Hawkes jump-diffusions and finance: a brief history and review.[1]

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ABSTRACT

A brief history of diffusions in Finance is presented, followed by an even briefer discussion of jump-diffusions that involve Poisson or Lévy jumps. The main purpose of the paper is then to discuss applications of self-exciting and mutually-exciting Hawkes point processes. After an outline of the basic properties of this class of processes, there is a review of some recent articles that show how incorporating them as contagious jumps into Financial diffusions may improve model fit, forecasting, pricing, hedging and portfolio management.

KEYWORDS Hawkes process; jump-diffusion; pricing; hedging; portfolio
JEL CLASSIFICATION G00; G11; G15; G17

1. Introduction

Brownian motion has an interesting history of applications in Finance and Physics. In particular the famous Black-Scholes-Merton pricing formula, introduced in the 1970's, sometimes including Poisson jumps introduced by Merton (1976), has been a significant workhorse in commercial use. Hawkes (1971a, 1971b) introduced self-exciting and mutually exciting generalisations of Poisson processes a couple of years earlier. They were not envisaged at that time to have anything to do with Finance, indeed they were not used much at all for any applications, except Seismology, for about 30 years. However, in the last 15 years or so applications have mushroomed in the literature in many fields of research, including a wide range of applications within Finance. This article reviews recent publications about a particular kind of Financial application: so-called Hawkes jump-diffusions.

Section 2 gives a brief history of Brownian motion and other diffusion processes that have been used in Finance. Section 3 briefly mentions jump-diffusions with Poisson or Lévy jumps. There is indeed a large literature on this topic but we do not wish to delay getting to the main purpose of this note: Hawkes jump-diffusions. These are discussed in section 5 after an introduction to Hawkes processes in Section 4. Section 6 concludes.

2. Diffusion models in Finance

2.1 Brownian Motion

Diffusion models in finance started with the simplest Brownian motion model introduced in the PhD thesis *Théorie de la Spéculation* by Bachelier (1900): his seminal thesis marked the beginning of financial mathematics. His interest was in applying the theory to pricing, but it remained essentially unknown to the finance literature until Paul Samuelson rediscovered it in the 1950s (alerted to it by statistician L. J. Savage). Samuelson (1972) describes Bachelier's pioneering work on Brownian motion as superior to Einstein's (1906) much praised later independent derivation of

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Brownian motion. Einstein was interested in the rather different problem of the irregular movement of microscopic particles suspended in a liquid, as observed by the Scottish botanist Robert Brown (1828). More generally, he hoped to apply the kinetic theory of matter as a means of verifying the existence of atoms.

The methods of Bachelier and Einstein were not totally rigorous because a proper theory of probability was not available at the time. Wiener (1923) provided a more rigorous treatment of Brownian Motion, including showing that a Brownian path was nowhere differentiable. A more complete and rigorous definition of probability theory was developed by Kolmogorov (1931, 1933) that allowed proper formal treatment of many stochastic processes. Davis (2006) describes in detail Bachelier's thesis and further advances in probability, such as martingales and stochastic integration, made by people such as Feller, Lévy, Doob and Itô.

As much as he admired Bachelier's introduction of the mathematics of Brownian Motion, and his innovative application to finance, Samuelson (1972) was very critical of the financial application: as the individual steps formed a Gaussian random walk it meant negative prices and unlimited liability were both possible. Both of these problems disappear if you use instead Geometric Brownian motion, as introduced by Samuelson (1965).

Davis (2006) also describes the more complicated business of applying advances in probability theory to the economics of finance. This includes the famous Black-Scholes (1973) option pricing theory, based on Geometric Brownian motion, and Merton's (1973) generalisations, including allowing for stochastic interest rates and the idea that prices should be expressed in terms of some *numéraire asset* (in his case a zero-coupon bond).

2.2 Other diffusion processes

2.2.1 Ornstein–Uhlenbeck process.

Diffusion processes other than Brownian motion or Geometric Brownian motion have found application in Finance. This process, Uhlenbeck and Ornstein (1930), is an example of a Gaussian Markov process that, in contrast to the Wiener process, is *mean-reverting*. Its original application to Physics was to the motion of a Brownian particle subject to friction. It is the continuous-time analogue of the discrete-time AR(1) process. It satisfies the stochastic differential equation

$$dx_t = -\theta x_t dt + \sigma dW_t, \tag{1}$$

where W_t is a standard Wiener process. A drift term can be added to give the equation

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t, \tag{2}$$

which in Finance is known as the Vasicek (1977) model. Then the mean reversion property is

$$E(x_t \mid x_0) = \mu + (x_0 - \mu)e^{-\theta t}. \tag{3}$$

The process is commonly used to model interest rates and exchange rates. A multidimensional version of the model can be obtained simply by writing x_t, W_t as vector processes and σ as a matrix.

