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GENERALIZATION OF EDELSTEIN'S FIXED POINT THEOREM

1. Introduction

Let (X_i,d_i), $i = 1,\dots,n$ be metric spaces. Let T_i, $i=1,\dots,n$ be transformations mapping of $B \overset{df}{=} X_1 \times \dots \times X_n$ into X_i. For any positive number a we define (cf. also [3])

$$Z_a = \left\{(x_1,\dots,x_n) \in B : d_i\left[x_i, T_i(x_1,\dots,x_n)\right] \leqslant a,\ i=1,\dots,n\right\}.$$

In [1] the following fixed point theorem has been proved, generalizing the Banach principle for contraction maps (cf. [4]):

Let E be a metric space and T an operator which transforms E into itself. Suppose that $d[T(x), T(y)] < d(x,y)$, $x \neq y$, $x,y \in E$. Assume that there exists $x \in E$ such that the sequence at iterates $\{T^m(x)\}$ contains a subsequence $\{T^{m_r}(x)\}$ convergent to a point $u \in E$. Then u is a unique fixed point of T.

The purpose of the present paper is to prove (using the notation of the sets Z_a) a fixed point theorem which generalizes the Edelstein's theorem and the result in [5].

2. Edelstein's fixed point theorem

Let $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$, $x_i, y_i \in X_i$, $i=1,\dots,n$, and let

$$\Delta = \left\{(x,y) \in B \times B:\ x_i = y_i,\ i=1,\dots,n\right\}.$$

We shall prove the following theorem.

T h e o r e m. Let (X_i, d_i), $i=1,\dots,n$ be metric spaces. Suppose that the transformations $T_i : B \to X_i$, $i=1,\dots,n$, fulfil the following conditions

$$(1) \qquad d_i\left[T_i(x), T_i(y)\right] < \sum_{k=1}^{n} a_{ik} d_k(x_k, y_k) \quad \text{in} \quad Y = B \times B - \Delta,$$

$$(2) \qquad |\lambda_i| \leqslant 1, \quad i=1,\dots,n,$$

where $a_{ik} > 0$, $i,k = 1,\dots,n$ and λ_i, $i=1,\dots,n$ are the characteristic roots of the matrix (a_{ik}), $i,k = 1,\dots,n$. Assume that there exists a point $u = (u_1,\dots,u_n) \in B$ such that the sequence at iterates $\left\{T_i^m(u)\right\}$ contains a subsequence $\left\{T_i^{m_r}(u)\right\}$ convergent to $z_i \in X_i$, $i = 1,\dots,n$. Then $z = (z_1,\dots,z_n)$ is a unique fixed point of the system of equations

$$(3) \qquad x_i = T_i(x), \quad i=1,\dots,n.$$

P r o o f. From (2) and Perron's theorem ([2], p.354), it follows that there exist positive numbers q_i, $i = 1,\dots,n$ such that

$$(4) \qquad \sum_{k=1}^{n} a_{ik} q_k \leqslant q_i, \quad i = 1,\dots,n.$$

Suppose that there exists an integer v, $1 \leqslant v \leqslant n$ such that $z_v \neq T_v(z)$. We define the functions

$$f_i(x,y) = \frac{d_i\left[T_i(x),\ T_i(y)\right]}{\sum_{k=1}^{n} a_{ik} d_k(x_k, y_k)}, \quad i = 1,\dots,n, \ (x,y) \in Y = B \times B - \Delta.$$

The functions f_i, $i = 1,\dots,n$ are continuous in Y. We see that $f_i(z, T(z)) < Q < 1$ $(T(z) = (T_1(z),\dots,T_n(z)))$. Hence there exists a neighbourhood U of $(z, T(z))$ such that for $(x,y) \in U$ we have $f_i(x,y) < Q$, $i = 1,\dots,n$. Also there exist neighbourhoods U_1 and U_2 of z and $T(z)$ respectively, such that $U_1 \times U_2 \subset U$ and for $r \geqslant s$ we have $T^{m_r}(u) \in U_1$ and $T^{m_r+1}(u) \in U_2$. Consequently, for $r \geqslant s$ and $i = 1,\dots,n$ we obtain

$$(5) \quad d_i\left[T_i^{m_r+1}(u),\ T^{m_r+2}(u)\right] < Q \sum_{k=1}^{n} a_{ik} d_k\left[T_k^{m_r}(u),\ T_k^{m_r+1}(u)\right].$$

Since the system of inequalities (4) is homogeneous, we may assume that

$$(6) \qquad q_i \geqslant d_i\left[T_i^{m_s}(u),\ T_i^{m_s+1}(u)\right], \quad i = 1,\dots,n .$$

From (1), (5), (6) and (4) we obtain

$$(7) \quad d_i\left[T_i^{m_{s+r}}(u),\ T_i^{m_{s+r}+1}(u)\right] \leqslant Q^r \ q_i, \quad r=1,2,\dots,\ i=1,\dots,n.$$

Let now a_m be a decreasing sequence of positive numbers tending to zero. We denote

$$Z_m = \left\{ y \in B : d_i(y_i,\ T_i(y)) \leqslant a_m, \quad i = 1,\dots,n \right\}.$$

Since $0 \leqslant Q < 1$, (7) implies that the sets Z_m are non empty. Let m be a fixed positive integer and consider the corresponding set Z_m. We have

$$d_i\left[z_i, T_i(z)\right] \leqslant d_i\left[z_i,\ T_i^{m_r}(u)\right] + d_i\left[T_i^{m_r}(u),\ T_i^{m_r+1}(u)\right] +$$

$$+ d_i\left[T_i^{m_r+1}(u),\ T_i(z)\right].$$

Let $\varepsilon > 0$ be arbitrary. Since

$$T_i^{m_r}(u) \xrightarrow[r\to\infty]{} z_i, \ T_i^{m_r+1}(u) \xrightarrow[r\to\infty]{} T_i(z), \quad i = 1,\dots,n$$

and

$$\left(T_1^{m_r}(u),\ldots, T_n^{m_r}(u)\right) \in Z_m$$

for all sufficiently large values of r, there exists a integer N such that for $r > N$

$$d_i\left[z_i, T_i(z)\right] \leqslant 2\varepsilon + a_m, \quad i = 1,\ldots,n.$$

Since ε is arbitrary, it follows that $d_i\left[z_i, T_i(z)\right] \leqslant a_m$ and $z = (z_1,\ldots,z_n) \in Z_m$. Therefore $z \in Z_m$ for every integer m and consequently $z_i = T_i(z_1,\ldots,z_n)$, $i = 1,\ldots,n$. This contradiction proves that $z = (z_1,\ldots,z_n)$ is a fixed point of the system of equations (3).

Now we shall prove that $z = (z_1,\ldots,z_n)$ is a unique fixed point of the system of equations (3). Suppose that there exists $b = (b_1,\ldots,b_n)$, $b \neq z$ such that $b_i = T_i(b)$, $i = 1,\ldots,n$. We see that

$$f_i(b,z) < Q < 1, \quad i = 1,\ldots,n.$$

We can prove (as in the preceding case) that for $m > N$ and $r = 1,2,\ldots,$ we have

$$d_i(b_i,z_i) \leqslant d_i\left[T_i^{m+r}(b), T_i^{m+r}(z)\right] \leqslant Q^r \max_i(d_i\left[T_i^m(b), T_i^m(z)\right]).$$

Passing to the limit as $r \to \infty$, we obtain

$$d_i(b_i,z_i) = 0, \quad i = 1,\ldots,n,$$

and $b = z$. This contradiction proves the theorem.

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