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STOCHASTIC SEMIGROUPS AND THEIR APPLICATIONS TO BIOLOGICAL MODELS

Dedicated to Professor Agnieszka Plucińska
on the occasion of her 80th birthday

Abstract. Some recent results concerning generation and asymptotic properties of stochastic semigroups are presented. The general results are applied to biological models described by piecewise deterministic Markov processes: birth-death processes, the evolution of the genome, genes expression and physiologically structured models.

1. Introduction

The main subject of our paper is stochastic semigroups. The stochastic semigroups are strongly continuous semigroups of positive linear operators acting on the space $L^1(E, \Sigma, m)$ and preserving the set of densities. Such semigroups have been intensively studied because they play a special role in applications. The book of Lasota and Mackey [28] is an excellent survey of many results on this subject. Stochastic semigroups are generated by e.g. partial differential equations (transport equations). Equations of this type appear also in the theory of stochastic processes (diffusion processes and jump processes), in the theory of dynamical systems and in population dynamics.

In this paper we present some recent results concerning the generation and the long-time behaviour of stochastic semigroups and illustrate them by some biological applications. Presented results are based on pa-

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pers [9, 12, 31, 34, 35, 36, 38, 43] but the paper contains also some results which have not been published yet, and new proofs of previous results. The organization of the paper is as follows. Section 2 contains the definitions of stochastic (Markov) operators, stochastic semigroups, and piecewise deterministic Markov processes. We also recall results concerning existence of stochastic semigroups. In Section 3 we study asymptotic properties of stochastic semigroups: asymptotic stability and sweeping. Theorems concerning asymptotic stability and sweeping allow us to formulate the Foguel alternative. This alternative says that under suitable conditions a stochastic semigroup is either asymptotically stable or sweeping. In Section 4 we give sufficient and necessary conditions for asymptotic stability and sweeping of stochastic semigroups on the space l^1. Section 5 contains applications of general results to piecewise deterministic Markov processes arising from biological models (genes expression and physiologically structured models). We also present a coagulation-fragmentation model of phytoplankton dynamics [4]. This model leads to a nonlinear stochastic semigroup. The stochastic version of this model based on a point process and its convergence to a superprocess is given in [39, 40].

2. Stochastic semigroups

Let (E,Σ,m) be a measure space with σ-finite measure m. An operator P on $L^1 = L^1(E,\Sigma,m)$ is called positive if $Pf \geq 0$ for $f \geq 0$. A positive linear operator P is called *stochastic (or Markov)* if $\|Pf\| = \|f\|$ for each non-negative function $f \in L^1$ and it is called *substochastic* if $\|Pf\| \leq \|f\|$ for $f \in L^1$. The family $\{P(t)\}_{t\geq 0}$ of linear operators on L^1 is called a *substochastic (stochastic) semigroup* if it is a strongly continuous semigroup and $P(t)$ is a substochastic (stochastic) operator on L^1 for every t. Let us denote by D the subset of the space L^1 which contains all densities

$$D = \{f \in L^1 :\ f \geq 0,\ \|f\| = 1\}.$$

Then a strongly continuous semigroup of linear operators $\{P(t)\}_{t\geq 0}$ on L^1 is a stochastic semigroup iff $P(t)(D) \subset D$ for $t \geq 0$.

Let $\{P(t)\}_{t\geq 0}$ be a substochastic semigroup. The infinitesimal generator (briefly the *generator*) of $\{P(t)\}_{t\geq 0}$ is by definition the operator A with domain $\mathcal{D}(A) \subseteq L^1$ defined as

$$\mathcal{D}(A) = \{f \in L^1 : \lim_{t\downarrow 0} \frac{1}{t}(P(t)f - f) \text{ exists in } L^1\},$$

$$Af = \lim_{t\downarrow 0} \frac{1}{t}(P(t)f - f), \quad f \in \mathcal{D}(A).$$

The operator A is closed with $\mathcal{D}(A)$ being dense in L^1 and for $\lambda > 0$ the resolvent operator is

$$R(\lambda, A)f = (\lambda - A)^{-1}f = \int_0^\infty e^{-\lambda s} P(s)f\,ds \quad \text{for} \quad f \in L^1.$$

The operator $\lambda R(\lambda, A)$ is substochastic and $R(\mu, A) \le R(\lambda, A)$ for $\mu > \lambda$.

In applications an unbounded operator is usually given and one needs to know whether it generates a stochastic (or substochastic) semigroup. A typical situation considered in this paper is in the form of an abstract Cauchy problem

$$u'(t) = Au(t) + Bu(t), \quad u(0) = f \in D. \tag{2.1}$$

In the case of a general L^1-space we have the following result about perturbations of substochastic semigroups [5, 7, 45] which is a generalization of the Kato approach [23] to the study of forward Kolmogorov equations on l^1. We resume the l^1 framework at the end of this section.

Theorem 1. *Assume that $(A, \mathcal{D}(A))$ is the generator of a substochastic semigroup on L^1 and $B\colon \mathcal{D}(A) \to L^1$ is a positive operator such that*

$$\int_E (Af(x) + Bf(x))\, m(dx) \le 0 \quad \textit{for} \quad f \in \mathcal{D}(A), f \ge 0. \tag{2.2}$$

Then for each $r \in (0, 1)$ the operator $(A+rB, \mathcal{D}(A))$ is the generator of a substochastic semigroup $\{P_r(t)\}_{t\ge0}$ on L^1 and the family of operators $\{P(t)\}_{t\ge0}$ defined by

$$P(t)f = \lim_{r\uparrow1} P_r(t)f, \quad f \in L^1, t > 0,$$

is a substochastic semigroup on L^1 with generator $(C, \mathcal{D}(C))$ being an extension of the operator $(A + B, \mathcal{D}(A))$:

$$\mathcal{D}(A) \subseteq \mathcal{D}(C) \quad \textit{and} \quad Cf = Af + Bf \quad \textit{for } f \in \mathcal{D}(A).$$

The generator $(C, \mathcal{D}(C))$ is characterized through the resolvent operator

$$R(\lambda, C)f = \lim_{N\to\infty} R(\lambda, A) \sum_{n=0}^{N} (BR(\lambda, A))^n f, \quad f \in L^1, \lambda > 0, \tag{2.3}$$

and the operator $BR(\lambda, A)$ is substochastic for each $\lambda > 0$.

The semigroup $\{P(t)\}_{t\ge0}$ from Theorem 1 is called the *minimal semigroup* related to $A + B$. The name is justified by the fact that it is the smallest positive semigroup generated by an extension of $(A + B, \mathcal{D}(A))$: if $\{T(t)\}_{t\ge0}$ is another such a semigroup then $T(t)f \ge P(t)f$ for all $f \in \mathcal{D}(A)$, $f \ge 0$. The minimal semigroup $\{P(t)\}_{t\ge0}$ satisfies the integral equation

$$P(t)f = S(t)f + \int_0^t P(t-s)BS(s)f\,ds$$

for any $f \in \mathcal{D}(A)$ and $t \geq 0$, where $\{S(t)\}_{t\geq 0}$ is the semigroup generated by $(A, \mathcal{D}(A))$, and it is also given by the Dyson-Phillips expansion

$$P(t)f = \sum_{n=0}^{\infty} S_n(t)f, \quad f \in \mathcal{D}(A),\ t \geq 0,$$

where

$$S_0(t)f = S(t)f, \quad S_{n+1}(t)f = \int_0^t S_n(t-s)BS(s)f\,ds, \quad n \geq 0.$$

Since equation (2.1) plays a crucial role in applications, it is very important to know when the trajectories of the semigroup $\{P(t)\}_{t\geq 0}$, i.e. the functions $t \mapsto P(t)f$, can be treated as (generalized) solutions of this equation. This problem is non-trivial and we refer the reader to [7] for its thorough study. If the generator of $\{P(t)\}_{t\geq 0}$ is the closure (the minimal closed extension) of the operator $(A+B, \mathcal{D}(A))$ then $\{P(t)\}_{t\geq 0}$ is the only substochastic semigroup of which the generator is an extension of $(A+B, \mathcal{D}(A))$ and in this case the trajectories of this semigroup are solutions of (2.1). Our next result provides several sufficient and necessary conditions for the generator of the minimal semigroup $\{P(t)\}_{t\geq 0}$ to be the closure of the operator $(A+B, \mathcal{D}(A))$. It should be noted that in the case of l^1 space Theorem 2 (or rather Corollary 1) gives the Kato result [23, Theorem 3] (see also [21, Section 23.12]).

Theorem 2. [43, 44] *Let $\lambda > 0$. Under the assumptions of Theorem 1 the following conditions are equivalent:*

(1) *The generator of $\{P(t)\}_{t\geq 0}$ is the closure of $(A+B, \mathcal{D}(A))$.*
(2) *For all $f \in L^1$*

$$\lim_{n\to\infty} \|(BR(\lambda, A))^n f\| = 0.$$

(3) *If for some $f \in L^\infty$, $f \geq 0$, we have $(BR(\lambda, A))^* f = f$ then $f = 0$, where $(BR(\lambda, A))^*$ denotes the adjoint of $BR(\lambda, A)$.*
(4) *$m\{x \in E : f_\lambda(x) > 0\} = 0$, where*

$$f_\lambda(x) = \lim_{n\to\infty} (BR(\lambda, A))^{*n} \mathbf{1}(x).$$

(5) *The operator $BR(\lambda, A)$ is mean ergodic, i.e., for every $f \in L^1$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} (BR(\lambda, A))^k f \quad \text{exists in } L^1.$$

Remark 1. It might be difficult to check conditions (1) or (2). However, one can check (5) by showing that there is a mean ergodic substochastic operator K such that $Kf \geq BR(\lambda, A)f$ for all $f \in L^1$, $f \geq 0$, or that there is $f \in L^1$, $f > 0$ a.e., such that $BR(\lambda, A)f \leq f$ (see [31, 43]).

Remark 2. Note that (see e.g. [44]) the generator of $\{P(t)\}_{t\geq 0}$ is the operator $(A+B,\mathcal{D}(A))$ if and only if for some $\lambda>0$

$$\lim_{n\to\infty}\|(BR(\lambda,A))^n\|=0.$$

In particular, if B is a bounded operator then this condition holds.

