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ORTHOGONAL STABILITY OF THE CAUCHY FUNCTIONAL EQUATION ON BALLS IN NORMED LINEAR SPACES

Abstract. We study the stability of some functional equations postulated for orthogonal vectors in a ball centered at the origin. The maps considered are defined on a finite-dimensional normed linear space with Birkhoff-James orthogonality and take their values in a real sequentially complete linear topological space. The main results establish the stability of the corresponding conditional Cauchy functional equation on a half-ball and in uniformly convex spaces on a whole ball. The methods used in the first part of the paper are similar to those from [10]. Since, however, now in a general structure, some additional problems arise, we need several new tools.

1. Introduction

In the present paper we deal with the stability of functional equations, in particular the Cauchy functional equation, postulated for pairs of orthogonal vectors from a ball centered at the origin in a normed linear space with Birkhoff–James orthogonality. R. Ger and J. Sikorska in [2] considered the stability of the Cauchy functional equation postulated for orthogonal vectors only and defined on the whole space. F. Skof in [11], [12] and F. Skof & S. Terracini in [13] dealt with the stability of the Cauchy and quadratic equations on an interval. Z. Kominek in [6] studied the stability of the Cauchy equation on an N-dimensional cube in the space $\mathbb{R}^N$. Unifying these investigations, the author studied in [10] the stability on a ball centered at the origin in a finite-dimensional inner product space.

As in earlier papers, because of the methods used in our proofs, we restrict ourselves to orthogonality in a finite-dimensional space.

Let X be a normed linear space X of dimension ≥ 2. The orthogonality relation in X is defined as follows.

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DEFINITION 1.1. A vector $x \in X$ is *orthogonal in the sense of Birkhoff–James* to a vector $y \in X$ ($x \perp y$) if and only if

$$\|x + \mu y\| \geq \|x\| \quad \text{for all} \quad \mu \in \mathbb{R}.$$

Many properties of the Birkhoff–James orthogonality can be found e.g. in [1], [3], [4], [5], [7], [9], [14], [15]. For the convenience of the reader we give here those that are used in the present paper more often:

(i) if $x, y \in X \setminus \{0\}$, $x \perp y$, then x and y are linearly independent;
(ii) if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$ (*homogeneity* of the relation);
(iii) if P is a two-dimensional subspace of X, $x \in P$, $\lambda > 0$, then there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

Let B denote a ball in X. Later on we shall also use the following

DEFINITION 1.2. (i) We say that a function $f : B \to Y$ is *additive (on the ball B)* if and only if for all $x, y \in B$ such that $x + y \in B$ we have $f(x + y) = f(x) + f(y)$. (ii) A function $f : B \to Y$ is *quadratic (on the ball B)* if and only if for all $x, y \in B$ such that $x + y$, $x - y \in B$ we have $f(x+y) + f(x-y) = 2f(x) + 2f(y)$. (iii) We say that a function $f : B \to Y$ is *orthogonally additive (on the ball B)* if and only if for all $x, y \in B$ such that $x + y \in B$ and $x \perp y$ we have $f(x + y) = f(x) + f(y)$.

Throughout the paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ denote the sets of all positive integers, all nonnegative integers and all real numbers, respectively, and $\perp$ stands for the orthogonality relation in the sense of Birkhoff-James.

2. Auxiliary results

On account of the results of J. Rätz ([9]) and Gy. Szabó ([15]) we have the following theorems.

THEOREM 2.1 ([9, p.41, Corollary 7]). *Let $(X, \|\cdot\|)$, $\dim X \geq 2$, be a real normed linear space with Birkhoff–James orthogonality, and let $(Y, +)$ be a uniquely 2-divisible abelian group. Then every orthogonally additive mapping $f : X \to Y$ has the form $f = g + h$ with g quadratic and h additive.*

THEOREM 2.2 ([9, p.47, Theorem 16], [15, p.95, Theorem 1.8]). *Let $(X, \|\cdot\|)$, $\dim X \geq 2$, be a real normed linear space with a norm that does not come from an inner product and with Birkhoff–James orthogonality, and let $(Y, +)$ be an abelian group. Then each orthogonally additive mapping $f : X \to Y$ is additive.*

REMARK 2.3. In general, the converse of Theorem 2.1 is not true. In particular, a quadratic function need not to be orthogonally additive.

REMARK 2.4. Under the assumptions of Theorem 2.2, every even orthogonally additive mapping is the zero function.

In what follows, we show that in every finite-dimensional normed linear space there exists a basis that consists of vectors orthogonal in the sense of Birkhoff–James. We start with quoting a theorem by R.C. James from [5].

THEOREM 2.5 ([5, p.268, Theorem 2.2]). *Any element of a normed linear space is orthogonal to some hyperplane through the origin.*

REMARK 2.6. For a given hyperplane H may not exist an element x with $x \perp H$. This problem of finding an element $x \in X$ with $x \perp H$ is equivalent to finding an element x with $|f(x)| = \|f\|\,\|x\|$ for a given functional f.

Using the above theorems we construct an orthogonal basis in X. Let $\{\alpha_{1,1}, \dots, \alpha_{1,N}\}$ be an arbitrary basis in X, i.e.

$$X = \operatorname{lin}\{\alpha_{1,1}, \dots, \alpha_{1,N}\} =: H_1.$$

By induction we construct a basis $\{u_1, \dots, u_N\}$ consisting of orthogonal vectors. Let $u_1 := \alpha_{1,1}$. From Theorem 2.5 there exists a hyperplane H_2 through the origin ($\dim H_2 = N - 1$) such that

$$u_1 \perp H_2, \quad H_2 = \operatorname{lin}\{\alpha_{2,2}, \dots, \alpha_{2,N}\}$$

for some vectors $\alpha_{2,2}, \dots, \alpha_{2,N} \in X$. Then

$$u_1 \perp \alpha_{2,i} \ \text{ for all } \ i \in \{2, \dots, N\}.$$

Define $u_2 := \alpha_{2,2}$. There exists a hyperplane $H_3 \subset H_2$ through the origin ($\dim H_3 = N - 2$) such that

$$u_2 \perp H_3, \quad H_3 = \operatorname{lin}\{\alpha_{3,3}, \dots, \alpha_{3,N}\}$$

for some $\alpha_{3,3}, \dots, \alpha_{3,N} \in X$. We have

$$u_2 \perp \alpha_{3,i} \ \text{ for all } \ i \in \{3, \dots, N\}$$

and

$$u_1 \perp \alpha_{3,i} \ \text{ for all } \ i \in \{3, \dots, N\} \ \text{(since } H_3 \subset H_2\text{)}.$$

