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REPRESENTATION THEOREMS FOR HYPERGRAPH SATISFIABILITY

Abstract

Given a set of propositions, one can consider its **inconsistency hypergraph**. Then the satisfiability of sets of clauses with respect to that hypergraph (see [1], [6]) turns out to be the usual satisfiability. The problem is which hypergraphs can be obtained from sets of formulas as inconsistency hypergraphs. In the present paper it is shown that this can be done for all hypergraphs with countably many vertices and pairwise incomparable edges. Then, a general method of transforming the combinatorial problems into the satisfiability problem is shown.

1. Preliminaries

Let us recall some definitions and facts which can also be found in Cowen [1] and Kolany [6].

A hypergraph is a structure $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set and $\mathcal{E}$ is a family of nonempty subsets of $\mathcal{V}$. The elements of $\mathcal{V}$ will be called **vertices**, and the elements of the set $\mathcal{E}$, **edges** of the hypergraph $\mathcal{G}$. Sets of vertices will sometimes be called **clauses**.

A hypergraphs is **compact** iff every edge contains a finite one.

A hypergraphs is **locally finite** iff every vertex belongs to a finite number of edges only.

Notice that a graph is a hypergraph with at most two-element edges.

Here, the vertices of a fixed hypergraph $\mathcal{G}$ will be interpreted as some elementary propositions, and the edges of $\mathcal{G}$ will be inconsistent sets of them. This interpretation leads to the following generalization of satisfiability of families of disjunctions (comp. [1]).

DEFINITION 1.1. A set of vertices σ **satisfies** a family of clauses $\mathcal{A}$ (wrt. $\mathcal{G}$) iff
1. σ does not contain any edge,
2. σ meets all clauses of $\mathcal{A}$, that is $\sigma \cap A \neq \emptyset$, for all A in $\mathcal{A}$.

If some σ satisfies the family $\mathcal{A}$, we say that $\mathcal{A}$ is **satisfiable** with respect to $\mathcal{G}$, or $\mathcal{G}$-satisfiable, for brevity.

Sets σ which do not contain edges will be called **consistent**. A set $\sigma \subset \mathcal{V}$ is **inconsistent** iff it is not consistent.

As it was stated above the notion of the hypergraph satisfiability is a generalization of the satisfiability in the sense of Classical Propositional Calculus. To see this let us consider a hypergraph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$, where $\mathcal{V}_0$ is the set of all propositional variables and their negations, and $\mathcal{E}_0$ consists of all pairs $\{p, \neg p\}$, where p is a variable. Then, given a family X of disjunctions of propositional variables and their negations, X is satisfiable in the usual sense iff X is $\mathcal{G}_0$-satisfiable, where elements of X are treated as sets of their disjuncts.

The hypergraph introduced above will be called **the graph of the Classical Propositional Calculus**, or **the CPC-graph**, for short.

Hypergraph satisfiability has the following property, analogous to the compactness property of CPC-satisfiability (see [6]).

THEOREM 1.2. *Let $\mathcal{G}$ be a compact hypergraph. Then a family of finite clauses $\mathcal{A}$ is $\mathcal{G}$-satisfiable iff every finite subfamily of $\mathcal{A}$ is $\mathcal{G}$-satisfiable.*

By the above, the unsatisfiability of a family of finite clauses $\mathcal{A}$ is reduced to the unsatisfiability of some finite subfamily of $\mathcal{A}$. It also turns out that compact hypergraps are the only hypergraphs with satisfiablity having the above compactness property, (see [6]). There are also other properties of hypergraph satisfiability which are analogue to those of usual satisfiability. They can be found in [1], [2] and [6].

Another property of hypergraph satisfiability which will be used further is the following

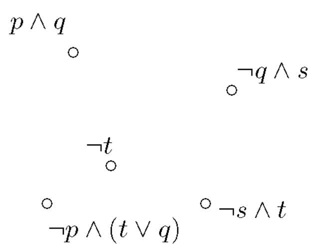
FACT 1.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a hypergraph and let $e \in \mathcal{E}, f \subset \mathcal{V}, e \subset f$. Then, for every family $\mathcal{A}$ of clauses, $\mathcal{A}$ is $\mathcal{G}$-satisfiable iff it is $\mathcal{G}_f$-satisfiable, where $\mathcal{G}_f = (\mathcal{V}, \mathcal{E} \cup \{f\})$.

2. Inconsistency hypergraphs

The case of the CPC-graph shows that the usual satisfiability is a particular case of the hypergraph satisfiability. In this paragraph we will show something opposite in some sense.

