Title: Asymptotic estimates for solutions of the second boundary value problem for parabolic equations

Author: Jan Chabrowski

Citation style: Chabrowski Jan. (1978). Asymptotic estimates for solutions of the second boundary value problem for parabolic equations. "Demonstratio Mathematica" (Vol. 11, nr 4 (1978) s. 997-1013), doi 10.1515/dema-1978-0415

DEMONSTRATIO MATHEMATICA
Vol. XI No 4 1978

Jan Chabrowski

ASYMPTOTIC ESTIMATES FOR SOLUTIONS OF THE SECOND BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS

Let Ω be an unbounded domain in R_n. We denote the boundary of Ω by $\partial\Omega$. We consider the second boundary value problem

$$(1)\quad \frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \left[a_{ij}(t,x)u_{x_i}\right]_{x_j} + \sum_{i=1}^{n} b_i(t,x)u_{x_i} + c(t,x)u$$

in $(0,\infty)\times\Omega$,

$$(2)\qquad \frac{du(t,x)}{dN(t,x)} = 0 \quad \text{for} \quad (t,x)\in(0,\infty)\times\partial\Omega,$$

$$(3)\qquad u(0,x) = \varphi(x) \quad \text{for} \quad x\in\Omega,$$

where $\frac{d}{dN(t,x)}$ denotes the inward conormal derivative to $(0,\infty)\times\partial\Omega$ at the point (t,x). The present work is concerned with the asymptotic estimates for solutions of weak solutions of the problem (1) - (3) under assumptions which allow the coefficients b_i and c to grow to infinity in various ways (see assumption (B)). In proving the main result (see Theorem 1) we make a crucial use of the Guščin form of Sobolev's inequality (see [3] and [5]). The method used here was inspired by the series of Guščin's papers on the second boundary value problem for parabolic equations (see [2], [3], [4] and [6]).

In the sequel we shall use the following notations:

$$D_T = (0,T)\times\Omega, \quad D_{T,R} = (0,T)\times(\Omega\cap(|x|<R)).$$

For any cylinder $(0,T)\times A$, where A is an open domain in R_n, by $W_2^{1,0}((0,T)\times A)$, $W_2^{1,1}((0,T)\times A)$ we denote the well-known Sobolev spaces (see [1]).

Let $\varrho(t,x)$ be a measurable function in $(0,T)\times A$. We denote by

$$\varrho_n(t,x) = \frac{1}{h}\int_t^{t+h} \varrho(s,x)ds$$

the Stieklov's average of the function ϱ with respect to t. Of course if $\varrho\in W_2^{1,0}((0,T)\times A)$ then $\varrho_h \in W_2^{1,1}((0,T-h)\times A)$ if $h>0$ and $\varrho_h \in W^{1,1}((|h|,T)\times A)$ if $h<0$. Further additional informations on this subject can be found in [1] or [4].

Throughout the paper we make the following assumptions concerning the coefficients of (1):

(A) The symmetric matrix $\{a_{ij}(t,x)\}$ is uniformly positive definite in $(0,\infty)\times\Omega$, i.e., there are positive numbers λ_0 and λ_1 such that

$$(4) \qquad \lambda_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t,x)\,\xi_i\xi_j \leq \lambda_1|\xi|^2$$

for almost all $(t,x)\in(0,\infty)\times\Omega$ and each $\xi\in R_n$. We assume also that a_{ij}, b_i and c are measurable in $(0,\infty)\times\Omega$ and moreover the coefficients b_i and c are essentially bounded in $D_{T,R}$ for each $T>0$ and $R>0$.

(B) There exists a positive function $H(t,x)\in C^1([0,\infty)\times\Omega)$ such that $\sup_{t>0}\int_\Omega H(t,x)dx < \infty$ and

$$(5)\qquad -2\sum_{i,j=1}^{n} a_{ij}(t,x)H_{x_i}H_{x_j} - \frac{2}{\lambda_0}\sum_{i=1}^{n} b_i(t,x)^2H^2 + $$
$$+\delta\int_{\Omega}\sum_{i=1}^{n} b_i(t,x)H_{x_i}H - c(t,x)H^2 - HH_t \geqslant 0$$

for all $(t,x) \in (0,\infty)\times\Omega$ and $\delta = 0,1$.