Define $u_3 := \alpha_{3,3}$. By the induction step there exists a hyperplane H_k through the origin ($\dim H_k = N - k + 1$) such that

$$u_{k-1} \perp H_k, \quad H_k = \operatorname{lin}\{\alpha_{k,k}, \dots, \alpha_{k,N}\}$$

for some vectors $\alpha_{k,k}, \dots, \alpha_{k,N} \in X$. Then

$$u_{k-1} \perp \alpha_{k,i} \ \text{ for all } \ i \in \{k, \dots, N\}.$$

Moreover,

$$u_j \perp \alpha_{k,i} \ \text{ for all } \ j \in \{1, \dots, k-2\} \ \text{ and for all } \ i \in \{k, \dots, N\}.$$

Define $u_k := \alpha_{k,k}$. In such a way we construct vectors till $u_{N-1} := \alpha_{N-1,N-1}$. We have $\dim \mathrm{lin}\{u_{N-1}, \alpha_{N-1,N}\} = 2$. There exists a vector u_N such that $u_{N-1} \perp u_N$ (cf. [9, p.37, Lemma 1]). In such a way we have defined a sequence of vectors $u_1, \ldots, u_N$ with the properties:

$$u_i \perp u_j \ \text{ for } \ i, j \in \{1, \ldots, N\},\ i < j.$$

Any two vectors orthogonal in the sense of Birkhoff–James are linearly independent, so $X = \mathrm{lin}\{u_1, \ldots, u_N\}$, and we have the desired basis.

3. Stability on a half-ball

Assume that the domain $(X, \|\cdot\|)$ is a real normed linear space, $\dim X = N \geq 2$, the target space Y is a real uniformly complete linear topological space, and V is a nonempty, bounded subset of Y which is convex and symmetric with respect to zero. Let, moreover, B stand for a ball in X centered at the origin. Without loss of generality we may assume that it is the unit ball.

The aim of this section is to prove the following

THEOREM 3.1. *If $f : B \to Y$ fulfils the condition*

$$(x, y, x + y \in B,\ x \perp y) \implies f(x + y) - f(x) - f(y) \in V, \tag{1}$$

then there exist an additive function $a : X \to Y$, a quadratic function $q : X \to Y$ and a constant $k = k(N)$ such that

$$f(x) - a(x) - q(x) \in k \operatorname{seq\,cl} V\,,\ \ x \in \frac{1}{2}B.$$

REMARK 3.2. The above result is slightly weaker than expected one. From Theorem 2.1 every orthogonally additive function is a sum of an additive and a quadratic one. But the converse, in general, is not true. This result, as well as the lack of uniqueness, is caused by a restriction from the whole space to a ball. Some details will be presented after Lemma 3.7.

The proof of this theorem is a consequence of several lemmas.

LEMMA 3.3. *Let $f : B \to Y$ be an odd mapping satisfying* (1). *Then for every $x \in \frac{1}{2}B$ and $\lambda \in \mathbb{R}$ such that $\lambda x,\ (1 + \lambda x) \in \frac{1}{2}B$ we have*

$$f(x + \lambda x) - f(x) - f(\lambda x) \in 3V. \tag{2}$$

Proof. Take $x \in \frac{1}{2}B$ and $\lambda \in \mathbb{R}$ such that $\lambda x,\ (1+\lambda)x \in \frac{1}{2}B$. The following four cases will be considered:

(i) $\lambda > 0$, (ii) $\lambda = 0$, (iii) $-1 < \lambda < 0$, (iv) $\lambda \leq -1$.

(i) There exists $y \in X$ such that $x \perp y$ and $x+y \perp \lambda x - y$.We check that $y,\ x+y,\ \lambda x - y \in B$. The above orthogonality relations with $\mu = 1$ give $\|x\| \leq \|x+y\|$ and $\|x+y\| \leq \|(1+\lambda)x\|$. Moreover, $\|y\| \leq \|x+y\| + \|-x\| \leq (2+\lambda)\|x\|$

and $\|\lambda x - y\| \le \|\lambda x\| + \|-y\| \le (2+2\lambda)\|x\|$. Hence, if $\|x\| < \frac{1}{2(1+\lambda)}$ then $\|x+y\| < \frac{1}{2}$, $\|y\| < \frac{2+\lambda}{2(1+\lambda)} < 1$, $\|\lambda x - y\| < \frac{2+2\lambda}{2(1+\lambda)} = 1$, so $x+y$, y, $\lambda x - y \in B$ and we can apply (1). Namely:

$$\begin{aligned} &f(x+\lambda x) - f(x+y) - f(\lambda x - y) \in V, \\ &f(x+y) - f(x) - f(y) \in V, \\ &f(\lambda x - y) - f(\lambda x) + f(y) \in V, \end{aligned}$$

whence (2) follows immediately.

(ii) If $\lambda = 0$ then (2) is surely satisfied since $f(0) \in V \subset 3V$.

(iii) Using (i) and the oddness of f we can write:

$$\begin{aligned} f(x+\lambda x) - f(x) - f(\lambda x) &= f(x+\lambda x) + f(-\lambda x) - f(x) \\ &= f(x+\lambda x) + f\left(\left(-\frac{\lambda}{1+\lambda}\right)(1+\lambda)x\right) \\ &\quad - f\left((1+\lambda)x + \left(-\frac{\lambda}{1+\lambda}\right)(1+\lambda)x\right) \in 3V. \end{aligned}$$

(iv) Using again (i) and the oddness of f we get

$$f(x+\lambda x) - f(x) - f(\lambda x) = f((-1-\lambda)(-x)) + f(-x) - f((-\lambda)(-x)) \in 3V.$$

This completes the proof.

