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A THEOREM OF THE HAHN-BANACH TYPE

Let Y be a linear subspace of a linear space X over the rationals $\mathbb{Q}$ and let $C \subseteq X$ be $\mathbb{Q}$-convex. Moreover, let $\mathcal{F}$ be a family of subsets of a linear space E over $\mathbb{Q}$ having the binary intersection property. Suppose that F is a $\mathbb{Q}$-concave set-valued function defined on C and assuming values in $\mathcal{F}$. We give some conditions under which every additive selection of the restriction of F to $Y \cap C$ can be extended to an additive selection of F.

1. Let X be a linear space over the set of rational numbers $\mathbb{Q}$ and let $A \subseteq X$ be a set. We say that A is *$\mathbb{Q}$-radial at a point* $a \in A$ iff for every $x \in X, x \neq 0$ there exists an $\varepsilon > 0$ such that $a + \lambda x \in A$ for every $\lambda \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}$.

A non-empty set $A \subseteq X$ is called *$\mathbb{Q}$-convex* iff $\lambda x + (1-\lambda)y \in A$ for all $x, y \in A$ and $\lambda \in \mathbb{Q} \cap [0,1]$. A functional $p : A \to \mathbb{R}$ defined on a $\mathbb{Q}$-convex set A is called *J – convex* iff

$$p\left(\frac{x+y}{2}\right) \leq \frac{p(x)+p(y)}{2} \quad \text{for } x, y \in A. \tag{1}$$

The proof of the following theorem can be found in [2] (Theorem 10.1.1) where X is n–dimensional euclidean space $\mathbb{R}^n$. The proof in the general case differs from that one only formally.

THEOREM A. *Let $D \subseteq X$ be a $\mathbb{Q}$-convex and $\mathbb{Q}$-radial at a point $x_0 \in D$. Assume that $Y \subseteq X$ is a linear subspace over $\mathbb{Q}$ of X, $x_0 \in Y$ and $p : D \to \mathbb{R}$ is a J-convex function. If $f : Y \to \mathbb{R}$ is an additive function fulfilling*

$$f(x) \leq p(x) \quad \textit{for } x \in D \cap Y, \tag{2}$$

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then there exists an additive function $g : X \to \mathbb{R}$ *such that* $g|_Y = f$ *and*

$$g(x) \le p(x) \quad \textit{for } x \in D.$$

Suppose that the hypotheses of Theorem A hold. Put

$$C := (D - x_0) \cap (x_0 - D)$$

and

$$q(x) := p(x_0 + x) - f(x_0), \quad \text{for } x \in C.$$

We can observe that C is symmetric, $\mathbb{Q}$−convex, and $\mathbb{Q}$−radial at 0 and q is J-convex on C. Moreover the inequalities

$$(2') \qquad f(x) \le q(x) \quad \text{for } x \in C \cap Y,$$

$$(3') \qquad g(x) \le q(x) \quad \text{for } x \in C,$$

hold. Setting in (2′) $x = 0$, we get $0 = f(0) \le q(0)$. Next putting in (1) for the functional q, $y = -x$, $x \in C$ we obtain

$$0 \le q(0) \le \frac{1}{2}q(x) + \frac{1}{2}q(-x).$$

Consequently

$$-q(-x) \le q(x) \quad \text{for all } x \in C.$$

Now we can introduce a set-valued function on C with compact and convex values in $\mathbb{R}$ by the formula

$$F(x) = [-q(-x), q(x)], \quad x \in C.$$

It is easy to check that the set-valued function F fulfils the following conditions:

$$(1'') \qquad F(\lambda x + (1 - \lambda)y) \subseteq \lambda F(x) + (1 - \lambda)F(y)$$

for $x, y \in C$ and for $\lambda \in \mathbb{Q} \cap [0, 1]$,

$$(2'') \qquad f(x) \in F(x) \quad \text{for } x \in C \cap Y,$$

$$(3'') \qquad g(x) \in F(x) \quad \text{for } x \in C.$$

In addition F is an odd set-valued function, i.e.,

$$(3) \qquad F(-x) = -F(x) \quad \text{for } x \in C.$$

Conversely, if the set-valued function F fulfils conditions (1″), (2″) and (3″), then for q the relations (1), (2′) and (3′) hold.