We introduce the concept of a weak solution of the problem (1)-(3).

Suppose that the function φ (see the initial condition (3)) is measurable in Ω and such that

$$\int_{\Omega}\varphi(x)^2H(0,x)dx < \infty \quad \text{and} \quad \int_{\Omega}\varphi(x)^2H(0,x)^2dx < \infty .$$

A function $u(t,x)$ defined in $(0,\infty)\times\Omega$ is said to be a weak solution of the problem (1)-(3) if it satisfies the following requirements:

(i) the integrals $\int_{\Omega}u(t,x)^2H(t,x)dx$ and $\int_{\Omega}u(t,x)^2H(t,x)^2dx$ are continuous on $[0,\infty)$.

(ii) $u \in W_2^{1,0}(D_{T,R})$ for every $T > 0$ and $R > 0$.

(iii) $$-\int_{D_T}\varrho(t,x)_t u(t,x)dtdx + \int_{D_T}\sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_i}\varrho_{x_j}dtdx -$$
$$-\int_{D_T}\sum_{i=1}^{n} b_i(t,x)u_{x_i}\varrho\, dtdx - \int_{D_T}c(t,x)u\varrho dtdx = \int_{\Omega}\varrho(0,x)\varphi(x)dx$$

for any $T > 0$ and for any function $\varrho \in W_2^{1,1}(D_T)$ such that $\varrho(T,x) = 0$ for $x \in \Omega$ end $\varrho(t,x) = 0$ for $t \in (0,T)$, $|x| > R$ for some positive constant R.

The fact that u is a weak solution of (1)-(3) can be stated in another way.

Lemma 1. Let $u(t,x)$ be a solution of the problem (1)-(3). Then

$$(7)\quad \int_\Omega u(t,x)\varrho(t,x)dx = \int_0^t\int_\Omega u(\tau,x)\varrho_\tau(\tau,x)d\tau dx -$$

$$-\int_0^t\int_\Omega \sum_{i,j=1}^n a_{ij}(\tau,x)u_{x_j}\varrho_{x_j}\, d\tau dx + \int_0^t\int_\Omega \sum_{i=1}^n b_i(t,x)u_{x_i}\varrho\, d\tau dx +$$

$$+\int_0^t\int_\Omega c(\tau,x)u(\tau,x)\varrho(\tau,x)d\tau dx - \int_\Omega \varphi(x)\varrho(0,x)dx$$

for almost all $t \in (0,\infty)$ and for any function $\varrho \in W_2^{1,1}((0,\infty)\times\Omega)$ vanishing for $|x| > R$ for some $R > 0$.

r o o f . Let $\varrho \in W_2^{1,1}((0,\infty)\times\Omega)$ and $\varrho(t,x) = 0$ for $|x| > R$ and $t \in (0,\infty)$. Fix $t_1 > 0$ and set

$$\zeta_\varepsilon(t) = \begin{cases} 1 & \text{for} \quad t < t_1 - \varepsilon \\ \dfrac{t_1 - t}{\varepsilon} & \text{for} \quad t_1 - \varepsilon \le t < t_1 \\ 0 & \text{for} \quad t_1 \le t. \end{cases}$$

As the test function in (6) we take

$$\bar{\varrho}(t,x) = \varrho(t,x)\,\zeta_\varepsilon(t).$$

Taking $\varepsilon \to 0$ in (6) we get (7).

Before stating the main result we prove the following

L e m m a 2. Let $u(t,x)$ be a solution of the problem (1) - (3). Then

$$(8)\qquad \int_\Omega u(t,x)^2 H(t,x)dx \le \int_\Omega \varphi(x)^2 H(0,x)dx$$

for $t \ge 0$.