LEMMA 3.4. *Let $f : B \to Y$ be an odd mapping satisfying* (1). *Then there exist an additive function $a : X \to Y$ and a constant $k_1 = k_1(N)$ such that for any $x \in \frac{1}{2}B$ one has*

$$a(x) - f(x) \in k_1 \operatorname{seq} \operatorname{cl} V.$$

Proof. Let $u_1, \dots, u_N$ be an orthogonal basis in X, $\|u_i\| = \frac{1}{2}$, $i \in \{1, \dots, N\}$. Each $x \in X$ we can write as $x = \sum_{i=1}^{N} \alpha_i u_i$ for some (uniquely determined) $\alpha_1, \dots, \alpha_N \in \mathbb{R}$. Further, we decompose each α_i into its integral part n_i and its mantissa m_i, so that $\alpha_i = n_i + m_i$, $i \in \{1, \dots, N\}$. Then $x = \sum_{i=1}^{N} (n_i u_i + m_i u_i)$.

Define a function $F : X \to Y$ by

$$F(x) := \sum_{i=1}^{N} \Big(n_i f(u_i) + f(m_i u_i) \Big).$$

Fix $x \in \frac{1}{2^N} B$. Since $u_i \perp \operatorname{lin}\{u_{i+1}, \dots, u_N\}$, then, by induction for every $i \in \{1, \dots, N\}$, we have

$$\begin{aligned}\|\alpha_i u_i\| &\le \|\alpha_i u_i + \alpha_{i+1} u_{i+1} + \ldots + \alpha_N u_N\| \\ &\le \|\alpha_1 u_1 + \ldots + \alpha_N u_N\| + \sum_{j=1}^{i-1} \|-\alpha_j u_j\| \le 2^{i-1}\|x\|.\end{aligned}$$

Hence $\|\alpha_i u_i\| < \frac{1}{2^{N-i+1}}$, $i \in \{1, \ldots, N\}$, so $\alpha_i u_i, \sum_{j=i}^{N} \alpha_j u_j \in \frac{1}{2^{N-i+1}} B \subset \frac{1}{2} B$ for $i \in \{1, \ldots, N\}$.

Observe that for any $x \in \frac{1}{2^N} B$ we have

$$\begin{aligned}F(x) - f(x) = &\sum_{i=1}^{N} \Big(n_i f(u_i) + f(m_i u_i) - f(n_i u_i + m_i u_i)\Big) \\ &+ \left(\sum_{i=1}^{N} f(\alpha_i u_i) - f\Big(\sum_{i=1}^{N} \alpha_i u_i\Big)\right).\end{aligned}$$

It is easy to show that

$$\sum_{i=1}^{N} f(\alpha_i u_i) - f\Big(\sum_{i=1}^{N} \alpha_i u_i\Big) \in (N-1)V$$

(it follows from the fact that for every $i \in \{1, \ldots, N-1\}$ we have $\alpha_i u_i \perp \sum_{j=i+1}^{N} \alpha_j u_j$, so that $f(\sum_{j=i}^{N} \alpha_j u_j) - f(\alpha_i u_i) - f(\sum_{j=i+1}^{N} \alpha_j u_j) \in V$).

Denote: $A_i := n_i f(u_i) + f(m_i u_i) - f(n_i u_i + m_i u_i), \ i \in \{1, \ldots, N\}$.

Observe that since we have $\|\alpha_i u_i\| < \frac{1}{2}$ and $\|u_i\| = \frac{1}{2}$, we get $|\alpha_i| < 1$ for all $i \in \{1, \ldots, N\}$. Considering two cases: $0 \le \alpha_i < 1$ and $-1 < \alpha_i < 0$, we get $A_i = 0$ and $A_i \in 3V$, respectively. As a consequence we obtain

$$F(x) - f(x) \in (4N-1)\, V, \quad x \in \frac{1}{2^N} B. \tag{3}$$

We show now that for any $x, y \in X$ we have

$$F(x+y) - F(x) - F(y) \in 6N\, V.$$

For this end, take $x, y \in X$ and write them in the form

$$x = \sum_{i=1}^{N} \alpha_i u_i = \sum_{i=1}^{N} (n_i u_i + m_i u_i),$$

$$y = \sum_{i=1}^{N} \beta_i u_i = \sum_{i=1}^{N} (k_i u_i + l_i u_i),$$

for some (uniquely determined) reals α_i, β_i, $i \in \{1, \ldots, N\}$; n_i, k_i stand here for the integral parts, and m_i, l_i – for mantissas of α_i and β_i ($i \in \{1, \ldots, N\}$), respectively.

Let F_i, $i \in \{1, \ldots, N\}$ stand for the i-th summand in the definition of function F.

Fix $i \in \{1, \ldots, N\}$. Assume first that $m_i + l_i < 1$. Then, by Lemma 3.3, we have

$$F_i(x+y) - F_i(x) - F_i(y) = f((m_i + l_i)u_i) - f(m_i u_i) - f(l_i u_i) \in 3\,V.$$

Let now $1 \leq m_i + l_i < 2$. Since $(m_i - 1)u_i \in B$, we have

$$\begin{aligned} &F_i(x+y) - F_i(x) - F_i(y) \\ &= f(u_i) + f\Big((m_i + l_i - 1)u_i\Big) - f(m_i u_i) - f(l_i u_i) \\ &= \Big(f(u_i) + f\Big((m_i - 1)u_i\Big) - f(m_i u_i)\Big) \\ &\quad + \Big(f\Big((m_i + l_i - 1)u_i\Big) - f\Big((m_i - 1)u_i\Big) - f(l_i u_i)\Big) \in 6\,V. \end{aligned}$$

Hence,

$$F(x+y) - F(x) - F(y) = \sum_{i=1}^{N} \Big(F_i(x+y) - F_i(x) - F_i(y)\Big) \in 6N\,V.$$

According to [8], there exists an additive function $a : X \to Y$ such that for all $x \in X$

$$a(x) - F(x) \in 6N \operatorname{seq} \operatorname{cl} V. \tag{4}$$

Moreover, from Lemma 3.3, we have

$$2^n f\Big(\frac{1}{2^n} x\Big) - f(x) \in 3(2^n - 1)\,V, \quad x \in \frac{1}{2}B, \quad n \in \mathbb{N}. \tag{5}$$

