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ON TWO PROPERTIES OF STRUCTURALLY COMPLETE LOGICS

This is an abstract of the paper which will appear in Reports on Mathematical Logic.

Preliminary notions

Let $\mathcal{S} = \langle S, f_1, \ldots, f_n \rangle$ sentential language C, possibly with index, denotes a standard consequence. Sb is the consequence operation determined by the rule of substitution. By $L(C)$ we denote the set of all Lindenbaum's extensions of the consequence C ($L(C) = \{X \subset S : C(X) \neq S$ and $C(X \cup \{\alpha\}) = S$ for every $\alpha \notin X\}$). $End(S)$ is the set of all endomorphisms of $\mathcal{S}$. Let $\overrightarrow{\mathcal{U}}$ be the consequence operation induced by a matrix $\mathcal{U}$. The symbol $E(\mathcal{U})$ stands for the set of all formulas which are valid in $\mathcal{U}$. We also write $\mathcal{U} \subset\cdot\, \mathcal{U}$, iff $\mathcal{U}$ is a submatrix of $\mathcal{U}_1$. In this paper we assume that in every functionally complete matrix $\mathcal{U}\langle \mathcal{A}, D\rangle$ the set D is proper non-empty subset of the domain A of $\mathcal{A}$. By a rule of inference we mean a non-empty subset of $2^S \times S$. A rule r is finitary iff for every $X \subset S$ and every $\alpha \in S$: if $\langle X, \alpha\rangle \in r$, then X is finite. A rule r is elementary iff $r = \{\langle h(X), h(\alpha)\rangle : h \in End(S)\}$ for some X and for some α. In turn, C_R^A stands for the consequence operation determined by A ($A \subset S$) and by the set of the rules R. For simplicity the symbol C_R will be used instead of $C_R^{\emptyset}$. Two particular sets of rules will be used: MP – the set which contains only the modus ponens and the Gödel's rule. CL stands for the set of all classical tautologies. I is the set of all theses of the intuitionistic logic and J denotes the set of all theses of the Johansson's minimal logic. JG is the set of all theses of the weakest logic, in the family of logics containing Johansson's minimal logic, for which Glivenko's theorem holds (cf. [7], p. 46). $S4$ is the set of all theses of the well-known modal logic. We say that C_1 is C_2 proper ($C_1 \in P(C_2)$) if and only if: $C_1(X) = S \Rightarrow h(X) \not\subset C_2(\emptyset)$, for every $X \subset S$ and for every $h \in End(S)$.

1. Structural completeness versus Tarski's property

She notion of structural completeness has been introduced by W. A. Pogorzelski [6] and it reads as follows: the consequence C is structurally complete ($C \in SCpl$) iff every structural and permissible rule of C is derivable in it.

If the set X is the only one Lindenbaum's extension of the consistent consequence C_1 then we say that the consequence C_1 has Tarski's property in relation to X (cf. [8], [1]).

LEMMA 1.1. *For every consequences* C_1, C_2, C_3, C_4 *such that* $C_4 \leq C_1 \leq C_2$, $C_3 \in P(C_2)$ *and* $C_2(X) = C_3(C_4(X))$ *for every* $X \subset S$ *we get:* $C_1 \in SCpl \Rightarrow L(C_1) = L(C_2)$.

PROOF. Let $C_1 \in SCpl$. Since $C_1 \leq C_2, C_1(X) \neq S$ and $C_2(X \cup \{\alpha\}) = S$ for every $\alpha \notin X$. Therefore, what we need to prove is $C_2(X) \neq S$.