P r o o f . Fix two numbers $0 \le t_1 < t_2$ and let $\varrho \in W_2^{1,0}((t_1,t_2)\times\Omega)$ and $\varrho(t,x) = 0$ for $t_1 < t < t_2$ and $|x| > R$. Set

$$\bar{\eta}(t,x) = \begin{cases} \eta(t,x) & \text{for} \quad (t,x) \in (t_1,t_2)\times\Omega \\ 0 & \text{elsewhere.} \end{cases}$$

Putting in (6) the Stieklov average $\bar{\eta}_{-h}$ we obtain

$$\int_0^\infty\int_\Omega \frac{\bar{\eta}(t-h,x)-\bar{\eta}(t,x)}{h}\, u(t,x)dtdx + \int_0^\infty\int_\Omega \sum_{i,j=1}^n a_{ij}(t,x)\left[\bar{\eta}_{-h}(t,x)\right]_{x_i} \times$$

$$\times\, u_{x_j}(t,x)dtdx - \int_0^\infty\int_\Omega \sum_{i,j=1}^n b_i(t,x)u_{x_i}(t,x)\,\bar{\eta}_{-h}(t,x)dtdx -$$

$$- \int_0^\infty\int_\Omega c(t,x)u(t,x)\,\bar{\eta}_{-h}(t,x)dtdx = 0.$$

Hence

$$\int_{t_1}^{t_2}\int_\Omega \Bigg\{ \eta(t,x)\,\frac{u(t+h,x)-u(t,x)}{h} + \sum_{i,j=1}^n \eta_{x_i}(t,x)\left[a_{ij}(t,x)u_{x_j}(t,x)\right]_h -$$

$$- \sum_{i=1}^n \left[b_i(t,x)u_{x_i}(t,x)\right]_h \eta(t,x) - \left[c(t,x)u(t,x)\right]_h \times$$

$$\times\, \eta(t,x)\Bigg\}\, dtdx = 0.$$

It is clear that the last equality can be written in the form

$$(9)\quad \int_{t_1}^{t_2}\int_\Omega \Bigg\{ u_h(t,x)_t\,\eta(t,x) + \sum_{i,j=1}^n \eta_{x_i}(t,x)\left[a_{ij}(t,x)u_{x_j}(t,x)\right]_h -$$

$$- \sum_{i=1}^n \left[b_i(t,x)u_{x_i}(t,x)\right]_h \eta(t,x) - \left[c(t,x)u(t,x)\right]_h \eta(t,x)\Bigg\}\, dtdx = 0.$$

Substituting into (9) the function

$$\eta(t,x) = u_h H \zeta^2,$$

where $\zeta \in C^1(R_n)$, $0 \leq \zeta \leq 1$ in R_n, $\zeta(x) = 1$ for $|x| < R$, $\zeta = 0$ for $|x| > R+1$ and $|\zeta_x|$ is bounded independently of R, we derive

$$(10)\quad \frac{1}{2}\int_\Omega u(t_2,x)^2 H(t_2,x)\zeta(x)^2 dx - \frac{1}{2}\int_\Omega u(t_1,x)^2 H(t_1,x)\zeta(x)^2 dx +$$
$$+ \int_{t_1}^{t_2}\int_\Omega \Big\{ \sum_{i,j=1}^n (a_{ij}u_{x_j})_h (u_h H\zeta^2)_{x_i} - \sum_{i=1}^n (b_i u_{x_i})_h u_h H\zeta^2 -$$
$$- (cu)_h u_h H\zeta^2 - u_h H_t \zeta^2 \Big\} dtdx = 0.$$

Since $\| (u-u_h)H^{\frac{1}{2}}\zeta\|_{L^2(D_T)} \underset{h\to 0}{\longrightarrow} 0$ for any $T > 0$, there exist a dense set ξ in $[0,\infty)$ and a subsequence $h_k \to 0$ such that

$$\Big\| \big[u(t,\cdot) - u_{h_k}(t,\cdot)\big] H(t,\cdot)\zeta(\cdot)\Big\|_{L^2(\Omega)} \underset{h_k\to 0}{\longrightarrow} 0$$

for all $t \in \xi$. Now taking $t_1, t_2 \in \xi$ and $h = h_k$ in (10) and passing to the limit we get

$$(11)\quad \int_\Omega u(t_2,x)^2 H(t_2,x)\zeta(x)^2 dx - \int_\Omega u(t_1,x)^2 H(t_1,x)\zeta(x)^2 dx =$$