Corollary 1. *Assume that $(A,\mathcal{D}(A))$ is the generator of a substochastic semigroup on L^1 and $B\colon\mathcal{D}(A)\to L^1$ is a positive operator such that*

$$\int_E (Af(x)+Bf(x))\,m(dx)=0 \quad \textit{for} \quad f\in\mathcal{D}(A), f\geq 0. \tag{2.4}$$

Let $\{P(t)\}_{t\geq 0}$ be the minimal semigroup related to $A+B$. Then $\{P(t)\}_{t\geq 0}$ is stochastic if and only if one of the equivalent conditions of Theorem 2 holds.

Proof. From (2.3) and (2.4) it follows that for each nonnegative $f\in L^1$ we have (see [43, Section 3])

$$\lambda\|R(\lambda,C)f\|=\|f\|-\lim_{n\to\infty}\|(BR(\lambda,A))^nf\|.$$

The substochastic semigroup $\{P(t)\}_{t\geq 0}$ is stochastic if and only if the operator $\lambda R(\lambda,C)$ is stochastic for some $\lambda>0$ (see e.g. [28, Section 7.8]). ∎

A lot of biological processes changes in a jump or switch-like way, e.g. the size of a cell during division and activation or inactivation of genes. A typical approach is to use piecewise deterministic stochastic processes with jumps. We now consider piecewise deterministic Markov processes introduced by Davis [13] and show how they are connected with substochastic semigroups. We restrict our considerations to processes without “active boundaries”. Let E be a Borel subset of a Polish space (separable complete metric space) and $\mathcal{B}(E)$ be the Borel σ-algebra. We consider three characteristics $(\pi,\varphi,\mathcal{J})$:

- A *semi-flow* $\pi\colon\mathbb{R}_+\times E\to E$ on E, i.e. $\pi_0x=x$, $\pi_{t+s}x=\pi_t(\pi_sx)$ for $x\in E$, $s,t\in\mathbb{R}_+$, and the mapping $(t,x)\mapsto\pi_tx$ is continuous.
- A *jump rate function* $\varphi\colon E\to\mathbb{R}_+$ which is Borel measurable and such that for every $x\in E$, $t>0$, the function $s\mapsto\varphi(\pi_sx)$ is integrable on $[0,t)$. We additionally assume that

$$\lim_{t\to\infty}\int_0^t\varphi(\pi_sx)ds=+\infty \quad \text{for all } x\in E. \tag{2.5}$$

- A *jump distribution* $\mathcal{J}\colon E\times\mathcal{B}(E)\to[0,1]$ which is a stochastic transition kernel such that $\mathcal{J}(x,\{x\})=0$ for all $x\in E$.

We first briefly describe the construction of the piecewise deterministic Markov process (PDMP) $X = \{X(t)\}_{t\ge 0}$ with characteristics $(\pi, \varphi, \mathcal{J})$ (see e.g. [13, 14] or [43, Section 5.1] for details). Define the function

$$F_x(t) = 1 - \exp\{-\int_0^t \varphi(\pi_s x)ds\}, \quad t \ge 0, x \in E, \tag{2.6}$$

and note that the assumptions imposed on φ imply that F_x is a distribution function of a positive and finite random variable for every $x \in E$. Let $t_0 = 0$ and let $X(0) = X_0$ be an E-valued random variable. For each $n \ge 1$ one can chose the n-th *jump time* t_n as a positive random variable satisfying

$$\Pr(t_n - t_{n-1} \le t | X_{n-1} = x) = F_x(t), \quad t \ge 0,$$

and define

$$X(t) = \begin{cases} \pi_{t-t_{n-1}}(X_{n-1}) & \text{for } t_{n-1} \le t < t_n, \\ X_n & \text{for } t = t_n, \end{cases}$$

where the n-th *post-jump position* X_n is an E-valued random variable such that

$$\Pr(X_n \in B | X(t_n-) = x) = \mathcal{J}(x, B),$$

and $X(t_n-) = \lim_{t\uparrow t_n} X(t) = \pi_{t_n - t_{n-1}}(X_{n-1})$. In this way, the trajectory of the process is defined for all $t < t_\infty := \lim_{n\to\infty} t_n$ and t_∞ is called the explosion time. To define the process for all times, we set $X(t) = \Delta$ for $t \ge t_\infty$, where $\Delta \notin E$ is some extra state representing a cemetery point for the process. The PDMP $\{X(t)\}_{t\ge 0}$ is called *non-explosive* if $\mathbb{P}_x(t_\infty = \infty) = 1$ for all $x \in E$, where $\mathbb{P}_x$ is the distribution of the process starting at $X(0) = x$.

We next recall the relation between PDMPs and substochastic semigroups on $L^1 = L^1(E, \mathcal{B}(E), m)$, where m is a σ-finite measure. We assume that there is a stochastic operator P on L^1 which is the transition operator corresponding to $\mathcal{J}$, i.e.

$$\int_E \mathcal{J}(x, B) f(x) m(dx) = \int_B Pf(x) m(dx) \quad \text{for all } B \in \mathcal{B}(E), f \in D,$$

and that there is a substochastic semigroup $\{S(t)\}_{t\ge 0}$ on L^1 satisfying

$$\int_E e^{-\int_0^t \varphi(\pi_r x)dr} \mathbf{1}_B(\pi_t x) f(x)\, m(dx) = \int_B S(t)f(x)\, m(dx) \tag{2.7}$$

for all $t \ge 0$, $f \in D$, $B \in \mathcal{B}(E)$. Suppose that the generator $(A, \mathcal{D}(A))$ of $\{S(t)\}_{t\ge 0}$ with

$$\mathcal{D}(A) \subseteq L^1_\varphi := \{f \in L^1 : \int_E \varphi(x)|f(x)| m(dx) < \infty\}$$

is such that

$$\int_E Af(x)\,m(dx) = -\int_E \varphi(x)f(x)\,m(dx) \quad \text{for } f \in \mathcal{D}(A), f \geq 0. \tag{2.8}$$

Since P is a stochastic operator, the operator B defined by $Bf = (P\varphi)f = P(\varphi f)$, $f \in \mathcal{D}(A)$, is positive and condition (2.4) holds, by (2.8). Consequently, there is the minimal semigroup $\{P(t)\}_{t\geq 0}$ on L^1 related to $A + P\varphi$, by Theorem 1. Corollary 1 together with Remark 1 provide the following

Corollary 2. [43] *If the operator K defined by*

$$Kf = \lim_{\lambda \downarrow 0} P(\varphi R(\lambda, A))f, \quad f \in L^1, \tag{2.9}$$

is mean ergodic then the minimal semigroup $\{P(t)\}_{t\geq 0}$ is stochastic.

The next result gives a probabilistic description of the analytic notions.

Theorem 3. [43] *Let t_∞ be the explosion time for $X = \{X(t)\}_{t\geq 0}$ and let $\mathbb{E}_x$ be the expectation with respect to the law $\mathbb{P}_x$ of the proces X starting at $X(0) = x$. Then the following hold:*

(1) *For any $\lambda > 0$*

$$\lim_{n\to\infty} (P(\varphi R(\lambda, A)))^{*n}\mathbf{1}(x) = \mathbb{E}_x(e^{-\lambda t_\infty}) \quad m - a.e.\ x.$$

(2) *For any $B \in \mathcal{B}(E)$, $f \in \mathcal{D}(A)$, $f \geq 0$, and $t > 0$*

$$\int_B P(t)f(x)m(dx) = \int_E \mathbb{P}_x(X(t) \in B, t < t_\infty)f(x)m(dx).$$

(3) *The operator K as defined in* (2.9) *is stochastic and it is the transition operator corresponding to the discrete-time Markov process $(X_n)_{n\geq 0}$ with stochastic kernel*

$$\mathcal{K}(x, B) = \int_0^\infty \mathcal{J}(\pi_s x, B)\varphi(\pi_s x)e^{-\int_0^s \varphi(\pi_r x)dr}ds, \quad x \in E, B \in \mathcal{B}(E).$$

We conclude from Corollary 1 and Theorem 3 that the minimal semigroup $\{P(t)\}_{t\geq 0}$ is stochastic if and only if

$$m\{x \in E : \mathbb{P}_x(t_\infty < \infty) > 0\} = 0.$$

In that case, if the distribution of $X(0)$ has a density $f \in \mathcal{D}(A)$ then $P(t)f$ is the density of $X(t)$ for all $t > 0$.

Remark 3. The assumption on the semi-flow that $\pi_t(E) \subseteq E$ for all $t \geq 0$ can be relaxed, by allowing "active boundaries" [13, 14]. One can consider a flow on an open set E^o and allow some points to exit from E^o or to enter into E^o through the boundary ∂E^o of E^o. Then the state space E is defined as $E = E^o \cup (\partial^- E^o \setminus \partial^+ E^o)$ where $\partial^\pm E^o = \{x \in \partial E^o : x = \pi_{\pm t} y$ for some $y \in E^o, t > 0\}$. The jump distribution $\mathcal{J}(x, \cdot)$ is supposed to be defined for all

$x \in E \cup \partial^+ E^o$. Let $t_*(x) = \inf\{t > 0 : \pi_t x \in \partial E^o\}$ for $x \in E$ be the exit time from E^o. Assuming that $F_x(t_*(x)) = 1$ for all $x \in E$, where F_x is as in (2.6), the process X can be defined similarly to that without boundaries. We will encounter an example of such a process in Section 5.2.

Particular examples of PDMP are so called *pure jump-type* Markov processes, when between jumps the process does not change its values. A pure jump-type Markov process on a countable set E is called a continuous-time Markov chain. Since the semi-flow satisfies $\pi_t x = x$ for all $t \ge 0, x \in E$, we have $F_x(t) = 1 - \exp\{-\varphi(x)t\}$, where F_x is as in (2.6). Thus condition (2.5) holds iff $\varphi(x) > 0$ for all x. It is convenient to allow the process X to start from x satisfying $\varphi(x) = 0$. In that case we take the first jump time $t_1 = \infty$, define $X(t) = x$ for all $t \ge 0$, and set $\mathcal{J}(x, \{x\}) = 1$ (such a state x is called absorbing). One can combine φ and $\mathcal{J}$ into a rate kernel $\varphi(x)\mathcal{J}(x, B)$. In applications the rate kernel is usually given and one can decompose it as above.