Suppose, to the contrary, that $C_3(C_4(X)) = S$. Since $C_3 \in P(C_2)$ then we get $h(C_4(X)) \not\subset C_2(\emptyset)$ for every $h \in End(S)$. Consider the rule $r_1 = \{\langle h(C_4(X)), \beta\rangle : \beta \in S$ and $h \in End(S)\}$. The rule r_1 is structural and permissible in C_1 but it is not derivable in C_1, so $C_1 \notin SCpl$ which is impossible. Therefore $L(C_1) \subset L(C_2)$. Now let $X \in L(C_2)$. We have $C_2(X \cup \{\alpha\}) = S$ for every $\alpha \notin X$. Since $C_1 \leq C_2$, then $C_1(X) \neq S$. Assume to the contrary that for some $\alpha \in S - X$ we have $C_1(X \cup \{\alpha\}) \neq S$. In this case we consider the rule $r_2 = \{\langle h(C_4(X \cup \{\alpha\})), \beta\rangle : \beta \in S$ and $h \in End(S)\}$. By assumption of the lemma we get $C_3(C_4(X \cup \{\alpha\})) = S$. Since $C_3 \in P(C_2)$ then we obtain: $h(C_4(X \cup \{\alpha\})) \not\subset C_2(\emptyset)$ for every $h \in End(S)$. Thus the structural rule r_2 is permissible in C_1. By assumption we have $C_4 \leq C_1$ so $C_1(C_4(X \cup \{\alpha\})) \neq S$ and therefore r_2 is not derivable in C_1 which contradicts the assumption.

Putting $C_3 = C_2$ and $C_4 = IdId(X) = X$ for every $X \subset S$) we get:

COROLLARY 1.2. *Let* $C_1 \leq C_2$ *and* $C_2 \in P(C_2)$. *If* $C_1 \in SCpl$, *then* $L(C_1) = L(C_2)$.

The assumption $C_2 \in P(C_2)$ is fulfilled, for example, when $C_2 = \overrightarrow{\mathcal{U}}$ and $\mathcal{U}$ is a functionally complete matrix. If we add to the assumptions of Lemma 1.1 that C_2 is Post-complete (i.e. $C_2(\{\alpha\}) = S$ for every $\alpha \in S - C_2(\emptyset)$), then $L(C_1) = \{C_2(\emptyset)\}$.

The assumption of Lemma 1.1 saying that $C_1 \in SCpl$, cannot be omitted. For example, if $C_3 = C_2 = C^{CL}_{MP}$, $C_4 = Id$, then $C_1 \notin SCpl$ and $L(C_1) \neq L(C_2)$ because $S - \{p_1\} \in L(C_1) - L(C_2)$. By Lemma 1.1 we get:

THEOREM 1.3. *If C is structurally complete, $Sb \leq C$ and $C(CL) = CL$, then C has Tarski's property (i.e. $L(C) = \{CL\}$).*

This theorem can be generalized. For instance $\overrightarrow{\mathcal{U}} \circ Sb$ is Post-complete for any functionally complete matrix corresponding to $\mathcal{S}$, thus if C is structurally complete, $Sb \leq C$ and $C(\overrightarrow{\mathcal{U}}(\emptyset)) = \overrightarrow{\mathcal{U}}(\emptyset)$, then by Lemma 1.1 putting $C_1 = C$, $C_2 = \overrightarrow{\mathcal{U}} \circ Sb$, $C_3 = \overrightarrow{\mathcal{U}}$ and $C_4 = Sb$ we get: $L(C) = \{\overrightarrow{\mathcal{U}}(\emptyset)\}$.

These results give the connection between $SCpl$ and Tarski's property.

2. Structural completeness and finite model property

Let $K_C = \{\mathcal{U} : \mathcal{U}$ is a finite matrix corresponding to $\mathcal{S}$ such $C \leq \overrightarrow{\mathcal{U}}\}$ that $C \leq \overrightarrow{\mathcal{U}}\}$. By a theory of C we mean any set X of formulas satisfying $X = C(X)$. We say that X has the finite model property corresponding to C (X has $fmp(C)$) iff $X = \bigcap\{E(\mathcal{U}) : X \subset E(\mathcal{U})$ and $\mathcal{U} \in K_C\}$.