$$= -2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^n a_{ij}u_{x_j}u_{x_i} H\zeta^2\, dtdx - 2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^n a_{ij}u_{x_j} uH_{x_i}\zeta^2 dtdx -$$

$$- 4\int_{t_1}^{t_2}\int_{\Omega}\sum_{i,j=1}^{n} a_{ij}u_{x_j}uH\zeta\zeta_{x_i}\,dtdx + 2\int_{t_1}^{t_2}\int_{\Omega}\sum_{i=1}^{n} b_i u_{x_i}uH\zeta^2 dtdx +$$

$$+ 2\int_{t_1}^{t_2}\int_{\Omega} cu^2H\zeta^2 dtdx + 2\int_{t_1}^{t_2}\int_{\Omega} u^2H_t\zeta^2 dtdx.$$

Observe that the following inequalities

$$\sum_{i,j=1}^{n} a_{ij}u_{x_j}uH_{x_i}\zeta^2 \leq \frac{\varepsilon_1}{2}\sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j}\zeta^2H + \frac{1}{2\varepsilon_1 H}\sum_{i,j=1}^{n} a_{ij}H_{x_i}H_{x_j}u^2\zeta^2,$$

$$\sum_{i,j=1}^{n} a_{ij}u_{x_j}uH\zeta\zeta_{x_i} \leq \frac{\varepsilon_2}{2}\sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j}H\zeta^2 + \frac{1}{2\varepsilon_2}\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}Hu^2$$

and

$$\sum_{i=1}^{n} b_i u_{x_i}uH\zeta \leq \varepsilon_3 |\nabla_x u|^2H\zeta^2 + \frac{1}{\varepsilon_3}\sum_{i=1}^{n} b_i^2u^2H\zeta^2$$

hold for any positive numbers ε_1, ε_2 and ε_3, which will be chosen later.

Substituting these inequalities into (11) we obtain

$$\int_{\Omega} u(t_2,x)^2H(t_2,x)\zeta(x)^2dx + (2\lambda_0 - 2\varepsilon_3 - 2\lambda_0\varepsilon_2 - \lambda_0\varepsilon_1)\times$$

$$\int_{t_1}^{t_2}\int_{\Omega}|\nabla_x u|^2H\zeta^2 dtdx + \int_{t_1}^{t_2}\int_{\Omega}\Big[-\frac{1}{\varepsilon_1 H}\sum_{i,j=1}^{n} a_{ij}H_{x_i}H_{x_j} - \frac{2}{\varepsilon_3}\sum_{i=1}^{n} b_i^2H -$$

$$- 2cH - 2H_t\Big]u^2\zeta^2\,dtdx \leq \int_{\Omega} u(t_1,x)^2H(t_1,x)\zeta(x)^2dx +$$

$$+ 2\varepsilon_2\int_{t_1}^{t_2}\int_{\Omega}\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}\,Hu^2dtdx.$$

Now taking $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \frac{1}{8}$, $\varepsilon_3 = \frac{\lambda_0}{2}$ and letting $R \to \infty$ we infer from the last inequality that

$$\int_\Omega u(t_2,x)^2 H(t_2,x)dx + \frac{\lambda_0}{4}\int_{t_1}^{t_2}\int_\Omega |\nabla_x u|^2 H dtdt +$$

$$+\int_{t_1}^{t_2}\int_\Omega u^2\left[-\frac{2}{H}\sum_{i,j=1}^n a_{ij}H_{x_i}H_{x_j} - \frac{4}{\lambda_0}\sum_{i=1}^n b_i^2 H - 2cH - 2H_t\right]dtdx \leqslant$$

$$\leqslant \int_\Omega u(t_1,x)^2 H(t_1,x)dx.$$

According to the assumption (B) the expression in brackets is nonnegative, hence

$$\int_\Omega u(t_2,x)^2 H(t_2,x)dx \leqslant \int_\Omega u(t_1,x)^2 H(t_1,x)dx$$

for almost all $t_1 \leqslant t_2$ and by the continuity of $\int_\Omega u(t,x)^2 H(t,x)dx$ with respect to t we obtain (8).