Remark 4. The semigroup $\{S(t)\}_{t\ge 0}$, given by $S(t)f = e^{-\varphi t} f$, $t \ge 0$, $f \in L^1$, has the generator $(A, \mathcal{D}(A))$ of the form $Af = -\varphi f$, $f \in \mathcal{D}(A) = L^1_\varphi$, which trivially satisfies (2.8). For each $\lambda > 0$ we have

$$\varphi R(\lambda, A) f = \frac{\varphi}{\lambda + \varphi} f, \quad f \in L^1.$$

If P is a stochastic operator then the adjoint operator of $P(\varphi R(\lambda, A))$ is of the form

$$(P(\varphi R(\lambda, A)))^* f = \frac{\varphi}{\lambda + \varphi} P^* f \quad \text{for} \quad f \in L^\infty.$$

Consequently, any fixed point $f \in L^\infty$ of this operator is the solution of $\varphi P^* f - \varphi f = \lambda f$.

We now consider the case of $E = \mathbb{N} = \{0, 1, \ldots\}$, where we use the notation $l^1 = L^1(E, \Sigma, m)$ with Σ being the family of all subsets and m the counting measure on E. In the discrete state space setting we represent any function f as a sequence $x = (x_i)_{i\in\mathbb{N}}$.

A matrix $Q = [q_{ij}]$ is called a *Kolmogorov matrix* if its entries have the following properties

(i) $q_{ij} \ge 0$ for $i \ne j$,
(ii) $\sum_{i=0}^\infty q_{ij} = 0$ for $j = 0, 1, 2, \ldots$.

A matrix $Q = [q_{ij}]$ is called a *sub-Kolmogorov matrix* if it satisfies condition (i) and the condition

(ii′) $\sum_{i=0}^\infty q_{ij} \le 0$ for $j = 0, 1, 2, \ldots$.

Consider the system of equations

$$x_i'(t) = \sum_{j=0}^{\infty} q_{ij} x_j(t), \quad i = 0, 1, 2, \dots,$$

where Q is a sub-Kolmogorov matrix. Let

$$\mathcal{D}_0(Q) = \{x \in l^1 : \sum_{j=0}^{\infty} |q_{jj}||x_j| < \infty\}. \tag{2.10}$$

The set $\mathcal{D}_0(Q)$ is dense in the space l^1 and the matrix Q defines a linear operator on $\mathcal{D}_0(Q)$ with values in l^1, since $\mathcal{D}_0(Q) \subseteq \{x \in l^1 : Qx \in l^1\}$. Let A be the diagonal part of Q, i.e. $A = [a_{ij}]$, $a_{jj} = q_{jj}$ and $a_{ij} = 0$ for $i \neq j$, and let $B = [b_{ij}]$ be the off-diagonal part of Q, i.e. $B = Q - A$. The operator A with domain $\mathcal{D}(A) = \mathcal{D}_0(Q)$ is the generator of a substochastic semigroup on l^1 and the operator $B\colon \mathcal{D}(A) \to l^1$ is positive. By Theorem 1, there is the minimal substochastic semigroup $\{P(t)\}_{t\geq 0}$ related to Q.

Suppose now that Q is a Kolmogorov matrix. Since $-q_{jj} = \sum_{i\neq j} q_{ij} \geq 0$ for all j, we can define the jump rate function $\varphi = (\varphi_j)$ by $\varphi_j = -q_{jj}$. Note that $l^1_\varphi = \mathcal{D}_0(Q)$. The jump distribution $\mathcal{J}$ is defined as follows. For k with $\varphi_k > 0$ set $\mathcal{J}(k, \{j\}) = q_{jk}/\varphi_k$ for $j \neq k$ and $\mathcal{J}(k, \{k\}) = 0$, and for each k such that $\varphi_k = 0$ set $\mathcal{J}(k, \{k\}) = 1$ and $\mathcal{J}(k, \{j\}) = 0$ for $j \neq k$. The operator $P\colon l^1 \to l^1$ is given by $(Px)_j = \sum_k \mathcal{J}(k, \{j\}) x_k$. We have $(Qx)_j = -\varphi_j x_j + P(\varphi x)_j$ for all $j \geq 0$, $x \in l^1$, and $(Q^* x)_j = -\varphi_j x_j + \varphi_j (P^* x)_j$ for all $j \geq 0$, $x \in l^\infty$, where Q^* denotes the transpose of the matrix Q. By Remark 4, condition (3) of Theorem 2 can be reformulated as: any nonnegative solution $x \in l^\infty$ of $Q^* x = \lambda x$ must be the zero solution. In addition to Corollary 1 we have (see also [11, Section 8.3])

Theorem 4. [23] *Let Q be a Kolmogorov matrix and let $\lambda > 0$ be a positive constant. The minimal semigroup related to Q is a stochastic semigroup on l^1 iff the equation $Q^* x = \lambda x$ has no nonzero solution $x \in l^\infty$.*

If the minimal semigroup $\{P(t)\}_{t\geq 0}$ related to a Kolmogorov matrix Q is a stochastic semigroup then the matrix Q is called *non-explosive*, as it corresponds to a non-explosive continuous-time Markov chain.

Finally, observe that an arbitrary stochastic operator $P\colon l^1 \to l^1$ is an integral operator, as defined in the next section. Indeed, for each i the function $f \mapsto (Pf)_i$ is a continuous linear functional from l^1 to $\mathbb{R}$. Thus there is a sequence $(k_{ij})_{j\in\mathbb{N}} \in l^\infty$ such that

$$(Pf)_i = \sum_{j=0}^{\infty} k_{ij} f_j = \int_{\mathbb{N}} k_{ij} f_j \, m(dj).$$

3. Asymptotic properties

An operator $P\colon L^1(E)\to L^1(E)$ is called an *integral* or *kernel* operator if there exists a measurable function $k\colon E\times E\to[0,\infty)$ such that

$$Pf(x)=\int_E k(x,y)f(y)\,m(dy)$$

for every density f. One can check that if the operator P satisfies condition $P(D)\subset D$ then

$$\int_E k(x,y)\,m(dx)=1$$

for almost all $y\in E$.

A semigroup $\{P(t)\}_{t\ge0}$ is called integral, if for each $t>0$, the operator $P(t)$ is an integral operator. A substochastic semigroup $\{P(t)\}_{t\ge0}$ is called *partially integral* if there exists a measurable function $k\colon(0,\infty)\times X\times X\to[0,\infty)$, called a *kernel*, such that

$$P(t)f(x)\ge\int_E k(t,x,y)f(y)\,m(dy)$$

for every density f and

$$\int_E\int_E k(t,x,y)\,m(dy)\,m(dx)>0$$

for some $t>0$.

A density f_* is called *invariant* if $P(t)f_*=f_*$ for each $t>0$. A stochastic semigroup $\{P(t)\}_{t\ge0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty}\|P(t)f-f_*\|=0\quad\text{for}\quad f\in D.$$

A stochastic semigroup $\{P(t)\}_{t\ge0}$ is called *sweeping* with respect to a set $B\in\Sigma$ if for every $f\in D$

$$\lim_{t\to\infty}\int_B P(t)f(x)\,m(dx)=0.$$

Theorem 5. [36] *Let E be a metric space and $\Sigma=\mathcal{B}(E)$ be the σ–algebra of Borel subsets of E. We assume that a partially integral stochastic semigroup $\{P(t)\}_{t\ge0}$ with the kernel k has the following properties:*

(a) *for every $f\in D$ we have $\int_0^\infty P(t)f\,dt>0$ a.e.,*
(b) *for every $y_0\in E$ there exist $\varepsilon>0$, $t>0$, and a measurable function $\eta\ge0$ such that $\int\eta\,dm>0$ and*

$$k(t,x,y)\ge\eta(x)$$

for $x \in E$ and $y \in B(y_0, \varepsilon)$, where $B(y_0, \varepsilon)$ is the open ball with center y_0 and radius ε. If the semigroup $\{P(t)\}_{t\ge 0}$ has no invariant density then it is sweeping with respect to compact sets.

For any $f \in L^1(E)$ the *support* of f is defined up to a set of measure zero by the formula

$$\operatorname{supp} f = \{x \in E : f(x) \neq 0\}.$$

Theorem 6. [35] *Let E be a metric space and $\Sigma = \mathcal{B}(E)$. Let $\{P(t)\}_{t\ge 0}$ be a substochastic semigroup on $L^1(E)$ which has the only one invariant density f_* and let $S = \operatorname{supp} f_*$. Assume that $\{P(t)\}_{t\ge 0}$ is a partially integral semigroup with the kernel $k(t, x, y)$ such that*

$$\int_S \int_S k(t_0, x, y)\, m(dx)\, m(dy) > 0$$

for some $t_0 > 0$. Moreover, we assume that for some $t_1 > 0$

(a) *there does not exist a nonempty measurable set $B \subsetneq E \setminus S$ such that $P^*(t_1)\mathbf{1}_B \ge \mathbf{1}_B$ and*

(b) *for every $y_0 \in E \setminus S$ there exist $\varepsilon > 0$ and a measurable function $\eta \ge 0$ such that $\int_{E\setminus S} \eta\, dm > 0$ and*

$$k(t_1, x, y) \ge \eta(x) \tag{3.1}$$

for $x \in E$ and $y \in B(y_0, \varepsilon)$, where $B(y_0, \varepsilon)$ is the open ball with center y_0 and radius ε.

Then for every $f \in D$ there exists a constant $c(f)$ such that

$$\lim_{t\to\infty} \mathbf{1}_S P(t) f = c(f) f_*$$

and for every compact set $F \in \Sigma$ and $f \in D$ we have

$$\lim_{t\to\infty} \int_{F\cap E\setminus S} P(t) f(x)\, m(dx) = 0.$$

If a substochastic semigroup $\{P(t)\}_{t\ge 0}$ on $L^1(E)$ has the only one invariant density f_* and $\operatorname{supp} f_* = E$ then $\{P(t)\}_{t\ge 0}$ is a stochastic semigroup and we have the following

Theorem 7. [34] *Let $\{P(t)\}_{t\ge 0}$ be a partially integral stochastic semigroup. Assume that the semigroup $\{P(t)\}_{t\ge 0}$ has a unique invariant density f_*. If $f_* > 0$ a.e., then the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable.*

If E is a compact space then from Theorem 5 and Theorem 7 it follows

Corollary 3. *Let E be a compact metric space and Σ be the σ–algebra of Borel sets. Let $\{P(t)\}_{t\ge 0}$ be a stochastic semigroup which satisfies conditions:*

(a) *for every $f \in D$ we have $\int_0^\infty P(t) f\, dt > 0$ a.e.,*

(b) *for every* $y_0 \in E$ *there exist* $\varepsilon > 0$, $t > 0$, *and a measurable function* $\eta \geq 0$ *such that* $\int \eta \, dm > 0$ *and*

$$P(t)f(x) \geq \eta(x) \int_{B(y_0,\varepsilon)} f(y)\, m(dy)$$

for $x \in E$, *where* $B(y_0, \varepsilon)$ *is the open ball with center* y_0 *and radius* ε.

