J. reine angew. Math. **775** (2021), 197–257
DOI 10.1515/crelle-2020-0039

Journal für die reine und angewandte Mathematik

Eigenfunction concentration via geodesic beams

By *Yaiza Canzani* at Chapel Hill and *Jeffrey Galkowski* at London

Abstract. We develop new techniques for studying concentration of Laplace eigenfunctions ϕ_λ as their frequency, λ, grows. The method consists of controlling $\phi_\lambda(x)$ by decomposing ϕ_λ into a superposition of geodesic beams that run through the point x. Each beam is localized in phase-space on a tube centered around a geodesic whose radius shrinks slightly slower than $\lambda^{-\frac{1}{2}}$. We control $\phi_\lambda(x)$ by the L^2-mass of ϕ_λ on each geodesic tube and derive a purely dynamical statement through which $\phi_\lambda(x)$ can be studied. In particular, we obtain estimates on $\phi_\lambda(x)$ by decomposing the set of geodesic tubes into those that are non-self-looping for time T and those that are. This approach allows for quantitative improvements, in terms of T, on the available bounds for L^∞-norms, L^p-norms, pointwise Weyl laws, and averages over submanifolds.

1. Introduction

On a smooth, compact, Riemannian manifold (M^n, g) with no boundary, we consider sequences of Laplace eigenfunctions $\{\phi_\lambda\}$ solving

$$(-\Delta_g - \lambda^2)\phi_\lambda = 0, \quad \|\phi_\lambda\|_{L^2(M)} = 1.$$

From a quantum mechanics point of view, $|\phi_\lambda(x)|^2$ represents the probability density for finding a quantum particle of energy λ^2 at the point $x \in M$. As a result, understanding how ϕ_λ concentrates across M is an important problem in the mathematical physics community.

In this article, we construct tools to examine the behavior of ϕ_λ by decomposing it into geodesic beams. To study how ϕ_λ concentrates near $x \in M$, we rewrite ϕ_λ as a sum of functions, each of which is microlocalized to a shrinking neighborhood of a geodesic that runs through x. The analysis of this decomposition, including a precise description of the L^∞-behavior of each geodesic beam, yields a bound on $\phi_\lambda(x)$ in terms of the local structure of the L^2-mass of ϕ_λ along each of the geodesic tubes starting at x. In addition, through an application of Egorov's Theorem, we obtain estimates on the growth of $\phi_\lambda(x)$ that rely only

The corresponding author is Jeffrey Galkowski.

Jeffrey Galkowski is grateful to the National Science Foundation for support under the Mathematical Sciences Postdoctoral Research Fellowship DMS-1502661. Yaiza Canzani is grateful to the Alfred P. Sloan Foundation.

on the dynamical behavior of geodesics emanating from x, and not on any other geometric structure of (M, g). Throughout the article, we refer to the tools developed here as *geodesic beam techniques*.

The term geodesic beam is inspired by Gaussian beams. Recall that, on the round sphere, these are eigenfunctions that concentrate in a $\lambda^{-\frac{1}{2}}$ neighborhood of a closed geodesic that have a Gaussian profile transverse to the geodesic. Gaussian beams have been extensively studied in the math and physics literature (see, e.g., [2, 4, 5, 19, 36, 42, 43, 58, 64]). Notably, Ralston [41] constructed quasimodes associated to stable periodic orbits modelled on Gaussian beams. These references concern modes associated to a single closed geodesic. In contrast, the methods developed here decompose functions into linear combinations of what we call geodesic beams. Each building block is similar to a Gaussian beam in that it is associated to a geodesic and concentrates in a small neighborhood thereof. However, three facts crucial to our construction are: that geodesic beams are only locally defined, that the geodesic need not close, and that they do not need to have a Gaussian profile transverse to the geodesic.

In this article we build the geodesic beam tools and illustrate their application by obtaining quantitative improvements to L^∞-norms for eigenfunctions on certain integrable geometries (see Section 5).

In addition, the techniques developed in this paper have remarkable implications in the study of L^∞-norms and averages of eigenfunctions, L^p-norms, and pointwise Weyl Laws. (See Section 1.2, Section 1.3, Section 1.4, respectively.) However, all of these applications require some additional non-trivial input, e.g., controlling looping behavior of geodesics in [12], understanding the local geometry of overlapping tubes in [14], and reduction of Weyl remainders to quasimode estimates in [15]. *We stress that the crucial technique in each application is that of geodesic beams, which are developed in this article*. We briefly describe the applications to L^∞-norms, averages, L^p-norms, and Weyl Laws now.

L^∞-norms. Beginning in the 1950s, the works [3, 32, 40] of Levitan, Avakumović, and Hörmander prove the estimate

$$\|\phi_\lambda\|_{L^\infty(M)} = O(\lambda^{\frac{n-1}{2}}) \quad \text{as } \lambda \to \infty;$$

known to be saturated on the round sphere. This bound was improved to $o(\lambda^{\frac{n-1}{2}})$ by Sogge, Toth, Zelditch and the second author [25, 26, 47, 49–51] under various dynamical assumptions at x. Notably, [49] was the first to study L^∞-bounds under purely local dynamical assumptions. When (M, g) has no conjugate points, a quantitative improvement of the form

$$\|\phi_\lambda\|_{L^\infty} = O\left(\frac{\lambda^{\frac{n-1}{2}}}{\sqrt{\log \lambda}}\right)$$

has been known since the classical work of Bérard [6, 10, 44]. However, until the present time, no quantitative improvements were available without *global geometric* assumptions on (M, g). In Section 1.2 we present applications of our geodesic beam techniques giving such improvements.

Averages. Another measure of eigenfunction concentration is the average over a submanifold $H \subset M$ of codimension k. In this case, the general bound

$$\int_H \phi_\lambda \, d\sigma_H = O(\lambda^{\frac{k-1}{2}})$$

was proved by Zelditch [63] and is saturated on the round sphere. This generalized the work of Good and Hejhal [28, 31]. Chen and Sogge [17] were the first to obtain a refinement on the standard bounds. This work has since been improved under various assumptions by Sogge, Xi, Zhang, Wyman, Toth, and the authors [13,16,48,59–62]. As before, none of these results obtain quantitative improvements without global geometric assumptions on (M, g). In Section 1.2 we present applications of our geodesic beam techniques giving such improvements.

L^p-norms. Since the seminal work of Sogge [46], it has been known that

$$\|\phi_\lambda\|_{L^p(M)} = O(\lambda^{\delta(p,n)}),$$

where $\delta(p, n)$ depends on how p compares to the critical exponent $p_c = \frac{2(n+1)}{n-1}$. Namely, $\delta(p, n) = \frac{n-1}{2} - \frac{n}{p}$ if $p \geq p_c$ and $\delta(p, n) = \frac{n-1}{4} - \frac{n-1}{2p}$ if $2 \leq p \leq p_c$. When (M, g) has non-positive sectional curvature, Hassell and Tacy [30] gave quantitative gains over this estimate of the form $O(\lambda^{\delta(p,n)}/(\log\lambda)^{\sigma(p,n)})$ when $p > p_c$ and with $\sigma(p, n) = \frac{1}{2}$. Blair and Sogge [8, 9] also obtained an improvement when $2 < p \leq p_c$ for some $\sigma(p, n) > 0$ smaller than $\frac{1}{2}$. In Section 1.3 we will present applications of our geodesic beam techniques which yield $\sqrt{\log\lambda}$ improvements for L^p-norms with $p > p_c$, generalizing those of [30].

Weyl Laws. Let $\{\lambda_j^2\}_j$ be the Laplace eigenvalues of (M, g). It is well known that

$$\#\{j : \lambda_j \leq \lambda\} = \frac{\mathrm{vol}(B^n)\,\mathrm{vol}(M)}{(2\pi)^n}\lambda^n + E(\lambda)$$

with $E(\lambda) = O(\lambda^{n-1})$ as $\lambda \to \infty$, where $B^n \subset \mathbb{R}^n$ is the unit ball. Indeed, this is the integrated version of the more refined statement proved by Hörmander in [32] which says that

$$\sum_{\lambda_j \leq \lambda} |\phi_{\lambda_j}(x)|^2 = \frac{\mathrm{vol}(B^n)}{(2\pi)^n}\lambda^n + E(\lambda, x) \quad \text{for all } x \in M,$$

with $E(\lambda, x) = O(\lambda^{n-1})$ uniform for $x \in M$. This estimate has been improved by Sogge and Zelditch [49] and Bérard [6] under various dynamical assumptions. In Section 1.4 we present improvements of these results based on geodesic beam techniques.

1.1. Main results: Localizing eigenfunctions near geodesic tubes. In this subsection we present Theorems 1 and 2, which are our main estimates for Laplace eigenfunctions. In Section 2 we present much more general versions of these two results, Theorems 10 and 11, that hold for quasimodes of more general operators.

In fact, we work in the semiclassical framework, writing $\lambda = h^{-1}$ and letting $h \to 0^+$. Then, relabeling $\phi_\lambda = \phi_h$, we study

$$(-h^2\Delta_g - 1)\phi_h = 0, \quad \|\phi_h\|_{L^2(M)} = 1. \tag{1.1}$$

This rescaling is useful because it allows us to work in compact subsets of phase space, and in particular, near the cosphere bundle S^*M where geodesic dynamics naturally take place.

Our main results give an estimate for ϕ_h near a point $x \in M$. We now introduce the necessary objects to state these estimates. We will work with a cover of S_x^*M by short geodesic tubes $\Lambda_\rho^\tau(R(h)) \subset T^*M$. This notation roughly means that the geodesic tube, $\Lambda_\rho^\tau(R(h))$, is the flowout of a ball of radius $R(h)$ around ρ for times $t \in [-\tau - R(h), \tau + R(h)]$. We will, in fact, take $\tau > 0$ small. This is similar to an $R(h)$ thickening (with respect to the Sasaki metric on T^*M) of the geodesic of length 2τ centered at $\rho \in S^*_xM$ (see (2.12) for a precise definition). We say that $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j=1}^{N_h}$ is a $(\tau, R(h))$-*cover* of S^*_xM if it covers $\Lambda^\tau_{S^*_xM}(\frac{1}{2}R(h))$ (see Definition 3 for the definition of a cover and (2.11) for the definition of $\Lambda^\tau_{S^*_xM}(\frac{1}{2}R(h))$).

In addition, a δ-*partition* of S^*_xM associated to the $(\tau, R(h))$-cover is a collection of functions $\{\chi_j\}_{j=1}^{N_h} \subset S_\delta(T^*M;[0,1])$ so that each χ_j is supported in the tube $\Lambda^\tau_{\rho_j}(R(h))$ and with the property that $\sum_{j=1}^{N_h}\chi_j \geq 1$ on $\Lambda^\tau_{S^*_xM}(\frac{1}{2}R(h))$. (See Appendix A.2 for a description the symbol class S_δ, and Definition 3 for the definition of a δ-partition.)

The functions χ_j are used to microlocalize ϕ_h to the tubes $\Lambda^\tau_{\rho_j}(R(h))$. We refer to $Op_h(\chi_j)\phi_h$ as a *geodesic beam through* x. They are constructed in Proposition 3.4 and have the additional property that $Op_h(\chi_j)$ nearly commutes with $(-h^2\Delta_g - 1)$ near x (so that these localizers do not destroy the property of being a quasimode locally near x). (See also Step 2 in the proof of Theorem 10.) The fact that $Op_h(\chi_j)$ nearly commutes with $(-h^2\Delta_g - 1)$ requires that we work with geodesic tubes of positive length, τ, independent of h rather than localizing to balls of radius $R(h)$ centered in S^*_xM.

In the following result, we control $\phi_h(x)$ by the L^2-mass of the geodesic beams through x.

Theorem 1. *Let $x \in M$. There exist $\tau_0 = \tau_0(M,g) > 0$, $R_0 = R_0(M,g) > 0$, $C_n > 0$ depending only on n, so that the following holds.*

Let $0 < \tau \leq \tau_0$, $0 \leq \delta < \frac{1}{2}$, and $8h^\delta \leq R(h) \leq R_0$. Let $\{\chi_j\}_{j=1}^{N_h}$ be a δ-partition for S^*_xM associated to a $(\tau, R(h))$-cover. Let $N > 0$. Then there are $h_0 = h_0(M, g, \{\chi_j\}, \delta) > 0$ and $C_N > 0$ with the property that for any $0 < h < h_0$ and ϕ_h satisfying (1.1)*,*

$$\|\phi_h\|_{L^\infty(B(x,h^\delta))} \leq C_n\tau^{-\frac{1}{2}}h^{\frac{1-n}{2}}R(h)^{\frac{n-1}{2}}\sum_{j=1}^{N_h}\|Op_h(\chi_j)\phi_h\|_{L^2(M)} + C_Nh^N\|\phi_h\|_{L^2(M)}.$$

Moreover, the constants h_0 and C_N are uniform for χ_j in bounded subsets of S_δ.

Crucially, this estimate makes no assumptions on the geometry of M or the dynamics of the geodesic flow. Information on the dynamics of the geodesic flow will later allow us to control the L^2-mass of the geodesic beams (see Theorem 2).

This result is a consequence of the more general and stronger result given in Theorem 10 below. (See Remark 6 for the proof.) Indeed, the latter is stated as a bound for $\int_H u_h\,d\sigma_H$, where $H \subset M$ is a general submanifold and u_h is a quasimode for a pseudodifferential operator with a real, classically elliptic symbol with respect to which H is conormally transverse. Note that when $H = \{x\}$, we have $\int_H u_h\,d\sigma_H = u_h(x)$. See Section 2 for a detailed description.

One can conclude from Theorem 1 that, in order to have maximal sup-norm growth at a point, an eigenfunction must have a component with L^2-norm bounded from below that is distributed in the same way as the canonical example on the sphere (up to scale h^δ for all $\delta < \frac{1}{2}$). Indeed, if one restricts attention to (τ, r) covers of S^*_xM without too many overlaps (see Definition 4) it follows from Theorem 1 that there exists $C_n > 0$, so that for all $\varepsilon > 0$, if

$$\#\left\{j : \varepsilon^2R(h)^{n-1} \leq \|Op_h(\chi_j)\phi_h\|^2_{L^2(M)} \leq \frac{R(h)^{n-1}}{\varepsilon^2}\right\} \leq \varepsilon^2N_h,$$

then $\|\phi_h\|_{L^\infty(B(x,h^\delta))} \leq \varepsilon C_n\tau^{-\frac{1}{2}}h^{\frac{1-n}{2}}$.

To understand Theorem 1 heuristically, one should think of $\|Op_h(\chi_j)\phi_h\|_{L^2(M)}$ as measuring the L^2-mass of ϕ_h on the tube of radius $R(h)$ around a geodesic that runs through the point x. Since vol(supp $\chi_j) \asymp R(h)^{n-1}$, an individual term in the sum in Theorem 1 is then

$$R(h)^{\frac{n-1}{2}}\|Op_h(\chi_j)\phi_h\|_{L^2(M)} \asymp \left(\frac{\|Op_h(\chi_j)\phi_h\|^2_{L^2(M)}}{\mathrm{vol}(\mathrm{supp}\,\chi_j)}\right)^{\frac{1}{2}} \mathrm{vol}(\mathrm{supp}\,\chi_j),$$

where vol is the volume measure on S^*_xM induced by the Sasaki metric on T^*M. In particular, the sum on the right of the estimate in Theorem 1 can be interpreted as $\int_{S^*_xM} |\frac{d\mu}{d\,\mathrm{vol}}|^{\frac{1}{2}}\, d\,\mathrm{vol}$, where μ is the measure giving the distribution of the mass squared of ϕ_h on S^*_xM. This statement can be made precise by using defect measures (see [13, Theorem 6]), but the results using defect measures can only be used to obtain $o(1)$ improvements on eigenfunction bounds.

We emphasize now that Theorem 1 is the key estimate for the proofs of all the applications to L^∞-norms, L^p-norms, and Weyl Laws stated in Sections 1.2, 1.3, 1.4, respectively.

At first sight it may seem that it is not easy to extract information from the upper bound provided in Theorem 1. However, the strength of this bound is showcased in our next result, Theorem 2. The latter combines the analytical bound of Theorem 1 together with Egorov's Theorem to obtain a purely dynamical statement. Indeed, $\phi_h(x)$ is controlled by covers of $\Lambda^\tau_{S^*_xM}(\frac{1}{2}R(h))$ by "good" tubes that are non-self-looping under the geodesic flow,

$$\varphi_t := \exp(tH_{|\xi|_g})$$

(where $H_{|\xi|_g}$ is the Hamiltonian vector field of $|\xi|_g$), and "bad" tubes whose number is small.

Definition 1 (non-self-looping sets). For $0 < t_0 < T_0$, we say that $A \subset T^*M$ is $[t_0, T_0]$ *non-self-looping* if

$$\bigcup_{t=t_0}^{T_0} \varphi_t(A) \cap A = \emptyset \quad \text{or} \quad \bigcup_{t=-T_0}^{-t_0} \varphi_t(A) \cap A = \emptyset. \tag{1.2}$$

The goal of our next result is to obtain quantitative control of $\phi_h(x)$ by splitting the geodesic tubes into "good" tubes $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j\in\mathcal{G}_\ell}$ that are $[t_\ell, T_\ell]$ non-self-looping and "bad" tubes $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j\in\mathcal{B}}$ that may be self-looping. The quantitative control is then given in terms of t_ℓ, T_ℓ, $|\mathcal{G}_\ell|$, and $|\mathcal{B}|$. Recall that $\tau > 0$ is a small parameter so the tubes $\Lambda^\tau_\rho(R(h))$ do not see the global dynamical structure of the geodesic flow. It is only when $T_\ell \gg \tau$ that one encounters this information.

It is convenient to work with covers by tubes for which the number of overlaps is controlled. Indeed, we say that a $(\tau, R(h))$- covering by tubes is a $(\mathfrak{D}, \tau, R(h))$-good covering, if it can be split into $\mathfrak{D} > 0$ families of disjoint tubes. See Definition 4 for a precise definition. In Proposition 3.3 we prove that one can always work with $(\mathfrak{D}_n, \tau, R(h))$-good coverings, where $\mathfrak{D}_n$ only depends on n.

In what follows we write $\Lambda_{\max}$ for the maximal expansion rate of the flow and $T_e(h)$ for the Ehrenfest time

$$T_e(h) := \frac{\log h^{-1}}{2\Lambda_{\max}}$$

(see (2.14)).

Theorem 2. *Let $x \in M$, $0 < \delta < \frac{1}{2}$. There exist positive constants $h_0 = h_0(M, g, \delta)$, $\tau_0 = \tau_0(M, g)$, $R_0 = R_0(M, g)$, and C_n depending only on n, so that for all $0 < \tau \leq \tau_0$ and $0 < h < h_0$ the following holds.*

Let $8h^\delta \leq R(h) \leq R_0$, and $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j=1}^{N_h}$ be a $(\mathfrak{D}, \tau, R(h))$-good cover for $S^*_x M$ for some $\mathfrak{D} > 0$. Let $0 \leq \alpha < 1 - 2\limsup_{h\to 0} \log R(h)/\log h$ and suppose there exists a partition of $\{1, \dots, N_h\}$ into $\mathcal{B}$ and $\{\mathcal{G}_\ell\}_{\ell\in\mathcal{L}}$ such that for every $\ell \in \mathcal{L}$ there exist $T_\ell = T_\ell(h) > 0$ and $t_\ell = t_\ell(h) > 0$ with $t_\ell(h) \leq T_\ell(h) \leq 2\alpha T_e(h)$ such that

$$\bigcup_{j\in\mathcal{G}_\ell} \Lambda^\tau_{\rho_j}(R(h)) \textit{ is } [t_\ell, T_\ell] \textit{ non-self-looping.}$$

Then, for all $N > 0$, there exists $C_N = C_N(M, g, N, \tau, \delta) > 0$ so that for ϕ_h solving (1.1)*,*

$$\|\phi_h\|_{L^\infty(B(x,h^\delta))} \leq C_n \mathfrak{D} \tau^{-\frac{1}{2}} h^{\frac{1-n}{2}} R(h)^{\frac{n-1}{2}} \left(|\mathcal{B}|^{\frac{1}{2}} + \sum_{\ell\in\mathcal{L}} \frac{|\mathcal{G}_\ell|^{\frac{1}{2}} t_\ell^{\frac{1}{2}}}{T_\ell^{\frac{1}{2}}} \right) \|\phi_h\|_{L^2(M)} + C_N h^N \|\phi_h\|_{L^2(M)}.$$

Remark 1. Note that, since the tubes $\Lambda^\tau_{\rho_j}(R(h))$ are essentially time τ flowouts of balls around ρ_j with radius $R(h)$, if the ball of radius $R(h)$ around ρ_j is $[t - \tau, T + \tau]$ non-self-looping, then $\Lambda^\tau_{\rho_j}(R(h))$ is $[t, T]$ non-self-looping. Therefore, we could replace the non-self-looping assumption on $\Lambda^\tau_{\rho_j}(R(h))$ in Theorem 2 by an analogous non-self-looping assumption on $B(\rho_j, R(h))$. Note, however, that these balls *cannot* be replaced by balls inside $S^*_x M$. We need them to have full dimension so that smooth cutoffs can be supported inside $\Lambda^\tau_\rho(R(h))$. Moreover, it is necessary that they encode quantitative information on how geodesics near the center of $\Lambda^\tau_\rho(R(h))$ return close to x.

This result is a consequence of the more general and stronger result given in Theorem 11. See Remark 7 for the proof. As with the previous theorem, the generalization is stated for averages over submanifolds of quasimodes of general operators. See Section 2 for a detailed explanation. For examples where Theorem 2 is applicable see Section 1.2.2 and Section 1.5.

We note that Theorem 2 distinguishes much finer features than that of self-conjugacy with maximal multiplicity. Indeed, the theorem can be used to obtain estimates at points *all* of whose geodesics return; provided the geodesics through the point have some additional non-recurrent structure (e.g., the umbilic points on the triaxial ellipsoid; see Section 1.5). In particular, this estimate distinguishes recurrent structure and non-recurrent structure as in Definition 2. At this point, we do not know to what extent it distinguishes periodic structure from recurrent structure.

Theorem 2 reduces estimates on $\phi_h(x)$ to the construction of covers of $\Lambda^\tau_{S^*_x M}(\frac{1}{2}R(h))$ by sets with appropriate structure. Here $\Lambda^\tau_{S^*_x M}(\frac{1}{2}R(h))$ denotes a $\frac{1}{2}R(h)$ thickening of the set of geodesics through x, see (2.11). If there is a cover of $\Lambda^\tau_{S^*_x M}(\frac{1}{2}R(h))$ by "good" sets $\{G_\ell\}_{\ell\in L}$ and a "bad" set B, with every G_ℓ being $[t_\ell(h), T_\ell(h)]$ non-self-looping, the estimate reads

$$\|\phi_h\|_{L^\infty(B(x,h^\delta))} \leq C_n \mathfrak{D} \tau^{-\frac{1}{2}} h^{\frac{1-n}{2}} \left([\mathrm{vol}(B)]^{\frac{1}{2}} + \sum_{\ell\in\mathcal{L}} \frac{[\mathrm{vol}(G_\ell)]^{\frac{1}{2}} t_\ell^{\frac{1}{2}}}{T_\ell^{\frac{1}{2}}} \right) \|\phi_h\|_{L^2(M)},$$

where vol denotes the volume induced on $S^*_x M$ by the Sasaki metric on T^*M, and where we write $\mathrm{vol}(A) = \mathrm{vol}(A \cap S^*_x M)$ for $A \subset T^*M$. The additional structure required on the sets G_ℓ and B is that they consist of a union of tubes $\Lambda^\tau_{\rho_i}(R(h))$ and that $T_\ell(h) < 2(1 - 2\delta)T_e(h)$.

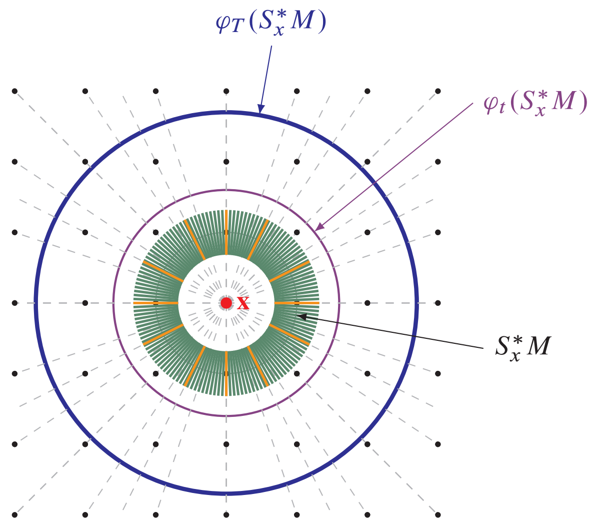

Figure 1

With this in mind, Theorem 2 should be thought of as giving a non-recurrent condition on S_x^*M which guarantees quantitative improvements over the standard bounds (see Definition 2 for a precise explanation of what we mean by non-recurrent structure). In particular, taking T_ℓ, t_ℓ, G_ℓ and B to be h-independent can be used to recover the dynamical consequences in [13,25] (see [24] and Section 1.6).

In Section 5 we illustrate how to build covers by good and bad tubes in some integrable geometries, and how to use them to obtain quantitative improvements over the known L^∞-bounds. In the figure we illustrate how to cover S_x^*M with "good" tubes (green) and "bad" tubes (orange) for a point x on the square flat torus. The grid represents the integer lattice on the universal cover of the torus. In Figure 1, there is *only one index* i.e. $\ell = 1$, and we chose $t_\ell = t = 1.6$, $T_\ell = T = 2.7$, $\tau = 0.2$, and $R = 0.01$. In the figure, the length of the green/orange tubes is $2(\tau + R)$. Note that some of the green tubes *are not* $[3\tau, T]$ non-self-looping but are $[t, T]$ non-self-looping, e.g., the tube at angle $\pi/4$. In practice, to obtain quantitative gains, one needs to work with $T \to \infty$. The figure is drawn for *one* relatively small T because choosing a larger T makes the figure illegible. A tube is "bad" if the geodesic generated by it returns to x in time between t and T. Note, in addition, that t_ℓ must be positive since our tubes have finite, positive width in the flow direction. Also, a set *may* be $[t_0, T]$ self-looping, but not $[\tilde{t}_0, T]$ self-looping for some $\tilde{t}_0 > t_0$, e.g., a neighborhood, $U \setminus V \subset T^*M$, where U is a neighborhood around an unstable hyperbolic closed geodesic in phase space and V is a slightly smaller neighborhood. While, at the moment we do not have examples where it is necessary to send $t_\ell \to \infty$ with h, we anticipate this will be useful in the future.

To understand why it is in general useful to have families of tubes $\mathcal{G}_\ell$ with different looping times, $[t_\ell, T_\ell]$, we consider the following setup. We assume that the geodesic flow is exponentially contracting in the sense that

$$\|d\varphi_t|_{S_x^*M}\| \le Ce^{-Ct}.$$

For simplicity, let $\dim M = 2$. The way in which we work with the assumption on the geodesic flow is that the flow out of an arc of length R in S_x^*M will have length $e^{-CT}R$ upon return to S_x^*M at time T. We, in general, do not have information about the place to which the arc returns. Suppose we want to cover $S^*_x M$ with tubes of radius R and divide them into $[t_0, T(h)]$ non-self-looping collections $\mathcal{G}_\ell$ such that Theorem 2 gives a $\log h^{-1}$ gain. Note that, for simplicity, we identify each tube with the arc of length R that is formed by its intersection with $S^*_x M$. Since $R \geq h^\delta$, and, in order to get a $\log h^{-1}$ improvement, we must take $T(h) \sim \log h^{-1}$, we have $R \geq e^{-CT(h)}$.

To simplify the situation further, we discretize the time and imagine that the return map, Φ, has the properties above. To produce a non-self-looping collection, we start with an arc A_0 of length ~ 1. To construct a $[t_0, T(h)]$ non-self-looping set, G_0, we let

$$A_1 := \bigcup_{t_0 \leq k \leq T(h)} \Phi^k(A_0) \cap A_0, \quad G_0 := A_0 \setminus A_1.$$

Since we do not know the directions in which A_0 returns, A_1 a priori consists of intervals of size $e^{-C}, e^{-2C}, \dots, e^{-CT(h)}$. Hence, A_1 has volume $\sim e^{-C}$ and is $[t_0, T(h)]$ self-looping. In order to get a $T(h)^{-1}$ improvement with only one $T_\ell(h) = T(h)$, any set which is $[t_0, T(h)]$ self-looping must have volume $\leq CT(h)^{-1}$. Since A_1's volume is $\gg T(h)^{-1}$, we must iterate this process by putting

$$A_\ell := \bigcup_{t_0 \leq k \leq T(h)} \Phi^k(A_{\ell-1}) \cap A_{\ell-1}, \quad G_{\ell-1} := A_{\ell-1} \setminus A_\ell.$$

A priori, A_ℓ has volume $\sim e^{-C\ell}$, is $[t_0, T(h)]$ self-looping, and consists of intervals of size $e^{-C\ell}, e^{-C(\ell+1)}, \dots, e^{-C(T(h)+\ell)}$. Therefore, in order to gain $T(h)^{-1}$ in our estimates, we must iterate until $e^{-C\ell} \sim T(h)^{-1}$. That is, $\ell(h) \sim \log T(h)$. Note that in this case the smallest arc in $A_{\ell(h)}$ has length

$$e^{-C(T(h)+\ell(h))} \sim h^C T(h)^{-C}.$$

Now, depending on C, this may be $\ll h^\delta$, which is the scale of our cover. There are a two ways around this. We could shrink $T(h)$ so that this scale is above R. However, this would be somewhat unnatural since then our dynamical gain would necessarily depend on the contraction rate. So that we may use our original $T(h)$, while still having a scale above h^δ, we shrink the non-self-looping times at each step so that G_ℓ is $e^{-\frac{C\ell}{2}} T(h)$ non-self-looping. In doing this, we have that G_ℓ is $[t_0, e^{-\frac{C\ell}{2}} T(h)]$ non-self-looping and has volume $\sim e^{-C\ell}$. In addition, the minimum size of an interval in A_ℓ is $e^{-\sum_{j=0}^{\ell} e^{-Cj/2} T(h)}$. Iterating until $\ell \sim \log T(h)$, then enables us to obtain our estimates.

In the following subsections, Section 1.2, Section 1.3, Section 1.4, we showcase a few of the many applications of Theorem 2 in obtaining quantitative improvements for L^∞-norms, L^p-norms, pointwise Weyl laws, and averages over submanifolds.

1.2. Improvements to L^∞-norms and averages. In this subsection we introduce some of the applications of geodesic beam techniques to the study of the L^∞-norms of ϕ_h, and of the averages $\int_H \phi_h \, d\sigma_H$ over a submanifold $H \subset M$. The goal is to obtain quantitative improvements on the known bounds [32, 63]

$$\phi_h(x) = O(h^{\frac{1-n}{2}}) \quad \text{and} \quad \int_H \phi_h(x)\, d\sigma_H = O(h^{\frac{1-k}{2}}), \tag{1.3}$$

where k is the codimension of H. These bounds are sharp since they are, for example, saturated on the round sphere. Note that the right-hand estimate includes the left if we take $H = \{x\}$. In Section 1.2.1 we present applications of our geodesic beam techniques to studying eigenfunction growth on manifolds with no conjugate points, or whose geometries satisfy a weaker condition. These results, and many more, can be found in [12]. In Section 1.2.2 we present applications to obtaining quantitative improvements of L^∞-norms in integrable geometries. The proofs of these and more general results are presented in Section 5.

1.2.1. Results under conjugate point assumptions. It is known that the L^∞-bound in (1.3) is saturated on the round sphere if one chooses ϕ_h to be a zonal harmonic that peaks at the given point $x \in S^n$. This phenomenon is possible since all geodesics through x are closed. In addition, on the sphere every point is maximally self-conjugate. In general, a point $x \in M$ is said to be conjugate to $y \in M$ if there exists a unit speed geodesic γ joining x and y, together with a non-trivial Jacobi field along γ that vanishes at x and y. The number of such Jacobi fields that are linearly independent is called the multiplicity of x with respect to y and is always bounded by $n-1$. When the multiplicity equals $n-1$ the point x is said to be maximally conjugate to y. As a consequence of our geodesic beam techniques, we obtain quantitative improvements on the L^∞-norm of an eigenfunction near a point x that, loosely speaking, is not maximally self-conjugate.