To state our main result we shall need the following assumption on Ω:

Let g be an increasing, continuous and positive function on $[0, \infty)$ such that

$$g(v) \geqslant Cv^{1-\varepsilon_0} \quad \text{for any} \quad v \leqslant \delta_0,$$

where C, δ_0 and ε_0 are some positive constants and $\varepsilon_0 \leqslant \frac{1}{n}$.

Let Ω be an unbounded domain in R_n. We say that Ω possesses property $\mathfrak{U}(g)$, and write $\Omega \in \mathfrak{U}(g)$, if $l(v) \geqslant g(v)$, where

$$l(v) = \inf_{mes_n Q = V} mes_{n-1}(\partial Q \cap \Omega)$$

and mes_k denotes k - dimensional measure.

The following form of Sobolev inequality was proved by Guščin (see [2] and [5]):

If $\Omega \in \mathfrak{U}(g)$ then for each function $f \in W_2^1(\Omega) \cap L_1(\Omega)$ the inequality

$$(12) \qquad \int_\Omega |\nabla f(x)|^2 dx \geqslant K \frac{\int_\Omega f(x)^2 dx}{P\left[\frac{(2\int_\Omega f(x)dx)^2}{\int_\Omega f(x)^2 dx}\right]}$$

holds, where $K = \frac{16}{\ln^2 2}$

and

$$P(v) = \int_0^v \frac{1}{\zeta}\left(\int_0^\zeta \frac{\theta}{g(\theta)^2}\,d\theta\right) d\zeta.$$

The examples of domains having property $\mathfrak{U}(g)$ can be found in [2], [5].

T h e o r e m 1. If u is a solution of the problem (1) - (3) and $\Omega \in \mathfrak{U}(g)$, then

$$(13) \qquad \int_\Omega u(t,x)^2 H(t,x)^2 dx \leqslant \frac{1}{J^{-1}\left(\frac{\lambda_0 Kt}{2}\right)},$$

where J^{-1} denotes the inverse function of the function J given by the formula

$$J(w) = \int_0^W \frac{P\left(\frac{4M\int_\Omega \varphi(x)^2 H(0,x)^2 dx}{\theta}\right)}{\theta}\,d\theta,$$

where $M = \sup_{t>0} \int_\Omega H(t,x)dx.$

P r o o f . The first part of the proof follows similar lines to that of Lemma 2. We use the equality (9) with $\eta = u_h H^2 \varsigma^2$, where ς is the function introduced in the proof of Lemma 2. Letting $h \to 0$ we obtain

$$(14)\quad \int_\Omega u(t_2,x)^2 H(t_2,x)^2 \varsigma(x)^2 dx - \int_\Omega u(t_1,x)^2 H(t_1,x)^2 \varsigma(x)^2 dx =$$

$$= -\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^n a_{ij} u_{x_i} (uH^2)_{x_j} \varsigma^2 dtdx - 4\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^n a_{ij} u_{x_j} uH^2 \times$$

$$\times \varsigma \varsigma_{x_i} dtdx + 2\int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^n b_i u_{x_i} uH^2 \varsigma^2 dtdx + \int_{t_1}^{t_2}\int_\Omega cu^2 H^2 \varsigma^2 dtdx +$$

$$+ 2\int_{t_1}^{t_2}\int_\Omega u^2 HH_t \varsigma^2 dtdx.$$

We now observe that the equality (14) can be written in the form

$$(15)\quad \int_\Omega u(t_2,x)^2 H(t_2,x)^2 \varsigma(x)^2 dx - \int_\Omega u(t_1,x)^2 H(t_1,x)^2 \varsigma(x)^2 dx =$$

$$= -\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^n a_{ij} (uH)_{x_i} (uH)_{x_j} \varsigma^2 dtdx +$$

$$+ 2\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^n a_{ij} u^2 H_{x_i} H_{x_j} \varsigma^2 dtdx - 4\int_{t_1}^{t_2}\int_\Omega \sum_{i,j=1}^n a_{ij} u_{x_j} uH^2 \varsigma\varsigma_{x_i} dtdx +$$

$$+ \int_{t_1}^{t_2}\int_\Omega \sum_{i=1}^n b_i u_{x_i} uH^2 \varsigma^2 dtdx + \int_{t_1}^{t_2}\int_\Omega cu^2 H^2 \varsigma^2 dtdx + 2\int_{t_1}^{t_2}\int_\Omega u^2 HH_t \varsigma^2 dtdx.$$