Then the semigroup $\{P(t)\}_{t\geq 0}$ *is asymptotically stable.*

From Theorems 5 and 7 it also follows

Corollary 4. *Let* E *be a metric space and* Σ *be the* σ*–algebra of Borel sets. Let* $\{P(t)\}_{t\geq 0}$ *be an integral stochastic semigroup with a continuous and positive kernel* $k(t, x, y)$ *for* $t > 0$. *If the semigroup* $\{P(t)\}_{t\geq 0}$ *has an invariant density, then it is asymptotically stable, and if* $\{P(t)\}_{t\geq 0}$ *has no invariant density, then it is sweeping with respect to compact sets.*

The property that a stochastic semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping from a sufficiently large family of sets (e.g. from all compact sets) is called the *Foguel alternative* [28].

4. Foguel alternative on discrete state space

In this section we restrict our considerations to substochastic semigroups on l^1, where criteria for asymptotic stability, sweeping and the Foguel alternative are relatively simple. Using our general approach to stochastic semigroups we prove the following well-known result concerning irreducible Markov chains.

Theorem 8. *Let* $\{P(t)\}_{t\geq 0}$ *be a stochastic semigroup on* l^1 *generated by the equation*

$$x'(t) = Qx(t).$$

Let us assume that the entries of the matrix Q *satisfy the following condition*
(T) *for every* $i, j \in \mathbb{N}$, $i \neq j$ *there exists a sequence of distinct positive integers* $i_0, i_1, \ldots, i_r$ *such that* $i_0 = j$, $i_r = i$ *and*

$$q_{i_r i_{r-1}} \cdots q_{i_2 i_1} q_{i_1 i_0} > 0. \tag{4.1}$$

Then the semigroup $\{P(t)\}_{t\geq 0}$ *satisfies the Foguel alternative:*

(a) *if the semigroup* $\{P(t)\}_{t\geq 0}$ *has an invariant density, then it is asymptotically stable,*
(b) *if the semigroup* $\{P(t)\}_{t\geq 0}$ *has no invariant density, then for every* $x \in l^1$ *and* $i \in \mathbb{N}$ *we have*

$$\lim_{t\to\infty} (P(t)x)_i = 0. \tag{4.2}$$

Proof. The semigroup $\{P(t)\}_{t\ge 0}$ can be written in the form

$$(P(t)x)_i = \sum_{j=0}^{\infty} p_{ij}(t)x_j.$$

Suppose first that $q_{mn} > 0$ for some $m, n \in \mathbb{N}$. We show that $p_{mn}(t) > 0$ for $t > 0$. Let $x(t)$ be the solution of the equation $x' = Qx$ with the initial condition $x_n(0) = 1$ and $x_k(0) = 0$ for $k \ne n$. Then $p_{mn}(t) = x_m(t)$. Since Q is a Kolmogorov matrix we have $q_{ij} \ge 0$ for $i \ne j$. Since $x(t)$ satisfies the equation $x' = Qx$ we have

$$x_n'(t) \ge q_{nn}x_n(t), \tag{4.3}$$

$$x_m'(t) \ge q_{mn}x_n(t) + q_{mm}x_m(t). \tag{4.4}$$

From inequality (4.3) it follows that $x_n(t) \ge e^{q_{nn}t}$. This and inequality (4.4) imply

$$\big(e^{-q_{mm}t}x_m(t)\big)' = e^{-q_{mm}t}\big(x_m'(t) - q_{mm}x_m(t)\big) \ge e^{-q_{mm}t}q_{mn}e^{q_{nn}t} > 0.$$

Thus

$$e^{-q_{mm}t}x_m(t) > x_m(0) = 0 \quad \text{for } t > 0,$$

and consequently $p_{nm}(t) = x_m(t) > 0$ for $t > 0$. Let us fix $t > 0$ and $i, j \in \mathbb{N}$. From the previous arguments and from (T) it follows that there exists a sequence of different positive integers $i_0, i_1, \ldots, i_r$ such that $i_0 = j$, $i_r = i$ and $p_{i_k i_{k-1}}\big(\frac{t}{r}\big) > 0$ for $k = 1, 2, \ldots, r-1$. Since $P(t) = P^r\big(\frac{t}{r}\big)$ we have $p_{ij}(t) > 0$ for $t > 0$. Thus $p_{ij}(t) > 0$ for arbitrary $i, j \in \mathbb{N}$, $i \ne j$ and $t > 0$. From inequality (4.3) it follows immediately that $p_{ii}(t) > 0$ for $t > 0$ and $i \in \mathbb{N}$. Hence, for every $t > 0$ the matrix $[p_{ij}(t)]$ has all entries positive. Thus, the result follows from Corollary 4. ■

Remark 5. The advantage of formulating Theorem 8 in the form of an alternative is that in order to show asymptotic stability we do not need to prove the existence of an invariant density. It is enough to check that condition (b) does not hold. Then the semigroup $\{P(t)\}_{t\ge 0}$ automatically is asymptotically stable. This idea enables us to give an almost immediate proof of Theorem 9 concerning asymptotic stability of a genome evolution model, where it is not easy to show that the Markov chain has a stationary distribution.

Example 1. A general birth-death process on $\mathbb{N} = \{0, 1, \ldots\}$ is described by the following system of equations

$$x_i'(t) = -a_i x_i(t) + b_{i-1}x_{i-1}(t) + d_{i+1}x_{i+1}(t) \tag{4.5}$$

for $i \geq 0$, where $b_{-1} = d_0 = 0$, $b_i \geq 0$, $d_{i+1} \geq 0$ for $i \geq 0$, $a_0 = b_0$, $a_i = b_i + d_i$ for $i \geq 1$. Let us assume that the system (4.5) generates a stochastic semigroup $\{P(t)\}_{t\geq 0}$ and that $d_i > 0$ for $i > 0$.

First, suppose that $b_i > 0$ for each i. Since $q_{i,i+1} = d_{i+1} > 0$, $q_{i+1,i} = b_i > 0$ for $i \geq 0$, condition (T) holds and Theorem 8 applies. Hence, the semigroup $\{P(t)\}_{t\geq 0}$ satisfies the Foguel alternative.

We now suppose that there exists $n > 0$ such that $b_n = 0$, $b_i > 0$ for $i \neq n$. It is easily seen that condition (T) does not hold. We claim that Theorem 6 applies. Thus, for each $\bar{x} \in l^1$ there exists a constant $c(\bar{x})$ such that the solution of (4.5) with the initial condition $x(0) = \bar{x}$ satisfies

$$\lim_{t\to\infty} x_i(t) = c(\bar{x})x_i^* \quad \text{for } i \leq n,$$
$$\lim_{t\to\infty} x_i(t) = 0 \qquad\quad \text{for } i > n,$$

where $x^* = (x_i^*)$ is an invariant density with $\operatorname{supp} x^* = \{0, 1, \ldots, n\}$. Indeed, since $\mathbb{N}$ is a discrete topological space condition (3.1) holds. Moreover the only sets which satisfy the inequality $P^*(t)\mathbf{1}_B \geq \mathbf{1}_B$ are $\emptyset$, $\mathbb{N}$, and $\mathbb{N}_n = \{0, 1, \ldots, n\}$. If the sequence $x^* = (x_i^*)$ is an invariant density and $S = \operatorname{supp} x^*$ then $P^*(t)\mathbf{1}_S \geq \mathbf{1}_S$. From that it follows that the semigroup has at most one invariant density, because in the opposite case we could find two invariant densities with disjoint supports, which is impossible in our case. Since $p_{ij}(t) = 0$ for $j \leq n < i$, we can restrict the semigroup $\{P(t)\}_{t\geq 0}$ to the space $L^1(\mathbb{N}_n, 2^{\mathbb{N}_n}, m)$. Then $\{P(t)\}_{t\geq 0}$ is a stochastic semigroup on this space and from the ergodic theorem for Markov chains on a finite space it follows that the semigroup has an invariant density.

We now give an application of Theorem 8 to a model describing the evolution of paralog families in a genome [38]. Two genes present in the same genome are said to be *paralogs* if they are genetically identical. It is not a precise definition of paralogs but it is sufficient for our purposes. We are interested in the size distribution of paralogous gene families in a genome. We divide genes into classes. The i-th class consists of all i-element paralog families. Let x_i be a number of families in the i-th class. Based on experimental data Słonimski et al. [42] suggested that

$$x_i \sim \frac{1}{2^i i}, \quad i = 2, 3, \ldots.$$

On the other hand, Huynen and van Nimwegen [22] claimed that

$$x_i \sim i^{-\alpha}, \quad i = 1, 2, 3, \ldots,$$

where $\alpha \in (2, 3)$ depends on the size of the genome and α decreases if the total number of genes increases. It is very difficult to decide which formula is correct if only experimental data are taken into account because one can compare only first few elements of both sequences. We construct a simple model of the evolution of paralog families which can help to solve this problem.

The model is based on three fundamental evolutionary events: gene loss, duplication, and accumulated change called for simplicity mutation. A single gene during the time interval of length Δt can be:

- *duplicated* with probability $d\Delta t + o(\Delta t)$ and duplication of it in a family of the i-th class moves this family to the $(i+1)$-th class,
- *removed* from the genome with probability $r\Delta t + o(\Delta t)$. For $i > 1$, removal of a gene from a family of the i-th class moves this family to the $(i-1)$-th class; removal of a gene from one-element family results in an elimination of this family from the genome. A removed gene is eliminated permanently from the pool of all genes,
- *changed* with probability $m\Delta t + o(\Delta t)$ and the gene starts a new one-element family and it is removed from the family to which it belonged.