Consider the set Ξ of unit speed geodesics on (M, g) and define

$$\mathcal{C}_x^{r,t} := \{\gamma(t) : \gamma \in \Xi,\ \gamma(0) = x,\ \exists\, n-1 \text{ conjugate points to } x \text{ in } \gamma(t-r, t+r)\}, \tag{1.4}$$

where we count conjugate points with multiplicity. Note that if $r_t \to 0^+$ as $|t| \to \infty$, then saying that $x \in \mathcal{C}_x^{r_t,t}$ for t large indicates that x behaves like a point that is maximally self-conjugate. This is the case for every point on the sphere. The following result applies under the assumption that this does not happen and obtains quantitative improvements in that setting. The obvious case where our next theorem applies is that of manifolds without conjugate points, where $\mathcal{C}_x^{r,t} = \emptyset$ for $0 < r < |t|$. In addition, the theorem applies to *all* non-trivial product manifolds $M = M_1 \times M_2$ (see Section 1.5).

Theorem 3 ([12, Theorem 1]). *Let $V \subset M$ and assume that there exist $t_0 > 0$ and $a > 0$ so that*

$$\inf_{x\in V} d(x, \mathcal{C}_x^{r_t,t}) \geq r_t \quad \textit{for } t \geq t_0$$

with $r_t = \frac{1}{a} e^{-at}$. Then there exist $C > 0$ and $h_0 > 0$ so that for $0 < h < h_0$ and $u \in \mathcal{D}'(M)$

$$\|u\|_{L^\infty(V)} \leq C h^{\frac{1-n}{2}} \left(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h} \|(-h^2\Delta_g - 1)u\|_{H_{\mathrm{scl}}^{(n-3)/2}(M)} \right).$$

For a definition of the semiclassical Sobolev spaces H^s_{scl} see (A.3). Here and below, when we write $\|v\|_{H^s_{\mathrm{scl}}}$ for some $v \in \mathcal{D}'$ with $v \notin H^s_{\mathrm{scl}}$, we define $\|v\|_{H^s_{\mathrm{scl}}} = \infty$.

Before stating our next theorem, we recall that if (M, g) has strictly negative sectional curvature, then it also has Anosov geodesic flow [1]. Also, both Anosov geodesic flow [37] and non-positive sectional curvature imply that (M, g) has no conjugate points.

Theorem 4 ([12, Theorems 3 and 4]). *Let (M, g) be a smooth, compact Riemannian manifold of dimension n. Let $H \subset M$ be a closed embedded submanifold of codimension k. Suppose one of the following assumptions holds:*

(A) *(M, g) has no conjugate points and H has codimension $k > \frac{n+1}{2}$.*

(B) *(M, g) has no conjugate points and H is a geodesic sphere.*

(C) *(M, g) is a surface with Anosov geodesic flow.*

(D) *(M, g) is non-positively curved and has Anosov geodesic flow, and H has codimension $k > 1$.*

(E) *(M, g) is non-positively curved and has Anosov geodesic flow, and H is totally geodesic.*

(F) *(M, g) has Anosov geodesic flow and H is a subset of M that lifts to a horosphere in the universal cover.*

Then there exists $C > 0$ so that for all $w \in C_c^\infty(H)$ the following holds. There is $h_0 > 0$ so that for $0 < h < h_0$ and $u \in \mathcal{D}'(M)$,

$$\left|\int_H wu \, d\sigma_H\right| \le C h^{\frac{1-k}{2}} \|w\|_\infty \bigg(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h} \|(-h^2\Delta_g - 1)u\|_{H_{\mathrm{scl}}^{(k-3)/2}(M)} \bigg). \tag{1.5}$$

Remark 2. Note that while $C > 0$ in (1.5) is independent of w, the choice of $h_0 > 0$ depends on high order derivatives of w.

To the authors' knowledge, the results in [12] improve and extend *all* existing bounds on averages over submanifolds for eigenfunctions of the Laplacian, including those on L^∞-norms (without additional assumptions on the eigenfunctions; see Remark 8 for more detail on other types of assumptions). Our estimates imply those of [13] and therefore give all previously known improvements of the form

$$\int_H u \, d\sigma_H = o(h^{\frac{1-k}{2}}).$$

Moreover, we are able to improve upon the results of [6, 10, 44, 48, 60, 62].

1.2.2. Integrable geometries. Next, we present a class of integrable geometries for which $\log h^{-1}$ improvements over the standard bounds are a consequence of Theorem 2 and its generalization, Theorem 11. We apply Theorem 11 to the case of Schrödinger operators, $-h^2\Delta_g + V$, acting on spheres of revolution where the bicharacteristic flow is integrable. When $V = 0$, these examples give manifolds with many conjugate points where we are able to obtain quantitatively improved L^∞-bounds away from the poles of S^2.

To state our results, we identify the surface of revolution M with $[0, \pi] \times S^1$ endowed with the metric $g(r, \theta) = dr^2 + \alpha(r)^2 d\theta^2$. We then consider operators of the form

$$P(h) = -h^2\Delta_g - V$$

with $V > 0$. The Hamiltonian for this problem is then

$$p(\theta, r, \xi_\theta, \xi_r) = \xi_r^2 + \frac{1}{\alpha(r)^2}\xi_\theta^2 - V(r)$$

and we assume that the map $r \mapsto \alpha(r)\sqrt{V(r)}$ has a single critical point at $r = r_s$ which is a non-degenerate maximum. In order that M be equivalent to a sphere, $\alpha(r)$ must satisfy $\alpha^{(2k)}(0) = 0$ and $\alpha^{(2k)}(\pi) = 0$ for all non-negative integers k.

Since $\{p, \xi_\theta\} = 0$, it follows that the pair (M, p) yields an integrable system on T^*M. Let $(\Theta, I) \in \mathbb{T}^2 \times \mathbb{R}^2$ be action-angle coordinates so that $T^*M = \bigsqcup_{I \in \mathbb{R}^2} \mathbb{T}_I$ is the foliation by Liouville tori (possibly with some singular elements). That is, in the (Θ, I) coordinates $p = p(I)$ and hence the Hamiltonian flow is given by

$$\varphi_t(\Theta, I) = (\Theta + t\partial_I p(I), I).$$

There is a single singular torus corresponding to the closed Hamiltonian bicharacteristic

$$\gamma_s := \{r = r_s\}.$$

In addition, we make the following assumption:

(i) The map $\{p = 0\} \ni I \mapsto \partial_I p(I) \in \mathbb{RP}^2$ is a diffeomorphism. When this is the case at I_0, we say p is iso-energetically non-degenerate at I_0 on $\{p = 0\}$.

Theorem 5. *Let α and V satisfy the assumptions above. Then, for*

$$P = -h^2\Delta_g - V(r) + hQ \tag{1.6}$$

with $Q \in \Psi^2(M)$ self-adjoint, and $K \subset [0, 2\pi] \times (0, \pi)$ compact, there exists $C > 0$ with the following properties. For all $L > 0$ there exists $h_0 > 0$ so that for $0 < h < h_0$, and $u \in \mathcal{D}'(M)$,

$$\|u\|_{L^\infty(K)} \le Ch^{-\frac{1}{2}}\left(\frac{\|u\|_{L^2(M)}}{L\sqrt{\log h^{-1}}} + \frac{L\sqrt{\log h^{-1}}\|Pu\|_{H^{-1/2}_{\mathrm{scl}}(M)}}{h}\right).$$

In particular, if

$$\|Pu\|_{H^{-1/2}_{\mathrm{scl}}(M)} = o\left(\frac{h\|u\|_{L^2(M)}}{\log h^{-1}}\right),$$

then

$$\|u\|_{L^\infty(K)} = o\left(\frac{h^{-\frac{1}{2}}}{\sqrt{\log h^{-1}}}\|u\|_{L^2(M)}\right).$$

Remark 3. Note that we make no assumptions on u. In particular, u need not be a joint eigenfunction of the quantum completely integrable system. Furthermore, the addition of the perturbation hQ (for Q general) destroys the quantum complete integrability of the operator.

1.3. Logarithmic improvements for L^p-norms. Since the work of Sogge [46] it has been known that

$$\|\phi_h\|_{L^p(M)} = O(h^{-\delta(p,n)}), \quad \delta(p,n) = \begin{cases} \frac{n-1}{2} - \frac{n}{p}, & p \ge p_c, \\ \frac{n-1}{4} - \frac{n-1}{2p}, & 2 \le p \le p_c, \end{cases}$$

where $p_c = \frac{2(n+1)}{n-1}$. This bound is saturated on the sphere by zonal harmonics when $p \ge p_c$ and by highest weight spherical harmonics (a.k.a. Gaussian beams) when $p \le p_c$. (See, e.g., [52] for a description of extremizing quasimodes.)

It is then natural to look for quantitative improvements on this bound under different geometric assumptions. When (M, g) has non-positive sectional curvature, a bound of the form

$$\|\phi_h\|_{L^p(M)} = O\left(\frac{h^{-\delta(p,n)}}{(\log h^{-1})^{\sigma(p,n)}}\right)$$

was proved by Hassell and Tacy [30], with $\sigma(p,n) = \frac{1}{2}$, for the case $p > p_c$. In the same setting, Blair and Sogge [8, 9] studied the $2 < p \le p_c$ case and obtained a logarithmic improvement for some $\sigma(p,n)$ that is smaller than $\frac{1}{2}$.

An application of Theorem 2 gives $(\log h^{-1})^{\frac{1}{2}}$ improvement when $p > p_c$ under very weak assumptions on the set of conjugate points of (M,g). Indeed, given $x \in M$, $r > 0$, and $t > 0$, we continue to write $\mathcal{C}_x^{r,t}$ for the set of points defined in (1.4). Note that if $r_t \to 0^+$ as $|t| \to \infty$, then saying that $y \in \mathcal{C}_x^{r_t,t}$ for t large indicates that y behaves like point that is maximally conjugate to x.

Theorem 6 ([14]). *Let $p > p_c$. Let $V \subset M$ and assume that there exist $t_0 > 0$ and $a > 0$ so that*

$$\inf_{x,y \in V} d\big(y, \mathcal{C}_x^{r_t,t}\big) \ge r_t \quad \textit{for } t \ge t_0,$$

with $r_t = \frac{1}{a}e^{-at}$. Then there exist $C > 0$ and $h_0 > 0$ so that for $0 < h < h_0$, and ϕ_h satisfying (1.1),

$$\|\phi_h\|_{L^p(V)} \le C\frac{h^{-\delta(p,n)}}{\sqrt{\log h^{-1}}}.$$

One should think of the assumption in Theorem 6 as ruling out maximal conjugacy of the points x and y uniformly up to time ∞.

Remark 4. There are estimates in terms of the dynamical properties of covers by tubes similar to Theorem 2 for each of the bounds in Theorems 3, 4, and 6. In particular, these estimates do *not* require global geometric assumptions on (M,g), instead only using dynamical properties near S_x^*M or SN^*H.

1.4. Logarithmic improvements for pointwise Weyl Laws. Let $\{h_j^{-2}\}_j$ be the eigenvalues of (M,g). It is well known that

$$\#\{j : h_j^{-1} \le h^{-1}\} = \frac{\operatorname{vol}(B^n)\operatorname{vol}(M)}{(2\pi)^n}h^{-n} + E(h)$$

with $E(h) = O(h^{1-n})$. Indeed, this result is the integrated version of the more refined statement proved by Hörmander in [32] which says that for all $x \in M$,

$$\sum_{h_j^{-1} \le h^{-1}} |\phi_{h_j}(x)|^2 = \frac{\operatorname{vol}(B^n)}{(2\pi)^n}h^{-n} + E(h,x), \tag{1.7}$$

with $E(h,x) = O(h^{1-n})$ uniformly for $x \in M$. When the set of looping directions over x has measure zero, Sogge and Zelditch [49] proved that $E(h,x) = o(h^{1-n})$. Also, Duistermaat and Guillemin [20] proved an integrated version of this result by showing that $E(h) = o(h^{1-n})$ if the set of closed geodesics in M has measure zero. In terms of quantitative improvements, Bérard [6] and Bonthonneau [10] proved that $E(h,x) = O(h^{1-n}/\log h^{-1})$ if (M,g) has no conjugate points. As before, another application of geodesic beam techniques is that $\log h^{-1}$ improvements can be obtained under weaker assumptions than having no conjugate points.

Theorem 7 ([15]). *Let $V \subset M$ and assume that there exist $t_0 > 0$ and $a > 0$ so that*

$$\inf_{x \in V} d(x, \mathcal{C}_x^{r_t,t}) \ge r_t \quad \textit{for } t \ge t_0,$$

with $r_t = \frac{1}{a}e^{-at}$. *Then there exist* $C > 0$ *and* $h_0 > 0$ *so that for* $0 < h < h_0$ *and* $E(h,x)$ *as in* (1.7),

$$\sup_{x\in V} E(h,x) \le \frac{Ch^{1-n}}{\log h^{-1}}.$$

We remark that there are generalizations of this result to Kuznecov sums estimates, where evaluation at x is replaced by an integral average over a submanifold H (see [63] for the first results in this direction). In addition, in the same way that Theorem 2 can be used to obtain quantitative improvements in L^∞-bounds in concrete geometric settings, the dynamical version of the estimate in Theorem 7 can be used to obtain improved remainder estimates for pointwise Weyl laws. We show, for example, that all non-trivial product manifolds satisfy the assumptions of Theorem 7 at *every* point in Section 1.5.

1.5. Examples. We now record some examples to which our theorems apply. We refer the reader to [12] for many more examples. First, note that Theorem 3 applies when M is a manifold without conjugate points. The following examples may (and typically do) have conjugate points.

1.5.1. Product manifolds.

Lemma 1.1. *Let* (M_i, g_i), $i = 1,2$, *be two compact Riemannian manifolds, and let* $M = M_1 \times M_2$ *be endowed with the product metric* $g = g_1 \oplus g_2$. *Then* $\mathcal{C}_x^{r,t} = \emptyset$ *for all* $x \in M$, $|t| > 0$, *and* $0 < r < t$.

Proof. Let $x = (x_1, x_2) \in M$ and let $\gamma(t)$ be a unit speed geodesic on M with $\gamma(0) = 0$. Then there are unit speed geodesics γ_1 and γ_2 in M_1 and M_2, respectively, such that $\gamma_1(0) = x_1$, $\gamma_2(0) = x_2$, and there exists $\theta_0 \in \mathbb{R}$ such that

$$\gamma(t) = (\gamma_1(t\cos\theta_0), \gamma_2(t\sin\theta_0)) \in M_1 \times M_2.$$

Moreover, for every $\theta \in \mathbb{R}$, the curve $\gamma_\theta := (\gamma_1(t\cos\theta), \gamma_2(t\sin\theta))$ is a unit speed geodesic. In particular, one perpendicular Jacobi field along $\gamma = \gamma_{\theta_0}$ is given by

$$J(t) = \partial_\theta \gamma_\theta\big|_{\theta=\theta_0} = t(-\sin\theta_0\dot{\gamma}_1(t\cos\theta_0), \cos\theta_0\dot{\gamma}_2(t\sin\theta_0)).$$

Thus, $\|J(t)\| = t$, and hence J vanishes only at $t = 0$. In particular, since there exists a Jacobi field vanishing only at $t = 0$, $\mathcal{C}_x^{r,t} = \emptyset$ for all $0 < r < |t|$. □

We point out that although $\mathcal{C}_x^{r,t}$ is empty for $0 < r < |t|$, M may, and often does, have self-conjugate points. For example, this is the case if $M_1 = S^{n_1}$ for $n_1 \ge 2$.

Corollary 8. *Let* (M_i, g_i), $i = 1,2$, *be two compact Riemannian manifolds of dimension* $n_i > 0$. *Let* $M = M_1 \times M_2$ *endowed with the metric* $g = g_1 \oplus g_2$. *Then there is* $C > 0$ *such that for all* $x \in M$ *and* $u \in \mathcal{D}'(M)$,

$$|u(x)| \le Ch^{\frac{1-(n_1+n_2)}{2}}\Big(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\|(-h^2\Delta_g - 1)u\|_{H_{\mathrm{scl}}^{(n_1+n_2-3)/2}(M)}\Big).$$

1.5.2. The triaxial ellipsoid. We consider the *triaxial ellipsoid*

$$M := \{x \in \mathbb{R}^3 : a^2x^2 + b^2y^2 + c^2z^2 = 1\}$$

with $0 < a < b < c$. It is well known that the four umbilic points (i.e. points at which the normal curvatures are equal in all directions) on M are maximally self-conjugate. In fact, for an umbilic point x_0, there is $T > 0$ such that every geodesic through x_0 returns to x_0 at time T. Nevertheless, Theorem 2 and its generalization, Theorem 11, are useful at these points. The reason for this is the presence of a hyperbolic closed geodesic through x_0 to which every other geodesic through x_0 exponentially converges forward and backward in time (up to reversal of the parametrization). In particular, letting (x_0, ξ_+) and (x_0, ξ_-) be the initial points of the hyperbolic geodesic, we have that the stable direction for ξ_+ is given by $T_{\xi_+} S^*_x M$ and the unstable direction for ξ_- is given by $T_{\xi_-} S^*_x M$ (see [38, Theorem 3.5.16]). Thus, for each $\delta > 0$ there is $C > 0$ such that if $d(\xi, \xi^{\pm}) > \delta$, then in for all $\mp t > 0$ one has that

$$\|d\varphi_t|_{T_\xi S^*_{x_0} M}\| \le Ce^{\pm Ct}.$$

This type of exponential convergence can be used (see [27], [12, Lemmas 3.1–3.2]) to generate covers and obtain

$$|u(x_0)| \le Ch^{-\frac{1}{2}}\bigg(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\|(-h^2\Delta_g - 1)u\|_{H^{-1/2}_{\mathrm{scl}}(M)}\bigg).$$

1.5.3. The spherical pendulum. One example to which Theorem 5 applies is that of $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ the standard sphere equipped with the round metric, g, and function $V \in C^\infty(S^2)$ given by $V(x_1, x_2, x_3) = 2x_3$. The quantum spherical pendulum is then the operator

$$P = -h^2\Delta_g + V.$$

Identifying the sphere with $M = [0, \pi]_r \times [0, 2\pi]_\theta$. The Hamiltonian is given by

$$p(\theta, r, \xi_\theta, \xi_r) = \xi_r^2 + \frac{1}{\sin^2 r}\xi_\theta^2 + 2\cos r - E,$$

with $E \in \mathbb{R}$. This Hamiltonian describes the movement of a pendulum of mass 1 moving without friction on the surface of a sphere of radius 1.

By [33] for $E \ge 14/\sqrt{17}$, p is iso-energetically non-degenerate for all I_0 on $\{p = 0\}$. It is easy to check by explicit computations that $E - 2\cos r > 0$ for $E > 2$ and the map given by $r \mapsto \sin r\sqrt{E - 2\cos r}$ has a single non-degenerate maximum on $[0, \pi]$. Therefore, taking $E = E_0 \ge 14/\sqrt{17}$ and $Q = h^{-1}(E_0 - E_h)$ in Theorem 5 yields the following Corollary 9.

Corollary 9. *Let $B > 0$, $E_0 \ge 14/\sqrt{17}$ and $\delta > 0$. There exists $C > 0$ such that for all $L > 0$ there exists $h_0 > 0$ so that the following holds. For all $u \in \mathcal{D}'(S^2)$, $0 < h < h_0$ and $E_h \in (E_0 - Bh, E_0 + Bh)$,*

$$\|u\|_{L^\infty(|x_3|<1-\delta)} \le Ch^{-\frac{1}{2}}\bigg(\frac{\|u\|_{L^2(S^2)}}{L\sqrt{\log h^{-1}}} + \frac{L\sqrt{\log h^{-1}}\|(P - E_h)u\|_{H^{-1/2}_{\mathrm{scl}}(S^2)}}{h}\bigg).$$

In particular, if $\|u\|_{L^2(S^2)} = 1$ and $Pu = o(h/\log h^{-1})_{L^2}$, then

$$\|u\|_{L^\infty(|x_3|<1-\delta)} = o\bigg(\frac{h^{-\frac{1}{2}}}{\sqrt{\log h^{-1}}}\bigg).$$

Note that if we define $\tilde{g} = g/\sqrt{E_0 - 2x_3}$ with $E_0 \geq 14/\sqrt{17}$, then Theorem 5 shows that the eigenfunctions ϕ_h for $(-h^2\Delta_{\tilde{g}} - 1)\phi_h = 0$ satisfy the bound

$$\|\phi_h\|_{L^\infty(|x_3|<1-\delta)} = o\bigg(\frac{h^{-\frac{1}{2}}}{\sqrt{\log h^{-1}}}\bigg)$$

for any $\delta > 0$.

1.6. Relations with previous dynamical conditions on pointwise estimates. In this subsection, we recall the previous dynamical conditions guaranteeing improved pointwise estimates [25, 45, 47, 49, 50, 50, 56]. We first define the *loop set at* x by

$$\mathcal{L}_x := \{\rho \in S^*_x M : \text{there exists } t \in \mathbb{R} \text{ such that } \varphi_t(\rho) \in S^*_x M\},$$

and recall that a point x is said to be *non-self-focal* if $\mathrm{vol}_{S^*_x M}(\mathcal{L}_x) = 0$. It is proved in [45, 49] that if x is non-self-focal, then

$$|\phi_h(x)| = o(h^{\frac{1-n}{2}}). \tag{1.8}$$

Next, define $T_\pm : S^*_x M \to [0, \infty]$ by

$$T_\pm(\rho) := \pm \inf\{\pm t > 0 : \varphi_t(\rho) \in S^*_x M\}$$

and $\Phi_\pm : T_\pm^{-1}(0, \infty) \to S^*_x M$ by

$$\Phi_\pm(\rho) = \varphi_{T_\pm(\rho)}(\rho).$$

We then define $\mathcal{R}_x$ as the recurrent set for Φ. In [25, 47, 56], it is shown that if $\mathrm{vol}_{S^*_x M}(\mathcal{R}_x) = 0$, then (1.8) continues to hold. In that case x is called *non-recurrent*. Finally, in [25, 50, 56] it is shown that there need only be no invariant $L^2(\mathrm{vol}_{S^*_x M})$ function for (1.8) to hold.

Definition 2. For the purposes of the present subsection, we will say that a point x is $(t_0, T(h))$ *non-looping via covers* if there is a $(\tau, R(h))$ cover for $S^*_x M$, $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j=1}^{N_h}$, and $\mathcal{B} \sqcup \mathcal{G} = \{1, \dots, N_h\}$, such that

$$\bigcup_{j \in \mathcal{G}} \Lambda^\tau_{\rho_j}(R(h)) \text{ is } [t_0, T(h)] \text{ non-self-looping} \quad \text{and} \quad |\mathcal{B}| \leq \frac{R(h)^{1-n}}{T(h)}.$$

(See also [15, Definition 2.1].) We will say that x *is* $T(h)$ *non-recurrent via covers* if there are sets of indices $\mathcal{G}_\ell \subset \{1, \dots, N_h\}$ and pairs of times (t_ℓ, T_ℓ) such that $\{1, \dots, N_h\} = \bigcup_\ell \mathcal{G}_\ell$ and

$$\bigcup_{j \in \mathcal{G}_\ell} \Lambda^\tau_{\rho_j}(R(h)) \text{ is } [t_\ell, T_\ell] \text{ non-self-looping} \quad \text{and} \quad \sum_\ell \frac{|\mathcal{G}_\ell|^{\frac{1}{2}} t_\ell^{\frac{1}{2}}}{T_\ell^{\frac{1}{2}}} \leq \frac{R(h)^{\frac{1-n}{2}}}{T(h)^{\frac{1}{2}}}.$$

(See also [15, Definition 2.2].)

First of all, we point out that x being $T(h)$ non-looping via covers implies that it is $T(h)$ non-recurrent via covers and that Theorem 2 states that if x is $T(h)$ non-recurrent via covers for some $T(h) \ll T_e(h)$, then there is $C > 0$ such that

$$|\phi_h(x)| \leq \frac{C h^{\frac{1-n}{2}}}{T(h)^{\frac{1}{2}}}.$$

In order to relate these two concepts to the concept of a non-self-focal point and a non-recurrent point respectively, we prove the following two lemmas in Appendix B

Lemma 1.2. *Suppose that the point x is non-self-focal. Then there are $t_0 > 0$ and $T : (0,1) \to (0,\infty)$ such that $\lim_{h\downarrow 0} T(h) = \infty$ and x is $(t_0, T(h))$ non-looping via covers.*

Lemma 1.3. *Suppose that x is non-recurrent. Then there is $T : (0,1) \to (0,\infty)$ such that $\lim_{h\downarrow 0} T(h) = \infty$ and x is $T(h)$ non-recurrent via covers.*

In particular, Lemmas 1.2 and 1.3 recover the fact that x being non-recurrent implies equation (1.8).

1.7. Outline of the paper. In Section 2 we present Theorems 10 and 11 which are the generalization of Theorems 1 and 2 to quasimodes of general pseudodifferential operators P. In Section 3 we perform the analysis of quasimodes for P and in particular prove Theorem 10. In Section 4 we give the proof of Theorem 11. In Section 5 we construct non-self-looping covers on spheres of revolution and prove Corollary 9. Finally, in Section 6, we prove that the Hamiltonian flow for $|\xi|_g^2 - 1$ can be replaced by that for $|\xi|_g - 1$. In Appendix A we present an index of notation and background on semiclassical analysis.

Acknowledgement. Thanks to Pat Eberlein, John Toth, Andras Vasy, and Maciej Zworski for many helpful conversations and comments on the manuscript. Thanks also to the anonymous referees for many suggestions which improved the exposition.

2. General results: Bicharacteristic beams

Our main estimate gives control on eigenfunction averages in terms of microlocal data. The ideas leading to the estimate build on the tools first constructed in [25] for sup-norms and generalized for use on submanifolds in [13].

Since it entails little extra difficulty, we work in the general setup of semiclassical pseudodifferential operators (see, e.g., [65] or [22, Appendix E] for a treatment of semiclassical analysis, see Section A.2 for a brief description of notation). Indeed, instead of only working with Laplace eigenfunctions, all our results can be proved for quasimodes of a pseudodifferential operator of any order that has real, classically elliptic symbol. We now introduce the necessary objects to state this estimate.

Let $H \subset M$ be a submanifold. For $p \in S^m(T^*M)$ define

$$\Sigma_{H,p} = \{p = 0\} \cap N^*H, \tag{2.1}$$

where N^*H is the conormal bundle to H and consider the Hamiltonian flow

$$\varphi_t := \exp(tH_p). \tag{2.2}$$

Here, and in what follows, H_p is the Hamiltonian vector field generated by p. In practice, we will prove our main result with H replaced by a family of submanifolds $\{H_h\}_h$ such that for all α multiindex there exists $\mathcal{K}_\alpha > 0$ such that for all $h > 0$,

$$|\partial_x^\alpha R_{H_h}| + |\partial_x^\alpha \Pi_{H_h}| \leq \mathcal{K}_\alpha \tag{2.3}$$

where R_{H_h} and Π_{H_h} denote the sectional curvature and the second fundamental form of H_h. Next, we assume that there is $\varepsilon > 0$ so that for all $h > 0$, the map $(-\varepsilon, \varepsilon) \times \Sigma_{H,p} \to M$,

$$(t, \rho) \mapsto \pi(\varphi_t(\rho)) \text{ is a diffeomorphism.} \tag{2.4}$$

We will say that a family of submanifolds $\{H_h\}_h$ is *regular* if it satisfies (2.3) and (2.4). In addition, we will prove uniform statements in a shrinking neighborhood of H_h. In particular, we prove estimates on $\tilde{H}_h$ where $\tilde{H}_h$ is another family of submanifolds such that

$$\sup_{\rho \in \Sigma_{H_h,p}} d(\rho, \Sigma_{\tilde{H}_h,p}) \le h^{\delta}, \quad |\partial_x^\alpha R_{\tilde{H}_h}| + |\partial_x^\alpha \Pi_{\tilde{H}_h}| \le 2\mathcal{K}_\alpha \tag{2.5}$$

for all $h > 0$. Note that when H_h is a family of points, the curvature bounds become trivial, and so in place of (2.5) we work with $d(x_h, \tilde{x}_h) < h^{\delta}$ and we may take $\mathcal{K}_0$ to be arbitrarily close to 0. It will often happen that the constants involved in our estimates depend on $\{H_h\}$ only through finitely many of the $\mathcal{K}_\alpha$ constants.

For $p \in S^m(T^*M)$, we say that p is *classically elliptic* if there exists $K_p > 0$ so that

$$|p(x, \xi)| \ge \frac{|\xi|^m}{K_p}, \quad |\xi| \ge K_p,\ x \in M. \tag{2.6}$$

In addition, for $p \in S^\infty(T^*M; \mathbb{R})$, we say that a submanifold $H \subset M$ of codimension k is *conormally transverse for* p if given $f_1, \dots, f_k \in C_c^\infty(M; \mathbb{R})$ locally defining H i.e. with

$$H = \bigcap_{i=1}^{k} \{f_i = 0\} \quad \text{and} \quad \{df_i\} \text{ linearly independent on } H,$$

we have

$$N^*H \subset \{p \ne 0\} \cup \bigcup_{i=1}^{k} \{H_p f_i \ne 0\},$$

where H_p is the Hamiltonian vector field associated to p, and N^*H is the set of conormal directions to H. Here, we interpret f_i as a function on the cotangent bundle by pulling it back through the canonical projection map. In addition, let $r_H : M \to \mathbb{R}$ be the geodesic distance to H; $r_H(x) = d(x, H)$. Then define $|H_p r_H| : \Sigma_{H,p} \to \mathbb{R}$ by

$$|H_p r_H|(\rho) := \lim_{t \to 0} |H_p r_H(\varphi_t(\rho))|. \tag{2.7}$$

A family of submanifolds $\{H_h\}_h$ is said to be *uniformly conormally transverse for* p if H_h is conormally transverse for p for all h and there exists $\mathfrak{I}_0 > 0$ so that for all $h > 0$,

$$\inf_{\rho \in \Sigma_{H,p}} |H_p r_{H_h}|(\rho) \ge \mathfrak{I}_0. \tag{2.8}$$

When $p(x, \xi) = |\xi|^2_{g(x)} - 1$, then $\Sigma_{H,p} = SN^*H$ and $|H_p r_H|(\rho) = 2$ for all $\rho \in SN^*H$.

Let $\{H_h\}_h \subset M$ be a regular and uniformly conormally transverse family of submanifolds. Then we may fix a family of regular hypersurfaces depending on h, $\mathcal{L}_h \subset T^*M$ such that

$$\mathcal{L}_h \text{ is uniformly transverse to } H_p \text{ with } \Sigma_{H_h,p} \subset \mathcal{L}_h \tag{2.9}$$

and so that with $\Psi : \mathbb{R} \times T^*M \to T^*M$ defined by $\Psi(t, q) = \varphi_t(q)$, there is $0 < \tau_{\text{inj}} \le 1$ (independent of h) so that

$$\Psi|_{(-\tau_{\text{inj}}, \tau_{\text{inj}}) \times \mathcal{L}_h} \text{ is injective} \tag{2.10}$$

for all $h > 0$.

Remark 5. Working with a family $\{\tilde{H}_h\}_h$, and obtaining uniform estimates for it, is needed in Theorem 1. In this case, $H_h = \{x\}$ for every h and $\tilde{H}_h$ is a point $\tilde{x}_h \in B(x, h^\delta)$. Moreover, it is often useful to allow H_h itself to vary with h (see, e.g., [14]). Note that any h-independent submanifold $H \subset M$ that is conormally transverse is automatically regular and uniformly conormally transverse. While in some applications it is useful to have h-dependent submanifolds H_h, as well as uniform estimates in a neighborhood of H_h, the reader may wish to ignore the dependence of H_h on h as well as letting $\tilde{H} = H$ for simplicity of reading.

