \rightarrow \exists y C)$.

($\Leftarrow$) Assume that $\alpha \not\Vdash \forall x(B \to \exists y C)$. Then we find a node $\beta \succeq \alpha$ and $b \in \mathrm{M}_\beta$ such that

$$\beta \Vdash B(b) \tag{1}$$

and

$$\beta \not\Vdash \exists y C(b), \text{ i.e. } \beta \not\Vdash C(b,c) \text{ for all } c \in \mathrm{M}_\beta. \tag{2}$$

Since $C \in \Phi_{n-2}$, we can apply induction hypothesis to (2) and get $\mathrm{M}_\beta \not\models C(b,c)$ for every $c \in \mathrm{M}_\beta$ and hence $\mathrm{M}_\beta \not\models \exists y C(b)$. On the other hand, by (1) and induction hypothesis for $B \in \Phi_{n-1}$, we get $\mathrm{M}_\beta \models B(b)$. Hence $\mathrm{M}_\beta \not\models B(b) \to \exists y C(b)$ and consequently $\mathrm{M}_\beta \not\models \forall x(B \to \exists y C)$. Of course, M_β is a model for $I\Delta_0$, so in M_β the formula $\forall x(B \to \exists y C)$ is (equivalent to a formula) in Π_n. But, by the assumption, $\mathrm{M}_\alpha \prec_n \mathrm{M}_\beta$ and hence also $\mathrm{M}_\alpha \not\models \forall x(B \to \exists y C)$. □

Notice that if $A \in \Phi_n$ then

$$A(0) \wedge \forall x(A(x) \to A(Sx)) \to \forall x A(x) \;\in\; \Phi_{n+2}.$$

So, using Lemma 1 one can easily prove, for any $(n+2)$-elementary extension model $\mathcal{K}$, that $\mathcal{K} \Vdash i\Phi_n$ iff $\mathcal{K}$ is $I\Pi_n$-normal. However, we can prove stronger results.

THEOREM 2. *Let $n \geq 2$. Let $\mathcal{K} \Vdash i\Delta_0$ be a n-elementary-extension model. Then, for every $\alpha \in K$: if $M_\alpha \models I\Pi_n$, then $\alpha \Vdash i\Phi_n$.*

PROOF. Assume that $n \geq 2$. Let α be an arbitrary node of $\mathcal{K}$ and let A be a formula in Φ_n.

Suppose $\alpha \not\Vdash A(0) \wedge \forall x(A(x) \to A(Sx)) \to \forall x A(x)$. Then, for some $\beta \succeq \alpha$ and $b \in \mathrm{M}_\beta$ we have

$$\beta \Vdash A(0) \wedge \forall x(A(x) \to A(Sx)) \text{ and } \beta \not\Vdash A(b).$$

So, since $A(b) \in \Phi_n$, by Lemma 1 we get $\mathrm{M}_\beta \not\models A(b)$ and hence $\mathrm{M}_\beta \not\models \forall x A(x)$. Obviously, $\mathcal{K}$ is $I\Delta_0$-normal, so considering the worlds of $\mathcal{K}$ we can assume that A is a Π_n formula. Of course, $\forall x A(x)$ is still in Π_n and, by the assumption, $\mathrm{M}_\alpha \prec_n \mathrm{M}_\beta$; so, consequently

$$\mathrm{M}_\alpha \not\models \forall x A(x). \tag{3}$$

On the other hand, using Lemma 1 we easily show that since $\beta \Vdash A(0)$, we have $\mathrm{M}_\beta \models A(0)$. Hence

$$\mathrm{M}_\alpha \models A(0). \tag{4}$$

It is easy to check that, since $\beta \Vdash \forall x(A(x) \rightarrow A(Sx))$, we have $\mathrm{M}_\beta \models \forall x(A(x) \rightarrow A(Sx))$. Hence, by the assumption that $\mathrm{M}_\alpha \prec_n \mathrm{M}_\beta$, we get

$$\mathrm{M}_\alpha \models \forall x(A(x) \rightarrow A(Sx)). \tag{5}$$

So, by (4) and (5), the premises of induction axiom for A are true in M_α and, by (3), the conclusion is false. So, since inside M_α we can identify A with a Π_n formula, this contradicts the assumption that $\mathrm{M}_\alpha \models I\Pi_n$. □

Theorem 2 provides us with a method of constructing Kripke models of the theories $i\Phi_n$: to obtain such a model it is enough to take an arbitrary (classical) model M of $I\Pi_n$, making it the root of the Kripke model being constructed, and consider an arbitrary family $\{\mathrm{M}_j : j \in J\}$ of n-elementary extensions of M. Then the elements of the family $\{\mathrm{M}_j : j \in J\}$ can be treated as the nodes and the relation $\prec_n$ induces the ordering of the frame of the considered model.