It is assumed that $\lim_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t} = 0$. Moreover, we assume that all elementary events are independent of each other. Let $s_i(t)$ be the number of i-element families in our model at the time t. It follows from the description of our model that

$$s_1'(t) = -(d+r)s_1(t) + 2(2m+r)s_2(t) + m\sum_{k=3}^{\infty} ks_k(t), \tag{4.6}$$

$$s_i'(t) = d(i-1)s_{i-1}(t) - (d+r+m)is_i(t) + (r+m)(i+1)s_{i+1}(t) \tag{4.7}$$

for $i \geq 2$. Let $s(t) = \sum_{i=1}^{\infty} s_i(t)$ be the total number of families. Then the sequence $(p_i(t))$, where $p_i(t) = s_i(t)/s(t)$, is the size distribution of paralogous gene families in a genome at time t.

The main result of the paper [38] is the following.

Theorem 9. *Let X be the space of sequences (x_i) which satisfy the condition $\sum_{i=1}^{\infty} i|x_i| < \infty$. There exists a sequence $(s_i^*) \in X$ such that for every solution $(s_i(t))$ of* (4.6) *and* (4.7) *with $(s_i(0)) \in X$ we have*

$$\lim_{t\to\infty} e^{(r-d)t} s_i(t) = Cs_i^* \tag{4.8}$$

for every $i = 1, 2, \ldots$ and C dependent only on the sequence $(s_i(0))$. Moreover, if $d = r$ then

$$\lim_{t\to\infty} s_i(t) = C\frac{\alpha^i}{i}, \tag{4.9}$$

where $\alpha = \dfrac{r}{r+m}$.

In the case of $d = r$ the total number of genes in a genome is constant. It means that the genome is in a stable state. In this case the distribution of paralog families is similar to that stated in Słonimski's conjecture, and both distributions are the same if $r = d = m$.

We now show that Theorem 9 follows from Theorem 8. The proof is different from the one presented in [38].

Proof of Theorem 9. First, we change variables. Let

$$y_i(t) = e^{(r-d)t} i s_i(t).$$

Then

$$y_1' = -(2d+m)y_1 + (m+r)y_2 + \sum_{k=1}^{\infty} m y_k, \tag{4.10}$$

$$y_i' = -(d+r+m+\tfrac{d-r}{i}) i y_i + d i y_{i-1} + (r+m) i y_{i+1} \tag{4.11}$$

for $i \geq 2$. We claim that the system (4.10)–(4.11) generates a stochastic semigroup on l^1. Indeed, the system (4.10)–(4.11) can be written in the following way:

$$y'(t) = Qy(t),$$

where $Q = (q_{ij})_{i,j\geq 1}$. The matrix Q is a Kolmogorov matrix. By Theorem 4, the minimal semigroup related to Q is stochastic if for any $\lambda > 0$ there is no non-zero solution of the equation $Q^*x = \lambda x$, where $x \in l^\infty$. Here

$$\begin{aligned}
(Q^*x)_1 &= -2dx_1 + 2dx_2,\\
(Q^*x)_2 &= (2m+r)x_1 - (r+2m+3d)x_2 + 3dx_3,\\
(Q^*x)_n &= mx_1 + (n-1)(r+m)x_{n-1} - [r(n-1) + d(n+1) + mn]x_n\\
&\quad + (n+1)dx_{n+1}
\end{aligned}$$

for $n \geq 3$. We consider the case of $d \neq 0$ (the case of $d = 0$ is trivial). The sequence $x = (x_i)_{i\geq 1}$ satisfies equation $Q^*x = \lambda x$ iff

$$\begin{aligned}
x_2 &= \Big(1 + \frac{\lambda}{2d}\Big)x_1,\\
x_3 &= \Big(1 + \frac{r+2m+\lambda}{3d}\Big)x_2 - \frac{r+2m}{3d}\,x_1,\\
x_{n+1} &= \Big(1 + \frac{(n-1)r+nm+\lambda}{(n+1)d}\Big)x_n - \frac{(n-1)(r+m)}{(n+1)d}x_{n-1} - \frac{m}{(n+1)d}\,x_1
\end{aligned}$$

for $n \geq 3$. Since the system is linear, it is sufficient to consider the case $x_1 > 0$. The above system of equations can be replaced by one equation

$$x_{n+1} = \Big(1 + \frac{\lambda}{(n+1)d}\Big)x_n + \frac{(n-1)(r+m)}{(n+1)d}(x_n - x_{n-1}) + \frac{m}{(n+1)d}(x_n - x_1)$$

for $n \ge 1$. Hence, the sequence (x_n) is increasing. Thus

$$x_{n+1} \ge \Big(1 + \frac{\lambda}{(n+1)d}\Big) x_n,$$

and consequently

$$x_n \ge x_1 \prod_{i=2}^{n} \Big(1 + \frac{\lambda}{di}\Big) \quad \text{for } n \ge 2.$$

Since the product $\displaystyle\prod_{i=1}^{\infty}(1+\lambda d^{-1} i^{-1})$ diverges, we have $x \notin l^\infty$.

Now let $y(0) \in D$ and $m > 0$. Then $y(t) \in D$ and since $\sum_{i=1}^{\infty} y_i(t) = 1$, we see from (4.10) that

$$y_1'(t) \ge -(2d+m) y_1(t) + m.$$

This implies that

$$\liminf_{t\to\infty} y_1(t) \ge \frac{m}{2d+m}.$$

It means that the semigroup $\{P(t)\}_{t\ge 0}$ generated by Q is not sweeping. According to Theorem 8 the semigroup $\{P(t)\}_{t\ge0}$ is asymptotically stable. Let $y^* = (y_i^*)$ be an invariant density for $\{P(t)\}_{t\ge0}$. If we return to the original system, we obtain (4.8) with $s_i^* = y_i^*/i$. If $r = d$ then the invariant density is of the form $y_i^* = \dfrac{m}{r}\Big(\dfrac{r}{r+m}\Big)^i$, which gives (4.9). ∎

We now present a result on asymptotic stability of stochastic semigroups on l^1 which is based on the Foguel alternative.

Theorem 10. *Let $Q = [q_{ij}]$, $i, j = 0, 1, 2, \ldots,$ be a non-explosive Kolmogorov matrix. We assume that there exist a sequence $v = (v_i)$ of nonnegative numbers and positive constants ε, m, and k such that*

$$\sum_{i=0}^{\infty} q_{ij} v_i \le \begin{cases} m, & \text{for } j \le k, \\ -\varepsilon, & \text{for } j > k. \end{cases} \tag{4.12}$$

Then the stochastic semigroup $\{P(t)\}_{t\ge0}$ related to Q is not sweeping from the set $\{0, 1, \ldots, k\}$. In particular, if the matrix Q satisfies conditions (T) *and* (4.12)*, then the semigroup $\{P(t)\}_{t\ge0}$ is asymptotically stable.*

Proof. Note that if the sequence $v = (v_i)$ satisfies condition (4.12), then the sequence $(v_i + 1)$ also satisfies this condition. Hence, we can assume that $v_i \ge 1$ for all i. Suppose contrary to our claim that the semigroup $\{P(t)\}_{t\ge0}$ is sweeping from the set $\{0, 1, \ldots, k\}$. Let $\{P_r(t)\}_{t\ge0}$ be the semigroup generated by the operator $Q_r = A + rB$, $r \in (0,1)$, with the domain

$$\mathcal{D}(Q_r) = \mathcal{D}_0(Q) = \{x \in l^1 : \sum_{j=0}^{\infty} |q_{jj}||x_j| < \infty\}.$$

Consider the space

$$l_v^1 = \{x \in l^1 : \sum_{i=0}^{\infty} v_i |x_i| < \infty\}$$

with the norm $\|x\| = \sum_{i=0}^{\infty} v_i |x_i|$. Let

$$\mathcal{D}_0^v(Q) = \{x \in l^1 : \sum_{i=0}^{\infty} v_i |q_{ii}||x_i| < \infty\}.$$

We now prove that if $x \in \mathcal{D}_0^v(Q)$, then $P_r(t)x \in \mathcal{D}_0^v(Q)$ for $t > 0$. In order to do it we define the matrix $\tilde{Q} = [\tilde{q}_{ij}]$, where $\tilde{q}_{ij} = q_{ij} v_i v_j^{-1}$ for $i \neq j$ and $\tilde{q}_{jj} = q_{jj} - m$. Observe that $\tilde{Q}$ is a sub-Kolmogorov matrix, because its entries lying outside the main diagonal are nonnegative and condition (4.12) implies

$$\sum_{i=0}^{\infty} \tilde{q}_{ij} = -m + \Big(\sum_{i=0}^{\infty} q_{ij} v_i\Big) v_j^{-1} \le -m + m v_j^{-1} \le 0.$$

Let $\{\tilde{P}_r(t)\}_{t\ge 0}$ be the substochastic semigroup generated by the operator $\tilde{Q}_r$ with the domain

$$\mathcal{D}(\tilde{Q}_r) = \mathcal{D}_0(\tilde{Q}) = \mathcal{D}_0(Q).$$

If $y \in \mathcal{D}_0(\tilde{Q})$ then $\tilde{P}_r(t)y \in \mathcal{D}_0(\tilde{Q})$ for $t \ge 0$. Let $H\colon l_v^1 \to l^1$ be the operator given by the formula $(Hx)_i = v_i x_i$ and let

$$U_r(t) = e^{mt} H^{-1} \tilde{P}_r(t) H.$$

Then $\{U_r(t)\}_{t\ge 0}$ is a strongly continuous semigroup of linear operators on l_v^1. Moreover, we have

$$\lim_{t\to 0^+} t^{-1}(U_r(t)x - x) = mx + H^{-1}\tilde{Q}_r H x = Q_r x$$

for $x \in \mathcal{D}_0^v(Q)$ and the domain of the infinitesimal generator C of the semigroup $\{U_r(t)\}_{t\ge 0}$ is the set $\mathcal{D}_0^v(Q)$. The operator Q_r with the domain $\mathcal{D}(Q_r)$ is the closure of the operator C in the space l^1. Thus the semigroup $\{U_r(t)\}_{t\ge 0}$ is the restriction of the semigroup $\{P_r(t)\}_{t\ge 0}$ to the space l_v^1. If $x \in \mathcal{D}_0^v(Q)$ then $U_r(t)x \in \mathcal{D}_0^v(Q)$, $\int_0^t U_r(\tau)x\, d\tau \in \mathcal{D}_0^v(Q)$, and

$$U_r(t)x = x + Q_r\Big(\int_0^t U_r(\tau)x\, d\tau\Big). \tag{4.13}$$

Let x be a density in l^1 such that $x \in D_0^v(Q)$. Let us fix $t > 0$ and set $y = \int_0^t U_r(\tau)x\,d\tau$. Then $y \in \mathcal{D}_0^v(Q)$ and, in particular, the double series