Given $A \subset T^*M$, define

$$\Lambda_A^\tau := \bigcup_{|t| \le \tau} \varphi_t(A).$$

For $R > 0$ and $A \subset \Sigma_{H,p}$ we define

$$\Lambda_A^\tau(r) := \Lambda_{A_R}^{\tau+r}, \quad A_r := \{\rho \in \mathcal{L}_h : d(\rho, A) < r\}, \tag{2.11}$$

where d denotes the distance induced by the Sasaki metric on T^*M (see, e.g., [7, Chapter 9] for an explanation of the Sasaki metric). In particular, the tube

$$\Lambda_\rho^\tau(r) := \bigcup_{|t| \le \tau + r} \varphi_t(\mathcal{L}_h \cap B(\rho, r)). \tag{2.12}$$

See Figure 2.

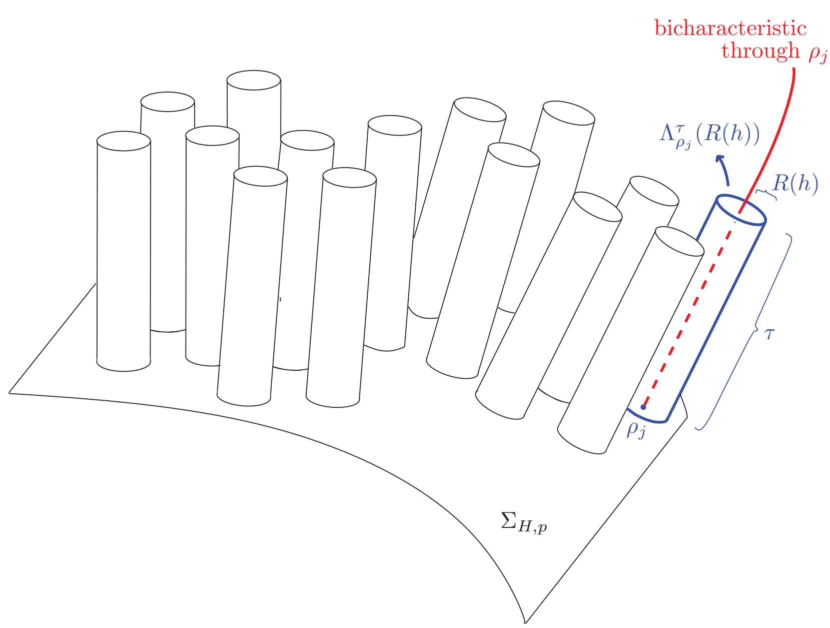

Figure 2. The tubes $\Lambda_{\rho_j}^\tau(R(h))$ through $\Sigma_{H,p}$.

Definition 3. Let $A \subset \Sigma_{H,p}$, $r > 0$, and $\{\rho_j(r)\}_{j=1}^{N_r} \subset A$. We say that the collection of tubes $\{\Lambda_{\rho_j}^\tau(r)\}_{j=1}^{N_h}$ is a (τ, r)-*cover* of a set $A \subset \Sigma_{H,p}$ provided

$$\Lambda_A^\tau\Big(\frac{1}{2}r\Big) \subset \bigcup_{j=1}^{N_r} \Lambda_{\rho_j}^\tau(r).$$

In addition, for $0 \le \delta \le \frac{1}{2}$ and $R(h) \ge 8h^\delta$, we say that a collection $\{\chi_j\}_{j=1}^{N_h} \subset S_\delta(T^*M;[0,1])$ is a *δ-partition for A associated to the $(\tau, R(h))$-cover* if $\{\chi_j\}_{j=1}^{N_h}$ is bounded in S_δ and

(i) $\operatorname{supp}\chi_j \subset \Lambda^\tau_{\rho_j}(R(h))$,

(ii) $\sum_{j=1}^{N_h}\chi_j \ge 1$ on $\Lambda_A^{\tau/2}(\frac{1}{2}R(h))$.

The main estimate is the following.

Theorem 10. *Let $P \in \Psi^m(M)$ have real, classically elliptic symbol $p \in S^m(T^*M;\mathbb{R})$. Let $\{H_h\}_h \subset M$ be a regular family of submanifolds of codimension k that is uniformly conormally transverse for p. There exist*

$$\tau_0 = \tau_0(M, p, \tau_{\mathrm{inj}}, \mathfrak{I}_0, \{H_h\}_h) > 0, \quad R_0 = R_0(M, p, k, \mathcal{K}_0, \tau_{\mathrm{inj}}, \mathfrak{I}_0) > 0,$$

$C_{n,k} > 0$ depending only on (n,k), and $C_0 > 0$ depending only on (M,p), so that the following holds.

Let $0 < \tau \le \tau_0$, $0 \le \delta < \frac{1}{2}$, and $8h^\delta \le R(h) \le R_0$. Let $\{\chi_j\}_{j=1}^{N_h}$ be a δ-partition for $\Sigma_{H,p}$ associated to a $(\tau, R(h))$-cover. Let $N > 0$ and $\{\tilde H_h\}_h \subset M$ be a family of submanifolds of codimension k satisfying (2.5). *There exists a constant $C > 0$, so that for every family $\{w_h\}_h$ with $w_h \in S_\delta \cap C_c^\infty(\tilde H_h)$ there are $C_N > 0$ and*

$$h_0 = h_0(M, P, \{\chi_j\}, \delta, \mathfrak{I}_0, \{H_h\}_h) > 0$$

with the property that for any $0 < h < h_0$ and $u \in \mathcal{D}'(M)$,

$$\begin{aligned} h^{\frac{k-1}{2}}\left|\int_{\tilde H_h} w_h u \, d\sigma_{\tilde H_h}\right| \le \frac{C_{n,k}}{\tau^{\frac12}\mathfrak{I}_0^{\frac12}}\|w_h\|_\infty R(h)^{\frac{n-1}{2}} &\sum_{j \in \mathcal{J}_h(w_h)} \|Op_h(\chi_j)u\|_{L^2(M)} \\ &+ Ch^{-1}\|w_h\|_\infty \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)} \\ &+ C_N h^N (\|u\|_{L^2(M)} + \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)}), \end{aligned}$$

where

$$\mathcal{J}_h(w_h) := \{j : \Lambda^\tau_{\rho_j}(2R(h)) \cap \pi^{-1}(\operatorname{supp} w_h) \ne \emptyset\}, \tag{2.13}$$

and $\pi : \Sigma_{\tilde H_h, p} \to \tilde H_h$ is the canonical projection. Moreover, the constants C, C_N, h_0 are uniform for χ_j in bounded subsets of S_δ. The constants τ_0, C, C_N, h_0 depend on $\{H_h\}_h$ only through finitely many of the constants $\mathcal{K}_\alpha$ in (2.3). *The constant C_N is uniform for $\{w_h\}_h$ in bounded subsets of S_δ.*

Remark 6 (Proof of Theorem 1). We emphasize now that Theorem 10 is the key analytical estimate of this article. In particular, Theorem 1 is a direct consequence of it. Indeed, we work with $P = -h^2\Delta_g - I$, $Pu = 0$. Let $H_h = \{x\}$ and $\tilde H_h = \{x_h\}$ with $x_h \in B(x, h^\delta)$. Let $w_h = 1$ for all h. In particular, $\mathcal{J}_h(w_h) = \{1, \dots, N_h\}$. Note that since $H_h = \{x\}$, it follows that $SN^*H = S_x^*M$. Also, in this case $\tau_{\mathrm{inj}}(\{x\})$ can be chosen uniform on M, and we have $H_p r_H = 2$ and $\mathfrak{I}_0 = 2$. Moreover, $\mathcal{K}_\alpha$ can be taken arbitrarily small. This yields $\tau_0 = \tau_0(M, g)$, $R_0 = R_0(M, g)$ and $h_0 = h_0(M, g, \{\chi_j\}, \delta)$. Theorem 1 follows.

We will next present Theorem 11 which combines Theorem 10 with an application of Egorov's Theorem to control eigenfunction averages using dynamical information at $\Sigma_{H,p}$. In fact, all the applications to obtaining quantitative improvements for L^∞-bounds and averages described in the introduction are reduced to a purely dynamical argument together with an application of Theorem 11.

As explained before Theorem 2, it will be convenient for us to work with covers by tubes without too much redundancy. We therefore introduce the following definition.

Definition 4. Let $A \subset \Sigma_{H,p}$, $r, \mathfrak{D} > 0$, and $\{\rho_j(r)\}_{j=1}^{N_r} \subset A$. The collection of tubes $\{\Lambda^\tau_{\rho_j}(r)\}_{j=1}^{N_r}$ is a $(\mathfrak{D}, \tau, r)$-*good cover* of a set $A \subset \Sigma_{H,p}$ provided that it is a (τ, r)-cover for A and there exists a partition $\{\mathcal{J}_\ell\}_{\ell=1}^{\mathfrak{D}}$ of $\{1, \dots, N_r\}$ so that for every $\ell \in \{1, \dots, \mathfrak{D}\}$,

$$\Lambda^\tau_{\rho_j}(3r) \cap \Lambda^\tau_{\rho_i}(3r) = \emptyset, \quad i, j \in \mathcal{J}_\ell,\ i \neq j.$$

In Proposition 3.3 we prove that there exists a $(\mathfrak{D}_n, \tau, r)$-good cover for $\Sigma_{H,p}$ where $\mathfrak{D}_\mathfrak{n}$ only depends on n. Thus, one can always work with such a cover.

We define the *maximal expansion rate* and the *Ehrenfest time* at frequency h^{-1}, respectively:

$$\Lambda_{\max} := \limsup_{|t|\to\infty} \frac{1}{|t|} \log \sup_{\{|p| \le \frac{1}{2}\}} \|d\varphi_t(x,\xi)\|, \quad T_e(h) := \frac{\log h^{-1}}{2\Lambda_{\max}}. \tag{2.14}$$

Note that $\Lambda_{\max} \in [0, \infty)$ and if $\Lambda_{\max} = 0$, we may replace it by an arbitrarily small positive constant.

The next theorem involves many parameters; their role is to provide flexibility when applying the theorem. This theorem controls averages over uniformly conormally transverse families of submanifolds in terms of families $\{\mathcal{G}_\ell\}_\ell$ of tubes that run conormally to the submanifolds and are $[t_\ell, T_\ell]$ non-self-looping. For an explanation on the roles of these tubes and non-looping times, see the text after Theorem 2.

Theorem 11. *Let $P \in \Psi^m(M)$ be a self-adjoint operator with classically elliptic symbol p. Let $\{H_h\}_h \subset M$ be a regular family of submanifolds of codimension k that is uniformly conormally transverse for p. Let $\{\tilde{H}_h\}_h$ be a family of submanifolds of codimension k satisfying* (2.5). *Let $0 < \delta < \frac{1}{2}$, $N > 0$ and $\{w_h\}_h$ with $w_h \in S_\delta \cap C_c^\infty(\tilde{H}_h)$. There exist positive constants $\tau_0 = \tau_0(M, p, \tau_{\text{inj}}, \mathfrak{I}_0, \{H_h\}_h)$, $R_0 = R_0(M, p, \mathcal{K}_0, k, \tau_{\text{inj}}, \mathfrak{I}_0)$, and $C_{n,k}$ depending only on n and k, $h_0 = h_0(M, P, \delta, \mathfrak{I}_0, \{H_h\}_h)$, and for each $0 < \tau \le \tau_0$ there are*

$$C = C(M, p, \tau, \delta, \mathfrak{I}_0, \{H_h\}_h), \quad C_N = C_N(M, P, N, \tau, \delta, \{w_h\}_h, \mathfrak{I}_0, \{H_h\}_h),$$

so that the following holds.

Let $8h^\delta \le R(h) < R_0$, $0 \le \alpha < 1 - 2\limsup_{h\to 0} \frac{\log R(h)}{\log h}$, and suppose $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j=1}^{N_h}$ is a $(\mathfrak{D}, \tau, R(h))$-good cover of $\Sigma_{H,p}$ for some $\mathfrak{D} > 0$. In addition, suppose there exist a subset $\mathcal{B} \subset \{1, \dots, N_h\}$ and a finite collection $\{\mathcal{G}_\ell\}_{\ell\in\mathcal{L}} \subset \{1, \dots, N_h\}$ with

$$\mathcal{J}_h(w_h) \subset \mathcal{B} \cup \bigcup_{\ell\in\mathcal{L}} \mathcal{G}_\ell,$$

where $\mathcal{J}_h(w_h)$ is defined in (2.13), *and so that for every $\ell \in \mathcal{L}$ there exist $t_\ell = t_\ell(h) > 0$ and $T_\ell = T_\ell(h)$ with $t_\ell(h) \le T_\ell \le 2\alpha T_e(h)$ so that*

$$\bigcup_{j\in\mathcal{G}_\ell} \Lambda^\tau_{\rho_j}(R(h)) \text{ is } [t_\ell, T_\ell] \text{ non-self-looping.}$$

Then, for $u \in \mathcal{D}'(M)$ and $0 < h < h_0$,

$$
\begin{aligned}
h^{\frac{k-1}{2}} \left| \int_{\tilde{H}_h} w_h u \, d\sigma_{\tilde{H}_h} \right| \leq{}& \frac{C_{n,k} \mathfrak{D} \|w_h\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac{1}{2}} \mathfrak{I}_0^{\frac{1}{2}}} \left(|\mathcal{B}|^{\frac{1}{2}} + \sum_{\ell \in \mathcal{L}} \frac{(|\mathcal{G}_\ell| t_\ell)^{\frac{1}{2}}}{T_\ell^{\frac{1}{2}}} \right) \|u\|_{L^2(M)} \\
&+ \frac{C_{n,k} \mathfrak{D} \|w_h\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac{1}{2}} \mathfrak{I}_0^{\frac{1}{2}}} \sum_{\ell \in \mathcal{L}} \frac{(|\mathcal{G}_\ell| t_\ell T_\ell)^{\frac{1}{2}}}{h} \|Pu\|_{L^2(M)} \\
&+ C h^{-1} \|w_h\|_\infty \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)} \\
&+ C_N h^N (\|u\|_{L^2(M)} + \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)}).
\end{aligned}
$$

Here, the constant C_N depends on $\{w_h\}_h$ only through finitely many S_δ seminorms of w_h. The constants τ_0, C, C_N, h_0 depend on $\{H_h\}_h$ only through finitely many of the constants $\mathcal{K}_\alpha$ in (2.3).

Remark 7 (Proof of Theorem 2). Note that making the same observations in Remark 6 it is straightforward to see that Theorem 2 is a generalization of Theorem 11. The only consideration is that the tubes are built using the geodesic flow, which is generated by the symbol $p(x,\xi) = |\xi|_{g(x)} - 1$ instead of $p_0(x,\xi) = |\xi|_{g(x)}^2 - 1$. We explain how to pass from one flow to the other in Section 6.

Remark 8. Note that in this paper we study averages of relatively weak quasimodes for the Laplacian with no additional assumptions on the functions. This is in contrast with results which impose additional conditions on the functions such as: that they be Laplace eigenfunctions that simultaneously satisfy additional equations [27,34,53,55]; that they be eigenfunctions in the very rigid case of the flat torus [11,29]; or that they form a density one subsequence of Laplace eigenfunctions [35].

Remark 9. We also note that the norm $C\|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)}$ in Theorems 11 and 10 may be replaced by $C_\varepsilon \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+\varepsilon)/2}(M)}$ for any $\varepsilon > 0$. However, for notational convenience we have chosen to use a sub-optimal Sobolev embedding to produce the $\|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)}$ term.

3. Estimates near bicharacteristics: Proof of Theorem 10

The proof of Theorem 10 relies on several estimates. In what follows we give an outline of the proof to motivate three propositions that together yield the proof of Theorem 10.

A note on notation. Throughout this section to ease notation we write

$$H, \tilde{H}, w, \qquad \text{instead of} \qquad H_h, \tilde{H}_h, w_h.$$

Proof of Theorem 10. Let $0 < \delta < \frac{1}{2}$. In what follows τ_0, R_0, ε_0 and h_0 are the constants given by Proposition 3.5. Let $8h^\delta \leq R(h) \leq R_0$, and $N > 0$. Let τ with $0 < \tau \leq \tau_0$ and $\{\rho_j\}_{j=1}^{N_h} \subset \Sigma_{H,p}$ be so that the tubes $\{\Lambda_{\rho_j}^\tau(R(h))\}_{j=1}^{N_h}$ form a $(\tau, R(h))$-covering of $\Sigma_{H,p}$. We divide the proof into three steps, each of which relies on a proposition.

Step 1: Localization near conormal directions. Let $\chi_0 \in C_c^\infty(\mathbb{R};[0,1])$ be a smooth cut-off function with $\chi_0(t)=1$ for $t\le\frac{1}{2}$ and $\chi_0(t)=0$ for $t\ge 1$. Let $K>0$ be defined as in (3.8) below and define

$$\beta_\delta(x',\xi') := \chi_0\Big(\frac{K|\xi'|_{\tilde{H}}}{h^\delta}\Big), \tag{3.1}$$

where $|\xi'|_{\tilde{H}}$ denotes the length of ξ' as an element of $T^*_{x'}\tilde{H}$ with respect to the Riemannian metric induced on $\tilde{H}$. In Proposition 3.2 we prove that for $w\in S_\delta\cap C_c^\infty(\tilde{H})$ there exists $C_N>0$, depending on P, finitely many seminorms of w, and finitely many of the constants $\mathcal{K}_\alpha$ in (2.3), so that for all $h>0$

$$\Big|\int_{\tilde{H}} wu\,d\sigma_{\tilde{H}}\Big| \le \|wOp_h(\beta_\delta)u\|_{L^1(\tilde{H})} + C_N h^N\big(\|u\|_{L^2(M)}+\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)}\big). \tag{3.2}$$

Step 2: Coverings by bicharacteristic beams. Let $\tilde{R}(h)=\frac{1}{2}R(h)$ and let $\tilde{\tau}=\frac{\tau}{4}$. In Proposition 3.3 we prove that there exist a constant $\mathfrak{D}_n$, depending only on n, points $\{\tilde{\rho}_j\}_{j=1}^{\tilde{N}_h}\subset\Sigma_{H,p}$, and a partition $\{\mathcal{J}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\{1,\dots,\tilde{N}_h\}$, so that

- $\Lambda^{\tilde{\tau}}_{\Sigma_{H,p}}(\frac{1}{2}\tilde{R}(h))\subset\bigcup_{j=1}^{\tilde{N}_h}\Lambda^{\tilde{\tau}}_{\tilde{\rho}_j}(\tilde{R}(h))$,
- $\Lambda^{\tilde{\tau}}_{\tilde{\rho}_j}(3\tilde{R}(h))\cap\Lambda^{\tilde{\tau}}_{\tilde{\rho}_\ell}(3\tilde{R}(h)))=\emptyset$, $j,\ell\in\mathcal{J}_i$, $j\ne\ell$.

That is, we work with a $(\mathfrak{D}_n,\tilde{\tau},\tilde{R}(h))$-good cover. In Proposition 3.4 we prove that there exists $C_0>0$ so that for $0<\varepsilon<\varepsilon_0$ and $0<h\le h_0$ there is a partition of unity $\{\chi_j^P\}_j$ for $\Lambda^{\tilde{\tau}}_{\Sigma_{H,p}}(\frac{1}{2}\tilde{R}(h))$ with

- $\chi_j^P\in S_\delta\cap C_c^\infty(T^*M;[-C_0h^{1-2\delta},1+C_0h^{1-2\delta}])$,
- $\operatorname{supp}\chi_j^P\subset\Lambda^{\tilde{\tau}+\varepsilon}_{\tilde{\rho}_j}(\tilde{R}(h))$,
- $\mathrm{MS_h}([P,Op_h(\chi_j^P)])\cap\Lambda^{\tilde{\tau}}_{\Sigma_{H,p}}(\varepsilon)=\emptyset$.

Indeed, this follows from applying Proposition 3.4 since $\tilde{R}(h)=\frac{1}{2}R(h)\ge\frac{1}{2}8h^\delta\ge 2h^\delta$. From now on we fix $\varepsilon>0$ so that $\varepsilon<\varepsilon_0$ and $\varepsilon<\frac{\tau}{4}$. See Appendix A.3 for background on microsupports.

Step 3: Estimates near bicharacteristics. In Proposition 3.5 we prove that there exist $C_{n,k}>0$, $C_N>0$, $h_0>0$, and $C>0$ so that for all $w\in S_\delta\cap C_c^\infty(\tilde{H})$ and $0<h<h_0$, if $\{\chi_j^P\}$ is as before, then

$$\begin{aligned} h^{\frac{k-1}{2}}\|wOp_h(\beta_\delta)u\|_{L^1(\tilde{H})} \le C_{n,k}\|w\|_\infty R(h)^{\frac{n-1}{2}} &\sum_{j\in\tilde{\mathcal{I}}_h(w)}\frac{\|Op_h(\chi_j^P)u\|_{L^2(M)}}{\tau^{\frac{1}{2}}|H_pr_H(\tilde{\rho}_j)|^{\frac{1}{2}}}\\ &+Ch^{-1}\|w\|_\infty\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)}\\ &+C_Nh^N\|w\|_\infty\|u\|_{L^2(M)}, \end{aligned} \tag{3.3}$$

where $\tilde{\mathcal{I}}_h(w)=\{j:\Lambda^{\tilde{\tau}}_{\tilde{\rho}_j}(\tilde{R}(h))\cap\pi^{-1}(\operatorname{supp}(w))\ne\emptyset\}$.

Remark 10. It is crucial that the cutoffs χ_j supported in disjoint tubes act almost orthogonally. This allows for efficient decomposition and recombination of estimates based on tubes and we use this fact throughout the text.

Next, let $\{\chi_\ell\}_{\ell=1}^{N_h}$ be a δ-*partition* associated to the $(\tau, R(h))$-cover $\{\Lambda^\tau_{\rho_\ell}(R(h))\}_{\ell=1}^{N_h}$ of $\Sigma_{H,p}$. We claim that for each $j \in \tilde{\mathcal{I}}_h(w)$,

$$\chi_j^P \le 2\sum_{\ell\in\mathcal{A}_j}\chi_\ell, \tag{3.4}$$

where

$$\mathcal{A}_j = \{\ell : \Lambda^{\tau/2}_{\tilde\rho_j}(\tilde R(h)) \cap \Lambda^\tau_{\rho_\ell}(R(h)) \neq \emptyset\}.$$

Indeed, this follows from two observations. The first one is that $\operatorname{supp}\chi_j^P \subset \Lambda^{\tau/2}_{\tilde\rho_j}(\tilde R(h))$ since $\varepsilon < \frac{\tau}{4}$. The second observation is that on $\Lambda^{\tau/2}_{\tilde\rho_j}(\tilde R(h))$ we have

$$\sum_{\ell=1}^{N_h}\chi_\ell = \sum_{\ell\in\mathcal{A}_j}\chi_\ell \ge 1$$

since $\sum_{\ell=1}^{N_h}\chi_\ell \ge 1$ on $\Lambda^{\tau/2}_{S^*_xM}(\tilde R(h))$ and $\operatorname{supp}\chi_\ell \subset \Lambda^\tau_{\rho_\ell}(R(h))$. Combining this with the fact that $\chi_j^P \le 1 + C_0h^{1-2\delta}$ yields the claim in (3.4).

Next, note that if $j \in \tilde{\mathcal{I}}_h(w)$, then $\mathcal{A}_j \subset \mathcal{J}_h(w)$, where

$$\mathcal{J}_h(w) = \{\ell : \Lambda^\tau_{\rho_\ell}(2R(h)) \cap \pi^{-1}(\operatorname{supp}(w)) \neq \emptyset\}.$$

This follows from the fact that if $\ell \in \mathcal{A}_j$, then $\Lambda^{\tau/2}_{\tilde\rho_j}(\tilde R(h)) \subset \Lambda^\tau_{\rho_\ell}(2R(h))$.

To complete the proof, we claim that there exists $C_n > 0$ depending only on n so that for every $\ell \in \{1, \dots, N_h\}$,

$$\#\{j \in \tilde{\mathcal{I}}_h(w) : \ell \in \mathcal{A}_j\} \le C_n. \tag{3.5}$$

Assuming the claim for now, we conclude from (3.4) that

$$\begin{aligned}\sum_{j\in\tilde{\mathcal{I}}_h(w)}\frac{\|Op_h(\chi_j^P)u\|_{L^2(M)}}{|H_pr_H(\tilde\rho_j)|^{\frac12}} &\le 4\mathfrak{I}_0^{-\frac12}\sum_{j\in\tilde{\mathcal{I}}_h(w)}\sum_{\ell\in\mathcal{A}_j}\|Op_h(\chi_\ell)u\|_{L^2(M)}\\ &\le 4C_n\mathfrak{I}_0^{-\frac12}\sum_{j\in\mathcal{J}_h(w)}\|Op_h(\chi_j)u\|_{L^2(M)}.\end{aligned}$$

Combining this with (3.3) and (3.2) finishes the proof of Theorem 10.

We now prove (3.5). Suppose that $\ell \in \mathcal{A}_j$. Then

$$B(\rho_\ell, R(h)) \cap B(\tilde\rho_j, \tilde R(h)) \cap \mathcal{L}_h \neq \emptyset.$$

In particular,

$$B(\tilde\rho_j, \tilde R(h)) \cap \mathcal{L}_h \subset B(\rho_\ell, 2R(h)) \cap \mathcal{L}_h.$$

Therefore, $\Lambda^{\tilde\tau}_{\tilde\rho_j}(\tilde R(h)) \subset \Lambda^{\tilde\tau}_{\rho_\ell}(2R(h))$. Thus, since the tubes $\{\Lambda^{\tilde\tau}_{\tilde\rho_j}(3\tilde R(h))\}_{j\in\mathcal{J}_i}$ are disjoint for each $i = 1, \dots, \mathfrak{D}_n$, there exists a constant $C_n > 0$, depending only on n, such that for every $\ell \in \{1, \dots, N_h\}$,

$$\#\{j : \ell \in \mathcal{A}_j\} \le \mathfrak{D}_n\frac{\sup_\ell \operatorname{vol}(\Lambda^{\tilde\tau}_{\rho_\ell}(2R(h))}{\inf_j \operatorname{vol}(\Lambda^{\tilde\tau}_{\tilde\rho_j}(\tilde R(h)))} \le C_n. \qquad \square$$

We proceed to state and prove all the propositions needed in the proof of Theorem 10.

3.1. Step 1: Localization near conormal directions. Our first result is quite general, and it shows that in order to study integral averages over $\tilde{H}$ of a function v it suffices to restrict ourselves to studying the conormal behavior of v. That is, the non-oscillatory behavior of v along $\tilde{H}$ is encoded in $Op_h(\beta_\delta)v$.

Lemma 3.1. *Let $0 \le \delta < \frac{1}{2}$, $N > 0$, and $w \in S_\delta \cap C_c^\infty(\tilde{H})$. Then there is $C_N > 0$, depending on finitely many seminorms of $w \in S_\delta$ and finitely many of the constants $\mathcal{K}_\alpha$ in* (2.3)*, so that for all $v \in \mathcal{D}'(\tilde{H})$,*

$$\left| \int_{\tilde{H}} w(1 - Op_h(\beta_\delta))(v)\, d\sigma_{\tilde{H}} \right| \le C_N h^N \|v\|_{L^2(\tilde{H})}.$$

Proof. Let $h > 0$. Here, we work in coordinates $(\bar{x}, x') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, where

$$\tilde{H} = \tilde{H}_h = \{\bar{x} = 0\}.$$

Let $\tilde{N}$ be so that $N < k - n + \tilde{N}(1 - 2\delta)$. Let $g_{\tilde{H}}$ denote the metric induced on $\tilde{H}$. Then integrating by parts with $L := \frac{1}{|\xi'|^2}(\sum_{j=1}^{n-k} \xi'_j h D_{x_j})$ gives

$$\begin{aligned}
&\int_{\tilde{H}} w(x)\,(1 - Op_h(\beta_\delta))v(x)\, d\sigma_{\tilde{H}}(x) \\
&\quad = \frac{1}{(2\pi h)^{n-k}} \iiint e^{\frac{i}{h}\langle x - x', \xi'\rangle} w(x)(1 - \beta_\delta(x, \xi'))v(x')\sqrt{|g_{\tilde{H}}(x')||g_{\tilde{H}}(x)|}\, dx\, dx'\, d\xi' \\
&\quad = \frac{1}{(2\pi h)^{n-k}} \iiint e^{\frac{i}{h}\langle x - x', \xi'\rangle} (L^*)^{\tilde{N}} \\
&\qquad\qquad \times \Big[w(x)(1 - \beta_\delta(x, \xi'))v(x')\sqrt{|g_{\tilde{H}}(x')||g_{\tilde{H}}(x)|}\, \Big]\, dx\, dx'\, d\xi' \\
&\quad \le C_N h^{k-n+\tilde{N}(1-2\delta)} \|v\|_{L^2(\tilde{H})}.
\end{aligned}$$

Here, C_N depends on the $C^{\tilde{N}}$-norm of w as well as finitely many of the constants $\mathcal{K}_\alpha$. The second fact follows since the transition maps for the coordinate change which flattens $\tilde{H}$ have $C^{\tilde{N}}$-norm bounded by finitely many of the constants $\mathcal{K}_\alpha$. □

We next apply Lemma 3.1 to the setup of Theorem 10.

Proposition 3.2. *Let P be as in Theorem* 10*. Let δ with $0 \le \delta < \frac{1}{2}$, let $N > 0$, and let $w \in S_\delta \cap C_c^\infty(\tilde{H})$. Then there exists $C_N > 0$, depending on P, finitely many seminorms of $w \in S_\delta$, and finitely many of the constants $\mathcal{K}_\alpha$ in* (2.3)*, so that for all $u \in \mathcal{D}'(M)$ and all $h > 0$,*

$$\left| \int_{\tilde{H}} w(1 - Op_h(\beta_\delta))(u)\, d\sigma_{\tilde{H}} \right| \le C_N h^N (\|u\|_{L^2(M)} + \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)}).$$

Proof. In order to use Lemma 3.1, we first bound $\|u\|_{L^2(\tilde{H})}$. For this, observe that since p is classically elliptic, by a standard elliptic parametrix construction (see, e.g., [22, Appendix E])

$$\|u\|_{H_{\mathrm{scl}}^{\frac{k+1}{2}}(M)} \le C(\|u\|_{L^2(M)} + \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)}),$$

where the constant C depends only on P. In particular, the semiclassical Sobolev estimates (see, e.g., [25, Lemma 6.1]) imply that

$$\|u\|_{L^2(\tilde{H})} \leq Ch^{-\frac{k}{2}}(\|u\|_{L^2(M)} + \|Pu\|_{H_{\text{scl}}^{(k-2m+1)/2}(M)}).$$

Using Lemma 3.1 then gives

$$\left|\int_{\tilde{H}} w(1 - Op_h(\beta_\delta))(u)\, d\sigma_{\tilde{H}}\right| \leq C_N h^N(\|u\|_{L^2(M)} + \|Pu\|_{H_{\text{scl}}^{(k-2m+1)/2}(M)}). \qquad \square$$

3.2. Step 2: Coverings by bicharacteristic beams. We first prove that there is a constant $\mathfrak{D}_n > 0$, depending only on n, so that for τ, r small enough, there is a $(\mathfrak{D}_n, \tau, r)$-good cover of $\Sigma_{H,p}$. We adapt the proof of [18, Lemma 2] to our purposes.