Notice also that

$$(16)\quad \sum_{i=1}^{n} b_i u_{x_i} uH^2 = \sum_{i=1}^{n} b_i (uH)_{x_i}\, uH - \sum_{i=1}^{n} b_i u^2 H_{x_i} H.$$

$$(17)\quad \sum_{i=1}^{n} b_i (uH)_{x_i}\, uH \leq \delta_1 \left|(uH)_x\right|^2 + \frac{1}{\delta_1} \sum_{i=1}^{n} b_i^2 u^2 H^2$$

for any $\delta_1 > 0$

$$(18)\quad \sum_{i,j=1}^{n} a_{ij} u_{x_i} uH^2 \zeta\zeta_{x_j} = \sum_{i,j=1}^{n} a_{ij} (uH)_{x_i} (uH)\zeta\zeta_{x_j} - $$

$$- \sum_{i,j=1}^{n} a_{ij} uH_{x_i} (uH)\zeta\zeta_{x_j}$$

$$(19)\quad \sum_{i,j=1}^{n} a_{ij} (uH)_{x_i} (uH)\zeta\zeta_{x_j} \leq \frac{\delta_2}{2} \sum_{i,j=1}^{n} a_{ij} (uH)_{x_i} (uH)_{x_j} \zeta^2 +$$

$$+ \frac{1}{2\delta_2} \sum_{i,j=1}^{n} a_{ij}\, \zeta_{x_i} \zeta_{x_j}\, u^2 H^2$$

for any $\delta_2 > 0$ and

$$(20)\quad \sum_{i,j=1}^{n} a_{ij} uH_{x_i} (uH)\zeta\zeta_{x_j} \leq \frac{\delta_3}{2} \sum_{i,j=1}^{n} a_{ij} H_{x_i} H_{x_j} u^2 \zeta^2 +$$

$$+ \frac{1}{2\delta_3} \sum_{i,j=1}^{n} a_{ij}\, \zeta_{x_i} \zeta_{x_j}\, u^2 H^2$$

for any $\delta_3 > 0$. Combining (16) - (20) with (15) we obtain

$$\int_\Omega u(t_2,x)^2H(t_2,x)^2\zeta(x)^2dx - \int_\Omega u(t_1,x)^2H(t_1,x)^2\zeta(x)^2dx +$$

$$+ (2\lambda_o - 2\delta_2\lambda_o - \delta_1)\int_{t_1}^{t_2}\int_\Omega |(uH)_x|^2\zeta^2 dtdx + \int_{t_1}^{t_2}\int_\Omega u^2\zeta^2\Big[(-2\delta_3 - 2)\times$$

$$\times\sum_{i,j=1}^n a_{ij}H_{x_i}H_{x_j} - \frac{2}{\delta_1}\sum_{i=1}^n b_i^2H^2 + 2\sum_{i=1}^n b_iH_{x_i}H - 2cH^2 - 2HH_t\Big]\,dtdx \leq$$

$$\leq\left(\frac{2}{\delta_3}+\frac{2}{\delta_2}\right)\int_{t_1}^{t_2}\int_\Omega\sum_{i,j=1}^n a_{ij}\zeta_{x_i}\zeta_{x_j}\,u^2H^2dtdx.$$

Taking $R\to\infty$, $\delta_3 = 1$, $\delta_2 = \frac{1}{2}$, $\delta_1 = \frac{\lambda_o}{2}$ we get

$$(21)\quad \int_\Omega u(t_2,x)^2H(t_2,x)^2dx - \int_\Omega u(t_1,x)^2H(t_1,x)^2dx +$$

$$+\frac{\lambda_o}{2}\int_{t_1}^{t_2}\int_\Omega |(uH)_x|^2dtdx + \int_{t_1}^{t_2}\int_\Omega u^2\Big[-4\sum_{i,j=1}^n a_{ij}H_{x_i}H_{x_j} -$$

$$-\frac{4}{\lambda_o}\sum_{i=1}^n b_i^2H^2 + 2\sum_{i=1}^n b_iH_{x_i}H - 2cH^2 - 2HH_t\Big]\,dtdx \leq 0.$$