Let $\mathcal{X}$ be a set of propositional formulas. The **inconsistency hypergraph** of $\mathcal{X}$ is the hypergraph whose vertices are the formulas of $\mathcal{X}$ and whose edges consist of minimal (in the sense of the inclusion) inconsistent subsets of $\mathcal{X}$. Let us notice that inconsistency hypergraphs must be compact, which follows from the well known **Compactness Theorem**, see [6].

Example 2.1. Let $X = \{p \wedge q, \neg p \wedge (t \vee q), \neg t, \neg q \wedge s, \neg s \wedge t\}$. Then the inconsistency hypergraph of X is $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{a, b, c, d, e\}$, $\mathcal{E} = \{\{a, b\}, \{a, d\}, \{b, c, d\}, \{c, e\}, \{d, e\}\}$, and where $a = p \wedge q, b = \neg p \wedge (t \vee q), c = \neg t, d = \neg q \wedge s, e = \neg s \wedge t$,



The following shows that the transition from sets of formulas to their inconsistency hypergraphs preserves satisfiability.

Fact 2.2. Let X be a set of formulas and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ its inconsistency hypergraph. Moreover, let $\mathcal{A}$ consist of finite subsets of $\mathcal{V}$. Then the set $\mathcal{X} = \{\bigvee A : A \in \mathcal{A}\}$ is satisfiable (in the sense of CPC) iff $\mathcal{A}$ is $\mathcal{G}$-satisfiable.

Proof. ($\Rightarrow$) Let us assume that $\mathcal{X}$ is satisfiable in the sense of CPC and let v be a valuation of variables which satisfies it. Then let σ be the set of all formulas in X which take 1 as the value under v. Then, naturally, σ $\mathcal{G}$-satisfies $\mathcal{A}$. Indeed. Firstly, σ contains no edge, since otherwise it would be CPC-inconsistent, which is not the case, for v satisfies σ. Now, let $A \in \mathcal{A}$. Then $v(\bigvee A) = 1$, hence $v(a) = 1$, for some $a \in A$. This, however means that $a \in \sigma$, which proves the desired result.

($\Leftarrow$) Let $\mathcal{A}$ be $\mathcal{G}$-satisfiable and let $\sigma \subset X$ $\mathcal{G}$-satisfy it. Then σ must be CPC-consistent, and hence CPC-satisfied by some valuation v of propositional variables. Given now $A \in \mathcal{A}$, because σ $\mathcal{G}$-satisfies $\mathcal{A}$, there exists $a \in \sigma \cap A$. Then, however, $v(a) = 1$, and hence $v(\bigvee A) = 1$, which completes the proof. □

The above shows that, in some cases, hypergraph satisfiability stems from the usual one by "forgetting" the nature of vertices. The question arises if it is always the case, i.e. if for every hypergraph, there exists a set of propositions whose inconsistency hypergraph has the same notion of satisfiability. Theorem 1.2 shows that it cannot be true in general – we have to restrict the considerations to the class of compact hypergraphs. Fact 1.3, in turn, proves that we may further restrict them to the class of hypergraphs with $\subset$-minimal edges which, in this situation, gives compact hypergraphs with pairwise incomparable edges. Such hypergraphs, in turn, can be identified with antichains in $Fin(\mathcal{V})\backslash\{\emptyset\}$, the family of all finite and nonempty subsets of a fixed $\mathcal{V}$.

3. Representation

Let $\mathcal{E}$ be an antichain in $Fin(\aleph_0)\backslash\{\emptyset\}$. Since the elements of $\mathcal{E}$ are finite, we may assume that $\mathcal{E}$ consists of ascending sequences of integers, i.e.

$$\mathcal{E} \subset \{(k_0, \ldots, k_n) : k_0 < \ldots < k_n < \omega, n < \omega\}.$$

Theorem 3.1. *For every antichain in $Fin(\aleph_0)$ there exists a sequence $\mathcal{F} = (\Phi_n : n < \omega)$ of propositions which satisfies*

$$\mathcal{F}''N \text{ is inconsistent iff } e \subset N, \text{ for some } e \in \mathcal{E},$$

where $N \subset \aleph_0$.

Proof. Let

$$\Phi_n = \neg p_n \wedge \bigwedge\!\!\!\bigwedge\{\bigvee\!\!\!\bigvee_{j<k} p_{n_j} : (n_0, n_1, \ldots, n_{k-1}, n) \in \mathcal{E}\}.$$

We shall prove that the above is the desired sequence. Let $e \subset N, e = (n_0, n_1, \ldots, n_{k-1}, n_k) \in \mathcal{E}, k \geq 0$, and let $n = n_k$. Then $\Phi_j \rightarrow \neg p_j, j < n$, as well as $\Phi_n \rightarrow \neg p_n \wedge (p_{n_0} \vee \ldots \vee p_{n_{k-1}})$ are tautologies. Since $n_0 < n_1 < \ldots < n_{k-1} < n_k = n$, the set $\{\Phi_{n_0}, \ldots, \Phi_{n_{k-1}}, \Phi_n\}$ is inconsistent. Thus also $\mathcal{F}''N$ is inconsistent.