Proposition 3.3. *There exist $\mathfrak{D}_n > 0$ depending only on n, $R_0 = R_0(n,k,\mathcal{K}_0) > 0$, and $0 < \tau_{\Sigma_{H,p}} < \frac{\tau_{\text{inj}}}{2}$ depending only on τ_{inj}, such that for $0 < r_1 < R_0$, $0 < r_0 \leq \frac{r_1}{2}$, and $0 < \tau < \tau_{\Sigma_{H,p}}$ there exist points $\{\rho_j\}_{j=1}^{N_{r_1}} \subset \Sigma_{H,p}$ and a partition $\{\mathcal{J}_i\}_{i=1}^{\mathfrak{D}_n}$ of $\{1,\dots,N_{r_1}\}$ so that*

- $\Lambda^\tau_{\Sigma_{H,p}}(r_0) \subset \bigcup_{j=1}^{N_{r_1}} \Lambda^\tau_{\rho_j}(r_1)$,
- $\Lambda^\tau_{\rho_j}(3r_1) \cap \Lambda^\tau_{\rho_\ell}(3r_1) = \emptyset$, $j, \ell \in \mathcal{J}_i$, $j \neq \ell$.

Proof. Let $\{\rho_j\}_{j=1}^{N_{r_1}}$ be a maximal $\frac{r_1}{2}$ separated set in $\Sigma_{H,p}$. Fix $i_0 \in \{1,\dots,N_{r_1}\}$ and suppose that $B(\rho_{i_0}, 3r_1) \cap B(\rho_\ell, 3r_1) \neq \emptyset$ for all $\ell \in \mathcal{L}_{i_0} \subset \{1,\dots,N_{r_1}\}$. Then, for all $\ell \in \mathcal{L}_{i_0}$, $B(\rho_\ell, \frac{r_1}{2}) \subset B(\rho_{i_0}, 8r_1)$. In particular,

$$\sum_{\ell \in \mathcal{L}_{i_0}} \operatorname{vol}\left(B\left(\rho_\ell, \frac{r_1}{2}\right)\right) \leq \operatorname{vol}(B(\rho_{i_0}, 8r_1)).$$

Now, there exist $\mathfrak{D}_n > 0$ and $R_0 > 0$ depending on (n,k) and a lower bound on the Ricci curvature of $\Sigma_{H,p}$, and hence on only $(n,k,\mathcal{K}_0)$, so that for $r_1 < R_0$,

$$\operatorname{vol}(B(\rho_{i_0}, 8r_1)) \leq \operatorname{vol}(B(\rho_\ell, 14r_1)) \leq \mathfrak{D}_n \operatorname{vol}\left(B\left(\rho_\ell, \frac{r_1}{2}\right)\right).$$

Hence,

$$\sum_{\ell \in \mathcal{L}_{i_0}} \operatorname{vol}\left(B\left(\rho_\ell, \frac{r_1}{2}\right)\right) \leq \operatorname{vol}(B(\rho_{i_0}, 8r_1)) \leq \frac{\mathfrak{D}_n}{|\mathcal{L}_{i_0}|} \sum_{\ell \in \mathcal{L}_{i_0}} \operatorname{vol}\left(B\left(\rho_\ell, \frac{r_1}{2}\right)\right)$$

and in particular, $|\mathcal{L}_{i_0}| \leq \mathfrak{D}_n$.

Now, suppose that

$$\Lambda^\tau_{\rho_k}(3r_1) \cap \Lambda^\tau_{\rho_{i_0}}(3r_1) \neq \emptyset.$$

Then there exists $q_k \in B(\rho_k, 3r_1) \cap \mathcal{L}_h$, $q_{i_0} \in B(\rho_{i_0}, 3r_1) \cap \mathcal{L}_h$ and $t_k, t_{i_0} \in [-\tau, \tau]$ so that

$$\varphi_{t_k - t_{i_0}}(q_k) = q_{i_0}.$$

Here, $\mathcal{L}_h$ is the hypersurface defined in (2.9). In particular, choosing $\tau_{\Sigma_{H,p}} < \frac{\tau_{\text{inj}}}{2}$, this implies that $q_k = q_{i_0}$, $t_k = t_{i_0}$ and hence $B(\rho_\ell, 3r_1) \cap B(\rho_{i_0}, 3r_1) \neq \emptyset$. This implies that $j \in \mathcal{L}_{i_0}$ and hence that there are at most $\mathfrak{D}_n$ such distinct j (including i_0).

At this point we have proved that each of the tubes $\Lambda^\tau_{\rho_j}(r_1)$ intersects at most $\mathfrak{D}_n - 1$ other tubes. We now construct the sets $\mathcal{J}_1, \dots, \mathcal{J}_{\mathfrak{D}_n}$ using a greedy algorithm. We will say that i intersects j if

$$\Lambda^\tau_{\rho_i}(r_1) \cap \Lambda^\tau_{\rho_j}(r_1) \neq \emptyset.$$

First place $1 \in \mathcal{J}_1$. Then suppose we have placed $j = 1, \dots, \ell$ in $\mathcal{J}_1, \dots, \mathcal{J}_{\mathfrak{D}_n}$ so that each of the sets $\mathcal{J}_i$ consists of disjoint indices. Then, since $\ell + 1$ intersects at most $\mathfrak{D}_n - 1$ indices, it is disjoint from $\mathcal{J}_i$ for some i. We add ℓ to $\mathcal{J}_i$. By induction we obtain the partition $\mathcal{J}_1, \dots, \mathcal{J}_{\mathfrak{D}_n}$.

Now, suppose $r_0 \leq r_1$ and that there exists $\rho \in \Lambda^\tau_{\Sigma_{H,p}}(r_0)$ so that $\rho \notin \bigcup_i \Lambda^\tau_{\rho_i}(r_1)$. Then there are $|t| < \tau + r_0$ and $q \in \mathcal{L}_h$ so that

$$\rho = \varphi_t(q), \quad d(q, \Sigma_{H,p}) < r_0, \quad \min_i d(q, \rho_i) \geq r_1.$$

In particular, by the triangle inequality, there exists $\tilde{\rho} \in \Sigma_{H,p}$ such that

$$d(\tilde{\rho}, \rho_i) \geq d(q, \rho_i) - d(q, \tilde{\rho}) > r_1 - r_0.$$

This contradicts the maximality of $\{\rho_j\}_{j=1}^{N_{r_1}}$ if $r_0 \leq \frac{r_1}{2}$. □

We proceed to build a δ-partition of unity associated to the cover we constructed in Proposition 3.3. The key feature in this partition will be that it is invariant under the bicharacteristic flow. Indeed, the partition is built so that its quantization commutes with the operator P in a neighborhood of $\Sigma_{H,p}$.

Proposition 3.4. *There exist constants* $\tau_1 = \tau_1(\tau_{\text{inj}}) > 0$ *and* $\varepsilon_1 = \varepsilon_1(\tau_1) > 0$, *and given* $0 < \delta < \frac{1}{2}$, $0 < \varepsilon \leq \varepsilon_1$ *there exists* $h_1 > 0$, *so that for any* $0 < \tau \leq \tau_1$, *and* $R(h) \geq 2h^\delta$, *the following holds.*

There exist $C_1 > 0$ *so that for all* $0 < h \leq h_1$ *and all* $(\tau, R(h))$*-covers of* $\Sigma_{H,p}$ *there exists a partition of unity* $\chi_j \in S_\delta \cap C_c^\infty(T^*M; [-C_1 h^{1-2\delta}, 1 + C_1 h^{1-2\delta}])$ *on* $\Lambda^\tau_{\Sigma_{H,p}}(\frac{1}{2}R(h))$ *for which*

- $\operatorname{supp} \chi_j \subset \Lambda^{\tau+\varepsilon}_{\rho_j}(R(h))$,
- $\mathrm{MS_h}([P, Op_h(\chi_j)]) \cap \Lambda^\tau_{\Sigma_{H,p}}(\varepsilon) = \emptyset$,

and the χ_j *are uniformly bounded in* S_δ.

Proof. Let $\mathcal{L}_h$ be as in (2.9) $\tau_1 < \frac{1}{2}\tau_{\text{inj}}$ and fix $0 < \tau \leq \tau_1$. Then let $\varepsilon_1 > 0$ be so small that $\Lambda^{\tau_1}_{\Sigma_{H,p}}(\varepsilon_1) \subset \Lambda^{2\tau_1}_{\mathcal{L}_h}(0)$, fix $0 < \varepsilon < \varepsilon_1$ and let h_1 be so small that $h^\delta \leq \varepsilon$ for all $0 < h \leq h_1$. For each $j \in \{1, \dots, N_h\}$ let

$$\mathcal{H}_j = \mathcal{L}_h \cap \Lambda^\tau_{\rho_j}(R(h)).$$

Let $\{\psi_j\} \subset C_c^\infty(\mathcal{L}_h; [0,1]) \cap S_\delta$ be a partition of unity on $\mathcal{L}_h \cap \Lambda^\tau_{\Sigma_{H,p}}(\frac{1}{2}R(h))$ subordinate to $\{\mathcal{H}_j\}_{j=1}^{N_h}$ that is uniformly bounded in S_δ. Then define $a_{j,0} \in S_\delta$ on $\Lambda^\tau_{\Sigma_{H,p}}(\varepsilon)$ by solving

$$a_{j,0}|_{\mathcal{L}_h} = \psi_j, \quad H_p a_{j,0} = 0 \text{ on } \Lambda^\tau_{\Sigma_{H,p}}(\varepsilon).$$

Clearly, $a_{j,0}$ defined in this way is a partition of unity for $\Lambda^\tau_{\Sigma_{H,p}}(\frac{1}{2}R(h))$. Furthermore, we can extend $a_{j,0}$ to T^*M as an element of S_δ so that

$$\operatorname{supp} a_{j,0} \subset \bigcup_{|t| \leq \tau + \varepsilon + R(h)} \varphi_t(\mathcal{H}_j) \subset \Lambda^{\tau+\varepsilon}_{\rho_j}(R(h)), \quad 0 \leq a_{j,0} \leq 1.$$

Note also that since $P \in \Psi^m(M)$ and $H_p a_{j,0} = 0$, for $b \in S_\delta$ with $\operatorname{supp} b \subset \Lambda^\tau_{\Sigma_{H,p}}(\varepsilon)$,

$$Op_h(b)[P, Op_h(a_{j,0})] \in h^{2-2\delta}\Psi_\delta(M).$$

We define $a_{j,k}$ by induction. Suppose we have $a_{j,\ell}$, $\ell = 0, \dots, k-1$, so that if we set

$$\chi_{j,k-1} := \sum_{\ell=0}^{k-1} h^{\ell(1-2\delta)} a_{j,\ell},$$

then

(A) $\sum_{j=1}^{N_h} \chi_{j,k-1} \equiv 1$ on $\Lambda^\tau_{\Sigma_{H,p}}(\frac{1}{2}R(h))$,
(B) $e_{j,k} := \sigma(h^{-1-k(1-2\delta)}[P, Op_h(\chi_{j,k-1})]) \in S_\delta$ on $\Lambda^\tau_{\Sigma_{H,p}}(\varepsilon)$.

Then, for every $k \geq 1$, define $a_{j,k} \in S_\delta$ by

$$a_{j,k}|_{\mathcal{L}_h} = 0, \quad H_p a_{j,k} = -i\, e_{j,k} \text{ on } \Lambda^\tau_{\Sigma_{H,p}}(\varepsilon). \tag{3.6}$$

Next extend $a_{j,k}$ to T^*M as an element of S_δ so that

$$\operatorname{supp} a_{j,k} \subset \bigcup_{|t| \leq \tau+\varepsilon+R(h)} \varphi_t(\mathcal{H}_j) \subset \Lambda^{\tau+\varepsilon}_{\rho_j}(R(h)).$$

Now, since $\sum_{j=1}^{N_h} \chi_{j,k-1} \equiv 1$ on $\Lambda^\tau_{\Sigma_{H,p}}(\frac{1}{2}R(h))$, by (B) we see that for $\rho \in \Lambda^\tau_{\Sigma_{H,p}}(\frac{1}{2}R(h))$,

$$\sum_{j=1}^{N_h} e_{j,k}(\rho) = \sigma\left(h^{-1-k(1-2\delta)}\left[P, Op_h\left(\sum_{j=1}^{N_h} \chi_{j,k-1}\right)\right]\right)(\rho) = 0.$$

In particular, (3.6) gives that $\sum_{j=1}^{N_h} a_{j,k} = 0$ on $\Lambda^\tau_{\Sigma_{H,p}}(\frac{1}{2}R(h))$. Therefore, since

$$\chi_{j,k} = \chi_{j,k-1} + h^{k(1-2\delta)} a_{j,k},$$

we conclude that

$$\sum_{j=1}^{N_h} \chi_{j,k} = 1 \quad \text{on } \Lambda^\tau_{\Sigma_{H,p}}(\tfrac{1}{2}R(h)),$$

and hence (A) is satisfied for $a_{j,\ell}$ with $\ell = 0, \dots, k$. To show that (B) is also satisfied, let $b \in S_\delta$ with $\operatorname{supp} b \subset \Lambda^\tau_{\Sigma_{H,p}}(\varepsilon)$. By assumption, we have

$$Op_h(b)[P, Op_h(\chi_{j,k-1})] \in h^{1+k(1-2\delta)}\Psi_\delta(M).$$

Also, using once again that $P \in \Psi^m(M)$ and that $H_p a_{j,k} = -i\, e_{j,k}$

$$Op_h(b)[P, Op_h(a_{j,k})] \in h\Psi_\delta(M) + h^{2-2\delta}\Psi_\delta(M).$$

Hence,

$$Op_h(b)[P, Op_h(\chi_{j,k})] = Op_h(b)[P, Op_h(\chi_{j,k-1} + h^{k(1-2\delta)} a_{j,k})] \in h^{1+k(1-2\delta)}\Psi_\delta(M),$$

and so, on $\Lambda^\tau_{\Sigma_{H,p}}(\varepsilon)$,

$$\begin{aligned}
&\sigma(h^{-1-k(1-2\delta)} Op_h(b)[P, Op_h(\chi_{j,k})]) \\
&\quad = \sigma\big(h^{-1-k(1-2\delta)} Op_h(b)\big([P, Op_h(\chi_{j,k-1})] + h^{k(1-2\delta)}[P, Op_h(a_{j,k})]\big)\big) \\
&\quad = b(e_{j,k} - e_{j,k}) = 0.
\end{aligned}$$

In particular,

$$Op_h(b)[P, Op_h(\chi_{j,k})] \in h^{1+(k+1)(1-2\delta)}\Psi_\delta(M), \tag{3.7}$$

and $e_{j,k+1} \in S_\delta$ on $\Lambda^\tau_{\Sigma_{H,p}}(\varepsilon)$ as claimed.

Finally, let

$$\chi_j \sim \sum_{\ell=0}^{\infty} h^{\ell(1-2\delta)} a_{j,\ell}.$$

Then, using (3.7),

$$\mathrm{MS_h}([P, Op_h(\chi_j)]) \cap \Lambda^\tau_{\Sigma_{H,p}}(\varepsilon) = \emptyset.$$

Now, note that by construction $\{\chi_j\}$ remains a partition of unity modulo $O(h^\infty)$ and by adding an h^∞ correction to teach term, we construct $\{\chi_j\}$ so that it forms a partition of unity. We also have by construction that $\chi_j \in C_c^\infty(T^*M; [-C_1 h^{1-2\delta}, 1 + C_1 h^{1-2\delta}])$ for some C_1 depending only on (M, p) and finitely many of the constants $\mathcal{K}_\alpha$. □

3.3. Step 3: Estimate near bicharacteristics. Let $h > 0$. Let $(x', \tilde{x})$ be Fermi coordinates near $\tilde{H} = \tilde{H}_h$ with corresponding dual coordinates $(\xi', \tilde{\xi})$. Then, since H is uniformly conormally transverse for p, $\tilde{H}$ and on $\Sigma_{\tilde{H},p}$, there exists j so that $H_p \tilde{x}_j \neq 0$. In particular,

$$dp,\ \{d\tilde{x}_i\}_{i=1}^k,\ \{d\xi_i'\}_{i=1}^{n-k} \text{ are linearly independent near } \Sigma_{H,p}.$$

Thus, there exist $y_1, \ldots, y_{n-1} \in C^\infty(T^*M; \mathbb{R})$ so that $(p, \tilde{x}, \xi', y)$ are coordinates on T^*M near $\Sigma_{\tilde{H},p}$ for which $\Sigma_{\tilde{H},p} = \{p = 0, \tilde{x} = 0, \xi' = 0\}$. In particular, there exists a constant $C > 0$ depending only on $(M, p, \mathcal{K}_0)$ so that

$$d((x_0, \xi_0), \Sigma_{\tilde{H},p})^2 \le C(p(x_0, \xi_0)^2 + |\tilde{x}_0|^2 + |\xi_0'|^2).$$

We define the constant $K > 0$ introduced in the definition (3.1) of β_δ to be large enough so that

$$\text{If } d((x_0, \xi_0), \Sigma_{\tilde{H},p}) \ge \frac{1}{2}h^\delta,\ (x_0', \xi_0') \in \operatorname{supp}\beta_\delta,\ \text{and } d(x, \tilde{H}) \le \frac{1}{K}h^\delta, \tag{3.8}$$
$$\text{then } |p(x_0, \xi_0)| \ge \frac{1}{3}h^\delta.$$

As introduced in Step 1 in the proof of Theorem 10, let $\chi_0 \in C_c^\infty(\mathbb{R}; [0, 1])$ be a smooth cut-off function with $\chi_0(t) = 1$ for $t \le \frac{1}{2}$ and $\chi_0(t) = 0$ for $t \ge 1$. Let $\beta_\delta(x', \xi')$ be defined as in (3.1). In what follows $\tau_1, \varepsilon_1, h_1$ are the positive constants given by Proposition 3.4.

Our next proposition estimates the main contribution to averages. In particular, we control the average near zero frequency by the L^2-mass along bicharacteristics co-normal to the submanifold H. One of the main estimates used in the proof of Proposition 3.5 is found in Lemma 3.8. In particular, p is factored as $e(x, \xi)(\xi_1 - a(x, \xi'))$ so that it can be treated using elementary estimates. This idea comes from [39] where, to the best of the authors' knowledge, it was first used to control L^∞-norms.

Proposition 3.5. *There exist two constants τ_0, $0 < \tau_0 \le \tau_1$, and ε_0, $0 < \varepsilon_0 \le \varepsilon_1$, with $\tau_0 = \tau_0(M, p, \tau_{\mathrm{inj}}, \Im_0)$ and $\varepsilon_0 = \varepsilon_0(\tau_0)$, $R_0 = R_0(M, p, k, \mathcal{K}_0, \tau_{\mathrm{inj}}, \Im_0) > 0$ and a constant $C_{n,k}$ depending only on n, k, and for each $0 < \delta < \frac{1}{2}$ there exists $0 < h_0 \le h_1$ so that the following holds.*

Let $0 < \tau \le \tau_0$, $0 < \varepsilon < \varepsilon_0$, $4h^\delta \le R(h) \le R_0$. Let $\mathfrak{D}_n$ be the constant from Proposition 3.3, let $0 < h < h_0$, and let $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j=1}^{N_h}$ be a $(\mathfrak{D}_n, \tau, R(h))$-good cover for $\Sigma_{H,p}$. In addition, let $\{\chi_j\}_{j=1}^{N_h}$ be the partition of unity built in Proposition 3.4. Then there exists a constant $C > 0$ so that for all $N > 0$ there is $C_N > 0$ with the following properties. For all $w = w(x'; h) \in S_\delta \cap C_c^\infty(\tilde{H})$, $0 < h \le h_0$, and $u \in \mathcal{D}'(M)$,

$$h^{\frac{k-1}{2}} \|wOp_h(\beta_\delta)u\|_{L^1(\tilde{H})} \le C_{n,k}\|w\|_\infty R(h)^{\frac{n-1}{2}} \sum_{j\in\mathcal{I}_h(w)} \frac{\|Op_h(\chi_j)u\|_{L^2(M)}}{\tau^{\frac{1}{2}}|H_p r_H(\rho_j)|^{\frac{1}{2}}} + Ch^{-1}\|w\|_\infty\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)} + C_N h^N \|w\|_\infty \|u\|_{L^2(M)},$$

where $\mathcal{I}_h(w) = \{j : \Lambda^\tau_{\rho_j}(R(h)) \cap \pi^{-1}(\operatorname{supp} w) \neq \emptyset\}$. Moreover, the constants C, C_N, h_0 are uniform for χ_j in bounded subsets of S_δ, uniform in $\tau, \varepsilon_0, \mathfrak{I}_0$ when these are bounded away from 0, and uniform for $\mathcal{K}_\alpha$-bounded.

Proof. We define $\tau_0 > 0$, $\varepsilon_0 > 0$ to be the constants given by Lemma 3.7 below. Let $\chi_0 \in C_c^\infty(\mathbb{R}; [0,1])$ be a smooth cut-off function with $\chi_0(t) = 1$ for $t \le \frac{1}{2}$ and $\chi_0(t) = 0$ for $t \ge 1$. We first decompose $\|wOp_h(\beta_\delta)u\|_{L^1(\tilde{H})}$ with respect to $\{\chi_j\}_{j=1}^{N_h}$. We write

$$Op_h(\beta_\delta) = \left[1 - \chi_0\left(\frac{Kd(x,\tilde{H})}{h^\delta}\right)\right]Op_h(\beta_\delta) + \chi_0\left(\frac{Kd(x,\tilde{H})}{h^\delta}\right)Op_h(\beta_\delta)\sum_{j=1}^{N_h} Op_h(\chi_j) + Op_h(\chi)$$

with

$$Op_h(\chi) = \chi_0\left(\frac{Kd(x,\tilde{H})}{h^\delta}\right)Op_h(\beta_\delta)\left(1 - \sum_{j=1}^{N_h} Op_h(\chi_j)\right).$$

First, note that $[1 - \chi_0(\frac{Kd(x,\tilde{H})}{h^\delta})]Op_h(\beta_\delta)u|_{\tilde{H}} \equiv 0$. Therefore,

$$\|Op_h(\beta_\delta)u\|_{L^1(\tilde{H})} \le \left\|Op_h(\beta_\delta)\sum_{j=1}^{N_h} Op_h(\chi_j)u\right\|_{L^1(\tilde{H})} + \|Op_h(\chi)u\|_{L^1(\tilde{H})}. \tag{3.9}$$

We first study the $\|Op_h(\chi)u\|_{L^1(\tilde{H})}$ term. To do this, let $\psi \in C_c^\infty(T^*M)$ be so that

$$|p(x,\xi)| \ge c|\xi|^m \quad \text{on } \operatorname{supp}(1-\psi).$$

Then, by a standard elliptic parametrix construction (see, e.g., [22, Appendix E]) together with the semiclassical Sobolev estimates (see, e.g., [25, Lemma 6.1]) there exist constants $C > 0$ and $0 < h_0 \le h_1$ so that the following holds. For all N there exists $C_N > 0$ such that for all $0 < h \le h_0$,

$$\begin{aligned}\|Op_h(1-\psi)Op_h(\chi)u\|_{L^2(\tilde{H})} &\le Ch^{-\frac{k}{2}}\|Op_h(1-\psi)Op_h(\chi)u\|_{H_{\rm scl}^{\frac{k+1}{2}}(M)} \\ &\le Ch^{-\frac{k}{2}}\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)} + C_N h^N\|u\|_{L^2(M)}.\end{aligned}$$

Together with Lemma 3.6 (below) applied to $\psi\chi$ and the fact that

$$\|Pu\|_{L^2(M)} \le \|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)}$$

this implies

$$\|Op_h(\chi)u\|_{L^2(\tilde{H})} \leq Ch^{-\frac{k}{2}-\delta}\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)} + C_N h^N \|u\|_{L^2(M)}. \tag{3.10}$$

Indeed, to see that Lemma 3.6 applies, let $(x_0, \xi_0) \in \operatorname{supp} \psi\chi$. Then observe that

$$\operatorname{supp} \chi \subset (\Lambda^{\tau}_{\Sigma_{H,p}}(2h^{\delta}))^c$$

and hence

$$d((x_0,\xi_0), \Sigma_{\tilde{H},p}) \geq h^{\delta}.$$

Next, note that, since $(x_0,\xi_0) \in \operatorname{supp} \beta_\delta$,

$$d((x_0,\xi_0), N^*\tilde{H}) \leq \frac{1}{K}h^{\delta}.$$

Therefore, since $d((x_0,\xi_0),\Sigma_{\tilde{H},p}) \geq h^{\delta}$, $d(x,\tilde{H}) \leq \frac{1}{K}h^{\delta}$, and $(x_0,\xi_0) \in \operatorname{supp}\beta_\delta$, by the definition (3.8) of K we obtain that $|p(x_0,\xi_0)| \geq \frac{h^\delta}{3}$ for all $0 < h \leq h_0$. To see that $|dp| > \frac{\mathfrak{I}_0}{2} > 0$ on $\operatorname{supp} \psi\chi$, we observe that $|H_p| > \mathfrak{I}_0 > 0$ on $\Sigma_{H,p}$. It follows from (3.9) and (3.10) that

$$\begin{aligned}\|wOp_h(\beta_\delta)u\|_{L^1(\tilde{H})} \leq{}& \left\| \sum_{j=1}^{N_h} wOp_h(\beta_\delta)Op(\chi_j)u \right\|_{L^1(\tilde{H})} \\ &+ C\|w\|_\infty h^{-\frac{k}{2}-\delta}\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)} \\ &+ C_N h^N \|w\|_\infty \|u\|_{L^2(M)}.\end{aligned} \tag{3.11}$$

By Proposition 3.3, or more precisely its proof, there exist a collection of balls $\{B_i\}_{i=1}^{M_h}$ in $\tilde{H}$ of radius $R(h) \leq R_0(n,k,\mathcal{K}_0)$ and constants $\alpha_{n,k}$ depending only on n,k, so that

$$\tilde{H} \subset \bigcup_{i=1}^{M_h} B_i$$

and each $x \in \tilde{H}$ lies in at most $\alpha_{n,k}$ balls B_i. Let $\{\psi_i\}_{i=1}^{M_h}$ be a partition of unity on $\tilde{H}$ subordinate to $\{B_i\}_{i=1}^{M_h}$. Then, by (3.11), for all $0 < h \leq h_0$,

$$\begin{aligned}\|wOp_h(\beta_\delta)u\|_{L^1(\tilde{H})} \leq{}& \sum_{i=1}^{M_h}\sum_{j=1}^{N_h} \|\psi_i wOp_h(\beta_\delta)Op(\chi_j)u\|_{L^1(\tilde{H})} \\ &+ Ch^{-\frac{k}{2}-\delta}\|w\|_\infty\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)} \\ &+ C_N h^N \|w\|_\infty \|u\|_{L^2(M)}.\end{aligned} \tag{3.12}$$

We next note that on $\tilde{H}$, the volume of a ball of radius r satisfies

$$|\operatorname{vol}_{\tilde{H}}(B(x,r)) - c_{n,k}r^{n-k}| \leq C_{\mathcal{K}_0} r^{n-k+1},$$

where $C_{\mathcal{K}_0} > 0$ is a constant depending only on $\mathcal{K}_0$ and $c_{n,k}$ is a constant that depends only on (n,k) (this can be seen by working in geodesic normal coordinates). Therefore, for some $c_{n,k} > 0$ and any $R(h) \leq R_0 = R_0(\mathcal{K}_0)$,

$$\|\psi_i wOp_h(\beta_\delta)Op(\chi_j)u\|_{L^1(\tilde{H})} \leq c_{n,k}R(h)^{\frac{n-k}{2}}\|\psi_i wOp_h(\beta_\delta)Op(\chi_j)u\|_{L^2(\tilde{H})}. \tag{3.13}$$

We next bound $\|\psi_i w Op_h(\beta_\delta) Op(\chi_j) u\|_{L^2(\tilde{H})}$. By Lemma 3.7 below there exist constants $C_{n,k} > 0$ depending only on (n,k), and $C > 0$ so that the following holds. For every $\tilde{N} > 0$ there exists $C_{\tilde{N}} > 0$, independent of (i,j), so that for all $0 < h \le h_0$,

$$
\begin{aligned}
(3.14)\qquad & \|\psi_i w Op_h(\beta_\delta) Op_h(\chi_j) u\|_{L^2(\tilde{H})} \\
&\le C_{n,k} \|w\|_\infty h^{\frac{1-k}{2}} R(h)^{\frac{k-1}{2}} \Bigg(\frac{\|Op_h(\chi_j)u\|_{L^2(M)}}{\tau^{\frac{1}{2}} |H_p r_H(\rho_j)|^{\frac{1}{2}}} \\
&\qquad + Ch^{-1} \|Op_h(\chi_j) Pu\|_{L^2(M)} \Bigg) + C_{\tilde{N}} h^{\tilde{N}} \|w\|_\infty \|u\|_{L^2(M)}.
\end{aligned}
$$

Also, note that if $j \notin \mathcal{I}_h(\psi_i w)$ for some $i \in \{1,\dots,M_h\}$, then

$$\Lambda^\tau_{\rho_j}(R(h)) \cap \pi^{-1}(\operatorname{supp} \psi_i w) = \emptyset.$$

Therefore, since $\operatorname{supp} \chi_j \subset \Lambda^\tau_{\rho_j}(R(h))$ for all j, for all N' there exists $C_{N'} > 0$ so that the following holds. For all $i \in \{1,\dots,M_h\}$ and $j \notin \mathcal{I}_h(\psi_i w)$,

$$\|\psi_i w Op_h(\beta_\delta) Op_h(\chi_j) u\|_{L^2(\tilde{H})} \le C_{N'} h^{N'} \|w\|_\infty \|u\|_{L^2(M)}.$$

In particular, since N_h and M_h grow like a polynomial power of h, we can choose N' so that

$$(3.15)\qquad \sum_{i=1}^{M_h} \sum_{j \notin I_h(\psi_i w)} \|\psi_i w Op_h(\beta_\delta) Op_h(\chi_j) u\|_{L^2(\tilde{H})} \le C_N h^N \|w\|_\infty \|u\|_{L^2(M)}.$$

Putting (3.13), (3.14) and (3.15) into (3.12), we find that for some adjusted $C_{n,k}$ and $0 < h \le h_0$,

$$
\begin{aligned}
&\|w Op_h(\beta_\delta) u\|_{L^1(\tilde{H})} \\
&\quad \le C_{n,k} \|w\|_\infty h^{\frac{1-k}{2}} R(h)^{\frac{n-1}{2}} \sum_{i=1}^{M_h} \sum_{j \in I_h(\psi_i w)} \Bigg(\frac{\|Op_h(\chi_j)u\|_{L^2(M)}}{\tau^{\frac{1}{2}} |H_p r_H(\rho_j)|^{\frac{1}{2}}} \\
&\qquad\qquad + Ch^{-1} \|Op_h(\chi_j) Pu\|_{L^2(M)} \Bigg) \\
&\qquad + Ch^{-\frac{k}{2}-\delta} \|w\|_\infty \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)} + C_N h^N \|w\|_\infty \|u\|_{L^2(M)}.
\end{aligned}
$$

We have used that both M_h and N_h grow like a polynomial power of h to collect all the $C_{\tilde{N}} h^{\tilde{N}} \|u\|_{L^2(M)}$ error terms in (3.14). Furthermore, since the balls $\{B_i\}$ are built so that every point in $\tilde{H}$ lies in at most $\alpha_{n,k}$ balls, and each ψ_i is supported on B_i, we have

$$
\begin{aligned}
(3.16)\qquad & \|w Op_h(\beta_\delta) u\|_{L^1(\tilde{H})} \\
&\le C_{n,k} \|w\|_\infty h^{\frac{1-k}{2}} R(h)^{\frac{n-1}{2}} \sum_{j \in \mathcal{I}_h(w)} \Bigg(\frac{\|Op_h(\chi_j)u\|_{L^2(M)}}{\tau^{\frac{1}{2}} |H_p r_H(\rho_j)|^{\frac{1}{2}}} \\
&\qquad + Ch^{-1} \|Op_h(\chi_j) Pu\|_{L^2(M)} \Bigg) \\
&\quad + Ch^{-\frac{k}{2}-\delta} \|w\|_\infty \|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)} + C_N h^N \|w\|_\infty \|u\|_{L^2(M)}.
\end{aligned}
$$

Now, since χ_j is supported in $\Lambda^{\tau}_{\rho_j}(R(h))$, and the tubes were built so that every point in $\Lambda^{\tau}_{\Sigma_{H,p}}(h^\delta)$ lies in at most $\beta_{n,k}$ tubes, we have $\sum_{j=1}^{N_h}|\chi_j|^2 \le \beta_{n,k}$. This implies

$$\sum_{j=1}^{N_h}\|Op_h(\chi_j)Pu\|^2_{L^2(M)} \le 2\beta_{n,k}\|Pu\|^2_{L^2(M)}.$$

Next, notice that since $\dim \Sigma_{H,p} = n-1$, we have $|\mathcal{J}_h(w)| \le c_{n,k}R(h)^{1-n}\operatorname{vol}(\Sigma_{H,p})$ for some $c_{n,k} > 0$ depending only on n, k. Therefore,

$$\begin{aligned}\sum_{j\in\mathcal{J}_h(w)}\|Op_h(\chi_j)Pu\|_{L^2(M)} &\le |\mathcal{J}_h(w)|^{\frac{1}{2}}\Bigg(\sum_{j=1}^{N_h}\|Op_h(\chi_j)Pu\|^2_{L^2(M)}\Bigg)^{\frac{1}{2}}\\ &\le c_{n,k}R(h)^{-\frac{n-1}{2}}\operatorname{vol}(\Sigma_{H,p})^{\frac{1}{2}}\|Pu\|_{L^2(M)}\end{aligned}$$

for some $c_{n,k} > 0$ depending only on n, k. Using this in (3.16) together with $\delta < \frac{1}{2}$, gives

$$\begin{aligned}\|wOp_h(\beta_\delta)u\|_{L^1(\tilde{H})} \le C_{n,k}\|w\|_\infty h^{\frac{1-k}{2}}R(h)^{\frac{n-1}{2}}&\sum_{j\in\mathcal{J}_h(w)}\frac{\|Op_h(\chi_j)u\|_{L^2(M)}}{\tau^{\frac{1}{2}}|H_pr_H(\rho_j)|^{\frac{1}{2}}}\\ &+ Ch^{-\frac{1+k}{2}}\|w\|_\infty\|Pu\|_{H_{\mathrm{scl}}^{(k-2m+1)/2}(M)}\\ &+ C_Nh^N\|w\|_\infty\|u\|_{L^2(M)},\end{aligned}$$

as claimed. Note that the constants C, C_N, h_0 are uniform for χ_j in bounded subsets of S_δ, and are also uniform in $\tau, \varepsilon_0, \mathfrak{I}_0$ when these are bounded away from 0. Furthermore, they depend only on finitely many of the constants $\mathcal{K}_\alpha$. □

We now state the following result which gives elliptic estimates in regions that are h^δ away from the characteristic variety of p.