Using now (3), (4) and (5) we get for any $x \in \frac{1}{2}B$

$$\begin{aligned} a(x) - f(x) &= 2^{N-1}\Big(a\Big(\frac{1}{2^{N-1}}x\Big) - F\Big(\frac{1}{2^{N-1}}x\Big)\Big) \\ &\quad + 2^{N-1}\Big(F\Big(\frac{1}{2^{N-1}}x\Big) - f\Big(\frac{1}{2^{N-1}}x\Big)\Big) \\ &\quad + 2^{N-1} f\Big(\frac{1}{2^{N-1}}x\Big) - f(x) \in \Big(2^N(5N+1) - 3\Big) \operatorname{seq} \operatorname{cl} V, \end{aligned}$$

so we have got the assertion of the lemma with $k_1 = 2^N(5N+1) - 3$.

LEMMA 3.5. *Let $f : B \to Y$ be an even mapping satisfying* (1). *Then for all vectors $x, y \in \frac{1}{2}B$ such that $x + y, x - y \in \frac{1}{2}B$ we have*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) \in k_2\,V \tag{6}$$

for some positive constant k_2.

Proof. We show first that

$$f(x) - 4^n f\left(\frac{1}{2^n}x\right) \in \frac{7}{3}(4^n - 1)\,V, \quad x \in \frac{1}{2}B, \quad n \in \mathbb{N}. \tag{7}$$

Take $x \in \frac{1}{2}B$. There exists $y \in X$ such that $x \perp y$ and $x+y \perp x-y$. As in Lemma 3.3 we get $\frac{x+y}{2}, \frac{y}{2}, \frac{x-y}{2} \in B$, so

$$f(x) - 4f\left(\frac{x}{2}\right) = \left(f(x) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right)\right)$$
$$+ \left(f\left(\frac{x+y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{y}{2}\right)\right) + \left(f\left(\frac{x-y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(-\frac{y}{2}\right)\right)$$
$$- 2\left(f\left(\frac{x}{2}\right) - f\left(\frac{x+y}{4}\right) - f\left(\frac{x-y}{4}\right)\right)$$
$$+ 2\left(f\left(\frac{y}{2}\right) - f\left(\frac{x+y}{4}\right) - f\left(\frac{y-x}{4}\right)\right) \in 7V,$$

and whence we have (7) with $n = 1$. The rest follows from an easy induction.

Fix now $x \in \frac{1}{2}B$ and $\lambda > 0$ such that λx, $(1+\lambda)x$, $(1-\lambda)x \in \frac{1}{2}B$. There exists $y \in X$ such that $x \perp y$ and $x+y \perp \lambda x - y$. We have $x+y$, $\lambda x - y$, $y \in B$. Moreover, $\|(1-\lambda)x + 2y\| \le \|x+y\| + \|\lambda x - y\| \le 3(1+\lambda)\|x\|$. So, $2y$, $(1-\lambda)x + 2y \in 2B$. By (1) and the evenness of f, we get

$$f(x+\lambda x) + f(x-\lambda x) - 2f(x) - 2f(\lambda x)$$
$$= \left(f(x+\lambda x) - 16f\left(\frac{x+\lambda x}{4}\right)\right) + \left(f(x-\lambda x) - 16f\left(\frac{x-\lambda x}{4}\right)\right)$$
$$- 2\left(f(x) - 16f\left(\frac{x}{4}\right)\right) - 2\left(f(\lambda x) - 16f\left(\frac{\lambda x}{4}\right)\right)$$
$$+ 16\left[\left(f\left(\frac{x+\lambda x}{4}\right) - f\left(\frac{x+y}{4}\right) - f\left(\frac{\lambda x - y}{4}\right)\right)\right.$$
$$+ 2\left(f\left(\frac{x+y}{4}\right) - f\left(\frac{x}{4}\right) - f\left(\frac{y}{4}\right)\right) + 2\left(f\left(\frac{\lambda x - y}{4}\right) - f\left(\frac{\lambda x}{4}\right) - f\left(\frac{-y}{4}\right)\right)$$
$$+ \left(f\left(\frac{x-\lambda x}{4}\right) + f\left(\frac{y}{2}\right) - f\left(\frac{x-\lambda x+2y}{4}\right)\right)$$
$$\left.+\left(f\left(\frac{x-\lambda x+2y}{4}\right) - f\left(\frac{x+y}{4}\right) - f\left(\frac{-\lambda x+y}{4}\right)\right) + 4f\left(\frac{y}{4}\right) - f\left(\frac{y}{2}\right)\right] \in 434V.$$

Actually, from the above, for any $x \in X$ and $\alpha, \beta \in \mathbb{R}$ such that αx, βx, $(\alpha+\beta)x$, $(\alpha-\beta)x \in \frac{1}{2}B$ we have

$$f(\alpha x + \beta x) + f(\alpha x - \beta x) - 2f(\alpha x) - 2f(\beta x) \in 434\,V. \tag{8}$$

Fix now $x, y \in \frac{1}{2}B$ such that $x+y, x-y \in \frac{1}{2}B$. Because of (8) it is enough to assume that x and y are linearly independent. Let u and v be vectors from the subspace $\operatorname{lin}\{x,y\}$, generated by x and y, such that $u \perp v$. Then $x = \alpha u + \beta v$, $y = \gamma u + \delta v$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We have $\|\alpha u\| \le \|\alpha u + \beta v\| = \|x\|$, $\|\beta v\| \le 2\|x\|$, $\|\gamma u\| \le \|\gamma u + \delta v\| = \|y\|$, $\|\delta v\| \le 2\|y\|$, $\|\alpha u + \gamma u\| \le \|x\| + \|y\|$, $\|\alpha u - \gamma u\| \le \|x\| + \|y\|$, $\|\beta v + \delta v\| \le 2\|x\| + 2\|y\|$, $\|\beta v - \delta v\| \le 2\|x\| + 2\|y\|$. Hence, αu, $\gamma u \in \frac{1}{2}B$, βv, δv, $\alpha u + \gamma u$, $\alpha u - \gamma u \in B$, $\beta v + \delta v$, $\beta v - \delta v \in 2B$. Using (1), (7) and (8), and taking care that the suitable vectors are in the domain, we get the desired assertion.