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} q_{ij}^r v_i y_j \tag{4.14}$$

is absolutely convergent. Here q_{ij}^r are entries of the matrix Q_r. We now estimate the following sum

$$\sum_{i=0}^{\infty} v_i (Q_r y)_i.$$

Since the series (4.14) is absolutely convergent, we have

$$\sum_{i=0}^{\infty} v_i (Q_r y)_i = \sum_{i=0}^{\infty} v_i \Big(\sum_{j=0}^{\infty} q_{ij}^r y_j\Big) = \sum_{j=0}^{\infty}\Big(\sum_{i=0}^{\infty} q_{ij}^r v_i\Big) y_j. \tag{4.15}$$

From (4.12) and (4.15) we obtain

$$\sum_{i=0}^{\infty} v_i (Q_r y)_i \le m \sum_{i=0}^{k} y_i - \varepsilon \sum_{i=k+1}^{\infty} y_i \le (m+\varepsilon)\sum_{i=0}^{k} y_i - \varepsilon \sum_{i=0}^{\infty} y_i. \tag{4.16}$$

From (4.13) and (4.16) it follows that

$$\begin{aligned}\sum_{i=0}^{\infty} v_i (U_r(t)x)_i \le \sum_{i=0}^{\infty} v_i x_i &+ (m+\varepsilon)\sum_{i=0}^{k}\Big(\int_0^t U_r(\tau)x\,d\tau\Big)_i \\ &- \varepsilon \sum_{i=0}^{\infty}\Big(\int_0^t U_r(\tau)x\,d\tau\Big)_i.\end{aligned}$$

If x is a density in l^1 and $x \in D_0^v(Q)$, then $\lim_{r\to 1} P_r(t)x = P(t)x$ and $P_r(t)x = U_r(t)x$. Thus

$$\begin{aligned}\sum_{i=0}^{\infty} v_i (P(t)x)_i &\le \sum_{i=0}^{\infty} v_i x_i + (m+\varepsilon)\sum_{i=0}^{k}\int_0^t (P(\tau)x)_i\,d\tau - \varepsilon \sum_{i=0}^{\infty}\int_0^t (P(\tau)x)_i\,d\tau \\ &= \sum_{i=0}^{\infty} v_i x_i + (m+\varepsilon)\sum_{i=0}^{k}\int_0^t (P(\tau)x)_i\,d\tau - \varepsilon t.\end{aligned}$$

Since we have assumed that the semigroup $\{P(t)\}_{t\ge 0}$ is sweeping from the set $\{0,1,\dots,k\}$, we have $\lim_{t\to\infty}(P(t)x)_i = 0$ for every $i \le k$ and $x \in l^1$. This implies that there exists $t_1 > 0$ such that

$$\int_0^t (P(\tau)x)_i\,d\tau \le \frac{\varepsilon t}{2k(m+\varepsilon)}$$

for $t > t_1$ and $i = 1, \ldots, k$. Thus

$$\sum_{i=0}^{\infty} v_i (P(t)x)_i \le \sum_{i=0}^{\infty} v_i x_i - \frac{\varepsilon t}{2} \tag{4.17}$$

for $t > t_1$. Since $x \in l_v^1$, we have $\sum_{i=0}^{\infty} v_i x_i < \infty$ and from (4.17) it follows that $\sum_{i=0}^{\infty} v_i (P(t)x)_i$ is a negative number for sufficiently large t, which contradicts the assumption that x is a density. Thus the semigroup $\{P(t)\}_{t\ge 0}$ is not sweeping from the set $\{0, 1, \ldots, k\}$. If, additionally, the semigroup $\{P(t)\}_{t\ge 0}$ satisfies condition (T) then the Foguel alternative holds and the semigroup is asymptotically stable. ∎

Example 2. Let us consider again a birth-death process as defined by equation (4.5) with $b_i > 0$ and $d_{i+1} > 0$ for all $i \ge 0$. Let us assume that there exists $\varepsilon > 0$ such that $b_i \le d_i - \varepsilon$ for $i \ge k$. Then the system (4.5) generates a stochastic semigroup. From Example 1 we know that condition (T) holds. Set $v_i = i$ for $i \ge 0$ and note that

$$\sum_{i=0}^{\infty} v_i q_{ij} = (j-1)d_j - j(b_j + d_j) + (j+1)b_j = b_j - d_j \le -\varepsilon$$

for $j \ge k$, which implies condition (4.12). Consequently, the stochastic semigroup generated by the system (4.5) is asymptotically stable.

5. Continuous state space models

5.1. Stochastic gene expression. We now present a simple model of gene expression introduced in the paper by Lipniacki et al. [29] and we recall some analytic results concerning this model obtained in the paper [12]. We consider the process of the regulation of a single gene. The model involves three processes: gene activation/inactivation, mRNA transcription/decay, and protein translation/decay. A gene can be in an active or inactive state and it can be transformed into an active state or into an inactive state, with intensities q_0 and q_1, respectively. In [29] the rates q_0 and q_1 depend only on the amount of the protein but [25] suggests that these rates can also depend on the number of mRNA molecules (see Figure 1 Mechanism III). Therefore, we assume that the rates q_0 and q_1 depend on the number of mRNA molecules $x_1(t)$ and on the number of protein molecules $x_2(t)$. If the gene is active then mRNA transcript molecules are synthesized at the rate R. The protein translation proceeds with the rate $Kx_1(t)$, where K is a constant. In addition, mRNA and protein molecules undergo the process of degradation. The mRNA and protein degradation rate are m and r, respectively. The state of the system is described by the triple $(x_1(t), x_2(t), \gamma(t))$, where $\gamma(t)$

is a random variables with values 1 if the gene is in the active state and 0 in the inactive state. The functions $x_1(t)$ and $x_2(t)$ satisfy the following equations

$$\frac{dx_1}{dt} = R\gamma(t) - mx_1, \tag{5.1}$$

$$\frac{dx_2}{dt} = Kx_1 - rx_2. \tag{5.2}$$

The switching function $\gamma(t)$ is a stochastic process with values in the set $\{0, 1\}$ and this process depends on the functions $x_1(t)$ and $x_2(t)$.

Equations (5.1)–(5.2) generate stochastic trajectories, which can be described as a piecewise deterministic Markov process

$$X(t) = (x_1(t), x_2(t), \gamma(t)) = (\mathrm{x}(t), \gamma(t)),\ t \geq 0,$$

with values in $\mathbb{R}^2 \times \{0, 1\}$ and the following characteristics $(\pi, \varphi, \mathcal{J})$. The semi-flow $\{\pi_t\}_{t\geq 0}$ is defined by $\pi_t(x, i) = (\pi_t^i(x), i)$ for all $(x, i) \in \mathbb{R}^2 \times \{0, 1\}$, where $\pi_t^i x$ denotes the solution $x^i(t)$ of the equation $x'(t) = g(x(t), i)$ with initial condition $x^i(0) = x$ and, for any $x = (x_1, x_2)$, the vector $g(x, i)$ in $\mathbb{R}^2$ has components

$$g_1(x_1, x_2, i) = Ri - mx_1, \quad g_2(x_1, x_2, i) = Kx_1 - rx_2.$$

Our state space will be $E = \mathcal{K} \times \{0, 1\}$, where $\mathcal{K}$ is the rectangle $[0, R/m] \times [0, KR/mr]$ such that $\pi_t^i(\mathcal{K}) \subseteq \mathcal{K}$ for all $t \geq 0$, $i = 0, 1$. The jump distribution $\mathcal{J}((x, i), \cdot)$ is the Dirac measure $\delta_{(x,1-i)}$ and it satisfies $\mathcal{J}((x, i), E) = 1$ for all $(x, i) \in E$. The jump rate function φ is defined by $\varphi(x, i) = q_i(x)$, where $q_0 = q_0(x)$ and $q_1 = q_1(x)$ are given non-negative continuous functions defined on $\mathcal{K}$. We assume that $q_i(x_*^i) > 0$, $i = 0, 1$, where $x_*^0 = (0, 0)$ and $x_*^1 = (R/M, KR/mr)$. Since $\pi_t^i(x) \to x_*^i$ as $t \to \infty$ and q_i is continuous for each i, condition (2.5) holds.

Let $L^1 = L^1(E, \mathcal{B}(E), m)$, where m is the product of the two-dimensional Lebesgue measure and the counting measure on $\{0, 1\}$. The transition operator P on L^1 corresponding to $\mathcal{J}$ is of the form $Pf(x, i) = f(x, 1 - i)$, $(x, i) \in E$, $f \in L^1$. Let $\{P_0(t)\}_{t\geq 0}$ be the stochastic semigroup on L^1 given by

$$P_0(t)f(x, i) = \mathbf{1}_E(\pi_{-t}^i(x), i) f(\pi_{-t}^i(x), i) \det[\frac{d}{dx}\pi_{-t}^i(x)]$$

and let $(A_0, \mathcal{D}(A_0))$ be its generator, which for sufficiently regular f is

$$A_0 f(x, i) = -\frac{\partial}{\partial x_1}(g_1(x, i)f(x, i)) - \frac{\partial}{\partial x_2}(g_2(x, i)f(x, i)).$$

Since φ is bounded, the operator A defined by $Af = A_0 f - \varphi f$, $f \in \mathcal{D}(A_0)$, is the generator of the substochastic semigroup $\{S(t)\}_{t\geq 0}$ satisfying (2.7)–

(2.8) and the minimal substochastic semigroup $\{P(t)\}_{t\geq 0}$ related to $A+P\varphi$ is stochastic. The functions $u_i(t,x_1,x_2) = P(t)f(x_1,x_2,i)$, $i=0,1$, satisfy the following Fokker–Planck system:

$$\begin{aligned}
\frac{\partial u_0}{\partial t} + \frac{\partial}{\partial x_1}(-mx_1u_0) + \frac{\partial}{\partial x_2}((Kx_1 - rx_2)u_0) &= q_1u_1 - q_0u_0,\\
\frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x_1}((R - mx_1)u_1) + \frac{\partial}{\partial x_2}((Kx_1 - rx_2)u_1) &= q_0u_0 - q_1u_1,
\end{aligned}$$

with the initial conditions: $u_0(0,x_1,x_2) = f(x_1,x_2,0)$ and $u_1(0,x_1,x_2) = f(x_1,x_2,1)$. Moreover, if f is the density of $X(0)$ then $P(t)f$ is the density of $X(t) = (\mathrm{x}(t),\gamma(t))$ and

$$\Pr(\mathrm{x}(t)\in B, \gamma(t)=i) = \iint\limits_B u_i(t,x_1,x_2)\,dx_1\,dx_2, \qquad i=0,1.$$

The main result of the paper [12] is the asymptotic stability of the semigroup $\{P(t)\}_{t\geq 0}$. The strategy of the proof is as follows. First, it is shown that the transition function of the related stochastic process has a kernel (integral) part. Then we find a set $\mathcal{E}\subset E$ on which the density of the kernel part of the transition function is positive. Next we show that the set $\mathcal{E}$ is an "attractor". Since the attractor $\mathcal{E}$ is a compact set, from Corollary 3 it follows that the semigroup is asymptotically stable.