Lemma 3.6. *Let $0 \le \delta < \frac{1}{2}$, $0 < k < n$. Let $\Theta : W \subset \mathbb{R}^n \to M$ be coordinates on M. Let $\chi \in S^{\mathrm{comp}}_\delta \cap C^\infty_c(T^*M; [-C_0h^{1-2\delta}, 1 + C_0h^{1-2\delta}])$ be so that there exist $c, h_1 > 0$ with*

$$\operatorname{supp}\chi \subset \{|p| \ge ch^\delta, |p| + |dp| > c\}$$

for $0 < h \le h_1$. Then there exists $C > 0$ such that for all $\tilde{\chi} \in S_\delta \cap C^\infty_c(T^*M; [0, 1])$ with $\tilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$, there exists $0 < h_0 < h_1$ so that the following holds. For all $N > 0$ there exists $C_N > 0$ such that for $0 < h < h_0$,

$$\|Op_h(\chi)u\|_{L^\infty_{\bar{x}}L^2_{x'}} \le Ch^{-\frac{k}{2}-\delta}\|Op_h(\tilde{\chi})Pu\|_{L^2_x} + C_Nh^N\|u\|_{L^2_x},$$

where $x = (x', \bar{x}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ are the coordinates induced by Θ. Moreover, C, C_N are uniform for $\tilde{\chi}$, χ in bounded subsets of S_δ, and for Θ in bounded subsets of C^∞.

Proof. First, let $\psi \in C^\infty_c(\mathbb{R})$ with $\psi \equiv 1$ on $[-1, 1]$. Then, using the standard elliptic parametrix construction [22, Appendix E] there exists $b_1 \in S^{\mathrm{comp}}_\delta$, $\sup|b_1| \le 2c^{-1} + C_1h^{1-2\delta}$, such that

$$Op_h(\chi)Op_h\bigg(1-\psi\bigg(\frac{2}{c}p\bigg)\bigg) = Op_h(b_1)Op_h(\tilde{\chi})P + O(h^\infty)_{\Psi^{-\infty}}. \tag{3.17}$$

Next, we show that there exists $b_2 \in S_\delta^{\mathrm{comp}}$ with $\sup|b_2| \le c^{-1}h^{-\delta} + C_1h^{1-3\delta}$ so that

$$Op_h(\chi)Op_h\Big(\psi\Big(\frac{2}{c}p\Big)\Big) = Op_h(b_2)Op_h(\tilde{\chi})P + O(h^\infty)_{\Psi^{-\infty}}. \tag{3.18}$$

Using that $|p| \ge ch^\delta$ on $\operatorname{supp}\chi$, one can carry out an elliptic parametrix construction in the second microlocal calculus associated to $p = 0$. Using a partition of unity, since $|dp| > \frac{c}{2}$ on $\operatorname{supp}\chi \cap \operatorname{supp}\psi(\frac{2}{c}p)$, we may assume that there exist an h-independent neighborhood V_0 of $\operatorname{supp}\chi$, $V_1 \subset T^*\mathbb{R}^n$ a neighborhood of 0, and a symplectomorphism $\kappa : V_1 \to V_0$ so that $\kappa^* p = \xi_1$. Let U be a microlocally unitary FIO quantizing κ. Then

$$\mathbf{P} := U^*PU = hD_{x_1} + hOp_h^L(\mathbf{r}), \quad \mathbf{r} \in S^{\mathrm{comp}}(\mathbb{R}^n),$$

where Op_h^L denotes the *left* quantization of $\mathbf{r}$. Moreover, there exist $\mathbf{a}, \tilde{\mathbf{a}} \in S_\delta^{\mathrm{comp}}(T^*\mathbb{R}^n)$ so that

$$Op_h^L(\mathbf{a}) = U^*Op_h(\chi)Op_h\Big(\psi\Big(\frac{2}{c}p\Big)\Big)U$$

and

$$Op_h^L(\tilde{\mathbf{a}}) = U^*Op_h(\tilde{\chi})U$$

with $\operatorname{supp}\mathbf{a} \subset \{|\xi_1| \ge ch^\delta\}$ and $\tilde{\mathbf{a}} \equiv 1$ on $\operatorname{supp}\mathbf{a}$. Now, for $\mathbf{b} \in S_\delta^{\mathrm{comp}}(T^*\mathbb{R}^n)$ supported on $|\xi_1| \ge ch^\delta$,

$$|\partial_x^\alpha \partial_\xi^\beta(\xi_1^{-1}\mathbf{b})| \le C_{\alpha\beta}h^{-(|\beta|+|\alpha|)\delta}|\xi_1|^{-1}.$$

Let $\mathbf{b}_0 = \frac{\mathbf{a}}{\xi_1}$. Then $\mathbf{b}_0 \in h^{-\delta}S_\delta^{\mathrm{comp}}$ and

$$\sup|\mathbf{b}_0| \le c^{-1}h^{-\delta}.$$

Observe that

$$Op_h^L(\mathbf{b}_0)Op_h^L(\tilde{\mathbf{a}})\mathbf{P} = Op_h^L(\mathbf{a}) + Op_h^L(\mathbf{e}_1) + O(h^\infty)_{\Psi^{-\infty}}$$

with $\operatorname{supp}\mathbf{e}_1 \subset \{|\xi_1| \ge ch^\delta\}$ and, since $\tilde{\mathbf{a}} \equiv 1$ on $\operatorname{supp}\mathbf{b}_0$,

$$\mathbf{e}_1 \sim \sum_{|\alpha|\ge 1} \frac{h^{|\alpha|}i^{|\alpha|}}{\alpha!}D_x^\alpha(\mathbf{b}_0)D_\xi^\alpha(\xi_1) + \sum_{|\alpha|\ge 0} \frac{h^{|\alpha|+1}i^{|\alpha|}}{k!}D_x^\alpha(\mathbf{b}_0)D_\xi^\alpha(\mathbf{r}).$$

In particular, $\mathbf{e}_1 \in h^{1-2\delta}S_\delta^{\mathrm{comp}}$. Then, setting $\mathbf{b}_\ell = -\frac{\mathbf{e}_\ell}{\xi_1} \in h^{\ell(1-2\delta)-\delta}S_\delta^{\mathrm{comp}}$, and

$$Op_h^L(\mathbf{e}_{\ell+1}) := Op_h^L(\mathbf{b}_\ell)Op_h^L(\tilde{\mathbf{a}})\mathbf{P} + Op_h^L(\mathbf{e}_\ell) + O(h^\infty)_{\Psi^{-\infty}}$$

we have $\mathbf{e}_{\ell+1} \in h^{(\ell+1)(1-2\delta)}S_\delta^{\mathrm{comp}}$ with $\operatorname{supp}\mathbf{e}_{\ell+1} \subset \{|\xi_1| \ge ch^\delta\}$. In particular, putting now $\mathbf{b} \sim \sum_\ell \mathbf{b}_\ell$,

$$Op_h^L(\mathbf{b})Op_h^L(\tilde{\mathbf{a}})\mathbf{P} = Op_h^L(\mathbf{a}) + O(h^\infty)_{\Psi^{-\infty}}.$$

It follows that

$$\begin{aligned} U\,Op_h^L(\mathbf{b})U^*Op_h(\tilde{\chi})P &= U\,Op_h^L(\mathbf{b})U^*U\,Op_h^L(\tilde{\mathbf{a}})U^*U\mathbf{P}U^* + O(h^\infty)_{\Psi^{-\infty}} \\ &= U\,Op_h^L(\mathbf{b})Op_h^L(\tilde{\mathbf{a}})\mathbf{P}U^* + O(h^\infty)_{\Psi^{-\infty}} \\ &= U\,Op_h^L(\mathbf{a})U^* + O(h^\infty)_{\Psi^{-\infty}} \\ &= Op_h(\chi)Op_h\Big(\psi\Big(\frac{2}{c}p\Big)\Big) + O(h^\infty)_{\Psi^{-\infty}}. \end{aligned}$$

In particular, there exists $b_2 \in h^{-\delta} S_\delta^{\mathrm{comp}}(T^*M)$ with $\sup |b_2| \le c^{-1}h^{-\delta} + C_1 h^{1-3\delta}$ so that

$$Op_h(b_2) = U\, Op_h^L(\mathbf{b})U^* + O(h^\infty)_{\Psi^{-\infty}}.$$

Therefore, as claimed in (3.18) that

$$Op_h(\chi)Op_h\left(\psi\left(\frac{2}{c}p\right)\right) = Op_h(b_2)Op_h(\tilde{\chi})P + O(h^\infty)_{\Psi^{-\infty}},$$

for all χ supported in V_0 and some suitable b_2 with $\|Op_h(b_2)\| \le 2c^{-1}h^{-\delta}$. Next, using that $Op_h(\tilde{\chi})Pu$ is compactly microlocalized, we apply the Sobolev Embedding [25, Lemma 6.1] (see also [65, Lemma 7.10]) in the $\bar{x}$ coordinates. Writing $b = b_1 + b_2$, we obtain using (3.17) and (3.18) that there exists $h_0 > 0$, and for all $N > 0$ there exists a constant $C_N > 0$ such that if $0 < h < h_0$, then for every $\bar{x}$,

$$\begin{aligned}\|Op_h(\chi)u(\bar{x},\cdot)\|_{L^2_{x'}} &= \|Op_h(b)Op_h(\tilde{\chi})Pu(\bar{x},\cdot)\|_{L^2_{x'}} + C_N h^N \|u\|_{L^2_x} \\ &\le 2c^{-1}C_k h^{-\frac{k}{2}-\delta}\|Op_h(\tilde{\chi})Pu\|_{L^2_x} + C_N h^N\|u\|_{L^2_x}.\end{aligned}$$

Since this is true for any $\bar{x}$, the claim follows. □

The following lemma contains the key new ideas used to prove our main theorems. In particular, it converts quantitative localization along a bicharacteristic into quantitative gains in averages. This idea is at the heart of the bicharacteristic beam techniques and originated in [25].

Lemma 3.7. *There exist $C_{n,k} > 0$, depending only on n and k, and positive constants $\tau_0 = \tau_0(M, p, \tau_{\mathrm{inj}}, \mathfrak{I}_0, \{H_h\}_h)$, $\varepsilon_0 = \varepsilon_0(\tau_0)$, $R_0 = R_0(M, p, k, \tau_{\mathrm{inj}}, \mathfrak{I}_0)$ so that the following holds. Let $0 < \tau \le \tau_0$, $0 \le \delta < \frac{1}{2}$, and $2h^\delta \le R(h) \le R_0$. Let γ be a bicharacteristic through $\Sigma_{H,p}$, and $\chi \in S_\delta \cap C_c^\infty(T^*M; [-C_1h^{1-2\delta}, 1 + C_1h^{1-2\delta}])$ with $\rho_\gamma := \gamma \cap \Sigma_{H,p} \in \operatorname{supp}\chi$,*

$$\operatorname{supp}(\chi) \subset \Lambda_{\rho_\gamma}^{\tau+\varepsilon_0}(R(h)), \tag{3.19}$$

and

$$\mathrm{MS_h}([P, Op_h(\chi)]) \cap \Lambda^\tau_{\Sigma_{H,p}}(\varepsilon_0) = \emptyset. \tag{3.20}$$

Then there are $C > 0$ and $h_0 > 0$ with the following properties. For every $N > 0$ there exists $C_N > 0$ such that, if $0 < h \le h_0$, then for $u \in \mathcal{D}'(M)$,

$$\begin{aligned}h^{k-1}\|Op_h(\beta_\delta)Op_h(\chi)u\|^2_{L^2(\tilde{H})} \le{}& C_{n,k}\frac{R(h)^{k-1}}{\tau|H_p r_H(\rho_\gamma)|}\|Op_h(\chi)u\|^2_{L^2(M)} \\ &+ CR(h)^{k-1}h^{-2}\|Op_h(\chi)Pu\|^2_{L^2(M)} \\ &+ C_N h^N\|u\|^2_{L^2(M)}.\end{aligned}$$

The constants τ_0, C, C_N, h_0 are uniform for χ in bounded subsets of S_δ, uniform for $\tau > 0$ and $\mathfrak{I}_0$ uniformly bounded away from zero, and only depend on $\{H_h\}_h$ through finitely many of the constants $\mathcal{K}_\alpha$ in (2.3).

Proof. The proof of this result relies heavily on Lemma 3.8 below. Let

$$\Theta : W \subset \mathbb{R}^n \to M$$

be coordinates on M. Let $h > 0$. We may adjust coordinates so that $\tilde{H} = \tilde{H}_h \subset \{x_1 = 0\}$, $dx_1|_{x_1=0} \in N^*\tilde{H}$, and $\frac{1}{2}H_p r_H \leq \partial_{\xi_1} p$, and so that the C^k-norm of the coordinate map Θ is bounded by finitely many of the constants $\mathcal{K}_\alpha$. Therefore, since $|\partial_{\xi_1} p(\rho_\gamma)| \geq \frac{1}{2}\mathfrak{I}_0$ by (2.8), we may apply Lemma 3.8 with $\mathfrak{I} := \frac{1}{2}\mathfrak{I}_0$. Let $r_0, \tilde{\tau}_0, C_0$, depending only on $(M, p, \mathfrak{I}_0, \Theta)$, be the constants from Lemma 3.8. Note that they are uniform for Θ in bounded sets of C^k. Therefore, they depend on $\{H_h\}_h$ through finitely many of the constants $\mathcal{K}_\alpha$. Let $r_1 = r_1(M, p, \mathfrak{I}_0, \Theta)$ be small enough so that for all $\rho \in \Sigma_{H,p}$,

$$\frac{\inf_{B(\rho,r_1)} |H_p r_H|}{\sup_{B(\rho,r_1)} |H_p r_H|} \geq \frac{1}{2}. \tag{3.21}$$

Let $r = \frac{1}{2}\min\{r_1, r_0\}$ and let $\{\rho_i\}_{i=1}^K \subset \Sigma_{H,p}$ be a maximal r separated set. Then, for all $q \in \Sigma_{H,p}$, there exists i so that $d(q, \rho_i) < r$ and in particular, $B(q, r) \subset B(\rho_i, 2r) \subset V_{\rho_i}$ where V_{ρ_i} is the subset from Lemma 3.8 associated to ρ_i.

Fix a point $\rho_0 \in \{\rho_i\}_{i=1}^K$. Without loss of generality assume that $d(\rho_\gamma, \rho_0) < r$. Next, let $0 < \tilde{\tau}_1 < \frac{\tau_{\text{inj}}}{2}$, $R_0 > 0$, $\varepsilon_0 > 0$ small enough (depending only on $(M, P, \mathfrak{I}_0, \tau_{\text{inj}})$) so that $\Lambda_{\rho_\gamma}^{\tilde{\tau}_1+\varepsilon_0}(R_0) \subset V_{\rho_0}$. Next, by letting

$$\tau_0 = \min\{\tilde{\tau}_0, \tilde{\tau}_1\}, \tag{3.22}$$

we have

$$\text{supp}(\chi) \subset \Lambda_{\rho_\gamma}^{\tau+\varepsilon_0}(R(h)) \subset V_{\rho_0}$$

for all $0 < \tau < \tau_0$ and h small enough. This will allow us to apply Lemma 3.8 to our χ.

We work in coordinates so that $\partial_{\xi_1} p(\rho_\gamma) \neq 0$, which we can assume since γ is a bicharacteristic through $\Sigma_{H,p}$ and $\rho_\gamma = \gamma \cap \Sigma_{H,p}$. In what follows we abuse notation slightly and redefine $\bar{x}$ as the normal coordinates to $\tilde{H}$ that are not x_1. With this notation $x = (x_1, \bar{x}, x')$.

Given a function $v_h \in C^\infty(M)$, we may bound $\|v_h\|_{L^2(M)}$ using the version of the Sobolev Embedding Theorem given in [25, Lemma 6.1] which gives, after setting $k = \ell$, that for all $\alpha > 0$ there exists $C_k > 0$ depending only on k so that

$$\|v_h(x_1, \bar{x}, \cdot)\|^2_{L^2_{x'}} \leq C_k h^{1-k}\Bigg(\alpha^{k-1}\|v_h(x_1, \cdot)\|^2_{L^2_{\bar{x},x'}} + \alpha^{-1-k}\sum_{i=2}^{k}\|(hD_{x_i})^k v_h(x_1, \cdot)\|^2_{L^2_{\bar{x},x'}}\Bigg). \tag{3.23}$$

We proceed to choose v_h so that

$$\|Op_h(\beta_\delta)(Op_h(\chi)u)(x_1, \bar{x}, \cdot)\|_{L^2_{x'}} = \|v_h(x_1, \bar{x}, \cdot)\|_{L^2_{x'}}, \tag{3.24}$$

and in such a way that the terms in (3.23) can be controlled efficiently. Let $0 < \tau < \tau_0$, and set $\tau_{\rho_0} := \tau|\partial_{\xi_1} p(\rho_0)|$.

Since γ is a bicharacteristic through $\Sigma_{H,p}$, we may define a function $a = a(x_1)$ so that $\xi - a(x_1)$ vanishes along γ. This is possible since we are working in coordinates so that $\partial_{\xi_1} p(\rho_\gamma) \neq 0$, and hence γ may be locally written (near ρ_γ) as $\gamma(x_1) = (x(x_1), a(x_1))$ for a and x smooth.

Define

$$\kappa(x, \xi) = \chi_0\left(\frac{|(x_1, \bar{x})|}{\varepsilon_0^2}\right)\chi_0\left(\frac{3|x_1|}{\tau_{\rho_0}}\right)\beta_\delta(x', \xi'),$$

where $\varepsilon_0 < 1$ is so that the coordinates are well defined if $|(x_1, \bar{x})| < \varepsilon_0$. Let

$$v_h := e^{-\frac{i}{h}\langle \bar{x}, \bar{a}(x_1)\rangle} Op_h(\kappa) Op_h(\chi) u,$$

where $\bar{a}(x_1) = (a_2(x_1), \dots, a_k(x_1))$ is so that $a(x_1) = (a_1(x_1), \bar{a}(x_1))$. The reason for working with this function v_h is that not only (3.24) is satisfied, but also

$$(hD_{x_i})^k v_h = e^{-\frac{i}{h}\langle \bar{x}, \bar{a}(x_1)\rangle}(hD_{x_i} - a_i)^k (Op_h(\kappa) Op_h(\chi) u)$$

for $i = 2, \dots, k$, and this will allow us to obtain a gain in the L^2-norm bound once we use that, by Lemma A.3, for (τ_0, ε_0) small enough (depending only on p),

$$\sup_{\Lambda_{\rho_\gamma}^{\tau_0+\varepsilon_0}(R(h))} \max_i |\xi_i - a_i(x_1)| \leq 3R(h). \tag{3.25}$$

We bound the terms in (3.23) by applying Lemma 3.8 with κ and χ. We first bound the non-derivative term on the right-hand side of (3.23).

By Lemma 3.8 we have that

$$\inf_{V_{\rho_0}} |\partial_{\xi_1} p| \geq \frac{3}{4} |\partial_{\xi_1} p(\rho_0)| \quad \text{on } \Lambda_{\rho_\gamma}^{\tau+\varepsilon_0}(R(h)).$$

This implies

$$\big(\Lambda_{\rho_\gamma}^{\tau+\varepsilon_0}(R(h)) \cap (\Lambda_{\Sigma_{H,p}}^{\tau}(\varepsilon_0))^c\big) \subset \Big\{|x_1| \geq \frac{3}{4}\tau_{\rho_0}\Big\}. \tag{3.26}$$

Let $b \in C_c^\infty(\mathbb{R}; [0, 1])$ with $b \equiv 1$ on $\{x_1 : |x_1| \leq \frac{\tau_{\rho_0}}{2}\}$, $\operatorname{supp} b \subset \{x_1 : |x_1| < \frac{3}{4}\tau_{\rho_0}\}$. By (3.19) and (3.20) we have

$$\mathrm{MS_h}([P, Op_h(\chi)]) \subset (\Lambda_{\rho_\gamma}^{\tau+\varepsilon_0}(R(h)) \cap (\Lambda_{\Sigma_{H,p}}^{\tau}(\varepsilon_0))^c).$$

Therefore, by (3.26),

$$\mathrm{WF_h}(b) \cap \mathrm{MS_h}([P, Op_h(\chi)]) = \emptyset. \tag{3.27}$$

Throughout the rest of the proof we write C, C_N for constants that are uniform as claimed. We also note that when bounding $\|Op_h(a)u\|_{L^2(M)}$ by $2\sup|a|\|u\|_{L^2(M)}$, h need only be taken small enough depending on finitely many seminorms of a in S_δ. Let $C_0 = C_0(M, P, \mathfrak{I}_0)$ as above and τ_0 as in (3.22). Applying Lemma 3.8 with κ, χ, b, $q = 1$, and using that $b \equiv 1$ on $|x_1| \leq \frac{\tau_{\rho_0}}{2}$, $\|Op_h(\kappa)\| \leq 2$ and $0 < \tau < \tau_0$, we have that there exists $h_0 > 0$ such that for all $0 < h < h_0$,

$$\begin{aligned} \|v_h(x_1, \cdot)\|_{L^2_{\bar{x}, x'}} &\leq 8\tau_{\rho_0}^{-\frac{1}{2}} \|b\, Op_h(\chi)u\|_{L^2(M)} \\ &\quad + 2C_0 \tau_{\rho_0}^{\frac{1}{2}} h^{-1} \|b\, POp_h(\chi)u\|_{L^2(M)} + C_N h^N \|u\|_{L^2(M)}. \end{aligned} \tag{3.28}$$

Next, note that

$$b\, POp_h(\chi) = bOp_h(\chi)P + b[P, Op_h(\chi)].$$

Therefore, since $|b| \leq 1$,

$$\|b\, POp_h(\chi)u\|_{L^2(M)} \leq 2\|Op_h(\chi)Pu\|_{L^2(M)} + \|b[P, Op_h(\chi)]u\|_{L^2(M)}.$$

Using the previous bound, equation (3.28) turns into

$$\|v_h(x_1,\cdot)\|_{L^2_{\bar{x},x'}} \le 16\tau_{\rho_0}^{-\frac{1}{2}}\|Op_h(\chi)u\|_{L^2(M)} + 4C_0\tau_{\rho_0}^{\frac{1}{2}}h^{-1}\|Op_h(\chi)Pu\|_{L^2(M)} + 2C_0\tau_{\rho_0}^{\frac{1}{2}}h^{-1}\|b[P,Op_h(\chi)]u\|_{L^2(M)} + C_Nh^N\|u\|_{L^2(M)}. \tag{3.29}$$

We proceed to bound the derivative terms in (3.23). For this, we first note that

$$\|(hD_{x_i})^k v_h(x_1,\cdot)\|_{L_{\bar{x},x'}} = \|Q_i\, Op_h(\kappa) Op_h(\chi) u(x_1,\cdot)\|_{L_{\bar{x},x'}}$$

after setting

$$Q_i := (hD_{x_i} - a_i)^k \quad \text{for } i = 2,\dots,k. \tag{3.30}$$

Writing $Q_i = Op_h(q_i)$ we get $q_i = (\xi_i - a_i)^k$ and Q_i commutes with $Op_h(\kappa)$ modulo $O(h)$. Note that there are no remainder terms since a_i is a function of only x_1. Then Lemma 3.8 gives that there exists $C_0 > 0$, independent of τ, and some $C, C_N > 0$ so that

$$\begin{aligned}\|(hD_{x_i})^k v_h(x_1,\cdot)\|_{L^2_{\bar{x},x'}} \le\ & 8\tau_{\rho_0}^{-\frac{1}{2}}\|b\, Q_i\, Op_h(\chi)u\|_{L^2(M)} \\ &+ 2C_0\tau_{\rho_0}^{\frac{1}{2}}h^{-1}\|b\, P Q_i\, Op_h(\chi)u\|_{L^2(M)} \\ &+ \|[Op_h(\kappa), Q_i]Op_h(\chi)u(x_1,\cdot)\|_{L^2_{\bar{x},x'}} \\ &+ C_Nh^N\|u\|_{L^2(M)}\end{aligned} \tag{3.31}$$

for all $0 < h < h_0$, where h_0 was possibly adjusted. We proceed to find efficient bounds for all the terms in (3.31). Throughout the rest of the proof we use C_0 for a positive constant that depends only on P and finitely may S_δ seminorms of (q,χ), possibly bigger than that above. We also write C_k for a positive constant that depends only on k. These constants may increase from line to line.

First, let $\tilde{\chi} \in S_\delta \cap C_c^\infty(T^*M;[0,1])$ with $\tilde{\chi} \equiv 1$ on supp χ and supp $\tilde{\chi} \subset \Lambda_{\rho_\gamma}^{\tau+\varepsilon_0}(R(h))$. Then note that by (3.25) and (3.30) there exists $C_N > 0$ such that

$$\begin{aligned}\|b\, Q_i\, Op_h(\chi)u\|_{L^2(M)} &\le \|b\, Q_i\, Op_h(\tilde{\chi})Op_h(\chi)u\|_{L^2(M)} + C_Nh^N\|u\|_{L^2(M)} \\ &\le C_kR(h)^k\|Op_h(\chi)u\|_{L^2(M)} + C_Nh^N\|u\|_{L^2(M)}\end{aligned} \tag{3.32}$$

for all $0 < h < h_0$ for h_0 small enough.

Second, using that

$$b\, P Q_i\, Op_h(\chi) = b\, Q_i\, Op_h(\chi)P + b[P,Q_i]Op_h(\chi) + b\, Q_i[P, Op_h(\chi)],$$

we claim that there exists $C_N > 0$ such that

$$\begin{aligned}&\|bPQ_i\, Op_h(\chi)u\|_{L^2(M)} \\ &\quad\le C_kR(h)^k\|Op_h(\chi)Pu\|_{L^2(M)} + C_0hR(h)^k\|Op_h(\chi)u\|_{L^2(M)} \\ &\qquad + \|b\, Q_i[P, Op_h(\chi)]u\|_{L^2(M)} + C_Nh^N\|u\|_{L^2(M)}.\end{aligned} \tag{3.33}$$

Indeed, the estimate on $b[P,Q_i]Op_h(\chi)$ was obtained as follows. We observe that

$$H_pq_i = k(\xi_i - a_i)^{k-1}H_p(\xi_i - a_i),$$

and since $H_p(\xi_i - a_i)$ vanishes on γ, $H_p q_i$ vanishes to order k on γ. Therefore, using $\tilde{\chi}$ as in (3.31), on supp $\tilde{\chi}$ we have $|H_p q_i| \le C_0 R(h)^k$ and there exists $C_N > 0$ such that

$$\begin{aligned}\|b[P, Q_i]Op_h(\chi)u\|_{L^2(M)} &\le C_0 h R(h)^k \|Op_h(\chi)u\|_{L^2(M)} \\ &\quad + \left\|\left([P, Q_i] - \frac{h}{i}Op_h(H_p q_i)\right)Op_h(\tilde{\chi})Op_h(\chi)u\right\|_{L^2(M)} \\ &\quad + C_N h^N \|u\|_{L^2(M)}.\end{aligned}$$

Finally, observe that $([P, Q_i] - \frac{h}{i}Op_h(H_p q_i))Op_h(\tilde{\chi}) \in h^2 R(h)^{k-2} S_\delta$ and hence the bound follows since $R(h) \ge 2h^\delta$ and $\delta < \frac{1}{2}$.

Finally, to bound the fourth term in (3.31) note that by [25, Lemma 6.1],

$$\|[Op_h(\kappa), Q_i]Op_h(\chi)u(x_1, \cdot)\|_{L^2_{\bar{x},x'}} \le C_{M,p,R_0} h^{-\frac{1}{2}} \|[Op_h(\kappa), Q_i]Op_h(\chi)u\|_{L^2(M)}.$$

Observe that $[Op_h(\kappa), Q_i]Op_h(\tilde{\chi}) \in hR(h)^{k-1} S_\delta$ since for $i = 2, \dots, k$ we have $\partial_{x_j} q_i = 0$ for $j \ne 1$, $\partial_{\xi_1}\kappa = 0$, $\partial_{\xi_j} q_i = 0$ for all $j \ne i$, and $\partial_{x_i}\kappa \in S_\delta$ because β_δ is a tangential symbol. We then obtain that there exists $C_N > 0$ such that

$$\begin{aligned}(3.34)\qquad &\|[Op_h(\kappa), Q_i]Op_h(\chi)u(x_1, \cdot)\|_{L^2_{\bar{x},x'}} \\ &\quad \le C h^{\frac{1}{2}} R(h)^{k-1} \|Op_h(\chi)u\|_{L^2(M)} + C_N h^N \|u\|_{L^2(M)}.\end{aligned}$$

By combining (3.32), (3.33), and (3.34) into (3.31), it follows that

$$\begin{aligned}(3.35)\qquad R(h)^{-k} &\|(hD_{x_i})^\ell v_h(x_1, \cdot)\|_{L^2_{\bar{x},x'}} \\ &\le \left(C_k \tau_{\rho_0}^{-\frac{1}{2}} + C_0 \tau_{\rho_0}^{\frac{1}{2}} + C h^{\frac{1}{2}} R(h)^{-1}\right)\|Op_h(\chi)u\|_{L^2(M)} \\ &\quad + C_k C_0 \tau_{\rho_0}^{\frac{1}{2}} h^{-1} \|Op_h(\chi)Pu\|_{L^2(M)} \\ &\quad + C_0 \tau_{\rho_0}^{\frac{1}{2}} h^{-1} \|b\, Q_i[P, Op_h(\chi)]u\|_{L^2(M)} + C_N h^N \|u\|_{L^2(M)}\end{aligned}$$

for some $C > 0$, $C_N > 0$, and for all $0 < h < h_0$ with h_0 small enough.