Notice that, for an arbitrary n-elementary extension model, if $\mathrm{M}_\alpha \models I\Pi_n$ for some $\alpha \in K$, then $\mathrm{M}_\beta \models I\Pi_n$ for all $\beta \succeq \alpha$. So, in particular, if the root of a n-elementary extension model $\mathcal{K}$ is a model of $I\Pi_n$, then $\mathcal{K}$ is $I\Pi_n$-normal. Indeed: Let $\mathcal{K}$ be n-elementary extension model with $\mathrm{M}_\alpha \models I\Pi_n$ and let $\beta \succeq \alpha$. Then, if $\mathrm{M}_\beta \models A(0) \wedge \forall x(A(x) \rightarrow A(Sx))$ and $\mathrm{M}_\beta \not\models \forall x A(x)$ for some Π_n formula A, then also $\mathrm{M}_\alpha \not\models \forall x A(x)$. But $\mathrm{M}_\alpha \models I\Pi_n$, so one of the premises of induction axiom for A must be false in M_α. We cannot have $\mathrm{M}_\alpha \not\models A(0)$, because $\mathrm{M}_\beta \models A(0)$. Therefore we must have $\mathrm{M}_\alpha \not\models \forall x(A(x) \rightarrow A(Sx))$ which, in turn, is impossible since, as one can easily check, it implies $\mathrm{M}_\beta \not\models \forall x(A(x) \rightarrow A(Sx))$.

Recall that every $I\Delta_0$-normal model is a model of $i\Delta_0$ and, similarly, every $I\Sigma_1$-normal model is a model of $i\Sigma_1$. Combining these facts with Theorem 2, we get

COROLLARY 3. *For every $n \geq 0$, every $I\Pi_n$-normal, n-elementary extension model is a model of $i\Phi_n$. Thus, $i\Phi_n$ is sound with respect to the class of $I\Pi_n$-normal, n-elementary extension Kripke models.*

Next we show that, if we restrict to the class of $(n+1)$-elementary extension models, every model of $i\Phi_n$ turns out to be $I\Pi_n$ normal.

THEOREM 4. *Let $n \geq 0$ and let $\mathcal{K}$ be a $(n+1)$-elementary extension model of $i\Delta_0$. Then, for every $\alpha \in K$, if $\alpha \Vdash i\Phi_n$ then $M_\alpha \models I\Pi_n$.*

PROOF. The theorem is obvious for $n = 0$. Assume that $n \geq 1$ and $\mathcal{K}$ is a $(n+1)$-elementary extension model of $i\Delta_0$. Let α be an arbitrary node of K with $\alpha \Vdash i\Phi_n$. We will show that for every Pi_n:

$$\mathrm{M}_\alpha \models A(0) \wedge \forall x(A(x) \to A(Sx)) \to \forall x A(x).$$

Let A be an arbitrary Π_n formula. Since $\mathcal{K} \Vdash i\Delta_0$, all the worlds of $\mathcal{K}$ are models of $I\Delta_0$. So, we may assume that A is actually in Φ_n. By the assumption we have $\alpha \Vdash A(0) \wedge \forall x(A(x) \to A(Sx)) \to \forall x A(x)$, so, in particular,

$$\alpha \not\Vdash A(0) \wedge \forall x(A(x) \to A(Sx)) \tag{6}$$

or

$$\alpha \Vdash \forall x A(x). \tag{7}$$

In the case (7), it is sufficient to observe that the formula $\forall x A(x)$ is still in Φ_n. Then, using Lemma 1, we get $\mathrm{M}_{alpha} \models \forall x A(x)$. Hence, the induction axiom for the formula A holds in M_α.

Assume (6). Then we have

$$\alpha \not\Vdash A(0) \quad \text{or} \quad \alpha \not\Vdash \forall x(A(x) \to A(Sx)).$$

When $\alpha \not\Vdash A(0)$, we get, by Lemma 1, $\mathrm{M}_\alpha \not\models \mathrm{A}(0)$. Hence the premise of the induction axiom for the formula A is false and, consequently, the induction axiom for A holds in M_α. Assume that $\alpha \not\Vdash \forall x(A(x) \to A(Sx))$. Then, for some $\beta \succeq \alpha$ and $b \in \mathrm{M}_\beta$,

$$\beta \Vdash A(b) \text{ and } \beta \not\Vdash A(Sb).$$

Since $A \in \Phi_n$, we get, by Lemma 1, $\mathrm{M}_\beta \models A(b) \wedge \neg A(Sb)$. Hence

$$\mathrm{M}_\beta \not\models \forall x(A(x) \to A(Sx)).$$

Observe that in M_β, the formula $\forall x(A(x) \to A(Sx))$ is equivalent to some Π_{n+1}-formula. By the assumption of the proposition, $\mathrm{M}_\alpha \prec_{\mathrm{n+1}} \mathrm{M}_\beta$, so also

$$\mathrm{M}_\alpha \not\models \forall x(A(x) \to A(Sx)).$$

But, again, in this case the induction axiom for the formula A holds in M_α. □

As an immediate consequence of Theorem 4 we get

COROLLARY 5. *For $n \geq 0$, every $(n+1)$-elementary extension model of $i\Phi_n$ is $\Pi\Pi_n$-normal.*

Recall that HA is the union of all its subtheories $i\Phi_n$, so, it seems now natural to ask about models of HA. However, although we have shown how to construct (non-trivial) Kripke models of each $i\Phi_n$, it is still an open problem how to construct non-trivial models of HA itself. The main idea of this paper relies on considering partially-elementary extension models as models for $i\Phi_n$'s. This idea, however, would lead in case of HA to considering PA-normal, elementary extension models which turn out to be trivial in the sense that the theory of every such a model is, in fact, classical and contains PA.

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