2.2.2 The Cox-Ingersoll-Ross model (CIR)

The Cox-Ingersoll-Ross (1985) model, often referred to as CIR, is obtained by a simple modification of the driving term in the Vasicek model, giving the stochastic differential equation

$$dx_t = \theta(\mu - x_t)dt + \sigma\sqrt{x_t}dW_t. \tag{4}$$

It is also used to describe the evolution of interest rates and the pricing of interest rate derivatives. Time varying functions replacing the fixed coefficients can be introduced in the model in order to make it consistent with a pre-assigned term structure of interest rates and volatilities, see for example Brigo and Mercurio (2001).

2.2.3 Heston model

Another approach models volatility itself as a separate but coupled process, hence getting a stochastic volatility class of models like the popular Heston (1993) stochastic volatility model.

$$\begin{aligned} dx_t &= rx_t dt + x_t\sqrt{V_t}\,dW_t^{(1)}; \\ dV_t &= k(\theta - V_t)dt + \sigma\sqrt{V_t}\,dW_t^{(2)}, \end{aligned} \tag{5}$$

where the two Wiener processes are possibly correlated. The second equation is a CIR model for the volatility. If the parameters obey the Feller condition ($2k\theta > \sigma^2$) then the volatility process V_t is strictly positive. A significant extension of both the CIR model and the Heston model to the case of stochastic mean and stochastic volatility is given by Chen (1996).

3. *Poisson and Lévy jump-diffusion*

Merton (1976) realized that diffusion could not describe some of the rapid changes that could be plainly seen from time to time in financial series. He introduced the idea that jumps in the form of a Poisson process, in conjunction with Geometric Brownian motion diffusion, gave a better description of many series and showed that option pricing could be improved by taking them into account. Cox and Ross (1976) showed that in this situation perfect hedging may still be achieved in certain circumstances.

Since then many papers have been written on Poisson jump-diffusions using some of the diffusions mentioned above to model various financial problems. The next step in modeling sophistication was to replace the Poisson process by more general Lévy jumps: an extensive treatment of their application to many financial problems is given in the comprehensive book by Cont and Tankov (2004, 2016). This excellent work combines many practical examples with clear explanations of the mathematical basis of their treatment.

However, Poisson and Lévy processes have well-known independence properties that do not explain the clustering of jumps that is often identified in financial series. This can to some extent be dealt with by underlying Markov processes which may have two or more states with different jump intensity levels, rather like a medical problem such as hayfever where more cases occur in certain weather/pollen conditions — the clustering is caused by changes in conditions, not by contagion (infection from other people). Jump intensity in financial series may well vary in periods of boom and bust, depression or high volatility.

Although much has been written about jump-diffusions of the kind discussed in this section, we will not discuss them further in this paper. Instead we concentrate on jump-diffusion processes in which the jumps have contagious characteristics: typically, various kinds of so-called Hawkes jump-diffusions.

4. Hawkes processes

Hawkes (1971a, 1971b, 1972) introduced a family of models for stochastic point processes called ‘self-exciting and mutually-exciting point processes’, the essential property of which was that the occurrence of any event increased the probability of further events occurring. This contagious behaviour is perhaps better understood in terms of a branching process or cluster process representation, (Hawkes and Oakes 1974). Although these processes did not become popular for many years, except among Seismologists, they became established in the literature under the name

Hawkes processes. Since about 2005, they have become popular in a very wide range of applications, including Finance. The mathematical properties of the original models have been extensively studied and the models themselves modified and extended.

Applications of Hawkes processes within the field of Finance are very diverse. We will not dwell on them here, but refer to the excellent survey by Bacry et al (2015) for an introduction to the theory and applications to Finance; see also Hawkes (2018). We give here a very brief definition of Hawkes processes before moving on to Section 5, where we discuss a very particular kind of application: Hawkes jump-diffusions.