5.2. A size structured model. We consider a model of a cellular population which was introduced for the first time probably by Bell and Anderson [10] and was studied and generalized in many papers (see e.g. [15, 17, 33, 37, 46]). In this model a cell is characterized by its size (maturity) which ranges from $x=a$ to $x=1$. Maturity increases with time t according to the equation

$$\frac{dx}{dt} = g(x). \tag{5.3}$$

We assume that an individual with the parameter x has k descendants and that $\mathcal{P}_k(x,B)$ is the probability that any of its descendants has the parameter in the set $B\in\mathcal{B}([a,1])$ at the birth. For example, if x is the age then $\mathcal{P}_k(x,B) = \mathbf{1}_B(0)$. If x is the size and if we consider the case of equal division [15, 17, 19, 26, 30] then

$$\mathcal{P}_2(x,B) = \begin{cases} 1, & \text{if} \quad x/2\in B,\\ 0, & \text{if} \quad x/2\notin B.\end{cases}$$

Models of unequal division have been investigated in many papers [3, 8, 18, 20, 24]. In the case of unequal division the transition function can be given by a stochastic kernel of the form

$$\mathcal{P}_2(x,B) = \int_B k(y,x)\,dy,$$

where k is a nonnegative measurable function such that $\int_a^1 k(y,x)\,dy = 1$ for all x. Our model [9, 37] includes both types of binary fission models.

Let $b_k(x)\Delta t$ be the probability that an individual with parameter x has k descendants in the time interval $[t, t+\Delta t]$. A cell with maturity x has

$$\mathcal{P}(x,B) = \sum_{k=1}^{\infty} k b_k(x)\mathcal{P}_k(x,B) \tag{5.4}$$

descendants with parameters in the set B in a unit of time, and

$$b(x) = \sum_{k=0}^{\infty} k b_k(x) \tag{5.5}$$

is the mean number of its descendants in a unit of time. By $\mu(x)$ we denote the rate of loss, by death or by division, of individuals with parameter x. Since in our model the maximal value of the parameter x is 1, we assume that

$$\int_a^1 \mu(x)\,dx = \infty, \tag{5.6}$$

which means that all cells die or divide before or at reaching the maximal maturity $x = 1$. We also assume that a mother of maturity x cannot have daughters with maturity exceeding $x - h$, that is,

$$\mathcal{P}(x,[a,x-h]) = 1 \qquad \text{for all} \quad x \in [a,1],$$

and that $\inf\{b(x)\colon x \in [1-h,1)\} > 0$, which means that cells with $x \ge 1-h$ can divide. Moreover, we assume that $g\colon [a,1] \to (0,\infty)$ is a continuously differentiable function, the functions $\mu\colon [a,1) \to [0,\infty)$ and $b_k\colon [a,1) \to [0,\infty)$ are continuous, that there is a constant $\bar{C} > 0$ such that

$$\bar{C}^{-1}\mu(x) \le b(x) \le \bar{C}\mu(x) \quad \text{for } x \ge 1-h,$$

and that for every $\overline{x} \in (a,1)$

$$\int_{\overline{x}}^1 \mathcal{P}(x,[a,\overline{x}))\,dx > 0. \tag{5.7}$$

If (5.7) were not satisfied, then for some $\overline{x}$ all daughters of any mother with $x \ge \overline{x}$ would also have maturity greater than $\overline{x}$, and thus we could restrict the set of parameters to the interval $[\overline{x}, 1]$.

Let us start with the Kolmogorov's backward equation

$$\frac{\partial v}{\partial t} = g(x)\frac{\partial v}{\partial x} - \mu(x)v(t,x) + \int_a^1 v(t,y)\mathcal{P}(x,dy).$$

By $u(t,x)$ we denote a function which describes the distribution of the population with respect to x. The number of individuals with the parameter x between x_1 and x_2 at a time t is given by the formula $\int_{x_1}^{x_2} u(t,x)\,dx$. The type of an evolution equation for u depends on the choice of the transition function $\mathcal{P}$. Let

$$\mathcal{P}^r(x,B) = \mathcal{P}(x, B \setminus \{a\}).$$

We assume that for each x the measure $\mathcal{P}^r(x,\cdot)$ is absolutely continuous with respect to the Lebesgue measure for a.e. x. Then, by the Radon–Nikodym theorem, there exists an operator P defined on the space

$$L^1_r[a,1] = \{f \in L^1[a,1] : \int_a^1 |f(x)|\mathcal{P}^r(x,[a,1])\,dx < \infty\},$$

and with values in the space $L^1[a,1]$ such that for each nonnegative function $f \in L^1_r[a,1]$ and each set $B \in \mathcal{B}([a,1])$ we have

$$\int_B Pf(x)\,dx = \int_a^1 \mathcal{P}^r(x,B)f(x)\,dx.$$

If the function $u(t,x)$ is sufficiently regular, then it satisfies the following equation

$$\frac{\partial u}{\partial t} + \frac{\partial (g(x)u)}{\partial x} = -\mu(x)u(t,x) + Pu(t,x), \tag{5.8}$$

with the boundary condition

$$g(a)u(t,a) = \int_a^1 \mathcal{P}(x,\{a\})u(t,x)\,dx \tag{5.9}$$

and the initial condition

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in [a,1]. \tag{5.10}$$

We have the following result on asynchronous exponential growth (AEG) of the population.

Theorem 11. [9] *Assume that one of the following conditions holds:*

(I) *there exists a measurable function* $q\colon [a,1]\times[a,1] \to [0,\infty)$ *such that*

$$\int_a^1\int_a^1 q(y,x)\,dx\,dy > 0 \quad \textit{and} \quad \mathcal{P}(x,B) \geq \int_B q(y,x)\,dy$$

for any $x \in [a,1]$ *and* $B \in \mathcal{B}([a,1])$,

(II) *there exist* $x_0 \in (a,1)$ *and* $\varepsilon > 0$ *such that* $\mathcal{P}(x,\{a\}) > 0$ *for* $x \in (x_0-\varepsilon, x_0+\varepsilon)$,

(III) *there exist* $x_0 \in (a,1)$, $\varepsilon > 0$, *and a* C^1*-function* $r\colon (x_0-\varepsilon, x_0+\varepsilon) \to [a,1]$ *such that* $g(r(x_0)) \neq r'(x_0)g(x_0)$ *and*

$$\mathcal{P}(x,\{r(x)\}) \geq \varepsilon \quad \text{for } x \in (x_0-\varepsilon, x_0+\varepsilon).$$

Then there exist a continuous function $f_*\colon [a,1] \to [0,\infty)$, $f_*(x) > 0$ *for* $x \in (a,1)$, *a continuous function* $w\colon [a,1] \to (0,\infty)$, *and* $\lambda \in \mathbb{R}$ *such that* $e^{-\lambda t}u(t,\cdot)$ *converges in* $L^1[a,1]$ *to* $f_* \int_a^1 u(0,x)w(x)\,dx$.

Sketch of the proof. Equation (5.8) can be written as an evolution equation $u'(t) = Au$. First, it is shown in [9] that A is an infinitesimal generator of a continuous semigroup $\{T(t)\}_{t\geq 0}$ of linear operators on $L^1[a,1]$. Then it is proved that there exist $\lambda \in \mathbb{R}$, and positive continuous functions v and w such that $Av = \lambda v$ and $A^*w = \lambda w$. From this it follows that the semigroup $\{P(t)\}_{t\geq 0}$ given by $P(t) = e^{-\lambda t}T(t)$ is a stochastic semigroup on the space $L^1([a,1],\mathcal{B}([a,1]),m)$, where m is a measure given by $m(B) = \int_B w(x)\,dx$. We can find $c > 0$ such that the function $f_* = cv$ is an invariant density with respect to $\{P(t)\}_{t\geq 0}$. If one of conditions I, II, III holds, then $\{P(t)\}_{t\geq 0}$ is partially integral. Finally, from Theorem 7 we conclude that this semigroup is asymptotically stable. Since the Lebesgue measure and the measure m are equivalent, we obtain that $e^{-\lambda t}u(t,\cdot)$ converges to $f_*\Phi(u(0,\cdot))$ in $L^1[a,1]$, where $\Phi(g) := \int_a^1 g(x)w(x)\,dx$. ∎

Remark 6. It should be noted that if $\mu(x) = b(x)$ for all $x \in [a,1)$, where b is as in (5.5) and μ satisfies (5.6), then the model considered in this section is a particular example of a piecewise deterministic Markov process with values in $E = [a,1)$ and with boundaries as described in Remark 3. To see this observe that equation (5.11) defines a semi-flow $\{\pi_t\}_{t\geq 0}$ whose trajectories can leave the interval E only through the point 1. For every $x \in E$ we have $t_*(x) = \inf\{t > 0 : \pi_t x = 1\}$ and

$$\int_0^{t_*(x)} b(\pi_s x)ds = \int_x^{\pi_{t_*(x)}x} \frac{b(y)}{g(y)}dy = \int_x^1 \frac{b(y)}{g(y)}dy.$$

Thus the integral is infinite by (5.6), which allows us to construct a PDMP with characteristics $(\pi, b, \mathcal{J})$, where $\mathcal{J}$ is such that $b(x)\mathcal{J}(x,B) = \mathcal{P}(x,B)$ and $\mathcal{P}$ is as in (5.4).