In the next part of the paper we consider the family $\mathcal{F}$ of subsets of a linear space over $\mathbb{Q}$. We assume that $\mathcal{F}$ has the binary intersection property. It means that every subfamily of $\mathcal{F}$, any two members of which intersect has non-empty intersection (see [3]).

2. In [1] a version of the Hahn-Banach theorem for subadditive set-valued function is proved. In this paper we are going to give a new version of the Hahn-Banach theorem for concave set-valued function i.e., fulfilling the inclusion $(1'')$.

THEOREM 1. *Let X be a linear space over $\mathbb{Q}$, let $C \subseteq X$ be $\mathbb{Q}$-convex, $\mathbb{Q}$-radial at a point $x_0 \in C$ and $C = -C$. Assume that Y is a linear subspace over $\mathbb{Q}$ of X, $x_0 \in Y$. Furthermore, assume that $\mathcal{F}$ is a family of non-empty subsets of a linear space E over $\mathbb{Q}$ having the binary intersection property and fulfilling the conditions:*

$$A \in \mathcal{F},\ u \in \mathrm{E}\ \Rightarrow A + u \in \mathcal{F} \tag{4}$$

$$A \in \mathcal{F},\ \mu \in \mathbb{Q} \cap (0, \infty)\ \Rightarrow \mu A \in \mathcal{F}. \tag{5}$$

If a set-valued function $F : C \to \mathcal{F}$ fulfils conditions $(1'')$, (3) and $f : Y \to E$ is an additive function, which is a selection of the restriction of F to $C \cap Y$ (i.e., $(2'')$ holds), then there exists an additive extension $g : X \to E$ of f fulfilling $(3'')$.

Proof. Denote by Ω the family of all additive maps $\phi : \mathrm{dom}\,\varphi \to E$ such that $Y \subseteq \mathrm{dom}\,\varphi \subseteq X$, where $\mathrm{dom}\,\varphi$ is a linear subspace of X over $\mathbb{Q}$, $\varphi(x) \in F(x)$ for $x \in \mathrm{dom}\,\varphi \cap C$ and $\varphi(x) = f(x)$ for $x \in Y$. The family Ω is non-empty because f belongs to it. In this family we introduce the partial order "$\prec$" defined by $\varphi \prec \psi$ iff $\mathrm{dom}\,\varphi \subseteq \mathrm{dom}\,\psi$ and $\psi|_{\mathrm{dom}\,\varphi}$ coincides with φ. The family Ω is inductive. To see that take a non-empty chain $\mathcal{C} \subseteq \Omega$. Set $\varphi_{\mathcal{C}}(x) = \varphi(x)$ if $x \in \mathrm{dom}\,\varphi$ and $\varphi \in \mathcal{C}$. It is easy to see that $\varphi_{\mathcal{C}} \in \Omega$. This function is the upper bound of $\mathcal{C}$. Applying the Kuratowski-Zorn lemma we can gain a maximal element in Ω. It suffices to show that an arbitrary φ belonging to Ω whose domain is different from whole X cannot be maximal in Ω. Take $z \in X \setminus \mathrm{dom}\,\varphi$. Let Z be a linear subspace over $\mathbb{Q}$ of X spanned by $\mathrm{dom}\,\varphi$ and z. Choose $x, y \in C \cap \mathrm{dom}\,\varphi$, $\lambda, \mu \in (0, \infty) \cap \mathbb{Q}$ such that $x + \mu z, y + \lambda z, x - \mu z, y - \lambda z \in C$ (such x, y, λ, μ exist because C is $\mathbb{Q}$–radial at x_0 and $x_0 \in \mathrm{dom}\,\varphi$). We have