4.1 Self-exciting point processes

An important property of a point process $N(t)$, defined as the number of events of some type in the time interval $(0,t)$, is the conditional intensity

$$\lambda_t = \lim_{\Delta\to 0} \mathbb{E}[N(t+\Delta)-N(t) \mid \mathcal{F}_t],$$

where the filtration $\mathcal{F}_t$ stands for the information available up to (but not including) time *t*. A fairly general self-exciting process can be defined in terms of an intensity of the form

$$\lambda_t = \mu(t) + \sum_{T_i<t} \gamma(t-T_i,\xi_i), \tag{6}$$

where $0<T_1<T_2<\ldots<T_n<\ldots$ are the times at which events occur. $\mu(t)$>0 provides a Poisson base level for the process: this may be an explicit function of time, possibly constant, or perhaps an exogenous economic function. The function $\gamma(t-T_i,\xi_i)\geq 0$, called the *(exciting) kernel* of the process, provides the contribution to the intensity at time *t* that is made by an event that occurs at a previous time $T_i<t$, which may have a *mark* ξ_i associated with it. Thus each event increases the intensity, which then decays according to the function γ until the next event occurs to push it up again.

Such a process is a marked self-exciting process, the mark being perhaps the magnitude of a price jump: for example, large price jumps may increase the intensity more than small jumps. A simple self-exciting process is one without marks, then the kernel becomes simply $\gamma(t-T_i)\geq 0$.

4.2 Mutually-exciting point processes

The processes become more useful when there are different types of events, such as price jumps for different stocks or different indices. Large (2007) identified 10 different types of event in an order book.

The basic model supposes that there are D different types of point process $\{N_i(t)\}_{i=1}^{D}$ with intensities given by

$$\lambda_{i:t} = \mu_i + \sum_{j=1}^{D}\int_0^t \gamma_{ij}(t-u)dN_j(u) = \mu_i + \sum_{j=1}^{D}\sum_{T_{j:r}<t}\gamma_{ij}(t-T_{j:r}), \tag{7}$$

where $T_{j:r}$ is the time at which the *r*th event of type *j* occurs. The function $\gamma_{ij}(\bullet)$ is a cross-exciting term with $\gamma_{ij}(t-T_{j:r})$ being the contribution to the intensity of type-*i* events made by a type-*j* event occurring at time $T_{j:r}$. $\gamma_{ii}(\bullet)$ is a self-exciting term for type-*i* events. This equation can also be expressed in matrix form as

$$\boldsymbol{\lambda}_t = \boldsymbol{\mu} + \int_0^t \boldsymbol{\Gamma}(t-u)\, d\mathbf{N}(u), \tag{8}$$

with the matrix $\Gamma(t-u)$ having elements $\gamma_{ij}(t-u)$ and the other terms being the obvious column vectors.

This basic model may be generalized with marks and exogenous factors in a manner similar to that for the univariate self-exciting model.

5. Hawkes jump-diffusions

As we discussed earlier, diffusion processes have often been used by themselves without reference to jumps. Similarly, Hawkes jump processes can be used without reference to any diffusion process. This may be particularly so in very high frequency Finance (such as tick-by-tick data) when every event may be thought of as a jump. Also, jumps may be extracted from otherwise continuous data and analysed separately.

In this section we will discuss the application of Hawkes jump-diffusion processes, i.e. some sort of Hawkes jump process incorporated in some sort of diffusion process of the kind discussed earlier. We shall be concerned with a number of questions:

- What type of diffusion?
- What are the details of the Hawkes jump model?
- How do the two processes work together?
- What financial problem are we trying to solve?
- Is the Hawkes jump-diffusion better at solving that problem than a simple diffusion, or Poisson jump-diffusion or Lévy jump-diffusion?

We cannot discuss every paper that deals with this topic but have chosen papers that between them deal with a wide range of financial processes. We cannot go deeply into the extensive and complex mathematics that is often involved, but try to explain clearly what models are being used and describe briefly some of the mathematical and computational methods that are used in solving the problems. Also, we tend to favour papers that are not totally theoretical, but include examples analysing real data and discussion of the implications for practical financial management.

When applying a jump-diffusion model to describe data series it will often, though not always, be necessary to identify the location of jumps in time and perhaps also the magnitude of the jumps. How this may be accomplished is an interesting topic but we will not address it here: if necessary, we just assume that the authors have somehow found a way to do it.

5.1 Interacting World markets

Aït-Sahalia et al. (2015) used a mutually-exciting jump-diffusion process to model market indices in six different parts of the World (US, UK, Japan, etc.). A jump in one region of the world increases the intensity of jumps both in the same region (self-excitation) as well as in other regions (cross-excitation), generating episodes of highly clustered jumps across world markets. The diffusion process is Geometric Brownian motion with stochastic volatility following a Heston process. Thus, for i=1 to m,

$$\begin{aligned} dX_{i,t} &= \mu_i dt + \sqrt{V_{i,t}}\, dW^X_{i,t} + Z_{i,t} dN_{i,t} \\ dV_{i,t} &= \kappa_i(\theta_i - V_{i,t}) + \eta_i \sqrt{V_{i,t}}\, dW^V_{i,t}. \end{aligned} \tag{9}$$

where subscript i refers to region i; W^X, W^V are independent multivariate Wiener processes; $N_{i,t}, Z_{i,t}$ are jump processes and jump magnitudes, respectively, with the $\{N_{i,t}\}\, i=1$ to m a set of mutually exciting Hawkes point processes with exponential kernels and therefore intensities satisfying differential equations

$$d\lambda_{i,t} = \alpha_i(\lambda_{i,\infty} - \lambda_{i,t})dt + \sum_{j=1}^{m} \beta_{i,j} dN_{j,t}. \qquad (10)$$

The jump magnitudes are assumed to have double exponential distributions, allowing for positive and negative jumps.