To prove the opposite, let us assume that $e \not\subseteq N$ for every $e \in \mathcal{E}$ and let us define $v(p_n) = 1$ iff $n \in e \backslash N$, for some $e \in \mathcal{E}, n < \omega$. We will prove that v satisfies all $\Phi_n, n \in N$. Indeed, let $n \in N$. Then clearly $n \notin e \backslash N$, for each $e \in \mathcal{E}$. Hence $v(p_n) = 0$, and $v(\neg p_n) = 1$. Now, let $e = (n_0, \ldots, n_{k-1}, n) \in \mathcal{E}$. Since $n \in N$, and $e \not\subseteq N$, some $n_j \in e \backslash N$, i.e. $v(p_{n_j}) = 1$. Hence $v(\bigvee_{j<k} p_{n_j}) = 1$, for each $(n_0, \ldots, n_{k-1}, n) \in \mathcal{E}$. Thus v satisfies Φ_n for every $n \in N$. $\Box$

One can easily notice that the above proof can be applied for antichains $\mathcal{E}$ in $Fin(\kappa) \backslash \{\emptyset\}, \kappa > \aleph_0$, provided that they are "locally finite", i.e. with $\bigcap \mathcal{E}_0 = \emptyset, \mathcal{E}_0 \subset \mathcal{E}, \mathcal{E}_0$ – infinite.

The same is no longer valid for uncountable sets without any restrictions. We have:

THEOREM 3.2. *Let $\mathcal{X}$ be an uncountable set of consistent propositions. Then there exists at least one consistent pair of propositions in $\mathcal{X}$.*

PROOF. Let $\mathcal{X}$ be an uncountable set of consistent propositional formulas and let us assume that every two propositions of $\mathcal{X}$ contradict each other. Since each element A of $\mathcal{X}$ is consistent, we may assume that it is of the form $A_1 \vee \ldots \vee A_k$, where $A_j, j = 1, \ldots, k$, is an elementary conjunction, i.e. conjunction of literals (variables and their negations), not containing pairs of opposite literals.

Let D_A be one of disjuncts of a proposition $A \in \mathcal{X}$ and let $\mathcal{X}' = \{D_A : A \in \mathcal{X}\}$. Since every pair of propositions of $\mathcal{X}$ is inconsistent, the set $\mathcal{X}'$ is uncountable and consists of consistent elementary conjunctions, every two of which contain a pair of opposite literals. Since the union of countable number of countable sets is countable, we may assume that $\mathcal{X}'$ consists of conjunctions with the same number $N < \omega$ of literals, each.

Let $\mathcal{X}_0 = \mathcal{X}'$ and let $C^{(0)} \in \mathcal{X}_0$. Since $\mathcal{X}_0$ is uncountable and $C^{(0)}$ contains only finitely many literals (in fact exactly N), there exist a literal k_0 of $C^{(0)}$ and an uncountable $\mathcal{X}_1 \subset \mathcal{X}_0$ whose each element contains k_0^* – the literal opposite to k_0. Let $C^{(1)} \in \mathcal{X}_1$. Again reasoning in the same manner as above, there is k_1 in $C^{(1)}$, different from k_0, and uncountable $\mathcal{X}_2 \subset \mathcal{X}_1$ with all elements containing k_1^*. Continuing this procedure, after the $(N+1)^{th}$ step we will obtain a sequence of distinct literals $(k_0, \ldots, k_N)$ and a sequence $(C^{(0)}, \ldots, C^{(N)})$ of elementary conjunctions of $\mathcal{X}'$ satisfying:

$$k_j \in lit(C^{(j)}), k_j^* \in lit(C^{(j+1)}), \ldots, lit(C^{(N)}),\ j = 0, \ldots, N-1,$$

where $lit(C)$ is the set of literals of C. This, however, means that $C^{(N)}$ contains $N+1$ distinct literals: $k_0^*, \ldots, k_{N-1}^*, k_N$, which is impossible, since the conjunctions of $\mathcal{X}'$ contain N literals each. The obtained contradiction completes the proof. $\square$

As it was shown in the latter theorem, in the case of hypergraphs with uncountably many vertices which are not locally finite, the satisfiability cannot be regarded as stemming from the "usual" one by "forgetting" the structure of vertices. It turns out, however, that the hypergraph satisfiability can be represented in classical terms.

Definition 3.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. Then by the **characteristic set** of $\mathcal{G}$, we denote the following set of propositions $\mathcal{X}(\mathcal{G}) = \{\bigvee \neg a : a \in e \in \mathcal{E}\}$.