The following lemma has been established as Lemma 9 in [10].

LEMMA 3.6. *Let $f : B \to Y$ satisfy the condition*

$$x,\ y,\ x+y,\ x-y \in B \implies f(x+y) + f(x-y) - 2f(x) - 2f(y) \in V.$$

Then there exist a quadratic mapping $q : X \to Y$ and a positive constant $k_3 = k_3(N)$ such that

$$q(x) - f(x) \in k_3 \operatorname{seq\,cl} V, \quad x \in B.$$

LEMMA 3.7. *Let $f : B \to Y$ be an even mapping satisfying* (1). *Then there exist a quadratic function $q : X \to Y$ and a positive constant $k_4 = k_4(N)$ such that*

$$q(x) - f(x) \in k_4 \operatorname{seq\,cl} V, \quad x \in \frac{1}{2}B.$$

Proof. The lemma is an immediate consequence of Lemma 3.5 and Lemma 3.6.

REMARK 3.8. Neither in Lemma 3.4 nor in Lemma 3.7 we get uniqueness of the function which approximates f. Moreover, in Lemma 3.7 we get a result slightly weaker than expected. Namely, we cannot expect that each function q satisfying the assertion of the lemma is orthogonally additive (what would mean, according to Remark 2.4, that for example in a normed linear space of dimension not less than 2, in which the norm does not come from an inner product, it is the zero function). The example below shows that, in general, a function q, which has all the properties occurring in Lemma 3.7, fails to be orthogonally additive.

EXAMPLE 3.9. Take $f : \mathbb{R}^2 \to \mathbb{R}$ which is even, orthogonally additive and assume that a norm $\|\cdot\|$ in $\mathbb{R}^2$ does not come from an inner product. Then for every $\varepsilon > 0$ we have

$$(x, y, x+y \in B,\ x \perp y) \implies |f(x+y) - f(x) - f(y)| \le \varepsilon.$$

Take an arbitrary $\varepsilon > 0$ and define $q : \mathbb{R}^2 \to \mathbb{R}$ as $q(x) := f(x) + c(x_1^2 + x_2^2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and with a real c such that $|c| \le \frac{4k_4\varepsilon}{\alpha}$, where α is a positive constant such that for all $x \in \mathbb{R}^2$ we have $(x_1^2 + x_2^2) \le \alpha\|x\|^2$

(the Euclidean norm and $\|\cdot\|$ are equivalent). Because f is quadratic, so is q. Moreover, for every $x \in \frac{1}{2}B$ we have

$$|f(x) - q(x)| = |c(x_1^2 + x_2^2)| < k_4\varepsilon.$$

However, the function q is not orthogonally additive on a half-ball. To see this, take arbitrary $x = (x_1, x_2)$ and $y = (y_1, y_2)$ from $\frac{1}{2}B$ such that $x \perp y$ and note that

$$q(x+y) - q(x) - q(y) = 2c(x_1y_1 + x_2y_2).$$

The above difference cannot be always zero. Otherwise, the orthogonality relation in the sense of Birkhoff–James would be equivalent to the orthogonality relation connected with some inner product defined on $\mathbb{R}^2$, which leads to a contradiction.

Now we have collected all tools needed and we are ready to prove Theorem 3.1. The proof works similarly to that of Theorem 1 from [10].

Proof of Theorem 3.1. Let f_o, $f_e : B \to Y$ denote the odd and even part of the function f, respectively. Then, if f fulfils (1), so do the functions f_o and f_e. From Lemma 3.4 we infer that there exist an additive function $a : X \to Y$ and a constant k_1 such that

$$a(x) - f_o(x) \in k_1 \operatorname{seq\,cl} V \quad \text{for all} \quad x \in \frac{1}{2}B,$$

and from Lemma 3.7 we get the existence of a quadratic function $q : X \to Y$ and a constant k_4 such that

$$q(x) - f_e(x) \in k_4 \operatorname{seq\,cl} V \quad \text{for all} \quad x \in \frac{1}{2}B.$$

Consequently,

$$a(x) + q(x) - f(x) \in (k_1 + k_4) \operatorname{seq\,cl} V \quad \text{for all} \quad x \in \frac{1}{2}B,$$

which gives the assertion of the theorem with $k = k_1 + k_4$ and completes the proof.

Similarly also to [10], we can establish stability results for Jensen, Pexider and exponential functional equations.

4. Stability in uniformly convex spaces

Let us recall the following

DEFINITION 4.1. A normed linear space $(X, \|\cdot\|)$ is called *uniformly convex* if and only if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $x, y \in X$ if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ then $\left\|\frac{1}{2}(x+y)\right\| < 1 - \delta(\varepsilon)$.

Assume now that $(X, \|\cdot\|)$ is a real uniformly convex space, $\dim X = N \geq 2$, with Birkhoff–James orthogonality relation and the sets Y, V and

B have the same properties as in paragraph 3. Let, moreover, S stand for the unit sphere in X.