By (3.27) we also know that there exists $C_N > 0$ and $h_0 > 0$ so that for all $0 < h < h_0$,

$$(3.36)\qquad \|b[P, Op_h(\chi)]u\|_{L^2(M)} + \|bQ_i[P, Op_h(\chi)]u\|_{L^2(M)} \le C_N h^N \|u\|_{L^2(M)}.$$

Feeding (3.36) into (3.29) and (3.35), and combining them in to (3.23), we have

$$\begin{aligned}R(h)^{1-k} h^{k-1} &\|v_h(x_1, \bar{x}, \cdot)\|^2_{L^2_{x'}} \\ &\le C_k \left(\|v_h(x_1, \cdot)\|^2_{L^2_{\bar{x},x'}} + R(h)^{-2k} \sum_{i=2}^{k} \|(hD_{x_i})^k v_h(x_1, \cdot)\|^2_{L^2_{\bar{x},x'}}\right) \\ &\le C_k \left(\tau_{\rho_0}^{-1} + C_0 \tau_{\rho_0} + C h R(h)^{-2}\right)\|Op_h(\chi)u\|^2_{L^2(M)} \\ &\qquad + C h^{-2} \|Op_h(\chi)Pu\|^2_{L^2(M)} + C_N h^N \|u\|_{L^2(M)}.\end{aligned}$$

Taking $\tau_0 \le C_0^{-1}(\sup_{\Sigma_{H,p}} |H_p r_H|)^{-1}$ and h_0 small enough so that $C h R(h)^{-2} \le \tau_{\rho_0}^{-1}$ proves the desired result because of (3.24). Also, note that, since $\rho_\gamma \in V_{\rho_0}$, in view of (3.21), we have

$$\frac{1}{2}|\partial_{\xi_1} p(\rho_0)| \le |\partial_{\xi_1} p(\rho_\gamma)| \le 2|\partial_{\xi_1} p(\rho_0)|.$$

We may therefore rewrite the bound for $\|v_h\|^2_{L^2(H)}$ in terms of $|H_p r_H(\rho_\gamma)|$ which completes the proof. □

In what follows we work with points $x \in \mathbb{R}^n$ and $(x,\xi) \in T^*\mathbb{R}^n$. We will isolate one position coordinate x_1 and write $(x,\xi) = (x_1, \tilde{x}, \xi_1, \tilde{\xi})$. This lemma is based on [25, Lemma 4.3] which in turn draws on the factorization ideas from [39].

Lemma 3.8. *Let $\Theta : W \subset \mathbb{R}^n \to M$ be coordinates on M, let $\rho_0 \in T^*\mathbb{R}^n$ and $\mathfrak{I} > 0$ be so that*

$$|\partial_{\xi_1} p(\rho_0)| \geq \mathfrak{I} > 0.$$

Then there exist $\tau_0 > 0$, $C_0 > 0$, $r_0 > 0$ depending only on $(M, p, \mathfrak{I}, \Theta)$ and $V_0 \subset T^*\mathbb{R}^n$ neighborhood of ρ_0, so that $B(\rho_0, r_0) \subset V_0$,

$$\frac{3}{4}|\partial_{\xi_1} p(\rho_0)| \leq \inf_{V_0} |\partial_{\xi_1} p| \leq \sup_{V_0} |\partial_{\xi_1} p| \leq \frac{4}{3}|\partial_{\xi_1} p(\rho_0)|,$$

and the following holds.

Let $0 \leq \delta < \frac{1}{2}$, $0 < \tau < \tau_0$. Let $I_\tau = \{x_1 : -\frac{\tau_{\rho_0}}{3} \leq x_1 \leq \frac{\tau_{\rho_0}}{3}\}$ with $\tau_{\rho_0} := \tau|\partial_{\xi_1} p(\rho_0)|$, and

$$\kappa = \kappa(x_1, \tilde{x}, \tilde{\xi}) \in S_\delta \cap C_c^\infty(I_\tau \times T^*\mathbb{R}^{n-1}).$$

Let $\chi \in S_\delta \cap C_c^\infty(V_0; [-2,2])$ and $q = q(x_1) \in C^\infty(\mathbb{R}; S^\infty(T^*\mathbb{R}^{n-1}))$. Then there is $C > 0$ such that for all $N > 0$, there is $C_N > 0$ and $h_0 > 0$ so that for all $0 < h \leq h_0$, and all x_1,

$$\begin{aligned}
&\|Op_h(q)Op_h(\kappa)Op_h(\chi)u(x_1,\cdot)\|_{L^2_{\tilde{x}}} \\
&\leq 4\tau_{\rho_0}^{-\frac{1}{2}}\|Op_h(\kappa)\|\|Op_h(q)Op_h(\chi)u\|_{L^2_x(|x_1|<\tau_{\rho_0}/2)} \\
&\quad + C_0\tau_{\rho_0}^{\frac{1}{2}}h^{-1}\|Op_h(\kappa)\|\|POp_h(q)Op_h(\chi)u\|_{L^2_x(|x_1|<\tau_{\rho_0}/2)} \\
&\quad + \|[Op_h(\kappa), Op_h(q)]Op_h(\chi)u(x_1,\cdot)\|_{L^2_{\tilde{x}}} + C_N h^N\|u\|_{L^2_x}.
\end{aligned}$$

Also, all constants are uniform when χ, κ, q are taken in bounded subsets of S_δ, Θ is taken in bounded subset of C^k, and when $\mathfrak{I}, \tau$ are taken uniformly bounded away from 0.

Proof. There exists an open neighborhood V_0 of the point ρ_0 so that $|\partial_{\xi_1} p| > \frac{\mathfrak{I}}{2}$ on V_0. Therefore, we may assume that there exist functions $e \in C^\infty(T^*\mathbb{R}^n)$ elliptic on V_0, and $a = a(x_1, \tilde{x}, \tilde{\xi}) \in C^\infty(\mathbb{R} \times S^0(T^*\mathbb{R}^{n-1}))$ so that for all $\psi \in C_c^\infty(V_0)$,

$$p(x,\xi)\psi(x,\xi) = e(x,\xi)(\xi_1 - a(x_1, \tilde{x}, \tilde{\xi}))\psi(x,\xi)$$

with e satisfying that for every α, β,

$$\begin{aligned}
&\|e^{-1}\|_\infty \leq C_1 = C_1(M, P, \mathfrak{I}), \qquad (3.37)\\
&\|\partial_x^\alpha \partial_\xi^\beta e(x,\xi)\|_\infty \leq C = C(M, P, \mathfrak{I}, \alpha, \beta, \Theta),
\end{aligned}$$

where $C(M, P, \mathfrak{I}, \alpha, \beta, \Theta)$ depends on Θ through finitely many C^k-norms. Moreover, there exists $r_0 = r_0(M, p, \mathfrak{I})$ so that $B(\rho_0, r_0) \subset V_0$.

Using this factorization, we see that there is $R \in S^0(T^*\mathbb{R}^n)$ so that for all $\psi \in S_\delta(V_0)$,

$$POp_h(\psi) = Op_h(e)(hD_{x_1} - Op_h(a))Op_h(\psi) + hOp_h(R)Op_h(\psi) + R_\infty,$$

where we write R_∞ for an $O(h^\infty)_{\Psi^{-\infty}}$ operator that may change from line to line but whose seminorms are bounded by those of the functions P, ψ, e, e^{-1}. Moreover, there exists an element $a_1 \in hC^\infty(\mathbb{R} \times S^0(T^*\mathbb{R}^{n-1}))$ so that for each fixed x_1 the operator

$$Op_h(a(x_1) + a_1(x_1)) : L^2_{\tilde{x}} \to L^2_{\tilde{x}}$$

is self-adjoint. Abusing notation slightly, we relabel $a+a_1$ as a and $Op_h(R)-Op_h(e)Op_h(a_1)$ as $Op_h(R)$. Then, for all $\psi \in S_\delta(V_0)$,

$$POp_h(\psi) = Op_h(e)(hD_{x_1} - Op_h(a))Op_h(\psi) + hOp_h(R)Op_h(\psi) + R_\infty.$$

Therefore, letting $Op_h(e)^{-1}$ denote a microlocal parametrix for $Op_h(e)$ on V_0, we have for all $\psi \in S_\delta(V_0)$,

$$(hD_{x_1} - Op_h(a))Op_h(\psi) = Op_h(e)^{-1}POp_h(\psi) + hOp_h(R_0)Op_h(\psi) + R_\infty, \tag{3.38}$$

where R_0 is such that $Op_h(R_0) = -Op_h(e)^{-1}Op_h(R)$. From the symbolic calculus together with (3.37) we see that for every α, β,

$$\|\partial_x^\alpha \partial_\xi^\beta R_0(x, \xi)\|_\infty \le C = C(M, P, \mathfrak{I}, \alpha, \beta, \Theta), \tag{3.39}$$

where C depends on Θ through finitely many C^k-norms. Shrinking V_0 (in a way depending only on $(M, p, \mathfrak{I})$ and the C^2-norm of Θ), if necessary, we may also assume that

$$\frac{3}{4}|\partial_{\xi_1} p(\rho_0)| \le \inf_{V_0} |\partial_{\xi_1} p| \le \sup_{V_0} |\partial_{\xi_1} p| \le \frac{4}{3}|\partial_{\xi_1} p(\rho_0)|.$$

Define

$$w := Op_h(q)Op_h(\chi)u, \tag{3.40}$$

with $Op_h(\psi) = Op_h(q)Op_h(\chi)$ we have by (3.38) that

$$(hD_{x_1} - Op_h(a))w = f$$

for

$$f := [Op_h(e)^{-1}POp_h(q)Op_h(\chi) + hOp_h(R_0)Op_h(q)Op_h(\chi)]u + R_\infty u. \tag{3.41}$$

Defining the operator $U(x_1, t)$ by

$$(hD_{x_1} - Op_h(a))U(x_1, t) = 0, \quad U(t, t) = \mathrm{Id},$$

we obtain that for all $x_1, t \in \mathbb{R}$,

$$w(x_1, \tilde{x}) = U(x_1, t)w(t, \tilde{x}) - \frac{i}{h}\int_{x_1}^t U(x_1, s)f(s, \tilde{x})\, ds.$$

Let $\varepsilon = \varepsilon(\tau)$ be defined as

$$\varepsilon := \frac{\tau_{\rho_0}}{3} = \frac{\tau|\partial_{\xi_1} p(\rho_0)|}{3},$$

and let $\Phi \in C_c^\infty(\mathbb{R}; [0, 3\varepsilon^{-1}])$ with $\operatorname{supp} \Phi \subset [0, \varepsilon]$ and $\int_{\mathbb{R}} \Phi = 1$. Then, integrating in t,

$$w(x_1, \tilde{x}) = \int_{\mathbb{R}} \Phi(t)U(x_1, t)w(t, \tilde{x})\, dt - \frac{i}{h}\int_{\mathbb{R}} \Phi(t)\int_{x_1}^t U(x_1, s)f(s, \tilde{x})\, ds\, dt. \tag{3.42}$$

Let τ_0 satisfy

$$\tau_0 < \sqrt{\frac{3}{2}} |\partial_{\xi_1} p(\rho_0)|^{-1} \| Op_h(R_0) \|^{-1}, \tag{3.43}$$

where $Op_h(R_0)$ is as in (3.38). Note that by (3.39) τ_0 only depends on $(M, P, \mathfrak{I}, \Theta)$.

From now on, we write

$$C = C(M, P, \mathfrak{I}, \varepsilon_0, \tau, \chi, q, \kappa, \Theta)$$

and

$$C_N = C_N(M, P, N, \tau, \mathfrak{I}, \varepsilon_0, \chi, q, \kappa, \Theta)$$

for constants depending on finitely many seminorms of the given parameters. To bound the first term in (3.42) we apply Cauchy–Schwarz and use that $U(x_1, t)$ is a unitary operator acting on $L^2_{\tilde{x}}$ to get

$$\Big\| \int_{\mathbb{R}} \Phi(t) Op_h(\kappa) U(x_1, t) w(t, \tilde{x})\, dt \Big\|_{L^\infty_{x_1} L^2_{\tilde{x}}} \leq \|\Phi\|_2 \|Op_h(\kappa)\| \|w\|_{L^2_{t,\tilde{x}}(|t| \leq \varepsilon)}.$$

To bound the second term in (3.42) we apply Minkowski's integral inequality, use that the support of Φ is contained in $[0, \varepsilon]$, and that $\operatorname{supp} \kappa \subset \{|x_1| < \varepsilon\}$ to get

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}} \Phi(t) \int_{x_1}^{t} Op_h(\kappa) U(x_1, s) f(s, \tilde{x})\, ds\, dt \Big\|_{L^\infty_{x_1} L^2_{\tilde{x}}} \\
&\leq \Big\| \int_{\mathbb{R}} \Phi(t) \Big(\int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} \mathbf{1}_{[-\varepsilon,\varepsilon]}(s) Op_h(\kappa) U(x_1, s) f(s, \tilde{x})\, ds \Big)^2 d\tilde{x} \Big)^{\frac{1}{2}} dt \Big\|_{L^\infty_{x_1}} \\
&\leq \|\mathbf{1}_{[-\varepsilon,\varepsilon]}(s)\|_{L^2_s} \|Op_h(\kappa)\| \|f\|_{L^2_{s,\tilde{x}}(|s| \leq \varepsilon)}.
\end{aligned}$$

Feeding these two bounds into (3.42), and using that $\Phi(t) \leq 3\varepsilon^{-1}$ and $\int_{\mathbb{R}} \Phi(t) dt = 1$ give

$$\|\Phi\|_{L^2(\mathbb{R})} \leq \sqrt{3} \varepsilon^{-\frac{1}{2}},$$

we obtain

$$\begin{aligned}
\|Op_h(\kappa) w(x_1, \cdot)\|_{L^2_{\tilde{x}}} \leq{}& \sqrt{3} \varepsilon^{-\frac{1}{2}} \|Op_h(\kappa)\| \|w\|_{L^2_x(|x_1| \leq \varepsilon)} \\
&+ \sqrt{2} \varepsilon^{\frac{1}{2}} h^{-1} \|Op_h(\kappa)\| \|f\|_{L^2_x(|x_1| \leq \varepsilon)}.
\end{aligned} \tag{3.44}$$

Finally, note that according to (3.41),

$$\begin{aligned}
\|f\|_{L^2_x(|x_1| \leq \varepsilon)} \leq{}& \|Op_h(e)^{-1} P Op_h(q) Op_h(\chi) u\|_{L^2_x(|x_1| \leq \varepsilon)} \\
&+ h \|Op_h(R_0) Op_h(q) Op_h(\chi) u\|_{L^2_x(|x_1| \leq \varepsilon)} \\
&+ C_N h^N \|u\|_{L^2_x} \\
\leq{}& C_0 \|P Op_h(q) Op_h(\chi) u\|_{L^2_x(|x_1| \leq 3\varepsilon/2)} \\
&+ h \|Op_h(R_0)\| \|Op_h(b) Op_h(q) Op_h(\chi) u\|_{L^2_x(|x_1| \leq 3\varepsilon/2)} \\
&+ C_N h^N \|u\|_{L^2_x}.
\end{aligned}$$

Using (3.37), we see that $C_0 > 0$ depends only $(M, P, \mathfrak{I})$. Therefore, since

$$Op_h(q) Op_h(\kappa) Op_h(\chi) = Op_h(\kappa) Op_h(q) Op_h(\chi) + [Op_h(q), Op_h(\kappa)] Op_h(\chi),$$

we may combine definition (3.40) of w with (3.44) to obtain

$$\begin{aligned}\|Op_h(q)Op_h(\kappa)Op_h(\chi)u(x_1,\cdot)\|_{L^2_{\tilde{x}}} &\le \sqrt{3}\varepsilon^{-\frac{1}{2}}\|Op_h(\kappa)\|\|Op_h(q)Op_h(\chi)u\|_{L^2_x(|x_1|\le\varepsilon)}\\ &\quad + C_0h^{-1}\varepsilon^{\frac{1}{2}}\|Op_h(\kappa)\|\|POp_h(q)Op_h(\chi)u\|_{L^2_x(|x_1|\le 3\varepsilon/2)}\\ &\quad + \sqrt{2}\varepsilon^{\frac{1}{2}}\|Op_h(R_0)\|\|Op_h(\kappa)\|\|Op_h(q)Op_h(\chi)u\|_{L^2_x(|x_1|\le 3\varepsilon/2)}\\ &\quad + C_Nh^N\|u\|_{L^2_x} + \|[Op_h(q),Op_h(\kappa)]Op_h(\chi)u(x_1,\cdot)\|_{L^2_{\tilde{x}}}.\end{aligned}$$

To finish the proof, we combine the first and third terms in the bound above using that

$$\sqrt{3}\varepsilon^{-\frac{1}{2}} = 3\tau_{\rho_0}^{-\frac{1}{2}}$$

and that (3.43) gives $\sqrt{2}\varepsilon^{\frac{1}{2}}\|Op_h(R_0)\| \le \tau_{\rho_0}^{-\frac{1}{2}}$. □

4. Non-looping propagation estimates: Proof of Theorem 11

The main result in this section is the proof of Theorem 11 which we present in what follows. The proof is based on an application of Egorov's Theorem (see Lemma 4.1) which in turn uses that cutoffs with disjoint support act almost orthogonally.

Proof of Theorem 11. By Theorem 10 there are constants τ_0, R_0, and $C_{n,k} > 0$ so that if $0 < \tau \le \tau_0$, $0 \le \delta < \frac{1}{2}$, $N > 0$, $8h^\delta \le R(h) < R_0$, then for $\{\Lambda^\tau_{\rho_j}(R(h))\}_j$ a $(\mathfrak{D}, \tau, R(h))$-good cover of $\Sigma_{H,p}$, and $\{\chi_j\}_j$ a δ-partition associated to the cover, there exist $C > 0$, $h_0 > 0$, so that for all $w = w(x';h) \in S_\delta \cap C^\infty_c(\tilde{H})$ there is $C_N > 0$ with the property that for any $0 < h < h_0$ and $u \in \mathcal{D}'(M)$,

$$\begin{aligned}(4.1)\qquad h^{\frac{k-1}{2}}\left|\int_{\tilde{H}} wu\,d\sigma_{\tilde{H}}\right| &\le \frac{C_{n,k}\|w\|_\infty}{\tau^{\frac{1}{2}}\mathfrak{I}_0^{\frac{1}{2}}}R(h)^{\frac{n-1}{2}}\sum_{j\in\mathcal{J}_h(w)}\|Op_h(\chi_j)u\|_{L^2(M)}\\ &\quad + Ch^{-1}\|w\|_\infty\|Pu\|_{H^{(k-2m+1)/2}_{\mathrm{scl}}(M)}\\ &\quad + C_Nh^N\big(\|u\|_{L^2(M)} + \|Pu\|_{H^{(k-2m+1)/2}_{\mathrm{scl}}(M)}\big).\end{aligned}$$

Suppose there exist $\mathcal{B} \subset \{1,\dots,N_h\}$ and a finite collection $\{\mathcal{G}_\ell\}_{\ell\in\mathcal{L}} \subset \{1,\dots,N_h\}$ satisfying $\mathcal{J}_h(w) \subset \mathcal{B} \cup \bigcup_{\ell\in\mathcal{L}}\mathcal{G}_\ell$, and with $\{\mathcal{G}_\ell\}_{\ell\in\mathcal{L}}$ having the non-self-looping properties described in the statement of the theorem. Furthermore, since we are working with a $(\mathfrak{D},\tau,R(h))$-good cover, we split each $\mathcal{G}_\ell$ into $\mathfrak{D}$ families $\{\mathcal{G}_{\ell,i}\}_{i=1}^{\mathfrak{D}}$ of disjoint tubes.

Note that

$$\begin{aligned}\sum_{j\in\mathcal{J}_h(w)}\|Op_h(\chi_j)u\|_{L^2(M)} &\le \sum_{\ell\in\mathcal{L}}\sum_{i=1}^{\mathfrak{D}}\sum_{j\in\mathcal{G}_{\ell,i}}\|Op_h(\chi_j)u\|_{L^2(M)}\\ &\quad + \sum_{j\in\mathcal{B}}\|Op_h(\chi_j)u\|_{L^2(M)}.\end{aligned}$$

Since $\bigcup_{j\in\mathcal{G}_\ell}\Lambda^\tau_{\rho_j}(R(h))$ is $[t_\ell(h),T_\ell(h)]$ non-self-looping, and the tubes in $\mathcal{G}_{\ell,i}$ are disjoint, we may apply Lemma 4.1 below to $\mathcal{G} = \mathcal{G}_{\ell,i}$ and $(t_j,T_j) = (t_\ell,T_\ell)$ for all $j \in \mathcal{G}_{\ell,i}$ together with Cauchy–Schwarz to get

$$\sum_{j\in\mathcal{G}_{\ell,i}} \|Op_h(\chi_j)u\|_{L^2(M)} \le \Big(\frac{t_\ell|\mathcal{G}_\ell|}{T_\ell}\Big)^{\frac12}\Big(\sum_{j\in\mathcal{G}_{\ell,i}}\frac{\|Op_h(\chi_j)u\|^2_{L^2(M)}T_\ell}{t_\ell}\Big)^{\frac12}$$
$$\le 2\Big(\frac{t_\ell|\mathcal{G}_\ell|}{T_\ell}\Big)^{\frac12}\Big(\|u\|^2_{L^2(M)}+\frac{T_\ell^2}{h^2}\|Pu\|^2_{L^2(M)}\Big)^{\frac12}.$$

On the other hand, using Cauchy–Schwarz and the fact that there are $\mathfrak{D}$ families of disjoint tubes,

$$\sum_{j\in\mathcal{B}}\|Op_h(\chi_j)u\|_{L^2(M)} \le 2\mathfrak{D}|\mathcal{B}|^{\frac12}\|u\|_{L^2(M)}.$$

Therefore, after adjusting $C_{n,k}$ in (4.1),

$$\begin{aligned}
&h^{\frac{k-1}{2}}\Big|\int_H wu\,d\sigma_H\Big|\\
&\le \frac{C_{n,k}\mathfrak{D}\|w\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac12}\mathfrak{I}_0^{\frac12}}\Big[\sum_{\ell\in\mathcal{L}}\Big(\frac{t_\ell|\mathcal{G}_\ell|}{T_\ell}\Big)^{\frac12}\Big(\|u\|^2_{L^2(M)}+\frac{T_\ell^2}{h^2}\|Pu\|^2_{L^2(M)}\Big)^{\frac12}\\
&\qquad\qquad + |\mathcal{B}|^{\frac12}\|u\|_{L^2(M)}\Big]\\
&\quad + Ch^{-1}\|w\|_\infty\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)} + C_N\big(\|u\|_{L^2(M)}+\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)}\big)\\
&\le \frac{C_{n,k}\mathfrak{D}\|w\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac12}\mathfrak{I}_0^{\frac12}}\Big[\sum_{\ell\in\mathcal{L}}\Big(\frac{t_\ell|\mathcal{G}_\ell|}{T_\ell}\Big)^{\frac12}\|u\|_{L^2(M)}\\
&\qquad\qquad + \sum_{\ell\in L}\Big(\frac{|\mathcal{G}_\ell|t_\ell T_\ell}{h^2}\Big)^{\frac12}\|Pu\|_{L^2(M)} + |\mathcal{B}|^{\frac12}\|u\|_{L^2(M)}\Big]\\
&\quad + Ch^{-1}\|w\|_\infty\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)} + C_N\big(\|u\|_{L^2(M)}+\|Pu\|_{H_{\rm scl}^{(k-2m+1)/2}(M)}\big). \quad\square
\end{aligned}$$

The next lemma relies on Egorov's Theorem to the Ehrenfest time (see, e.g., [21, Proposition 3.8] and [65]).

Lemma 4.1. *Assume that P is self-adjoint. Let $0\le\delta_0<\frac12$, $0<2\varepsilon_0<1-2\delta_0$, and let $\mathcal{G}$ be a set of indices with $|\mathcal{G}|\le h^{-N}$ for some $N>0$. For each $\ell\in\mathcal{G}$ let $0\le\delta_\ell\le\delta_0$, $0<\alpha_\ell<1-2\delta_\ell-2\varepsilon_0$, and $\chi_\ell\in S_{\delta_\ell}(T^*M)\cap C_c^\infty(T^*M;[-C_1h^{1-2\delta_0},1+C_1h^{1-2\delta_0}])$. In addition, for each $\ell\in\mathcal{G}$ let $t_\ell(h)>0$ and $0<T_\ell(h)\le 2\alpha_\ell T_e(h)$ be so that*

$$\bigcup_{k\in\mathcal{G}}\operatorname{supp}\chi_k\cap\varphi_{-t}(\operatorname{supp}\chi_\ell)=\emptyset \tag{4.2}$$

for all $t\in[t_\ell(h),T_\ell(h)]$ or $t\in[-T_\ell(h),-t_\ell(h)]$, and suppose that

$$\bigcup_{k\neq\ell}\operatorname{supp}\chi_k\cap\operatorname{supp}\chi_\ell=\emptyset. \tag{4.3}$$

Then there exists a constant $h_0>0$ so that for $0<h<h_0$,

$$\sum_{\ell\in\mathcal{G}}\frac{\|Op_h(\chi_\ell)u\|^2_{L^2(M)}T_\ell(h)}{t_\ell(h)}\le 4\|u\|^2_{L^2(M)}+4\max_{\ell\in\mathcal{G}}\frac{T_\ell(h)^2}{h^2}\|Pu\|^2_{L^2(M)}.$$

Moreover, the constant h_0 can be chosen to be uniform for χ_ℓ in bounded subsets of $S_\delta(T^*M)$ and $N < N_0$.

Proof. Throughout this proof it will be convenient to write $\|\cdot\|$ for $\|\cdot\|_{L^2(M)}$. Define $\tilde{\chi}$ by

$$Op_h(\tilde{\chi}) = \sum_{\ell\in\mathcal{G}} \sum_{k=\frac{-T_\ell}{2t_\ell}}^{\frac{T_\ell}{2t_\ell}} e^{\frac{ikt_\ell P}{h}} Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}}.$$

First, we claim that there exists $h_0 > 0$ so that for all $0 < h < h_0$,

$$\|Op_h(\tilde{\chi})u\|^2 \le \frac{3}{2}\|u\|^2. \tag{4.4}$$

Indeed, Egorov's Theorem [21, Proposition 3.9] gives that there exist $C_\chi > 0$ and $h_0 > 0$ so that for every k,

$$\begin{aligned} e^{\frac{ikt_\ell P}{h}} Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}} &= Op_h(\chi_{k,\ell}) + O(h^\infty)_{\Psi^{-\infty}}, \\ \chi_{k,\ell} &= \chi_\ell \circ \varphi_{kt_\ell} + r_{k,\ell}(h), \end{aligned} \tag{4.5}$$

where $r_{k,\ell} \in h^{1-d_{k,\ell}(h)-2\delta_\ell} S_{d_{k,\ell}(h)/2+\delta_\ell}$, $\operatorname{supp} r_{k,\ell} \subset \operatorname{supp} \chi_\ell \circ \varphi_{kt_\ell}$,

$$|r_{k,\ell}(h)| \le C_\chi h^{1-d_{k,\ell}(h)-2\delta_\ell} \quad \text{and} \quad d_{k,\ell}(h) \le |k| \frac{t_\ell}{T_e(h)}$$

for all $0 < h < h_0$. Note that since $\{\chi_\ell\}_{\ell\in\mathcal{G}} \mapsto \tilde{\chi}$ is a continuous map from

$$\prod_{\ell\in\mathcal{G}} S_{\delta_\ell}(T^*M) \to S_{\frac{1}{2}-\varepsilon_0}(T^*M),$$

the constant C_χ can be chosen to be uniform for $\{\chi_\ell\}_{\ell\in\mathcal{G}}$ in bounded subsets of $\Pi_\ell S_{\delta_\ell}(T^*M)$, and that then the same is true for h_0.

Now, let $\ell, m \in \mathcal{G}$ with $\ell \ne m$ and assume without loss that $T_\ell \le T_m$. Then, using (4.2) and (4.3), we have for $\frac{-T_\ell(h)}{2t_\ell} \le k \le \frac{T_\ell(h)}{2t_\ell}$, $\frac{-T_m(h)}{2t_m} \le j \le \frac{T_m(h)}{2t_m}$,

$$\varphi_{-kt_\ell}(\operatorname{supp}\chi_\ell) \cap \varphi_{-jt_m}(\operatorname{supp}\chi_m) = \operatorname{supp}\chi_\ell \cap \varphi_{kt_\ell - jt_m}(\operatorname{supp}\chi_m) = \emptyset.$$

In addition, using (4.2), we have if $\ell = m$, then for $\frac{-T_\ell(h)}{2t_\ell} \le k < j \le \frac{T_\ell(h)}{2t_\ell}$,

$$\varphi_{-kt_\ell}(\operatorname{supp}\chi_\ell) \cap \varphi_{-jt_\ell}(\operatorname{supp}\chi_m) = \operatorname{supp}\chi_\ell \cap \varphi_{(k-j)t_\ell}(\operatorname{supp}\chi_m) = \emptyset.$$

Thus, it follows from (4.5) that

$$\tilde{\chi} = \sum_{\ell\in\mathcal{G}} \sum_{k=-\frac{T_\ell}{2t_\ell}}^{\frac{T_\ell}{2t_\ell}} \chi_\ell \circ \varphi_{kt_\ell} + r(h)$$

with $|r(h)| \le C_\chi h^{2\varepsilon_0}$ for all $0 < h < h_0$, and C_χ, h_0 can be chosen uniform for $\{\chi_\ell\}_{\ell=1}^J$ in bounded subsets of S_{δ_0}. We have used that the support of the functions $r_{k,\ell}$ are disjoint, together with the fact that $2\varepsilon_0 < 1-\alpha_\ell-2\delta_\ell$ implies $2\varepsilon_0 < 1-d_{k,\ell}(h)-2\delta_\ell$, to get the bound on $r(h)$. This implies that

$$\tilde{\chi} \in S_{\frac{1}{2}-\varepsilon_0} \quad \text{and} \quad -C_\chi h^{2\varepsilon_0} \le \tilde{\chi} \le 1 + C_\chi h^{2\varepsilon_0} \tag{4.6}$$

for all $0 < h < h_0$.

Note that by the sharp Gårding inequality (4.6) yields

$$\langle (1 + C_\chi h^{2\varepsilon_0} - Op_h(\tilde{\chi})^* Op_h(\tilde{\chi}))u, u\rangle \geq -C_\chi h^{2\varepsilon_0}\|u\|_{L^2}^2,$$

which in turn gives

$$\|Op_h(\tilde{\chi})u\|^2 \leq (1 + 2C_\chi h^{2\varepsilon_0})\|u\|^2 \tag{4.7}$$

for all $0 < h < h_0$. Also, note that since $\varepsilon_0 > 0$, we may shrink h_0 so that (4.7) gives

$$\|Op_h(\tilde{\chi})u\|^2 \leq \frac{3}{2}\|u\|^2 \tag{4.8}$$

for $0 < h < h_0$ as claimed in (4.4).

Next, note that since the supports of the compositions $\chi_m \circ \varphi_{jt_m}$ and $\chi_\ell \circ \varphi_{kt_\ell}$ are disjoint for $(j, m) \neq (k, \ell)$, Egorov's Theorem also gives

$$\left\langle e^{\frac{ijt_m P}{h}} Op_h(\chi_m) e^{-\frac{ijt_m P}{h}} u, e^{\frac{ikt_\ell P}{h}} Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}} u\right\rangle = O_\chi(h^\infty)\|u\|^2, \tag{4.9}$$

where the constant in $O_\chi(h^N)$ depends only on the $|\alpha| \leq C_N\, n$ seminorms of χ, where C_N is a universal constant. It then follows from (4.8) and (4.9) that

$$\frac{3}{2}\|u\|^2 \geq \sum_{\ell\in\mathcal{G}} \sum_{k=-\frac{T_\ell}{2t_\ell}}^{\frac{T_\ell}{2t_\ell}} \left\|e^{\frac{ikt_\ell P}{h}} Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}} u\right\|^2 + O_\chi\Big(h^\infty \max_\ell |T_\ell|\Big)\|u\|^2, \tag{4.10}$$

as long as we work with $0 \leq h \leq h_0$ and h_0 small enough so that $r(h)$ can be absorbed by $\frac{3}{2}\|u\|^2$.