Data observed were simply daily returns, so that neither jumps nor detailed intra-day diffusions were actually observed. The models were fitted by a generalised method of moments (GMM) based on comparison of theoretical and observed moments such as first four moments, autocovariances and cross-covariances. This was a major task, so that in practice only univariate (m=1) and bivariate (m=2) cases were completed.

This procedure provided evidence of self-excitation in all six world markets, and of asymmetric cross-excitation, with the US market typically having more influence on the jump intensity of other markets than the reverse. The model clearly out performed one with Poisson jumps (zero βs in the above model). This was also shown by obtaining better tail distributions and predicted co-jumping: important for risk management and portfolio choice.

5.2 Pricing and hedging

Hainaut, D. (2016a) analysed the impact of volatility clustering in stock markets on the evaluation and risk management of equity indexed annuities (EIA). The reference index is modelled by a diffusion combined with a bivariate mutually-excited jump process. The index, S_t, has dynamics

$$\frac{dS_t}{S_t} = \mu_t + \sigma dW_t + (e^{J_t^+} - 1)dN_t^+ + (e^{J_t^-} - 1)dN_t^-, \qquad (11)$$

where N_t^+, N_t^- are, respectively, positive and negative jump processes with jump sizes J_t^+, J_t^-, each exponentially distributed. The clustering of shocks is modelled by mutually-exciting marked processes where the jump intensities have dynamics given by

$$d\lambda_t^i = \kappa_i(c_i - \lambda_t^i)dt + \delta_{ii} J_t^i dN_t^i + \delta_{ij} J_t^j dN_t^j, \text{ where } (i,j) = (+,-) \text{ or } (-,+). \qquad (12)$$

For example the intensity of positive jumps is excited both by previous positive jumps and by negative jumps, so that

$$d\lambda_t^+ = \kappa_+(c_+ - \lambda_t^+)dt + \delta_{++} J_t^+ dN_t^+ + \delta_{+-} J_t^- dN_t^-,$$

with a similar equation for the intensity of negative jumps obtained by interchanging + and – subscripts or superscripts.

The model is used to evaluate options embedded in simple variable annuities. The paper compares prices, one-year value at risk, and tail value at risk of simple EIAs, computed with different models.

Hainaut and Moraux (2019) combined a Hawkes jump-diffusion process with hidden Markov switching between three states. Within each state the diffusion is standard geometric Brownian motion. The jumps are driven by exponential Hawkes process with double exponential distribution of jumps sizes, so that the intensity is given by

$$d\lambda_t = \alpha(\theta_t - \lambda_t) + \eta dL_t, \qquad (13)$$

where L_t is the accumulated absolute jump size $L_t = \sum_{i=1}^{N_t} |J_i|$. (14)

However, the parameters of these processes are different within the three hidden Markov states, but for reasons of simplicity the parameters α, η and the distribution of jump sizes are taken to be the same for all three states. Then

$$d\ln S_t = \left(\mu_t - \frac{\sigma_t^2}{2} - \lambda_t E(e^J - 1) \right) dt + \sigma_t dW_t + JdN_t \quad . \tag{15}$$

The model is calibrated on daily returns of S&P 500 data using a complex iterative method that obtains initial parameter estimates from a peaks-over-threshold (POT) approach followed by a sequential Markov chain particle filter. The first state is judged to be one of strong economic growth: it has small volatility σ and average return on stocks of 23.5%. The third state is taken to be one of economic recession. with volatility above 40% and average stock return -38%. The second state is intermediate between these two. The self-exciting jumps occur mainly in times of economic recession and are rare in periods of economic growth. Jumps are estimated to be responsible for 4%, 25% of volatility in states two and three respectively but contribute virtually nothing in the first state of economic growth.

An analysis using AIC and BIC confirms that the model fits the data better than either a simple, non-switching, Hawkes jump-diffusion or a switching Geometric Brownian motion, without jumps.