PROPOSITION 4.2. *There exists an $n \in \mathbb{N}$ such that for every $x \in S$ there exists a decomposition*

$$x = \sum_{i=1}^{2^j} u_{j,i}\,, \quad 1 \le j \le n$$

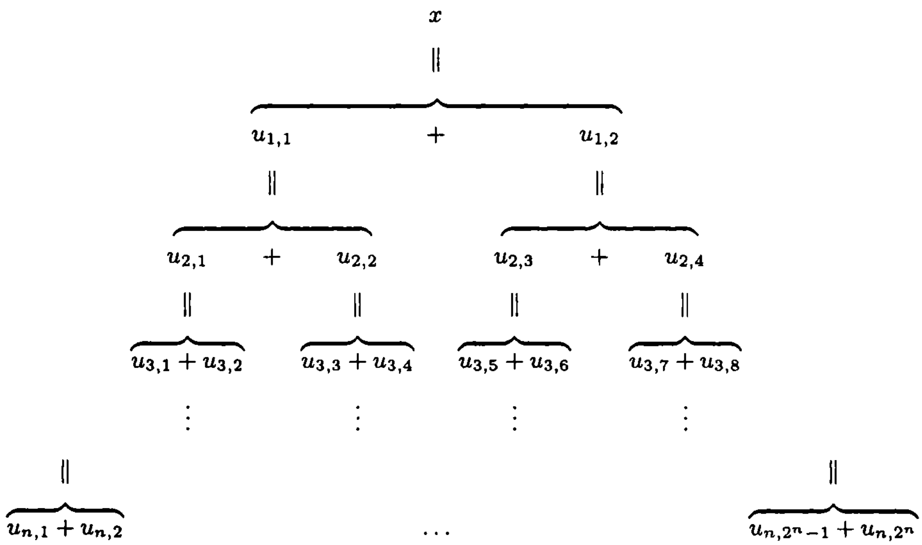
such that

$$\|u_{j,i}\| < 1 \ \ for \ \ 1 \le j \le n-1, \ \ 1 \le i \le 2^j,$$
$$\|u_{n,i}\| \le \frac{1}{2} \ \ for \ \ 1 \le i \le 2^n,$$

and moreover,

$$u_{j-1,i} = u_{j,2i-1} + u_{j,2i}\,, \quad 1 \le j \le n, \ \ 1 \le i \le 2^{j-1},$$
$$u_{j,2i-1} \perp u_{j,2i}\,, \quad \|u_{j,2i-1}\| = \|u_{j,2i}\|\,, \quad 1 \le j \le n, \ \ 1 \le i \le 2^{j-1}.$$

The following diagram shows the decomposition described in Proposition 4.2.



Proof. Take $x \in \frac{1}{2}S$. By [15] for every $y \in \frac{1}{2}S$ there exists a unique $\lambda \in \left[\frac{1}{3}, 3\right]$ which depends continuously on y and such that

$$x + \lambda y \perp x - \lambda y.$$

Define the function $f_x : \frac{1}{2}S \to \mathbb{R}$ as follows

$$f_x(y) := \|x + \lambda y\| - \|x - \lambda y\|$$

for all $y \in \frac{1}{2}S$ and with λ as described above. Since $f_x(x) = 2\|x\| > 0$ and $f_x(-x) = -2\|x\| < 0$ (in both cases $\lambda = 1$) and S is connected, there exists a $y_0 \in \frac{1}{2}S$ such that $f_x(y_0) = 0$. Hence, for a suitable λ we have $\|x + \lambda y_0\| = \|x - \lambda y_0\|$ and $x + \lambda y_0 \perp x - \lambda y_0$. From the last orthogonality relation we deduce, moreover, that $\|x+\lambda y_0\| \leq 2\|x\| = 1$, whence the vector $2x$, of norm 1, can be written as a sum $(x+\lambda y_0)+(x-\lambda y_0)$ of two orthogonal vectors of the same norm smaller than or equal to 1.

So far, we have proved that any vector of the norm 1 can be decomposed into two orthogonal vectors, both of the same norm, smaller than or equal to 1.

In what follows, we shall see that it is impossible to decompose a given vector of norm 1 into a sum of two orthogonal vectors of norms close to 1. In other words, we proceed to show that there exists $\delta > 0$ such that for every $x \in S$ we have

$$(x = u + v,\ u \perp v,\ \|u\| = \|v\|) \Longrightarrow \|u\| < 1 - \delta. \tag{9}$$

Suppose, on the contrary, that there exists a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in S$, and sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ of elements of the unit ball such that $x_n = u_n + v_n$, $u_n \perp v_n$ and $\|u_n\| = \|v_n\| \geq 1 - \delta_n$ with $\delta_n \to 0$.

Let $r_n := \|u_n\|$ and $S_n := \{u \in X : \|u\| = r_n\}$. Then $1 - \delta_n \leq r_n \leq 1$ and $r_n \to 1$. Let, moreover, $(w_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$ be such that $w_n \in S_n$, $\xi_n > 0$, $\xi_n \to 0$ and

$$(1 + \xi_n)w_n = u_n + v_n.$$

Compute

$$\left\|\frac{w_n}{r_n} - \frac{u_n}{r_n}\right\| = \frac{1}{r_n}\|v_n - \xi_n w_n\| \geq \frac{1}{r_n}\left(\|v_n\| - \xi_n\|w_n\|\right) = 1 - \xi_n.$$

So, there exists n_0 such that

$$\left\|\frac{w_n}{r_n} - \frac{u_n}{r_n}\right\| \geq \frac{1}{2} \quad \text{for all} \quad n \geq n_0.$$

Take $\eta > 0$. There exists $n_1 \geq n_0$ such that $\frac{1}{2}\xi_{n_1} \leq \eta$. We thus obtain

$$\begin{aligned}\left\|\frac{1}{2}\left(\frac{w_{n_1}}{r_{n_1}} + \frac{u_{n_1}}{r_{n_1}}\right)\right\| &= \frac{1}{r_{n_1}}\left\|u_{n_1} + \frac{1}{2}v_{n_1} - \frac{1}{2}\xi_{n_1}w_{n_1}\right\| \\ &\geq \frac{1}{r_{n_1}}\left\|u_{n_1} + \frac{1}{2}v_{n_1}\right\| - \frac{1}{2}\xi_{n_1} \geq 1 - \eta,\end{aligned}$$

which contradicts the uniform convexity of X. So, we have proved that there exists some $\delta > 0$ such that any vector $x \in S$ can be decomposed into a sum of two orthogonal vectors of equal norms less than $1 - \delta$. It is now easy to see that the desired n we reach after $\left[\frac{1}{2\delta}\right] + 1$ steps.

In what follows we shall again treat separately odd and even functions.