$$\frac{\lambda}{\lambda+\mu}\varphi(x) + \frac{\mu}{\lambda+\mu}\varphi(-y) = \varphi\Big(\frac{\lambda}{\lambda+\mu}x + \frac{\mu}{\lambda+\mu}(-y)\Big)$$

$$\in F\Big(\frac{\lambda}{\lambda+\mu}x + \frac{\mu}{\lambda+\mu}(-y)\Big) = F\Big(\frac{\lambda}{\lambda+\mu}(x+\mu z) + \frac{\mu}{\lambda+\mu}(-y-\lambda z)\Big)$$

$$\subseteq \frac{\lambda}{\lambda+\mu}F(x+\mu z) + \frac{\mu}{\lambda+\mu}F(-y-\lambda z).$$

Hence and by (3) we get

$$0 \in \frac{\lambda}{\lambda+\mu}[F(x+\mu z) - \varphi(x)] - \frac{\mu}{\lambda+\mu}[F(y+\lambda z) - \varphi(y)].$$

Thus

$$0 \in \frac{F(x+\mu z)-\varphi(x)}{\mu} - \frac{F(y+\lambda z)-\varphi(y)}{\lambda}$$

whence

$$\frac{F(x+\mu z)-\varphi(x)}{\mu} \cap \frac{F(y+\lambda z)-\varphi(y)}{\lambda} \neq \emptyset. \tag{6}$$

Similarly the relations

$$\begin{aligned} \frac{\lambda}{\lambda+\mu}\varphi(x)+\frac{\mu}{\lambda+\mu}\varphi(-y) &= \varphi\Big(\frac{\lambda}{\lambda+\mu}x+\frac{\mu}{\lambda+\mu}(-y)\Big) \\ \in F\Big(\frac{\lambda}{\lambda+\mu}x+\frac{\mu}{\lambda+\mu}(-y)\Big) &= F\Big(\frac{\lambda}{\lambda+\mu}(x-\mu z)+\frac{\mu}{\lambda+\mu}(-y+\lambda z)\Big) \\ &\subseteq \frac{\lambda}{\lambda+\mu}F(x-\mu z)-\frac{\mu}{\lambda+\mu}F(y-\lambda z) \end{aligned}$$

give

$$\frac{F(x-\mu z)-\varphi(x)}{-\mu} \cap \frac{F(y-\lambda z)-\varphi(y)}{-\lambda} \neq \emptyset. \tag{7}$$

We have also

$$\begin{aligned} \frac{\lambda}{\lambda+\mu}\varphi(-x)+\frac{\mu}{\lambda+\mu}\varphi(-y) &= \varphi\Big(\frac{\lambda}{\lambda+\mu}(-x)+\frac{\mu}{\lambda+\mu}(-y)\Big) \\ \in F\Big(\frac{\lambda}{\lambda+\mu}(-x)+\frac{\mu}{\lambda+\mu}(-y)\Big) &= F\Big(\frac{\lambda}{\lambda+\mu}(-x+\mu z)+\frac{\mu}{\lambda+\mu}(-y-\lambda z)\Big) \\ &\subseteq \frac{\lambda}{\lambda+\mu}F(-x+\mu z)-\frac{\mu}{\lambda+\mu}F(y+\lambda z). \end{aligned}$$

The same argument as above allows to get

$$\frac{F(x-\mu z)-\varphi(x)}{-\mu} \cap \frac{F(y+\lambda z)-\varphi(y)}{\lambda} \neq \emptyset. \tag{8}$$

Conditions (6), (7), (8) and the binary intersection property imply that there exists a $u \in E$ such that

$$u \in \bigcap \Big\{ \frac{F(x+\mu z)-\varphi(x)}{\mu} : x \in \operatorname{dom}\varphi \cap C, \mu \in \mathbb{Q}\setminus\{0\},\ x+\mu z \in C \Big\}.$$