THEOREM 2.1. *If $C \circ Sb$ is a structurally complete consequence operation and $C \circ Sb(\emptyset)$ has $fmp(C_\circ)$, then every theory of $C \circ Sb$ has $fmp(C_\circ)$.*

PROOF. Let $\overrightarrow{K}_{C_\circ}$ be the consequence operation determined by $K_{C_\circ}$ (see [10]), i.e. $\overrightarrow{K}_{C_\circ}(X) = \bigcap\{\overrightarrow{\mathcal{U}}(X) : \mathcal{U} \in K_{C_\circ}\}$. Since $C \circ Sb(\emptyset)$ has $fmp(C_\circ)$ then we have $C \circ Sb(\emptyset) = \overrightarrow{K}_{C_\circ}(\emptyset)$. Thus, each rule permissible for $\overrightarrow{K}_{C_\circ}$ is also permissible for $C \circ Sb$. But $\overrightarrow{K}_{C_\circ}$ is structural, $C \circ Sb$ is structurally complete and hence $\overrightarrow{K}_{C_\circ} \leq C \circ Sb$. Moreover, we have $\overrightarrow{K}_{C_\circ} \circ Sb(X) = \bigcap\{E(\mathcal{U}) : X \subset E(\mathcal{U}) \wedge \mathcal{U} \in K_{C_\circ}\}$ which follows from the equality $\overrightarrow{\mathcal{U}}(Sb(X)) = \bigcap\{E(\mathcal{U}_1) : \mathcal{U}_1 \subsetdot \mathcal{U}\}$ (comp. [9]) and from that $\overrightarrow{\mathcal{U}} \leq \mathcal{U}_1$ whenever $\mathcal{U}_1 \subsetdot \mathcal{U}$. We conclude that for every theory X of $C \circ Sb$,

$X = \overrightarrow{K}_{C_\circ}(Sb(X)) = \bigcap\{E(\mathcal{U} : X \subset E(\mathcal{U}) \wedge \mathcal{U} \in K_{C_\circ}$. Therefore every theory X of $C \circ Sb$ has $fmp(C_\circ)$.

We know (cf. [2],[5]) that there are theories of $C^I_{MP} \circ Sb$ which lack $fmp(C_{MP})$, but for example $C^J_{MP}(\emptyset)$ and $C^I_{MP}(\emptyset)$ have $fmp(C_{MP})$ (cf. [7]). Moreover, R. I. Goldblatt proved that C^{JG}_{MP} has also $fmp(C_{MP})$. Thus, by Theorem 2.1 a consequence C^A_{MP} if $C^A_{MP}(\emptyset)$ has $fmp(C_{MP})$ and there exists a set X being a theory of $C^A_{MP} \circ Sb$ which lacks $fmp(C_{MP})$, then $C^A_{MP} \circ Sb$ is not $SCpl$. (For example $C^J_{MP} \circ Sb$, $C^I_{MP} \circ Sb$, $C^{JG}_{MP} \circ Sb$ are not structurally complete).
By virtue of [3], we get a theory X of $C^{S4}_{MP\Box} \circ Sb$ which has not $fmp(C_{MP})$. Therefore, by Theorem 2.1, $C^A_{MP\Box} \circ Sb$ is not structurally complete if $A \subset X$ and $C^A_{MP\Box}(\emptyset)$ has $fmp(C_{MP})$. Hence for example $C^{S4}_{MP\Box} \circ Sb \notin SCpl$. Since $C^I_{MP} \circ Sb \notin SCpl$ and there are theories of $C^I_{MP} \circ Sb$ which lack $fmp(C^I_{MP})$ then the assumption saying that $C \circ Sb \in SCpl$ is essential. From Theorem 2.1 we get that the implication of Lemma 1.1 is not reversible. Indeed, C^I_{MP} is not $SCpl$. On the other hand it is easy to prove that $L(C^I_{MP}) = L(C^{CL}_{MP})$.

We say that C is finitely based by means of elementary rules iff $C = C_R$ for some finite set R of finitary and elementary rules. Let C be the structural consequence operation finitely based by means of elementary rules and X let be a theory of $C \circ Sb$. We say that X is finitely axiomatizable iff $X = C \circ Sb(X_f)$ for some finite set $X_f \subset X$. Theorem 2.1 and Harrop's theorem on decidability (cf. [4]) yield:

COROLLARY 2.2. *Let C be a structural consequence operation finitely based by means of elementary rules and such that $C(\emptyset)$ has $fmp(C)$. If $C \circ Sb \in SCpl$ then every finitely axiomatizable theory of $C \circ Sb$ is decidable.*

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