On the other hand, since the propagators $e^{\frac{ikt_\ell P}{h}}$ are unitary operators,

$$\begin{aligned} \left\|e^{\frac{ikt_\ell P}{h}} Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}} u\right\|^2 &= \left\|Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}} u\right\|^2 \\ &= \|Op_h(\chi_\ell)u\|^2 - I_{k,\ell} - II_{k,\ell}, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} I_{k,\ell} &= \big\langle Op_h(\chi_\ell)[u - e^{-\frac{ikt_\ell P}{h}} u], Op_h(\chi_\ell)u\big\rangle, \\ II_{k,\ell} &= \big\langle Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}} u, Op_h(\chi_\ell)[u - e^{-\frac{ikt_\ell P}{h}} u]\big\rangle. \end{aligned}$$

It follows from (4.11) that

$$\begin{aligned} &\sum_\ell \sum_{k=-\frac{T_\ell}{2t_\ell}}^{\frac{T_\ell}{2t_\ell}} \left\|e^{\frac{ikt_\ell P}{h}} Op_h(\chi_\ell) e^{-\frac{ikt_\ell P}{h}} u\right\|^2 \\ &\qquad = \sum_\ell \frac{T_\ell}{t_\ell}\|Op_h(\chi_\ell)u\|^2 - \sum_\ell \sum_{k=-\frac{T_\ell}{2t_\ell}}^{\frac{T_\ell}{2t_\ell}} I_{k,\ell} + II_{k,\ell}. \end{aligned} \tag{4.12}$$

Observe that

$$I_{k,\ell} = \frac{i}{h}\int_0^{kt_\ell} \big\langle Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu, Op_h(\chi_\ell)u\big\rangle\, ds = A_{k,\ell} + B_{k,\ell},$$

where

$$A_{k,\ell} := \frac{i}{h}\int_0^{kt_\ell} \big\langle e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu, e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} u\big\rangle\, ds,$$

$$B_{k,\ell} := \frac{i}{h}\int_0^{kt_\ell} \big\langle e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu, e^{\frac{isP}{h}} Op_h(\chi_\ell)(u - e^{-\frac{isP}{h}} u)\big\rangle\, ds.$$

To deal with the $A_{k,\ell}$ terms, note that

$$\begin{aligned}\sum_{k,\ell} A_{k,\ell} &\leq \frac{1}{h}\sum_{k,\ell}\int_0^{kt_\ell} \|e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu\|\|e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} u\|\, ds \\ &\leq \frac{1}{h}\Big(\sum_{\ell,k}\int_0^{kt_\ell} \|e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu\|^2\, ds\Big)^{\frac{1}{2}} \\ &\qquad \times \Big(\sum_{\ell,k}\int_0^{kt_\ell} \|e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} u\|^2\, ds\Big)^{\frac{1}{2}}.\end{aligned}$$

In addition, observe that for $v \in L^2$,

$$\sum_{\ell,k}\int_0^{kt_\ell} \|e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} v\|^2\, ds \leq \langle Lv, v\rangle, \tag{4.13}$$

with

$$L := \sum_{\ell,k}\int_0^{kt_\ell} e^{\frac{isP}{h}} Op_h(\chi_\ell)^* Op_h(\chi_\ell) e^{-\frac{isP}{h}}\, ds.$$

Also, another application of Egorov's Theorem gives

$$L = Op_h\Big(\sum_{\ell,k}\int_0^{kt_\ell} |\chi_\ell|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, ds\Big) + O(h^\infty)_{\Psi^{-\infty}},$$

where $\tilde{r}_{k,\ell}(s,h) \in h^{1-d_{k,\ell}(h)-2\delta_\ell} S_{d_{k,\ell}/2+\delta_\ell}$ with $\operatorname{supp} \tilde{r}_{k,\ell}(s,h) \subset \operatorname{supp} \chi_\ell \circ \varphi_s$ and

$$|\tilde{r}_{k,\ell}(s,h)| \leq C_\chi h^{1-d_{k,\ell}(h)-2\delta_\ell}.$$

Next, we claim that (4.2) gives

$$\Big|\int_0^{kt_\ell} |\chi_\ell|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, ds\Big| \leq t_\ell(1 + C_\chi h^{1-d_{k,\ell}(h)-2\delta_\ell}). \tag{4.14}$$

To see this, let $\rho \in T^*M$, $s, t \in [-\frac{T_\ell}{2}, \frac{T_\ell}{2}]$, be so that $\varphi_s(\rho) \in \operatorname{supp} \chi_\ell$ and $\varphi_t(\rho) \in \operatorname{supp} \chi_\ell$. Suppose $s \geq t$ and note that

$$\varphi_s(\rho) \in \varphi_{s-t}(\operatorname{supp} \chi_\ell) \cap \operatorname{supp} \chi_\ell.$$

Therefore, since $0 \leq s - t \leq T_\ell$, we obtain $0 \leq s - t \leq t_\ell$ from (4.2). This proves the claim.

In addition, we claim that combining (4.14) with (4.3) gives

$$\Big|\sum_{\ell,k}\int_0^{kt_\ell} |\chi_\ell|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, ds\Big| \leq \max_\ell T_\ell(h)(1 + C_\chi h^{1-\varepsilon_0}). \tag{4.15}$$

To see this, first observe that $\#\{k \in [-\frac{T_\ell}{2t_\ell}, \frac{T_\ell}{2t_\ell}]\} \leq \frac{T_\ell}{t_\ell}$. Together with (4.14) this implies

$$\left|\sum_k \int_0^{kt_\ell} |\chi_\ell|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, ds\right| \leq T_\ell(1 + C_\chi h^{1-\varepsilon_0}). \tag{4.16}$$

Second, note that

$$\operatorname{supp}\left(\sum_k \int_0^{kt_\ell} |\chi_\ell|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, ds\right) \subset \bigcup_{s=-T_\ell/2}^{T_\ell/2} \varphi_{-s}(\operatorname{supp} \chi_\ell).$$

Therefore, by (4.3) for $\ell \neq j$

$$\begin{aligned} \operatorname{supp}&\left(\sum_k \int_0^{kt_\ell} |\chi_\ell|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, ds\right) \\ &\cap \operatorname{supp}\left(\sum_k \int_0^{kt_j} |\chi_j|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, ds\right) = \emptyset. \end{aligned} \tag{4.17}$$

Combining (4.16) with (4.17), we obtain (4.15) as claimed.

Using (4.13) and (4.15) together with the same argument we used for $\tilde{\chi}$, for h_0 small enough (uniform for χ_ℓ in bounded subsets of S_{δ_ℓ}),

$$\sum_{\ell,k} \int_0^{kt_\ell} \left\| e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} v \right\|^2 ds \leq 2 \max_\ell T_\ell(h) \|v\|^2.$$

In particular,

$$\left|\sum_{\ell,k} A_{k,\ell}\right| \leq 2 \frac{\max_\ell T_\ell(h)}{h} \|Pu\| \|u\|.$$

We next turn to dealing with $B_{k,\ell}$. Note that

$$B_{k,\ell} = \frac{1}{h^2} \int_0^{kt_\ell} \int_0^s \left\langle e^{\frac{i(t-s)P}{h}} e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu, e^{\frac{itP}{h}} Op_h(\chi_\ell) e^{-\frac{itP}{h}} Pu \right\rangle dt\, ds.$$

Therefore, by a similar argument this time using

$$\left|\int_0^{kt_\ell} \int_0^{kt_\ell} |\chi_\ell|^2 \circ \varphi_s + \tilde{r}_{k,\ell}(s,h)\, dt\, ds\right| \leq k t_\ell^2 (1 + C_\chi h^{1-d_{k,\ell}(h)-2\delta_\ell}),$$

we obtain

$$\begin{aligned} \left|\sum_{\ell,k} B_{k,\ell}\right| &\leq \frac{1}{h^2} \sum_{\ell,k} \int_0^{kt_\ell} \int_0^s \|e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu\| \|e^{\frac{itP}{h}} Op_h(\chi_\ell) e^{-\frac{itP}{h}} Pu\|\, dt\, ds \\ &\leq \frac{1}{h^2} \sum_{\ell,k} \int_0^{kt_\ell} \int_0^{kt_\ell} \|e^{\frac{isP}{h}} Op_h(\chi_\ell) e^{-\frac{isP}{h}} Pu\|^2\, ds\, dt \\ &\leq 2 \frac{\max_\ell T_\ell^2(h)}{h^2} \|Pu\|^2. \end{aligned}$$

We have therefore shown that

$$\left|\sum_{\ell,k} I_{k,\ell}\right| \leq 2 \frac{\max_\ell T_\ell(h)}{h} \|Pu\| \|u\| + 2 \frac{\max_\ell T_\ell^2(h)}{h^2} \|Pu\|^2. \tag{4.18}$$

Next, note that

$$\begin{aligned}
II_{k,\ell} &= \big\langle Op_h(\chi_\ell)e^{\frac{-ikt_\ell P}{h}}u, Op_h(\chi_\ell)[u-e^{-\frac{ikt_\ell P}{h}}u]\big\rangle\\
&= \frac{i}{h}\int_0^{kt_\ell}\big\langle e^{\frac{ikt_\ell P}{h}}Op_h(\chi_\ell)e^{\frac{-ikt_\ell P}{h}}u, e^{\frac{ikt_\ell P}{h}}Op_h(\chi_\ell)e^{-\frac{isP}{h}}Pu\big\rangle ds\\
&\le \frac{1}{h}\int_0^{kt_\ell}\|e^{\frac{ikt_\ell P}{h}}Op_h(\chi_\ell)e^{\frac{-ikt_\ell P}{h}}u\|\,\|e^{\frac{ik(t_\ell-s)P}{h}}e^{\frac{iksP}{h}}Op_h(\chi_\ell)e^{-\frac{isP}{h}}Pu\|\,ds.
\end{aligned}$$

Then, by unitarity of $e^{-\frac{it_\ell-sP}{h}}$ and (4.13),

$$\Big|\sum_{\ell,k}II_{k,\ell}\Big| \le 2\frac{\max_\ell T_\ell}{h}\|Pu\|\|u\|. \tag{4.19}$$

In particular, from (4.18) and (4.19) we have

$$\begin{aligned}
\Big|\sum_{\ell,k}I_{k,\ell}+II_{k,\ell}\Big| &\le 4\frac{\max_\ell T_\ell}{h}\|Pu\|\|u\|+2\frac{\max_\ell T_\ell^2}{h^2}\|Pu\|^2\\
&\le 2\|u\|^2+4\frac{\max_\ell T_\ell^2}{h^2}\|Pu\|^2.
\end{aligned}\tag{4.20}$$

By possibly shrinking h_0, we may assume that the error term in (4.10) is smaller than $\frac{1}{2}\|u\|^2$ for $0<h<h_0$. We conclude from (4.10) together with (4.11), (4.12) and (4.20) that

$$2\|u\|^2 \ge \sum_\ell \frac{T_\ell(h)}{t_\ell}\|Op_h(\chi_\ell)u\|^2 - 2\|u\|^2 - 4\frac{\max_\ell T_\ell^2}{h^2}\|Pu\|^2. \tag{4.21}$$

Therefore, (4.21) gives

$$\sum_{\ell\in\mathcal{G}}\frac{\|Op_h(\chi_\ell)u\|^2 T_\ell(h)}{t_\ell} \le \Big(4\|u\|^2+4\frac{\max_\ell T_\ell^2}{h^2}\|Pu\|^2\Big)$$

for $0<h<h_0$. As noted right after (4.5) the constant h_0 can be chosen to be uniform for χ_ℓ in compact subsets of $S_{\delta_0}(T^*M)$. □

5. Quantitative improvements in integrable geometries

In this section, we focus on the special case of spheres of revolution

$$M=[0,2\pi]_\theta\times[0,\pi]_r$$

with Hamiltonian

$$p(\theta,r,\xi_\theta,\xi_r)=\xi_r^2+\frac{1}{\alpha(r)^2}\xi_\theta^2+V(r),$$

and operate under the assumptions of Theorem 5.

In this setting, one can explicitly describe the Liouville tori intersected with $\{p=0\}$ as

$$\mathbb{T}_{\xi_\theta}=\Big\{(\theta,r,\xi_r):\xi_r^2=V(r)-\frac{1}{\alpha(r)^2}\xi_\theta^2\Big\}.$$

In particular,

$$\mathbb{T}_{\xi_\theta} \cap S^*_{(\theta_0,r_0)}M = \left\{\xi_r = \pm\sqrt{V(r_0) - \frac{1}{\alpha(r_0)^2}\xi_\theta^2}\right\},$$

and for any $\delta > 0$ there is $c > 0$ so that if $r_0 \in [\delta, 2\pi - \delta]$, the two intersections are separated by at least

$$c\sqrt{\alpha(r_0)\sqrt{V(r_0)} - \xi_\theta}. \tag{5.1}$$

Let $R_1 > 0$ and define

$$A_{\pm,R_1} := \{(\theta, r, \xi_\theta, \xi_r) \in T^*M : \pm\xi_r \geq R_1\}.$$

Theorem 5 is a consequence of the following lemma which constructs non-looping covers together with Theorem 11.

Lemma 5.1. *Let the above assumptions hold. Fix $\delta > 0$ and let $\{\Lambda^\tau_{\rho_j}(R)\}_{j=1}^{N_R}$ be as in Proposition* 3.3. *Then there exists $\beta > 0$ so that if $r_0 \in [\delta, 2\pi - \delta]$ and $H = \{x\} = \{(r_0, \theta_0)\}$, the following holds. For all $0 < \tau < \tau_0$, $\alpha_1 > 0$, $0 < R \ll 1$, and $0 < T < cR^{\alpha_1 - 1}$, there exists $\mathcal{B} \subset \{1, \dots, N_R\}$ so that for $R_1 = R^{\alpha_1}$,*

$$|\mathcal{B}| \leq \beta T^3 R^{1-\alpha_1} + R^{-\alpha_1}$$

and for $j \notin \mathcal{B}$ with $\Lambda^\tau_{\rho_j}(R) \cap \Lambda^\tau_{A_{\pm,R_1} \cap \Sigma_{H,p}}(R) \neq \emptyset$,

$$d\left(\Lambda^\tau_{A_{\pm,R_1}\cap\Sigma_{H,p}}(R), \bigcup_{t\in[1,T]} \varphi_t(\Lambda^\tau_{\rho_j}(R))\right) \geq 2R.$$

In particular,

$$\bigcup_{j\notin\mathcal{B}} \Lambda^\tau_{\rho_j}(R) \text{ is } [1,T] \text{ non-self-looping}.$$

Proof. We start by removing tubes covering the intersection of an $R^{1-\alpha_1}$ neighborhood of $\xi_\theta = \sqrt{V(r_0)}\alpha(r_0)$ with $\Sigma_{H,p}$. This requires $R^{-\alpha_1}$ tubes of radius R. In particular, this covers an $R^{1-\alpha_1}$ neighborhood of the singular torus and we may restrict our attention to $A_{\pm,R_1}$.

We claim that there is $C > 0$ so that if ρ_1, ρ_2 are at least α away from the singular torus, then

$$|\Theta(\rho_1) - \Theta(\rho_2)| + |I(\rho_1) - I(\rho_2)| \leq C\alpha^{-1} d(\rho_1, \rho_2). \tag{5.2}$$

Indeed, by (e.g., [54, equation (3.37)], [57, Theorem 3.12], and [23, Theorem, p. 9]) there are Birkhoff normal form symplectic coordinates in a neighborhood of the stable bicharacteristic γ_s so that $\rho = (t, x, \tau, \xi) \in S^1 \times \mathbb{R} \times \mathbb{R}^2$ with γ_s given by $\{(t, 0, 0, 0) : t \in S^1\}$ so that

$$p(t, x, \tau, \xi) = \tau + f(x^2 + \xi^2, \tau),$$

$f \in C^\infty((-\delta, \delta)^2; \mathbb{R})$ for some $\delta > 0$ and

$$f(u, v) = \alpha(v)u + O(v^2) + O_v(u^2)$$

for some $\alpha \in C^\infty((-\delta, \delta); \mathbb{R})$.

In particular, we may work with action-angle coordinates (Θ, I) given by

$$I_1 = \tau, \quad I_2 = \frac{1}{2}(x^2 + \xi^2), \quad x = \sqrt{2I_2}\cos(\Theta_2), \quad \xi = \sqrt{2I_2}\sin(\Theta_2).$$

In these coordinates

$$p(\Theta, I) = I_1 + f(2I_2, I_1),$$

the action coordinate function $I_2(x, \xi)$ measures the squared distance from (x, ξ) to the singular torus, and we have

$$|\partial_{I,\Theta}\rho| \le \frac{C}{\sqrt{2I_2}} = C\alpha^{-1}.$$

This yields (5.2) as claimed.

Next, suppose

$$d(\rho, \Sigma_{H,p} \cap A_{\pm,R_1}) < 2R, \quad d(\varphi_t(\rho), \Sigma_{H,p} \cap A_{\pm,R_1}) < 2R.$$

There exists $\tilde{\rho} \in \Sigma_{H,p} \cap A_{\pm,R_1}$ with $d(\rho, \tilde{\rho}) < 2R$. Therefore, for some $C > 0$,

$$d(\varphi_t(\tilde{\rho}), \varphi_t(\rho)) < CRt$$

and hence, for $t \le T$,

$$d(\varphi_t(\tilde{\rho}), \Sigma_{H,p} \cap A_{\pm,R_1}) < (CT + 1)R.$$

Now, for $RT \ll R^{\alpha_1}$, by (5.1) since ρ is at least $R^{1-\alpha_1}$ away from the singular torus, the only intersection of $\mathbb{T}_{I_0(\tilde{\rho})}$ with

$$\{q : d(q, \Sigma_{H,p} \cap A_{\pm,R_1}) < (CT + 1)R\}$$

happens at q with $d(q, \tilde{\rho}) < (CT + 1)R$. In particular,

$$d(\varphi_t(\tilde{\rho}), \tilde{\rho}) < (CT + 1)R,$$

and hence by (5.2),

$$d(t\partial_I p(I_0), 2\pi\mathbb{Z}^2) < CTRR^{-1+\alpha_1}.$$

That is, $\tilde{\rho}$ is CTR^{α_1} close to a rational torus of period t. Thus, the same is true for the original ρ with possibly a different constant.

Now, the points that are CTR^{α_1} close to the intersection of $\Sigma_{H,p} \cap A_\pm$ with $\mathbb{T}_{I_0}$ can be covered by $CTR^{1-\alpha_1}$ tubes. Moreover, since p is isoenergetically non-degenerate, there is $c > 0$ so that the rational tori of period $\le T$, are separated by cT^{-2}. Hence, there are at most CT^2 such tori and we require $CT^3R^{1-\alpha_1}$ tubes. □

Proof of Theorem 5. Fix $L > 0$, $r_0 \in [\delta, 2\pi - \delta]$, $\theta_0 \in [0, \pi]$, and $\alpha_1 = \frac{1}{2}$. Then, for $0 < R \ll 1$ and $0 < T < R^{-\frac{1}{2}}$, we may apply Lemma 5.1. Let $\{\Lambda^\tau_{\rho_j}(R)\}_{j=1}^{N_R}$ be the cover of $\Sigma_{H,p}$ given by Proposition 3.3. Then there are $\mathcal{G}, \mathcal{B} \subset \{1, \dots, N_R\}$ so that

$$|\mathcal{B}| \le (\beta T^3 + 1)R^{-\frac{1}{2}}, \quad \{1, \dots, N_R\} \subset \mathcal{G} \cup \mathcal{B},$$

and

$$\bigcup_{j \in \mathcal{G}} \Lambda^\tau_{\rho_j}(R) \text{ is } [1, T] \text{ non-self-looping.}$$

Fix $0 < \varepsilon < \delta < \frac{1}{2}$, let $R = h^{\varepsilon}$ and $T = L^2 \log h^{-1}$. We next apply Theorem 11 with P as in (1.6), $\mathcal{G}_\ell = \mathcal{G}$, $T_\ell = T$ and $t_\ell = 1$ for all ℓ. Then there exist $C > 0$ independent of L, for any $N > 0$, $C_N > 0$, and $h_0 > 0$, so that for all $0 < h < h_0$,

$$\begin{aligned}
h^{\frac{1}{2}} \|u\|_{L^\infty(B((r_0,\theta_0),h^\delta))} &\le C h^{\frac{\varepsilon}{2}} \bigg(\bigg[(\log h^{-1})^{\frac{3}{2}} h^{-\frac{\varepsilon}{4}} + \frac{h^{-\frac{\varepsilon}{2}}}{L\sqrt{\log h^{-1}}} \bigg] \|u\|_{L^2(M)} \\
&\qquad + \frac{h^{-\frac{\varepsilon}{2}} L \sqrt{\log h^{-1}}}{h} \|Pu\|_{L^2(M)} \bigg) \\
&\qquad + C h^{-1} \|Pu\|_{H_{\mathrm{scl}}^{-1/2}(M)} \\
&\qquad + C_N h^N \big(\|u\|_{L^2(M)} + \|Pu\|_{H_{\mathrm{scl}}^{-1/2}(M)} \big) \\
&\le C \bigg(\beta \bigg[(\log h^{-1})^{\frac{3}{2}} h^{\frac{\varepsilon}{4}} + \frac{1}{L\sqrt{\log h^{-1}}} \bigg] \|u\|_{L^2(M)} \\
&\qquad + \frac{L\sqrt{\log h^{-1}}}{h} \|Pu\|_{H_{\mathrm{scl}}^{-1/2}(M)} \bigg). \qquad \square
\end{aligned}$$

6. Change of the Hamiltonian

When studying quasimodes for the Laplacian, it will be convenient to replace the operator $P_0 := -h^2 \Delta_g - 1$ by an operator whose dynamics agree with those of $p = |\xi|_g - 1$.

Lemma 6.1. *There exists $P \in \Psi^0(M)$ with real, classically elliptic symbol p such that $\{p = 0\} = S^*M$, $p = |\xi|_g - 1$ in a neighborhood of S^*M and there exist $Q \in \Psi^{-2}(M)$ and $E \in h^\infty \Psi^{-\infty}(M)$ satisfying*

$$P = QP_0 + E.$$

In particular, for all $s \in \mathbb{R}$ there exists a constant $C_s > 0$ depending only on s so that for all $N > 0$, there exist $C_{N,s} = C(N, s, M, g) > 0$ and $h_0 = h_0(N, s, M, g) > 0$ so that for $0 < h < h_0$ and $u \in \mathcal{D}'(M)$,

$$\|Pu\|_{H^s_{\mathrm{scl}}(M)} \le C_s \|P_0 u\|_{H^{s-2}_{\mathrm{scl}}(M)} + C_{N,s} h^N \|u\|_{H^{-N}_{\mathrm{scl}}(M)}.$$

Proof. Let $\psi_1 \in C_c^\infty(\mathbb{R}; [0,1])$ with $\operatorname{supp} \psi_1 \subset (-\frac{1}{2}, \frac{1}{2})$ and $\psi_1 \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}]$. Next, let $\psi_2 \in C_c^\infty(\mathbb{R}; [0,1])$ with $\psi_2 \equiv 1$ on $[-4, -\frac{1}{2}] \cup [\frac{1}{2}, 4]$ so that $\psi := \psi_1 + \psi_2$ has $\psi \equiv 1$ on $[-4, 4]$. Define

$$\tilde{P} = \tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3$$

with

$$\begin{aligned}
\tilde{P}_1 &:= \frac{1}{2} \psi_1(-h^2 \Delta_g), \\
\tilde{P}_2 &:= \psi_2(-h^2 \Delta_g) \sqrt{-h^2 \Delta_g}, \\
\tilde{P}_3 &:= 2(1 - \psi(-h^2 \Delta_g)).
\end{aligned}$$

Note that by the functional calculus [65, Theorem 14.9], $\tilde{P} \in \Psi(M)$ with symbol

$$\tilde{p} := \frac{1}{2} \psi_1(|\xi|_g^2) + \psi_2(|\xi|_g^2) |\xi|_g + 2(1 - \psi(|\xi|_g^2)).$$

In particular, $\tilde{p} = |\xi|_g$ in a neighborhood of S^*M.

Next, observe that

$$\begin{aligned}(\tilde{P}+I)(\tilde{P}-I) &= P_0 + h^2\Delta_g + \tilde{P}^2\\ &= P_0 - (I-\psi_2^2(-h^2\Delta_g))(-h^2\Delta_g)\\ &\quad + \tilde{P}_1^2 + \tilde{P}_3^2 + 2\tilde{P}_1\tilde{P}_2 + 2\tilde{P}_2\tilde{P}_3 + 2\tilde{P}_1\tilde{P}_3.\end{aligned}$$

Now, there exists $c>0$ so that

$$\mathrm{WF}_h(\tilde{P}_1)\cup \mathrm{WF}_h(\tilde{P}_3)\cup \mathrm{WF}_h(I-\psi_2^2(-h^2\Delta_g))\subset\{|\sigma(P_0)|>c\langle\xi\rangle^2\}.$$

In particular, by the elliptic parametrix construction (see, e.g., [22, Appendix E.2]) there is $Q_1\in\Psi^{-2}(M)$ so that

$$(\tilde{P}+I)(\tilde{P}-I) = Q_1P_0 + O(h^\infty)_{\Psi^{-\infty}}.$$

Now, $\sigma(\tilde{P}+I)>1$ therefore, $(\tilde{P}+I)^{-1}\in\Psi(M)$ and we have that

$$\tilde{P}-I = (\tilde{P}+I)^{-1}Q_1P_0 + O(h^\infty)_{\Psi^{-\infty}}$$

which completes the proof of the lemma after letting $Q=(\tilde{P}+I)^{-1}Q_1$ and $P=\tilde{P}-I$. □

Applying Theorem 11 to P from Lemma 6.1, where $P_0 := -h^2\Delta_g - 1$, and then estimating Pu by Lemma 6.1, we obtain the following theorem.

Theorem 12. *Let $\{H_h\}_h\subset M$ be a regular family of submanifolds of codimension k that is uniformly conormally transverse for p. Let $\{\tilde{H}_h\}_h$ be a family of submanifolds of codimension k satisfying* (2.5). *Let $0<\delta<\frac{1}{2}$, $N>0$ and $\{w_h\}_h$ with $w_h\in S_\delta\cap C_c^\infty(\tilde{H}_h)$. There exist positive constants $\tau_0=\tau_0(M,g,\tau_{\mathrm{inj}},\{H_h\}_h)$, $R_0=R_0(M,g,\mathcal{K}_0,k,\tau_{\mathrm{inj}})$, $C_{n,k}$, depending only on n and k, and $h_0=h_0(M,g,\delta,\{H_h\}_h)$ and for each $0<\tau\le\tau_0$ there exist $C=C(M,g,\tau,\delta,,\{H_h\}_h)>0$ and $C_N=C_N(M,g,N,\tau,\delta,\{w_h\}_h,\{H_h\}_h)>0$, so that the following holds.*

Let $8h^\delta\le R(h)< R_0$,$0\le\alpha<1-2\limsup_{h\to 0}\frac{\log R(h)}{\log h}$, and suppose $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j=1}^{N_h}$ is a $(\mathfrak{D},\tau,R(h))$ cover of SN^*H for some $\mathfrak{D}>0$. In addition, suppose there exist a subset $\mathcal{B}\subset\{1,\dots,N_h\}$ and a finite collection $\{\mathcal{G}_\ell\}_{\ell\in\mathcal{L}}\subset\{1,\dots,N_h\}$ with

$$\mathcal{J}_h(w_h)\subset\mathcal{B}\cup\bigcup_{\ell\in\mathcal{L}}\mathcal{G}_\ell,$$

where $\mathcal{J}_h(w_h)$ is defined in (2.13), *and so that for every $\ell\in\mathcal{L}$ there exist $t_\ell=t_\ell(h)>0$ and $T_\ell=T_\ell(h)\le 2\alpha T_e(h)$ so that*

$$\bigcup_{j\in\mathcal{G}_\ell}\Lambda^\tau_{\rho_j}(R(h))\ \textit{is}\ [t_\ell,T_\ell]\ \textit{non-self-looping for}\ \varphi_t:=\exp(tH_{|\xi|_g}).$$

Then, for $u\in\mathcal{D}'(M)$ and $0<h<h_0$,

$$\begin{aligned}h^{\frac{k-1}{2}}\left|\int_{\tilde{H}_h}w_h u\,d\sigma_{\tilde{H}_h}\right| &\le \frac{C_{n,k}\mathfrak{D}\|w_h\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac{1}{2}}}\Bigg(|\mathcal{B}|^{\frac{1}{2}}+\sum_{\ell\in\mathcal{L}}\frac{(|\mathcal{G}_\ell|t_\ell)^{\frac{1}{2}}}{T_\ell^{\frac{1}{2}}}\Bigg)\|u\|_{L^2(M)}\\ &\quad+\frac{C_{n,k}\mathfrak{D}\|w_h\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac{1}{2}}}\sum_{\ell\in\mathcal{L}}\frac{(|\mathcal{G}_\ell|t_\ell T_\ell)^{\frac{1}{2}}}{h}\|P_0u\|_{L^2(M)}\\ &\quad+Ch^{-1}\|w_h\|_\infty\|P_0u\|_{H^{(k-3)/2}_{\mathrm{scl}}(M)}\\ &\quad+C_Nh^N(\|u\|_{L^2(M)}+\|P_0u\|_{H^{(k-3)/2}_{\mathrm{scl}}(M)}).\end{aligned}$$

Here, the constant C_N depends on $\{w_h\}_h$ only through finitely many S_δ seminorms of w_h. The constants τ_0, C, C_N, h_0 depend on $\{H_h\}_h$ only through finitely many of the constants $\mathcal{K}_\alpha$ in (2.3).

A. Appendix

A.1. Index of notation. In general we denote points in T^*M by ρ. When position and momentum need to be distinguished, we write $\rho = (x, \xi)$ for $x \in M$ and $\xi \in T^*_x M$. Sets of indices are denoted in calligraphic font (e.g., $\mathcal{J}$). Next, we list symbols that are used repeatedly in the text along with the location where they are first defined.

$\mathcal{C}_x^{r,t}$	(1.4)	$\mathcal{H}_\Sigma$	(2.9)	β_δ	(3.1)
$\Sigma_{H,p}$	(2.1)	τ_{inj}	(2.10)	$\mathfrak{D}_n$	Proposition 3.3
φ_t	(2.2)	$\Lambda_A^\tau(r)$	(2.11)	Ψ_δ^k	(A.1)
$\mathcal{K}_\alpha$	(2.3)	$\Lambda_\rho^\tau(r)$	(2.12)	S_δ^k	(A.1)
r_H	(2.7)	$\mathcal{J}_h(w)$	(2.13)	H_{scl}^k	(A.3)
K_p	(2.6)	$T_e(h)$	(2.14)	MS_h	Definition 5
$\mathfrak{T}_0$	(2.8)	$\Lambda_{\max}$	(2.14)		

For the definition of $[t, T]$ non-self-looping, see (1.2). For that of $(\mathfrak{D}, \tau, r)$-good covers, see Definition 4.