Consequently

$$\varphi(x)+\lambda u \in F(x+\lambda z) \quad \text{for } x \in \operatorname{dom}\varphi, \lambda \in \mathbb{Q} \ \text{ such that } \ x+\lambda z \in C.$$

The function $\varphi_0 : Z \to E$ defined by $\varphi_0(x+\lambda z) := \varphi(x)+\lambda u$ is an additive extension of φ different from φ and fulfils the condition

$$\varphi_0(x) \in F(x) \quad \text{for } x \in Z \cap C.$$

Thus φ cannot be a maximal element in Ω. The proof is complete. ∎

Let X be a real linear space and $A \subseteq X$. We say that A is *radial at a point* $a \in A$ iff for every $x \in X$, $x \neq 0$ there exists an $\varepsilon > 0$ such that $a + \lambda x \in A$ for all $\lambda \in (-\varepsilon, \varepsilon)$.

Similar considerations as in the proof of Theorem 1 give the following result.

THEOREM 2. *Let X be a real linear space, C be a convex symmetric subset of X, and Y be a subspace of X. Assume that $\mathcal{F}$ is a family of non-empty subsets of a real linear space E having the binary intersection property and fulfilling condition* (4) *and*

$$A \in \mathcal{F},\ \mu \in (0, \infty) \Rightarrow \mu A \in \mathcal{F}.$$

If C is radial at a point $x_0 \in A, x_0 \in Y$ and a set-valued function $F : C \to \mathcal{F}$ is concave and fulfils condition (3), *$f : Y \to E$ is a linear function which is a selection of the restriction F to $Y \cap C$, then there exists a linear extension $g : X \to E$ of f fulfilling* ($3''$).

3. Let E denote *an ordered real linear space* i.e. E has a binary reflexive and transitive relation "$\leq$" such that

$$y_1 \leq y_2 \Rightarrow \lambda y_1 \leq \lambda y_2 \quad \text{for all} \quad y_1, y_2 \in E, \text{ and real } \lambda \geq 0,$$
$$y_1 \leq y_2 \Rightarrow y_1 + y_3 \leq y_2 + y_3 \quad \text{for all} \quad y_1, y_2, y_3 \in E.$$

We say that E has *the least upper bound property* (abbreviated: *l.u.b.p.*) iff every non-empty subset A of E which has an upper bound, has least upper bound.

As a consequence of Theorem 2 we can get the following theorem.

COROLLARY. *Let X be a linear space and let Y be a linear subspace of X. Assume that D is a convex subset of X, D is radial at $x_0 \in D, x_0 \in Y$ and $D = 2x_0 - D$. Moreover, assume that E is an ordered real linear space with l.u.b.p. If $p : D \to E$ is convex and $f : Y \to E$ is a linear function dominated by p on $Y \cap D$, then there is a linear extension $g : X \to E$ of f which is dominated by p on D.*

Proof. Consider the family $\mathcal{F}$ of all intervals $[a, b]$ in E, the set $C = (D - x_0) \cap (x_0 - D)$ and the set-valued function

$$F(x) := [-p(-x + x_0) + f(x_0), p(x + x_0) - f(x_0)].$$

One can easily see that all assumptions of Theorem 2 are fulfilled. Thus there exists a linear extension $g : X \to E$ of f such that

$$g(x) \in F(x) \quad \text{for } x \in C.$$

If $x \in x_0 + C$, then

$$\begin{aligned} g(x) &= g(x_0) + g(x - x_0) = f(x_0) + g(x - x_0) \\ &\leq f(x_0) + p(x - x_0 + x_0) - f(x_0) = p(x). \end{aligned}$$

This completes the proof. ∎

Our corollary gives Theorem 2.1 from [4] in the case $2x_0 - D = D$.

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