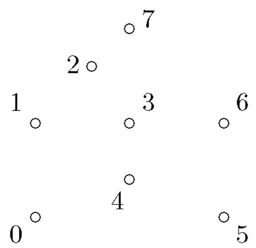
Since we make no stipulations on propositional variables, we may assume the vertices $\mathcal{V}$ as playing their role.

The following theorem, in a slightly different formulation, can be found in [4]. Some ideas of this kind also can be found in [3].

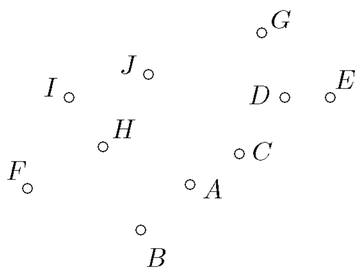
Theorem 3.4. *Let $\mathcal{A}$ be a family of finite clauses. Then $\mathcal{A}$ is $\mathcal{G}$-satisfiable iff the set $\mathcal{X}(\mathcal{G}) \cup \{\bigvee A : A \in \mathcal{A}\}$ is satisfiable in the sense of Classical propositional Calculus.*

Proof. Given a valuation satisfying the considered set of propositions, we take the set of all vertices which get the value of truth as the set satisfying $\mathcal{A}$ with respect to $\mathcal{G}$. On the other hand, if σ satisfies $\mathcal{A}$ wrt. $\mathcal{G}$, one has to assign the value of truth to those propositional variables (i.e. vertices) which are in σ. $\square$

Example 3.5. In order to determine the edge 2-colorability of the following graph $G = (V, \mathsf{E})$:



one has to show the usual 2-colorability of the hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:



where

$$\mathcal{V} = \{A, B, C, D, E, F, G, H, I, J\} = \mathsf{E},$$

and

$$\mathcal{E} = \{\{A, B\}, \{A, C\}, \{C, D, E\}, \{D, G\}, \{B, F, H\}, \{H, I\}, \{I, J\}\},$$

where $A = \{0, 1\}, B = \{0, 4\}, C = \{1, 2\}, D = \{2, 3\}, E = \{2, 7\}, F = \{3, 4\}, G = \{3, 6\}, H = \{4, 5\}, I = \{5, 6\}, J = \{6, 7\}$. This, applying Fact 2.2 and Theorem 3.1, reduces the problem to the satisfiability of the set $\{\Phi_A \vee \Phi_B, \Phi_A \vee \Phi_C, \Phi_C \vee \Phi_D \vee \Phi_E, \Phi_D \vee \Phi_G, \Phi_B \vee \Phi_H, \Phi_H \vee \Phi_I, \Phi_I \vee \Phi_J\}$, where $\Phi_A = \neg A, \Phi_B = A \wedge \neg B, \Phi_C = \neg C \wedge A, \Phi_D = \neg D, \Phi_E = \neg E \wedge (C \vee D), \Phi_F = \neg F, \Phi_G = \neg G \wedge D, \Phi_H = \neg H \wedge (B \vee F), \Phi_I = \neg I \wedge H, \Phi_J = \neg J \wedge I$ (here, we identify variables with vertices $\mathcal{V}$ of $\mathcal{G}$).

In view of Theorem 3.4, the same could be obtained by showing the satisfiability of

$$\{A \vee B, \neg A \vee \neg B, A \vee C, \neg A \vee \neg C, C \vee D \vee E, \neg C \vee \neg D \vee \neg E, D \vee G, \neg D \vee \neg G, B \vee F \vee H, \neg B \vee \neg F \vee \neg H, H \vee I, \neg H \vee \neg I, I \vee J, \neg I \vee \neg J\}.$$

References

[1] R. H. Cowen, *Hypergraph satisfiability*, **Reports on Mathematical Logic** 24 (1991), pp. 113–118.

[2] R. H. Cowen, *Combinatorial Analytic Tableaux*, **Reports on Mathematical Logic** 27 (1993), pp. 29–39.

[3] R. H. Cowen, *2-CNFs and Logical Embeddings*, preprint.

[4] V. Davydov, *Tautologies and Positive Solvability of Linear Homogenous Systems*, **Annals of Pure and Applied Logic** 57 (1992), pp. 27–43.

[5] A. Schrijver, *The Dependence of Some Logical Axioms on Disjoint Transversals and Linked Systems*, **Colloquium Mathematicum** XXXIX (1978), pp. 191–199.

[6] A. Kolany, *Satisfiability on hypergraphs*, **Studia Logica** 52 (1993), pp. 393–404.

[7] N. Linial and M. Tarsi, *Deciding hypergraph 2-colorability by H-resolution*, **Theoretical Computer Science** 38 (1985), pp. 343–347.

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