A.2. Notation from semiclassical analysis. We refer the reader to [65] or [22, Appendix E] for a complete treatment of semiclassical analysis, but recall some of the relevant notation here. We say $a \in C^\infty(T^*M)$ is a symbol of order m and class $0 \le \delta < \frac{1}{2}$, writing $a \in S^m_\delta(T^*M)$ if there exists $C_{\alpha\beta} > 0$ so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{m-|\beta|}, \quad \langle \xi \rangle := (1 + |\xi|_g^2)^{\frac{1}{2}}. \tag{A.1}$$

Note that we implicitly allow a to also depend on h, but omit it from the notation. We then define $S^\infty_\delta(T^*M) := \bigcup_m S^m_\delta(T^*M)$. We sometimes write $S^m(T^*M)$ for $S^m_0(T^*M)$. We also sometimes write S_δ for S^m_δ. Next, we say that $a \in S^{\text{comp}}_\delta(T^*M)$ if a is supported in an h-independent compact subset of T^*M.

Next, there is a quantization procedure $Op_h : S^m_\delta \to \mathcal{L}(C^\infty(M), \mathcal{D}'(M))$ and we say $A \in \Psi^m_\delta(M)$ if there exists $a \in S^m_\delta(T^*M)$ so that $Op_h(a) - A = O(h^\infty)_{\Psi^{-\infty}}$, where we say an operator is $O(h^k)_{\Psi^{-\infty}}$ if for all $N > 0$ there exists $C_N > 0$ so that

$$\|Au\|_{H^N(M)} \le C_N h^k \|u\|_{H^{-N}(M)},$$

and say an operator, A, is $O(h^\infty)_{\Psi^{-\infty}}$ if for all $N > 0$ there exists $C_N > 0$ so that

$$\|Au\|_{H^N(M)} \le C_N h^N \|u\|_{H^{-N}(M)}.$$

For $a \in S^{m_1}_\delta(T^*M)$ and $b \in S^{m_2}_\delta(T^*M)$, we have that

$$Op_h(a) Op_h(b) = Op_h(c), \quad c(x,\xi) \sim \sum_j h^j L_{2j}(a(x,\xi) b(y,\eta))\Big|_{\substack{x=y,\\ \xi=\eta}}, \tag{A.2}$$

where L_{2j} is a differential operator of order j in (x, ξ) and order j in (y, η).

There is a symbol map $\sigma : \Psi^m_\delta(M) \to S^m_\delta(T^*M)/h^{1-2\delta}S^{m-1}_\delta(T^*M)$ so that

$$\sigma(Op_h(a)) = a, \qquad \sigma(Op_h(a)^*) = \bar{a},$$
$$\sigma(Op_h(a)Op_h(b)) = ab, \quad \sigma([Op_h(a), Op_h(b)]) = -ih\{a,b\},$$

and

$$0 \to h^{1-2\delta}\Psi^{m-1}_\delta(M) \to \Psi^m_\delta(M) \overset{\sigma}{\to} S^m_\delta(M)/h^{1-2\delta}S^{m-1}_\delta(M) \to 0$$

is exact.

The main consequence of (A.2) that we will use is that if $p \in S^m(M)$ and $a \in S^k_\delta(T^*M)$, then

$$[Op_h(p), Op_h(a)] = \frac{h}{i}Op_h(H_p a) + h^{2-2\delta}Op_h(r)$$

with $r \in S^{m+k-2}_\delta(T^*M)$.

We define the semiclassical Sobolev spaces $H^s_{\mathrm{scl}}(M)$ by

$$H^s_{\mathrm{scl}}(M) := \{u \in \mathcal{D}'(M) : \|u\|_{H^s_{\mathrm{scl}}(M)} < \infty\}, \tag{A.3}$$

where

$$\|u\|_{H^s_{\mathrm{scl}}(M)} := \|Op_h(\langle\xi\rangle^s)u\|_{L^2(M)}.$$

A.3. Background on microsupports and Egorov's Theorem.

Definition 5. For a pseudodifferential operator $A \in \Psi^{\mathrm{comp}}_\delta(M)$, we say that A is microsupported in a family of sets $\{V(h)\}_h$ and write $\mathrm{MS_h}(A) \subset V(h)$ if

$$A = Op_h(a) + O(h^\infty)_{\Psi^{-\infty}}$$

and for all α, N, there exists $C_{\alpha,N} > 0$ so that

$$\sup_{(x,\xi)\in T^*M\setminus V(h)} |\partial^\alpha_{x,\xi}a(x,\xi)| \le C_{\alpha,N}h^N.$$

For $B(h) \subset T^*M$, will also write $\mathrm{MS_h}(A) \cap B(h) = \emptyset$ for $\mathrm{MS_h}(A) \subset (B(h))^c$.

Note that the notation $\mathrm{MS_h}(A) \subset V(h)$ is a shortening for $\mathrm{MS_h}(A) \subset \{V(h)\}_h$.

Lemma A.1. *Let* $0 \le \delta < \frac{1}{2}$ *and* $\delta' > \delta$, $c > 0$. *Suppose that* $A \in \Psi^{\mathrm{comp}}_\delta(M)$ *and that* $\mathrm{MS_h}(A) \subset V(h)$. *Then*

$$\mathrm{MS_h}(A) \subset \{(x,\xi) : d((x,\xi), V(h)^c) \le ch^{\delta'}\}.$$

Proof. Let $A = Op_h(a) + O(h^\infty)_{\Psi^{-\infty}}$. Suppose that $2r(h) := d(\rho_1, V(h)^c) \le ch^{\delta'}$ and let $\rho_0 \in V(h)^c$ with $d(\rho_1,\rho_0) \le r(h)$. Then, for any $N > 0$,

$$\begin{aligned}|\partial^\alpha a(\rho_1)| &\le \sum_{|\beta|\le N-1} |\partial^{\alpha+\beta}a(\rho_0)|r(h)^{|\beta|} + C_{|\alpha|+N}\sup_{|k|\le|\alpha|+N, T^*M}|\partial^k a|r(h)^N \\ &\le \sum_{|\beta|\le N-1}\sup_{V^c}|\partial^{\alpha+\beta}a(\rho)|r(h)^{|\beta|} + C_{\alpha N}h^{-N\delta}r(h)^N \\ &\le C_{\alpha NM}h^M + C_{\alpha N}h^{-N\delta}r(h)^N.\end{aligned}$$

So, letting $N \ge M(\delta'-\delta)^{-1}$,

$$|\partial^\alpha a(\rho_1)| \le C_{\alpha M}h^M. \qquad \square$$

Lemma A.2. *Let* $0 \le \delta < \frac{1}{2}$ *and* $A, B \in \Psi_\delta^{\mathrm{comp}}(M)$. *Suppose that* $\mathrm{MS_h}(A) \subset V(h)$ *and* $\mathrm{MS_h}(B) \subset W(h)$.

(1) *The statement* $\mathrm{MS_h}(A) \subset V(h)$ *is well defined. In particular, it does not depend on the choice of quantization procedure.*

(2) $\mathrm{MS_h}(AB) \subset V(h) \cap W(h)$

(3) $\mathrm{MS_h}(A^*) \subset V(h)$

(4) *If* $V(h) = \emptyset$, *then* $\mathrm{WF_h}(A) = \emptyset$.

(5) *If* $A = Op_h(a) + O(h^\infty)_{\Psi^{-\infty}}$, *then* $\mathrm{MS_h}(a) \subset \operatorname{supp} a$.

Proof. The proofs of (1)–(3) are nearly identical, relying on the asymptotic expansion for, respectively, the change of quantization, composition, and adjoint so we write the proof for only (2). Write $A = Op_h(a) + O(h^\infty)_{\Psi^{-\infty}}$ and $B = Op_h(b) + O(h^\infty)_{\Psi^{-\infty}}$. Then

$$Op_h(a)Op_h(b) = Op_h(a \# b) + O(h^\infty)_{\Psi^{-\infty}},$$

where

$$a \# b(x,\xi) \sim \sum_j h^j L_{2j} a(x,\xi) b(y,\eta)\Big|_{\substack{x=y\\ \xi=\eta}}$$

and L_{2j} are differential operators of order $2j$. Suppose that $\mathrm{MS_h}(A) \subset V$. Then, for any $N > 0$,

$$\sup_{V^c} |\partial^\alpha a| \le C_{\alpha N} h^N.$$

So, choosing $M > (N + \delta|\alpha|)(1-2\delta)^{-1}$,

$$|\partial^\alpha a \# b| \le \left| \partial^\alpha \sum_{j<M} h^j L_{2j} a(x,\xi) b(y,\eta)\Big|_{\substack{x=y\\ \xi=\eta}} \right| + C_{\alpha M} h^{M(1-2\delta)-|\alpha|\delta} \le C_{\alpha N} h^N.$$

In particular,

$$\sup_{V^c} |\partial^\alpha a \# b| \le C_{\alpha N} h^N.$$

An identical argument shows

$$\sup_{W^c} |\partial^\alpha a \# b| \le C_{\alpha N} h^N.$$

Statement (4) follows from the definition since if $V(h) = \emptyset$, $a \in h^\infty S_\delta$, and (5) follows easily from the definition. □

Lemma A.3. *Let* $\varphi_t := \exp(tH_p)$ *and* $\Sigma \subset T^*M$ *compact. There exist* $\delta > 0$ *small enough and* $C_1 > 0$ *so that uniformly for* $t \in [0,\delta]$, *and* $(x_i,\xi_i) \in \Sigma$,

$$\begin{aligned} &\frac{1}{2} d((x_1,\xi_1),(x_2,\xi_2)) - C_1 d((x_1,\xi_1),(x_2,\xi_2))^2 \\ &\quad \le d(\varphi_t(x_1,\xi_2), \varphi_t(x_2,\xi_1)) \\ &\quad \le 2d((x_1,\xi_1),(x_2,\xi_2)) + C_1 d((x_1,\xi_1),(x_2,\xi_2))^2, \end{aligned}$$

where d *is the distance induced by the Sasaki metric. Furthermore, if* $\varphi_t(x_i,\xi_i) = (x_i(t),\xi_i(t))$, *then*

$$d_M(x_1(t), x_2(t)) \le d_M(x_1,x_2) + C_1 d((x_1,\xi_1),(x_2,\xi_2))\delta,$$

where d_M *is the distance induced by the metric on* M.

Proof. By Taylor's theorem,

$$\varphi_t(x_1,\xi_1)-\varphi_t(x_2,\xi_2)=d_x\varphi_t(x_2,\xi_2)(x_1-x_2)+d_\xi\varphi_t(x_2,\xi_2)(\xi_1-\xi_2)\\ +O_{C^\infty}\Big(\sup_{q\in\Sigma}|d^2\varphi_t(q)|(|\xi_1-\xi_2|^2+|x_1-x_2|^2)\Big).$$

Now,

$$\varphi_t(x,\xi)=(x,\xi)+(\partial_\xi p(x,\xi)t,-\partial_x p(x,\xi)t)+O(t^2)$$

so

$$d_\xi\varphi_t(x,\xi)=(0,I)+t(\partial_\xi^2 p,-\partial_{\xi x}^2 p)+O(t^2)$$
$$d_x\varphi_t(x,\xi)=(I,0)+t(\partial_{x\xi}^2 p,-\partial_x^2 p)+O(t^2).$$

In particular,

$$\varphi_t(x_1,\xi_1)-\varphi_t(x_2,\xi_2)=((0,I)+O(t))(\xi_1-\xi_2)+((I,0)+O(t))(x_1-x_2)\\ +O((\xi_1-\xi_2)^2+(x_1-x_2)^2)$$

and choosing $\delta>0$ small enough gives the result. □

B. Proofs of Lemmas 1.2 and 1.3

Lemma B.1. *Let $t,T>0$ and suppose that $G\subset S_x^*M$ is a closed set that is $[t,T]$ non-self-looping. Then there is $R>0$ such that $B_{T^*M}(G,R)$ is $[t,T]$ non-self-looping.*

Proof. We will assume that $\varphi_s(G)\cap G=\emptyset$ for $s\in[t,T]$, the case of $s\in[-T,-t]$ being similar. Let $q\in G$. We claim there is $R_q>0$ such that

$$\bigcup_{s\in[t,T]}\varphi_t(B_{T^*M}(q,R_q))\cap B_{T^*M}(G,R_q)=\emptyset.$$

Suppose not. Then there are $q_n\to q$ and $s_n\in[t,T]$ such that $d(\varphi_{s_n}(q_n),G)\to 0$. Extracting subsequences, we may assume $s_n\to s\in[t,T]$ and $\varphi_{s_n}(q_n)\to\rho\in G$. But then $\varphi_s(q)=\rho$ and, in particular, G is not $[t,T]$ non-self-looping.

Now, $G\subset\bigcup_{q\in G}B(q,R_q)$ and hence, by compactness, there are q_i, $i=1,\dots,N$, such that

$$G\subset\bigcup_{i=1}^N B(q_i,R_{q_i}).$$

In particular, there is $0<R<\min_i R_{q_i}$ such that

$$B(G,R)\subset\bigcup_{i=1}^N B(q_i,R_{q_i}).$$

This implies that $B(G,R)$ is $[t,T]$ non-self-looping. □

Lemma B.2. *Let $\tau,\mathfrak{D},t,T>0$, $R(h)\geq 8h^\delta$, and $\{\Lambda_{\rho_j}^\tau(R(h))\}_{j\in\mathcal{G}}$ a $(\mathfrak{D},\tau,R(h))$-good cover of S_x^*M. Suppose that $G\subset S_x^*M$ is closed and $[t,T]$ non-self-looping. Then, for all $\varepsilon>0$, there is $R>0$ small enough such that for $R(h)<R$,*

$$\mathcal{G}:=\{j\in\mathcal{J}:\Lambda_{\rho_j}^\tau(R(h))\cap B_{S_x^*M}(G,R)\neq\emptyset\}$$

satisfies

$$\bigcup_{j\in\mathcal{G}} \Lambda^{\tau}_{\rho_j}(R(h)) \text{ is } [\max(t,3\tau),\max(t,3\tau,T)] \text{ non-self-looping} \tag{B.1}$$

and

$$|\mathcal{G}| \le \mathfrak{D} R(h)^{1-n}(\mathrm{vol}_{S^*_xM}(G)+\varepsilon). \tag{B.2}$$

Proof. By Lemma B.1, there is $R_0>0$ such that $B(G,R_0)$ is $[t,T]$ non-self-looping. Furthermore, since G is closed, there is $R_1>0$ such that

$$\mathrm{vol}_{S^*_xM}(B(G,R_1)) < \mathrm{vol}_{S^*_xM}(G)+\varepsilon.$$

Therefore, putting $R=\min(\frac{R_0}{4},\frac{R_1}{4})$, for $R(h)\le R$, and $j\in\mathcal{G}$,

$$\bigcup_{j\in\mathcal{G}} \Lambda^{\tau}_{\rho_j}(R(h))\cap S^*_xM \subset B_{T^*M}(G,\min(R_0,R_1)).$$

In particular, (B.1) and (B.2) hold. □

Proof of Lemma 1.2. Suppose that x non-self-focal. Let $\mathcal{L}^T_x := T_+^{-1}([0,T])$ and note that for all $T>0$, $\mathcal{L}^T_x$ is closed. Thus, by Lemma B.2 for all $T>0$ there is $R_0=R_0(T)>0$ such that for $R(h)\le R_0$, with $\tilde{\mathcal{B}} := \{j : \Lambda^{\tau}_{\rho_j}(R(h))\cap B_{S^*_xM}(\mathcal{L}^T_x,R_0)\}$, one has

$$|\tilde{\mathcal{B}}| \le \frac{R(h)^{1-n}}{T}.$$

Next, since $G := S^*_xM \setminus B(\mathcal{L}^T_x,R_0)$ is closed and $[\frac{\mathrm{inj}\,M}{2},T]$ non-self-looping, there is $R_1=R_1(T)>0$ such for $R(h)\le R_1$ and

$$\mathcal{G} = \{j : \Lambda^{\tau}_{\rho_j}(R(h))\cap B(G,R_1)\},$$

equation (B.1) holds with $t=\frac{\mathrm{inj}\,M}{2}$ and $T=T$. Putting

$$R(T) := \min(R_1(T),R_2(T)), \quad \mathcal{B} := \tilde{\mathcal{B}}\setminus\mathcal{G},$$

and defining

$$h_0(T)=\inf\{h>0 : R(h)>R(T)\}, \quad T(h)=\sup\{T>0 : h_0(T)>h\},$$

we have shown that x is $(\frac{\mathrm{inj}\,M}{2},T(h))$ non-looping. □

Proof of Lemma 1.3. Let $\mathcal{R}^{\pm,\delta,S}_x$ be the set of points $\rho\in S^*_xM$ for which there exists $0<\pm t\le S$ such that $\varphi_t(\rho)\in S^*_xM$ and $d(\varphi_t(\rho),\rho)\le\delta$. Then

$$\mathcal{R}_x = \bigcap_{\delta>0}\bigcup_{S>0}\mathcal{R}^{\delta,S}_x, \quad \mathcal{R}^{\delta,S}_x := \bigcap_{\pm}\mathcal{R}^{\pm,\delta,S}_x.$$

Note that $\mathcal{R}^{\delta,S}_x$ is closed for all δ, S, and that for all $\varepsilon>0$ there is $\delta>0$ such that for all $S>0$,

$$\mathrm{vol}_{S^*_xM}(\mathcal{R}^{S,\delta}_x) \le \mathrm{vol}_{S^*_xM}(\mathcal{R}_x)+\varepsilon.$$

Now, assume that x is non-recurrent. Then, for all $\varepsilon > 0$, there is a constant $\delta = \delta(\varepsilon) > 0$ such that for all $S > 0$,

$$\mathrm{vol}_{S_x^*M}(\mathcal{R}_x^{S,\delta}) \le \varepsilon.$$

Let $\{\rho_i\}_{i=1}^{N(\delta)} \subset S_x^*M$ be such that $S_x^*M \subset \bigcup_i B(\rho_i, \frac{\delta}{4})$ and $N(\delta) \le C\delta^{1-n}$.

Letting $G_0 := \mathcal{R}_x^{S,\delta}$, by Lemma B.2 there is a constant $R_0 = R_0(\varepsilon, S) > 0$ such that for $R(h) \le R_0$, defining $\tilde{\mathcal{G}}_0 := \{j : \Lambda_{\rho_j}^\tau(R(h)) \cap B_{S_x^*M}(G_i, R_0)\}$, we have

$$|\tilde{\mathcal{G}}_0| \le \mathfrak{D} R(h)^{1-n}\varepsilon.$$

Next, let $G_i := \overline{B_{S_x^*M}(\rho_i, \frac{\delta}{4})} \setminus B_{S_x^*M}(\mathcal{R}_x^{T,\delta}, R_0)$ so that G_i is closed and $[\frac{\mathrm{inj}\, M}{2}, S]$ non-self-looping. By Lemma B.2, there are $R_i = R_i(\varepsilon, S) > 0$ such that for $R(h) \le \min_i R_i$, if we set $\tilde{\mathcal{G}}_i := \{j : \Lambda_{\rho_j}^\tau(R(h)) \cap B_{S_x^*M}(G_i, R_i)\}$, then

$$|\tilde{\mathcal{G}}_i| \le R(h)^{1-n}\mathfrak{D}\delta^{n-1}, \quad i \ge 1,$$

and for $i \ge 1$,

$$\bigcup_{j \in \tilde{\mathcal{G}}_i} \Lambda_{\rho_j}^\tau(R(h)) \text{ is } [\mathrm{inj}\, M/2, S] \text{ non-self-looping.}$$

Then we have

$$\sum_{i=0}^{N} \sqrt{\frac{|\tilde{\mathcal{G}}_i| R(h)^{n-1}\,\mathrm{inj}\, M}{2S}} \le N(\delta)\delta^{\frac{n-1}{2}} \sqrt{\frac{\mathfrak{D}\,\mathrm{inj}\, M}{2S}} + \sqrt{\mathfrak{D}\varepsilon}.$$

Now, for $\varepsilon := \frac{1}{4\mathfrak{D}T}$ let $\delta := \delta(\varepsilon)$ and set

$$S := 2N^2(\delta)\delta^{n-1}\mathfrak{D}\,\mathrm{inj}\, M.$$

Working with $R_i = R_i(\varepsilon, S) = R_i(T)$ as defined before, we have

$$\sum_{i=0}^{N} \sqrt{\frac{|\tilde{\mathcal{G}}_i| R(h)^{n-1}\,\mathrm{inj}\, M}{2S}} \le \sqrt{\frac{1}{T}}.$$

Defining

$$h_0(T) = \inf\Big\{h > 0 : R(h) > \min_i R_i(T)\Big\}, \quad T(h) = \sup\{T > 0 : h_0(T) > h\},$$

we have shown that x is $(\frac{\mathrm{inj}\, M}{2}, T(h))$ non-recurrent. □

References

[1] *D. V. Anosov*, Geodesic flows on closed Riemannian manifolds of negative curvature, Trudy Mat. Inst. Steklov. **90** (1967), 209.

[2] *J. A. Arnaud*, Vi hamiltonian theory of beam mode propagation, Progr. Optics **11** (1973), 247–304.

[3] *V. G. Avakumović*, Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten, Math. Z. **65** (1956), 327–344.

[4] *V. M. Babič* and *V. S. Buldyrev*, Short-wavelength diffraction theory, Springer, Berlin 1991.

[5] *V. M. Babič* and *V. F. Lazutkin*, The eigenfunctions which are concentrated near a closed geodesic, in: Problems of mathematical physics. No. 2: Spectral theory, diffraction problems (Russian), Izdat. Leningrad. Univ., Leningrad (1967), 15–25.
[6] *P. H. Bérard*, On the wave equation on a compact Riemannian manifold without conjugate points, Math. Z. **155** (1977), no. 3, 249–276.
[7] *D. E. Blair*, Riemannian geometry of contact and symplectic manifolds, 2nd ed., Prog. Math. **203**, Birkhäuser, Boston 2010.
[8] *M. D. Blair* and *C. D. Sogge*, Refined and microlocal Kakeya–Nikodym bounds of eigenfunctions in higher dimensions, Comm. Math. Phys. **356** (2017), no. 2, 501–533.
[9] *M. D. Blair* and *C. D. Sogge*, Concerning Toponogov's theorem and logarithmic improvement of estimates of eigenfunctions, J. Differential Geom. **109** (2018), no. 2, 189–221.
[10] *Y. Bonthonneau*, The Θ function and the Weyl law on manifolds without conjugate points, Doc. Math. **22** (2017), 1275–1283.
[11] *J. Bourgain*, Eigenfunction bounds for the Laplacian on the n-torus, Int. Math. Res. Not. IMRN **1993** (1993), no. 3, 61–66.
[12] *Y. Canzani* and *J. Galkowski*, Improvements for eigenfunction averages: An application of geodesic beams, preprint 2019, `https://arxiv.org/abs/1809.06296v3`.
[13] *Y. Canzani* and *J. Galkowski*, On the growth of eigenfunction averages: Microlocalization and geometry, Duke Math. J. **168** (2019), no. 16, 2991–3055.
[14] *Y. Canzani* and *J. Galkowski*, Growth of high L^p norms for eigenfunctions: an application of geodesic beams, preprint 2020, `https://arxiv.org/abs/2003.04597`.
[15] *Y. Canzani* and *J. Galkowski*, Weyl remainders: An application of geodesic beams, preprint 2020, `https://arxiv.org/abs/2010.03969`.
[16] *Y. Canzani*, *J. Galkowski* and *J. A. Toth*, Averages of eigenfunctions over hypersurfaces, Comm. Math. Phys. **360** (2018), no. 2, 619–637.
[17] *X. Chen* and *C. D. Sogge*, On integrals of eigenfunctions over geodesics, Proc. Amer. Math. Soc. **143** (2015), no. 1, 151–161.
[18] *T. H. Colding* and *W. P. Minicozzi, II*, Lower bounds for nodal sets of eigenfunctions, Comm. Math. Phys. **306** (2011), no. 3, 777–784.
[19] *M. Dimassi*, *J.-C. Guillot* and *J. Ralston*, Gaussian beam construction for adiabatic perturbations, Math. Phys. Anal. Geom. **9** (2006), no. 3, 187–201.
[20] *J. J. Duistermaat* and *V. W. Guillemin*, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. **29** (1975), no. 1, 39–79.
[21] *S. Dyatlov* and *C. Guillarmou*, Microlocal limits of plane waves and Eisenstein functions, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 2, 371–448.
[22] *S. Dyatlov* and *M. Zworski*, Mathematical theory of scattering resonances, Grad. Stud. Math. **200**, American Mathematical Society, Providence 2019.
[23] *L. H. Eliasson*, Normal forms for Hamiltonian systems with Poisson commuting integrals—elliptic case, Comment. Math. Helv. **65** (1990), no. 1, 4–35.
[24] *J. Galkowski*, A microlocal approach to eigenfunction concentration, J. Equ. Dériv. Partielles (2018), Exposé No. 3
[25] *J. Galkowski*, Defect measures of eigenfunctions with maximal L^∞ growth, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 4, 1757–1798.
[26] *J. Galkowski* and *J. A. Toth*, Eigenfunction scarring and improvements in L^∞ bounds, Anal. PDE **11** (2017), no. 3, 801–812.
[27] *J. Galkowski* and *J. A. Toth*, Pointwise bounds for joint eigenfunctions of quantum completely integrable systems, Comm. Math. Phys. **375** (2020), no. 2, 915–947.
[28] *A. Good*, Local analysis of Selberg's trace formula, Lecture Notes in Math. **1040**, Springer, Berlin 1983.
[29] *E. Grosswald*, Representations of integers as sums of squares, Springer, New York 1985.
[30] *A. Hassell* and *M. Tacy*, Improvement of eigenfunction estimates on manifolds of nonpositive curvature, Forum Math. **27** (2015), no. 3, 1435–1451.
[31] *D. A. Hejhal*, Sur certaines séries de Dirichlet associées aux géodésiques fermées d'une surface de Riemann compacte, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 8, 273–276.
[32] *L. Hörmander*, The spectral function of an elliptic operator, Acta Math. **121** (1968), 193–218.
[33] *E. Horozov*, On the isoenergetical nondegeneracy of the spherical pendulum, Phys. Lett. A **173** (1993), no. 3, 279–283.
[34] *H. Iwaniec* and *P. Sarnak*, L^∞ norms of eigenfunctions of arithmetic surfaces, Ann. of Math. (2) **141** (1995), no. 2, 301–320.

[35] *J. Jung* and *S. Zelditch*, Number of nodal domains and singular points of eigenfunctions of negatively curved surfaces with an isometric involution, J. Differential Geom. **102** (2016), no. 1, 37–66.

[36] *J. B. Keller* and *W. Streifer*, Complex rays with an application to gaussian beams, JOSA **61** (1971), no. 1, 40–43.

[37] *W. Klingenberg*, Riemannian manifolds with geodesic flow of Anosov type, Ann. of Math. (2) **99** (1974), 1–13.

[38] *W. Klingenberg*, Riemannian geometry, 2nd ed., De Gruyter Stud. Math. **1**, Walter de Gruyter, Berlin 1995.

[39] *H. Koch*, *D. Tataru* and *M. Zworski*, Semiclassical L^p estimates, Ann. Henri Poincaré **8** (2007), no. 5, 885–916.

[40] *B. M. Levitan*, On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order, Izv. Akad. Nauk SSSR. Ser. Mat. **16** (1952), 325–352.

[41] *J. V. Ralston*, On the construction of quasimodes associated with stable periodic orbits, Comm. Math. Phys. **51** (1976), no. 3, 219–242.

[42] *J. V. Ralston*, Approximate eigenfunctions of the Laplacian, J. Differential Geom. **12** (1977), no. 1, 87–100.

[43] *J. V. Ralston*, Gaussian beams and the propagation of singularities, in: Studies in partial differential equations, MAA Stud. Math. **23**, Mathematical Association of America, Washington (1982), 206–248.

[44] *B. Randol*, The Riemann hypothesis for Selberg's zeta-function and the asymptotic behavior of eigenvalues of the Laplace operator, Trans. Amer. Math. Soc. **236** (1978), 209–223.

[45] *Y. G. Safarov*, Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition, Funct. Anal. Appl. **22** (1988), no. 3, 213–223.

[46] *C. D. Sogge*, Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal. **77** (1988), no. 1, 123–138.

[47] *C. D. Sogge*, *J. A. Toth* and *S. Zelditch*, About the blowup of quasimodes on Riemannian manifolds, J. Geom. Anal. **21** (2011), no. 1, 150–173.

[48] *C D. Sogge*, *Y. Xi* and *C. Zhang*, Geodesic period integrals of eigenfunctions on Riemannian surfaces and the Gauss–Bonnet theorem, Camb. J. Math. **5** (2017), no. 1, 123–151.

[49] *C. D. Sogge* and *S. Zelditch*, Riemannian manifolds with maximal eigenfunction growth, Duke Math. J. **114** (2002), no. 3, 387–437.

[50] *C. D. Sogge* and *S. Zelditch*, Focal points and sup-norms of eigenfunctions, Rev. Mat. Iberoam. **32** (2016), no. 3, 971–994.

[51] *C. D. Sogge* and *S. Zelditch*, Focal points and sup-norms of eigenfunctions II: The two-dimensional case, Rev. Mat. Iberoam. **32** (2016), no. 3, 995–999.

[52] *M. Tacy*, A note on constructing families of sharp examples for L^p growth of eigenfunctions and quasimodes, Proc. Amer. Math. Soc. **146** (2018), no. 7, 2909–2924.

[53] *M. Tacy*, L^p estimates for joint quasimodes of semiclassical pseudodifferential operators, Israel J. Math. **232** (2019), no. 1, 401–425.

[54] *J. A. Toth*, L^2-restriction bounds for eigenfunctions along curves in the quantum completely integrable case, Comm. Math. Phys. **288** (2009), no. 1, 379–401.

[55] *J. A. Toth* and *S. Zelditch*, L^p norms of eigenfunctions in the completely integrable case, Ann. Henri Poincaré **4** (2003), no. 2, 343–368.

[56] *D. G. Vasil'ev* and *Y. G. Safarov*, The asymptotic distribution of eigenvalues of differential operators, in: Spectral theory of operators (Novgorod, 1989), Amer. Math. Soc. Transl. Ser. 2 **150**, American Mathematical Society, Providence (1992), 55–110.

[57] *S. Vũ Ngọc*, Symplectic techniques for semiclassical completely integrable systems, in: Topological methods in the theory of integrable systems, Cambridge Science, Cambridge (2006), 241–270.

[58] *A. Weinstein*, Fourier integral operators, quantization, and the spectra of Riemannian manifolds, in: Géométrie symplectique et physique mathématique, Editions du Centre National de la Recherche Scientifique, Paris (1975), 289–298.

[59] *E. L. Wyman*, Integrals of eigenfunctions over curves in compact 2-dimensional manifolds of nonpositive sectional curvature, preprint 2017, `https://arxiv.org/abs/1702.03552`.

[60] *E. L. Wyman*, Period integrals in non-positively curved manifolds, preprint 2018, `https://arxiv.org/abs/1806.01424`.

[61] *E. L. Wyman*, Looping directions and integrals of eigenfunctions over submanifolds, J. Geom. Anal. **29** (2019), no. 2, 1302–1319.

[62] *E. L. Wyman*, Explicit bounds on integrals of eigenfunctions over curves in surfaces of nonpositive curvature, J. Geom. Anal. **30** (2020), no. 3, 3204–3232.

[63] *S. Zelditch*, Kuznecov sum formulae and Szegö limit formulae on manifolds, Comm. Partial Differential Equations **17** (1992), no. 1–2, 221–260.

[64] *S. Zelditch*, Gaussian beams on Zoll manifolds and maximally degenerate Laplacians, in: Spectral theory and partial differential equations, Contemp. Math. **640**, American Mathematical Society, Providence (2015), 169–197.
[65] *M. Zworski*, Semiclassical analysis, Grad. Stud. Math. **138**, American Mathematical Society, Providence 2012.

Yaiza Canzani, Department of Mathematics, University of North Carolina at Chapel Hill,
Chapel Hill, NC, USA
e-mail: canzani@email.unc.edu

Jeffrey Galkowski, Department of Mathematics, University College London,
London, United Kingdom
https://orcid.org/0000-0001-5228-4998
e-mail: j.galkowski@ucl.ac.uk

Eingegangen 20. März 2020, in revidierter Fassung 15. September 2020