LEMMA 4.3. *Let $f : B \to Y$ be an odd function satisfying* (1). *Then there exist an additive function $a : X \to Y$ and a constant $c_1 = c_1(N)$ such that*

$$f(x) - a(x) \in c_1 \operatorname{seq}\operatorname{cl} V \quad \textit{for all} \quad x \in B.$$

Proof. First we will show that for a positive constant m_1 one has

$$f(x) - 2f\left(\frac{x}{2}\right) \in m_1 V, \; x \in B. \tag{10}$$

Take $x \in B$ and using Proposition 4.2 write it in the form

$$x = \sum_{i=1}^{2^n} u_{n,i}, \quad u_{n,2k-1} \perp u_{n,2k}, \quad u_{n,2k-1}, u_{n,2k} \in \frac{1}{2}B, \quad k \in \{1, \ldots, 2^{n-1}\}.$$

From (1) and Lemma 3.3 we get (with $u_{0,1} := x$)

$$\begin{aligned} f(x) - 2f\left(\frac{x}{2}\right) &= \sum_{j=1}^{n} \sum_{i=1}^{2^{j-1}} \Big(f(u_{j-1,i}) - f(u_{j,2i-1}) - f(u_{j,2i}) \Big) \\ &+ \sum_{i=1}^{2^n} \left(f(u_{n,i}) - 2f\left(\frac{u_{n,i}}{2}\right)\right) \\ &+ 2\sum_{j=1}^{n} \sum_{i=1}^{2^{j-1}} \left(f\left(\frac{u_{j,2i-1}}{2}\right) + f\left(\frac{u_{j,2i}}{2}\right) - f\left(\frac{u_{j-1,i}}{2}\right)\right) \in 3(2^{n+1} - 1)\, V. \end{aligned}$$

So we have got (10) with $m_1 = 3(2^{n+1} - 1)$.

Let a be the function from Lemma 3.4. Take $x \in B$. We have

$$f(x) - a(x) = \left(f(x) - 2f\left(\frac{x}{2}\right)\right) + 2\left(f\left(\frac{x}{2}\right) - a\left(\frac{x}{2}\right)\right) \in c_1 \operatorname{seq}\operatorname{cl} V,$$

where $c_1 = m_1 + 2k_1$ with constants taken from Lemma 3.4 and (10). This finishes the proof.

LEMMA 4.4. *Let $f : B \to Y$ be an even mapping satisfying* (1). *Then there exist a quadratic function $q : X \to Y$ and a constant $c_2 = c_2(N)$ such that*

$$f(x) - q(x) \in c_2 \operatorname{seq}\operatorname{cl} V \quad \textit{for all} \quad x \in B.$$

Proof. It runs analogously to the previous one. Instead of (10) we prove, however

$$f(x) - 4f\left(\frac{x}{2}\right) \in m_2 V, \; x \in B,$$

with $m_2 = 3 \cdot 2^{n+2} - 5$, and we use Lemma 3.7. The constant c_2 is then equal to $m_2 + 4k_4$.

Proceeding as in the proof of the main result of the third paragraph, we use the above two lemmas in order to obtain the next result.

THEOREM 4.5. *If a function* $f : B \to Y$ *satisfies* (1) *then there exist an additive function* $a : X \to Y$, *a quadratic function* $q : X \to Y$ *and a constant* $c = c(N)$ *such that*

$$f(x) - a(x) - q(x) \in c\,\mathrm{seq}\,\mathrm{cl}\,V \quad \textit{for all} \quad x \in B\,.$$

5. Some examples

The assumption about the uniform convexity of the space is only a sufficient condition for the stability of the Cauchy equation on the whole ball. We will see in the following examples that this condition is not necessary. We will show it in two examples in two-dimensional spaces but these results can be generalized to $\mathbb{R}^n$ for arbitrary $n \geq 2$. Since the considerations concern concrete spaces, the estimating constants obtained are much better than in the general case of uniformly convex spaces.

EXAMPLE 5.1. We consider $X = l_2^1$; it means we take $\mathbb{R}^2$ with the norm defined as $\|(x,y)\| = |x| + |y|$ for all $(x,y) \in \mathbb{R}^2$.

One can check that the following vectors are orthogonal in the sense of Birkhoff–James, and, in general, the symmetric relations do not hold:

$$\begin{aligned}
(1,0) &\perp (a,b) && \text{if} \quad |a| \leq |b|,\\
(0,1) &\perp (a,b) && \text{if} \quad |a| \geq |b|,\\
(a,b) &\perp (1,1) && \text{if} \quad ab \leq 0,\\
(a,b) &\perp (1,-1) && \text{if} \quad ab \geq 0.
\end{aligned}$$

The homogeneity of Birkhoff–James orthogonality gives many other orthogonality relations.

We prove that for every $x \in l_2^1$, $\|x\| = 1$, there exist vectors $u, v, w \in l_2^1$ such that

$$x = (u+v) + w,\ \|u\| = \|v\| = \|w\| = \frac{1}{2},\ u \perp v,\ u+v \perp w. \tag{11}$$

We will show that every vector $(a,b) \in \mathbb{R}^2$ of norm 1 from the first quadrant (similar considerations are valid for the other quadrants) can be decomposed in this way. It is clear that such (a,b) is of the form $\left(\frac{1}{2}+d, \frac{1}{2}-d\right)$ or $\left(\frac{1}{2}-d, \frac{1}{2}+d\right)$ for some $d \in \left[0, \frac{1}{2}\right]$. If $d \in \left[0, \frac{1}{4}\right]$ then

$$\left(\frac{1}{2}+d, \frac{1}{2}-d\right) = \left(\frac{1}{2}, 0\right) + \left(d, \frac{1}{2}-d\right).$$

We have

$$\left(\frac{1}{2}, 0\right) \perp \left(d, \frac{1}{2}-d\right) \quad \text{and} \quad \left\|\left(\frac{1}{2}, 0\right)\right\| = \left\|\left(d, \frac{1}{2}-d\right)\right\| = \frac{1}{2}.$$

If $d \in \left[\frac{1}{4}, \frac{1}{2}\right]$ then

$$\left(\frac{1}{2}+d, \frac{1}{2}-d\right) = \left[\left(d, \frac{1}{2}-d\right) + \left(\frac{1}{4}, -\frac{1}{4}\right)\right] + \left(\frac{1}{4}, \frac{1}{4}\right).$$

Moreover,

$$\left(d, \frac{1}{2}-d\right) \perp \left(\frac{1}{4}, -\frac{1}{4}\right), \quad \left(d+\frac{1}{4}, \frac{1}{4}-d\right) \perp \left(\frac{1}{4}, \frac{1}{4}\right),$$

$$\left\|\left(d, \frac{1}{2}-d\right)\right\| = \left\|\left(\frac{1}{4}, -\frac{1}{4}\right)\right\| = \left\|\left(\frac{1}{4}, \frac{1}{4}\right)\right\| = \frac{1}{2}$$

$$\text{and} \quad \left\|\left(d+\frac{1}{4}, \frac{1}{4}-d\right)\right\| = 2d \le 1.$$

When we change the coordinates we have similar decompositions for vectors of the form $\left(\frac{1}{2}-d, \frac{1}{2}+d\right)$ for $d \in \left[0, \frac{1}{2}\right]$.