Hence, by the assumption (5), we obtain

$$\int_\Omega u(t_2,x)^2H(t_2,x)^2dx \leq -\frac{\lambda_o}{2}\int_{t_1}^{t_2}\int_\Omega |(uH)_x|^2\,dtdx +$$

$$+\int_\Omega u(t_1,x)^2H(t_1,x)^2dx$$

for any $0 \leq t_1 \leq t_2$, Applying (12) to the function (uH), we conclude

$$\int_\Omega u(t_2,x)^2H(t_2,x)^2dx \leq -\frac{\lambda_0 K}{2}\int_{t_1}^{t_2}\frac{\int_\Omega u(t,x)^2H(t,x)^2dx}{P\left[\frac{(2\int_\Omega|u(t,x)|H(t,x)dx)^2}{\int_\Omega u(t,x)^2H(t,x)^2dx}\right]}dt+$$

$$+\int_\Omega u(t_1,x)^2H(t_1,x)^2dx.$$

By Hölder's inequality and Lemma 2 we have

$$(21)\quad \int_\Omega |u(t,x)|\,H(t,x)dx \leq \left[\int_\Omega u(t,x)^2H(t,x)dx\right]^{\frac{1}{2}}\left[\int_\Omega H(t,x)dx\right]^{\frac{1}{2}} \leq$$

$$\leq M^{\frac{1}{2}}\left[\int_\Omega \varphi(x)^2H(0,x)dx\right]^{\frac{1}{2}}.$$

Using the fact that P is increasing we conclude that

$$\int_\Omega u(t_2,x)^2H(t_2,x)^2dx \leq -\frac{\lambda_0 K}{2}\int_{t_1}^{t_2}\frac{\int_\Omega u(t,x)^2H(t,x)^2dx}{P\left[\frac{4M\int_\Omega \varphi(x)^2H(0,x)dx}{\int_\Omega u(t,x)^2H(t,x)^2dx}\right]}dt+$$

$$+\int_\Omega u(t_1,x)^2H(t_1,x)^2dx$$

for any $0 \leq t_1 < t_2$. It is easily seen that the last inequality implies

$$D_-\left[\int_\Omega u(t,x)^2H(t,x)^2dx\right] \leq -\frac{\lambda_0 K}{2}\frac{\int_\Omega u(t,x)^2H(t,x)^2dx}{P\left[\frac{4M\int_\Omega \varphi(x)^2H(0,x)dx}{\int_\Omega u(t,x)^2H(t,x)^2dx}\right]},$$

where D_- denotes the left-hand lower Dini's derivative. On the other hand consider the ordinary differential equation

$$z'(t) = -\frac{\lambda_0' K}{2} \frac{z(t)}{P\left(\frac{K_1}{z(t)}\right)} \tag{22}$$

with the initial condition

$$z(0) = \int_{\Omega} u(0,x)^2 H(0,x)^2 dx = \int_{\Omega} \varphi(x)^2 H(0,x)^2 dx, \tag{23}$$

where

$$K_1 = 4M \int_{\Omega} \varphi(x)^2 H(0,x) dx.$$

It follows from the standard theorem on differential inequalities that (see theorem 9.5 in [7] p.27)

$$\int_{\Omega} u(t,x)^2 H(t,x)^2 dx \leq z(t)$$

for $t \geq 0$. Now solving the differential equation (22) with the condition (23) we obtain the estimate (13).

Theorem 2. Let u be a solution of the problem (1) - (3) and $\Omega \in \mathfrak{U}(g)$. If the assumption (5) is replaced by

$$2 \sum_{i,j=1}^{n} a_{ij}(t,x) H_{x_i} H_{x_j} - \frac{2}{\lambda_0} \sum_{i=1}^{n} b_i(t,x)^2 H + \tag{5'}$$

$$+ \delta \sum_{i=1}^{n} b_i(t,x) H_{x_i} H - cH^2 - HH_t \geq \alpha H^2$$

for all $(t,x) \in (0,\infty) \times \Omega$, $\delta = 0,1$ and for some positive constant α, then

$$(24)\qquad \int_\Omega u(t,x)^2H(t,x)^2dx \leq \min\left(\frac{1}{J^{-1}\left(\frac{\lambda_0 Kt}{2}\right)},\ e^{-2\alpha t}\int_\Omega \varphi(x)^2H(c,x)^2dx\right)$$

for all $(t,x) \in (0,\infty)\times\Omega$.