5.3. Piecewise deterministic growth/decay processes. In this section we describe two simple examples of one-dimensional piecewise deterministic Markov processes on $E = (0,\infty)$ derived from the mathematical study of

populations of cells. In their examination of the stable nature of the cell cycle, Lasota and Mackey [27] proposed a model for the cell division cycle based on three hypotheses:
1) there is an 'activator' x, which is necessary for mitosis, produced by cells according to

$$\frac{dx}{dt} = g(x), \tag{5.11}$$

2) a cell containing the amount x of the activator divides with probability $\varphi(x)\Delta t + o(\Delta t)$ during the time interval $[t, t+\Delta t]$, and
3) at division, each daughter cell receives exactly half of the level of the activator.

In [27] it was shown, under some regularity conditions, that the density f_n of the amount X_n of the activator at the moment of division in the n-th generation of cells satisfies $f_{n+1} = Kf_n$ with K being an integral operator

$$Kf(x) = 2q(2x)\int_0^{2x} \exp\Big\{-\int_y^{2x} q(z)dz\Big\} f(y)\,dy,$$

where $q(z) = \varphi(z)/g(z)$, and that K is asymptotically stable: there is a density $f_* \in D$ such that $Kf_* = f_*$ and

$$\lim_{n\to\infty} \|K^n f - f_*\| = 0 \quad \text{for all } f \in D. \tag{5.12}$$

Note that condition (5.12) implies that K is mean ergodic.

We assume that g and φ are nonnegative continuous functions, $g(x) > 0$ for $x > 0$, and that, for some $\bar{x} > 0$,

$$\int_{\bar{x}}^{\infty} \frac{1}{g(y)} dy = \int_{\bar{x}}^{\infty} \frac{\varphi(y)}{g(y)} dy = \infty.$$

The amount of the activator can be described as a piecewise deterministic Markov process $\{X(t)\}_{t\ge 0}$ with values in $E = (0, \infty)$ and the following characteristic $(\pi, \varphi, \mathcal{J})$. Equation (5.11) defines a semi-flow $\{\pi_t\}_{t\ge 0}$ such that $\pi_t(E) \subseteq E$ for all $t \ge 0$. Since $\int_0^t \varphi(\pi_s x) ds = \int_x^{\pi_t x} q(y) dy$ and $\pi_t x \to \infty$ for all x, condition (2.5) holds. The jump distribution $\mathcal{J}(x, \cdot)$ is $\delta_{x/2}$ for all x and the corresponding transition operator P on L^1 is $Pf(x) = 2f(2x)$ for $f \in L^1$. Allowing P to be any stochastic operator on L^1 and letting $\mathcal{J}(x, B) = P^* \mathbf{1}_B(x)$, we obtain a general piecewise deterministic growth process as studied in [31] through the evolution equation of the form

$$\frac{\partial u(t,x)}{\partial t} = -\frac{\partial}{\partial x}(g(x)u(t,x)) - \varphi(x)u(t,x) + P(\varphi u(t,\cdot))(x). \tag{5.13}$$

Conditions (2.7)–(2.8) hold with the operator A of the form

$$Af(x) = -\frac{d}{dx}(g(x)f(x)) - \varphi(x)f(x),$$

and the operator K in (2.9) is given by $K(f) = P(\varphi R_0)f$, where

$$R_0 f(x) = \frac{1}{g(x)} \int_0^x \exp\Big\{-\int_y^x q(z)dz\Big\} f(y)\, dy.$$

It might happen that the minimal semigroup $\{P(t)\}_{t\geq 0}$ related to $A + P\varphi$ is not stochastic [31, Example 4]. If K satisfies (5.12) with $f_* = Kf_*$, then the semigroup $\{P(t)\}_{t\geq 0}$ is stochastic and $P(t)R_0 f_* = R_0 f_*$ for all $t > 0$, if $R_0 f_* \in L^1$, which implies that $\{P(t)\}_{t\geq 0}$ has an invariant density. The asymptotic behavior of the semigroup $\{P(t)\}_{t\geq 0}$ can be studied with the help of the results from Section 3. The semigroup $\{P(t)\}_{t\geq 0}$ can have up to one invariant density and it is partially integral, if condition (I) or (III) of Theorem 11 holds with the interval $[a, 1]$ replaced by E and $\mathcal{P}$ by $\mathcal{P}(x, B) = \varphi(x)\mathcal{J}(x, B)$, $x \in E$.

The second example comes from the regulation of genes. Friedman et al. [16] have considered stochastic aspects of gene expression following from bursts of protein production, and derived an equation of the form (5.13) for the concentration of the protein molecules in a given cell, where $g(x) = -\gamma x$ with $\gamma > 0$ and

$$Pf(x) = \int_0^x f(x-y)\frac{1}{b}e^{-y/b}dy$$

with $b > 0$ the mean number of molecules per burst. This bursting type production was studied in [31, 32] in the context of piecewise deterministic Markov processes, where this time, instead of the growth, protein molecules undergo the process of degradation according to equation (5.11) with $g < 0$, and the degradation is randomly interrupted with intensity φ when a random amount of protein molecules is produced, independently of the current number of proteins, so that the operator P is of the form

$$Pf(x) = \int_0^x f(x-y)h(y)\, dy,$$

where h is a probability density on $(0, \infty)$.

5.4. Coagulation-fragmentation phytoplankton models. Mathematical modelling of plankton behaviour is a complex issue involving various mathematical tools including advection-diffusion-reaction equations, fragmentation-coagulation processes, point processes, superprocesses, and stochastic partial differential equations. A review of mathematical models of plankton dynamics can be found in [41]. Phytoplankton cells tend to form aggregates in which they live together like colonial organisms. Since the size of aggregates is important in the study of fish recruitment, the change of the size distribution of aggregates is a very interesting problem both from the biological and mathematical point of view.

We now describe a model of phytoplankton dynamics introduced by Arino and Rudnicki [4], which takes into account growth and death of aggregates as well as coagulation-fragmentation processes. Although coagulation is a complex physical process including turbulent shear, particle settling and Brownian motion, it seems that the main role is played by TEP (Transparent Exopolymer Particles). TEP are by-products of the growth of phytoplankton and their stickiness cause that cells remain together upon contact. On the other hand the low level of concentration of TEP leads to fragmentation of phytoplankton aggregates. In this model aggregates are structured by size, i.e. their mass, which is proportional to the number of cells. The division or death of individual cells change the size of aggregates. Apart from growth due to the division of cells within an aggregate, two main mechanisms are at work: splitting of a given aggregate into parts, which is called a fragmentation process, and coagulation (aggregation), by which two distinct aggregates join together to form a single one. In our model all factors mentioned above are hidden in the probability of aggregation, which makes mathematics much simpler. It is assumed that coagulation is a binary process and two distinct aggregates join together with some probability, which depends only on the size of aggregates.

In the model the size of an aggregate is denoted by x. An aggregate grows with rate $b(x)$ but it can die, for example, by sinking to a seabed or whatever cause, with mortality rate $d(x)$. It can break with rate $p(x)$ and the size y of its descendants is given by the conditional density $K(y,x)$. We assume that the ability to glue to another aggregate depends on the size and is given by the function $g(x)$. Let the function $u(t,x)$ be the density of the distribution of x, i.e.

$$\int_{x_1}^{x_2} u(t,x)\,dx$$

is the number of cells of size $x_1 < x < x_2$ at time t. Taking the sums of the variations due to growth and mortality, fragmentation and coagulation, one can check that u satisfies the equation

$$u'(t) = -a(x)u(t) + A_1u(t) + A_2u(t) + A_3u(t), \tag{5.14}$$

where

$$a(x) = d(x) + p(x) + g(x),$$

$$(A_1 f)(x) = -\frac{d}{dx}(b(x)f(x)),$$

$$(A_2 f)(x) = 2\int_x^\infty K(x,y)p(y)f(y)\,dy,$$

$$(A_3 f)(x) = \frac{\int_0^x f(x-y)f(y)(x-y)yg(x-y)g(y)\,dy}{x\int_0^\infty zg(z)f(z)\,dz}.$$

In equation (5.14) we have terms responsible for growth $A_1 u$, death $-d(x)u(t)$, fragmentation $-p(x)u(t) + A_2 u(t)$, and coagulation $-g(x)u(t) + A_3 u(t)$.

We consider the solutions of (5.14) as functions from $\mathbb{R}_+$ to the space $L^1(\mathbb{R}_+, m) = L^1(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), m)$, where the measure m is given by $dm = x\,dx$. In order to formulate some properties of solutions of (5.14) we assume that the functions p, d, g and K are sufficiently regular, otherwise some unwanted phenomena can occur, see e.g. Banasiak [6]. A strongly continuous semigroup of non-linear operators $\{P(t)\}_{t\ge 0}$ on the space which contains the set of densities D is called a non-linear stochastic semigroup if $P(t)(D) \subset D$ for $t \ge 0$.

Theorem 12. [4] *For each $u_0 \in L^1(\mathbb{R}_+, m)$ there exists a unique solution $u\colon [0,\infty) \to L^1(\mathbb{R}_+, m)$ of equation (5.14) such that $u(0) = u_0$. Let $P(t)u_0(x) = u(t,x)$ for $u_0 \in L^1(\mathbb{R}_+, m)$. Then $\{P(t)\}_{t\ge 0}$ is a strongly continuous semigroup of positive bounded operators on $L^1(\mathbb{R}^+, m)$. If $g(x) = xd(x)$ then $\{P(t)\}_{t\ge 0}$ is a non-linear stochastic semigroup.*

It is rather difficult to study the behaviour of solutions of equation (5.14) when time goes to infinity. Partial results can be obtained by studying the behaviour of moments $M_n(t)$ of solutions, i.e. $M_n(t) = \int_0^\infty x^n u(t,x)\,dx$, $n = 0, 1, 2, \ldots$. Ordinary differential equations for $M_n(t)$ have been derived in [4] and they can be used to give sufficient conditions for the existence of large aggregates, which is a property important from the biological point of view.

The model can be extended in two ways. A first way would be to include a space distribution of aggregates. Such a generalization was done in [39] and [40]. Alternatively, one can assume that during the division of cells some of them fall off the aggregates and enter the system as new aggregates, leaving the size of the original aggregate unchanged by cell division [1, 2, 8].

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