Similarly to the earlier considerations one should first treat separately odd and even functions. First we will show that if f is an odd mapping satisfying (1) then

$$f(x) - 2f\left(\frac{x}{2}\right) \in m_1 V, \quad x \in B$$

for some positive constant m_1. Namely, we have

$$\begin{aligned} &f(x) - 2f\left(\frac{x}{2}\right) \\ &\quad = \Big(f(u+v+w) - f(u+v) - f(w)\Big) + \Big(f(u+v) - f(u) - f(v)\Big) \\ &\qquad + \left(f(u) - 2f\left(\frac{u}{2}\right)\right) + \left(f(v) - 2f\left(\frac{v}{2}\right)\right) + \left(f(w) - 2f\left(\frac{w}{2}\right)\right) \\ &\qquad + 2\left(f\left(\frac{u}{2}\right) + f\left(\frac{v}{2}\right) - f\left(\frac{u+v}{2}\right)\right) \\ &\qquad + 2\left(f\left(\frac{u+v}{2}\right) + f\left(\frac{w}{2}\right) - f\left(\frac{u+v+w}{2}\right)\right) \in 15\,V. \end{aligned}$$

Now, as in the proof of Lemma 4.3, using Lemma 3.4, we infer that there exist an additive function $a : l_2^1 \to Y$ and a positive constant c_1 such that

$$f(x) - a(x) \in c_1 \operatorname{seq\,cl} V, \quad x \in B.$$

If now f is an even mapping satisfying (1), then

$$f(x) - 4f\left(\frac{x}{2}\right) \in m_2 V, \quad x \in B,$$

with some constant m_2, and there exist a quadratic mapping $q : l_2^1 \to Y$ (taken from Lemma 3.7) and a positive constant c_2 such that

$$f(x) - q(x) \in c_2 \operatorname{seq} \operatorname{cl} V, \quad x \in B.$$

Here we use similar computations to those used above for odd mappings.

Now, joining both previous results, we are able to state that there exist an additive function $a : l_2^1 \to Y$, a quadratic function $q : l_2^1 \to Y$ and a positive constant c such that

$$f(x) - a(x) - q(x) \in c \operatorname{seq} \operatorname{cl} V, \quad x \in B.$$

It is interesting to see that the method described for uniformly convex spaces, which seems to be quite natural for that situation, does not work here. Namely, we have the following situation.

Consider the unit sphere. In the first quadrant, for example, all vectors of the unit sphere can be decomposed into orthogonal vectors of equal norms as follows:

$$\left(\frac{1}{2} + d, \frac{1}{2} - d\right) = \begin{cases} \left(\frac{1}{2}, 0\right) + \left(d, \frac{1}{2} - d\right), \, d \in \left[0, \frac{1}{4}\right] \\ \left(\frac{1}{2}, \frac{1}{2} - 2d\right) + (d, d), \; d \in \left[\frac{1}{4}, \frac{1}{2}\right], \end{cases}$$

and analogously for vectors of the form $\left(\frac{1}{2} - d, \frac{1}{2} + d\right)$ for $d \in \left[0, \frac{1}{2}\right]$. All vectors of the unit sphere in the first quadrant between $\left(\frac{1}{4}, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, \frac{1}{4}\right)$ can be then decomposed into two orthogonal vectors of norm $\frac{1}{2}$. But later when we are going further towards the vertices of the unit sphere, the norm of the two vectors obtained in this decomposition increases, namely

$$\left\| \left(\frac{1}{2}, \frac{1}{2} - 2d\right) \right\| = \|(d, d)\| = 2d.$$

Finally at the vertices it is only possible to decompose into two vectors of norm 1, so we do not get (9) as we did in a uniformly convex space. Actually, also in such a space, it is impossible to decompose a vector of norm 1 into two orthogonal vectors, each of norm $\frac{1}{2}$. Since two nonzero vectors, orthogonal in the sense of Birkhoff–James, are linearly independent, this contradicts directly the strict convexity (and so the uniform convexity).

EXAMPLE 5.2. Consider now the space l_2^∞; it means, we take $\mathbb{R}^2$ with the norm defined as $\|(x, y)\| = \max\{|x|, |y|\}$ for all $(x, y) \in \mathbb{R}^2$.

It is interesting, in comparison with Example 5.1, that the orthogonal vectors now look as follows:

$$\begin{aligned} (a,b) &\perp (1,0) \quad \text{if} \quad |a| \le |b|, \\ (a,b) &\perp (0,1) \quad \text{if} \quad |a| \ge |b|, \\ (1,1) &\perp (a,b) \quad \text{if} \quad ab \le 0, \\ (1,-1) &\perp (a,b) \quad \text{if} \quad ab \ge 0. \end{aligned}$$

Again, in general, the symmetric relations do not hold.

Proceeding as earlier, we prove that for every $x \in l_2^1$, $\|x\| = 1$, there exist vectors $u, v, w \in l_2^1$ such that (11) holds. All the remaining computations are unchanged.

REMARK 5.3. If we consider n-dimensional spaces l_n^1 or l_n^∞ with $n > 2$ while decomposing a vector to the form (11) it is enough to fix $n-2$ coordinates and then, after necessary scaling down, to work with two other coordinates using Examples 5.1 or 5.2.

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