P r o o f . It follows from the inequality (21) and the condition (5) that

$$\int_\Omega u(t,x)^2H(t,x)^2dx - \int_\Omega u(t_1,x)^2H(t_1,x)^2dx +$$

$$+\frac{\lambda_0}{2}\int_{t_1}^{t_2}\int_\Omega |(uH)_x|^2\, dtdx + 2\alpha\int_{t_1}^{t_2}\int_\Omega u^2H^2dtdx \leq 0,$$

herce

$$\int_\Omega u(t_2,x)^2H(t_2,x)^2dx \leq -2\alpha\int_{t_1}^{t_2}\int_\Omega u(t,x)^2H(t,x)^2dtdx +$$

$$+\int_\Omega u(t_1,x)^2H(t_1,x)^2dx$$

for any $0 \leq t_1 < t_2$. Proceeding as at the end of the proof of Theorem 1 we obtain

$$\int_\Omega u(t,x)^2H(t,x)^2dx \leq e^{-2\alpha t}\int_\Omega \varphi(x)^2H(0,x)^2dx$$

for all $t \geq 0$ and the assertion follows.

To illustrate the estimates (13) and (24) we shall give two examples:

E x a m p l e 1. Suppose that

$$c \leq -M|x|^2, \quad b_i = 0$$

for $(t,x) \in (0,\infty)\times\Omega$. Set $H(t,x) = e^{-v|x|^2}$ then the inequality (5) has the form

$$\left(-8\nu^2 \sum_{i,j=1}^{n} a_{ij}x_ix_j - c\right)H^2 \geqslant (-8\nu^2\lambda_0|x|^2 + M|x|^2)M^2 \geqslant 0$$

for all $(t,x) \in (0,\infty)\times\Omega$, provided ν is sufficiently small. In this example Theorem 1 is applicable.

Example 2. Suppose that the coefficients b_i and c are bounded and set

$$H(t,x) = \left[\prod_{i=1}^{n} \cosh x_i\right]^{-1} e^{-\mu t}.$$

We can easily verify that

$$\left[-2\sum_{i,j=1}^{n} a_{ij}\operatorname{tgh}x_i\operatorname{tgh}x_j - \frac{2}{\lambda_0}\sum_{i=1}^{n} - \delta\sum_{i=1}^{n} b_i\operatorname{tgh}x_i - c + \mu\right]H^2 \geqslant \alpha H^2$$

for all $(t,x) \in (0,\infty)\times\Omega$, provided μ is sufficiently large and it is clear that we can apply Theorem 2.

REFERENCES

[1] O.A. Ladyženskaja, W.A. Solonnikov, N.N. Ural´cewa: Linear and quasilinear equations of parabolic type, Moscow 1967 (Russian).

[2] A.K. Guščin: Stability of solutions of parabolic equations in unbounded domain, Differencial´nye Uravnenija, T. VI, N° 4 (1970) 741-761 (Russian).

[3] A.K. Guščin: On estimates of solutions of the boundary problems for second order parabolic equations, Trudy Mat. Inst. Steklov. T. 126 (1973) 5-45 (Russian).

[4] A.K. Guščin: Some properties of a weak solutions of the second boundary value problem for parabolic equations, Mat. Sb. 97 (139), N 2(6) (1975) 242-261 (Russian).

[5] A.K. G u š č i n : On estimate of the Dirichlet's integral in unbounded domains, Mat. Sb. T. 99 (141) N^o 2 (1976) 282-294 (Russian).

[6] A.K. G u š č i n : Stability of solutions of the second boundary value problem for parabolic equations, Mat. Sb. T.101 (143), N^o 4 (12) (1976) 459-499 (Russian).

[7] J. S z a r s k i : Differential inequalities, Warszawa 1968.

INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, KATOWICE
Received November 14, 1977.