The paper ends with a substantial discussion of the application of the model to option pricing, the details of which are beyond the scope of this review.

Hainaut and Moraux (2018) use a simpler model with no switching between hidden economic states. Thus the above equations still hold, but the parameters μ, σ, θ remain constant instead of varying with time. Using daily returns of S&P 500 from 2005 to 2015 it is found that the volatility of the Brownian part is about 12% whereas, if the pure diffusion is fitted without jumps, the volatility rises to 21%: so the jumps contribute considerably to the volatility.

The main core of the paper is concerned with option prices and hedging strategies under this model and, in particular, the effect of contagious jumps on these. To this end a family of affine changes of measure is obtained that preserves the dynamics of prices under the risk neutral measure. There follows a substantial theoretical development that forms the basis of calculating pricing and hedging strategies, and subsequent simulation studies.

The practical problems to be thus studied are minimum variance hedging of an option using either the underlying asset or another derivative as the hedging instrument. The first conclusion is that jump clustering can cause huge losses in the absence of any hedge.

Secondly, hedging with the underlying does little to mitigate exposure to the risk of jump clustering; also there is not much difference between minimum variance hedging and delta hedging, except when the jump intensity is high.

Third, the situation may be much better if the hedging instrument is another derivative, typically with a longer maturity than the one we aim to protect. In this situation the minimum variance hedging strategy clearly outperforms a pure delta hedging strategy.

Ma et al (2017) considered vulnerable options in which the buyer is exposed to the default risk of the option writer, an example of counterparty risk. In order to price such an option Hawkes jump-diffusion models are proposed both for the price of a stock and the value of an option-writer's assets. The former includes two kinds of independent self-exciting jumps: one a sequence of jumps, N_t, arising from a systematic shock, such as a financial crisis, which influence all entities in the financial system; the other are idiosyncratic shocks, $N_t^{(1)}$, for the particular stock. A similar model for the option-writer's assets includes the same systematic shocks and a third set of shocks, $N_t^{(2)}$.

The intensities of these jump processes satisfy simple Hawkes models

$$\lambda_t = \lambda_0 + \theta \int_0^t e^{-\delta(t-s)}\, dN_s;$$
$$\lambda_t^{(1)} = \lambda_0^{(1)} + \theta_1 \int_0^t e^{-\delta_1(t-s)}\, dN_s^{(1)}; \quad \lambda_t^{(2)} = \lambda_0^{(2)} + \theta_2 \int_0^t e^{-\delta_2(t-s)}\, dN_s^{(2)}. \tag{16}$$

The stock price, S_t, and the value of the option writer's assets, A_t, satisfy jump-diffusions

$$\frac{dS_t}{S_{t-}} = (r - k_S(\lambda_t + \lambda_t^{(1)}))dt + \sigma_S dW_t^{(1)} + d\sum_{i=1}^{N_t + N_t^{(1)}} Y_i;$$
$$\frac{dA_t}{A_{t-}} = (r - k_A(\lambda_t + \lambda_t^{(2)}))dt + \sigma_A dW_t^{(2)} + d\sum_{i=1}^{N_t + N_t^{(2)}} Z_i. \tag{17}$$

r is a risk-free rate; $W_t^{(1)}, W_t^{(2)}$ are standard Wiener processes with correlation ρ. $\{Y_i\}$ is an i.i.d. sequence of jump sizes with mean k_S, that apply to both systematic and idiosyncratic jumps in the stock price, $\{Z_i\}$ is an i.i.d. sequence of jump sizes with mean k_A, that apply to both systematic and idiosyncratic jumps in the value of the option writer's assets.

The main aim of the paper is to price a put or call European option at time t. It is assumed that the option writer can only default at the strike time T, and does so if her/his assets are less than some critical value at that time. Formulae are given for these prices that reduce to the usual Black-Scholes formula as a special case when jumps do not occur. The general formulae depend on certain expectations that are not analytically evaluated: instead the jump processes are simulated and the expectations calculated from the consequent paths of the jump-diffusions.

Using these simulations, and assuming that any jumps have log-Normal distributions of jump size, the paper concludes with an extensive numerical study of the performance of the full model, and compared with various models (Black-Scholes, Merton, Klein, Tian) that either have Poisson jumps (therefore no jump clustering) or no jumps at all.

5.3 Portfolio optimization

Aït-Sahalia and Hurd (2016) considered a portfolio of assets represented by the standard mutually independent Geometric Brownian processes with constant volatilities and a mutually-exciting set of jump processes with exponential kernels. Only negative asset jumps are included, as these are considered to be the most important from both a portfolio risk management perspective and their contribution to mutual excitation. In this theoretical paper the main conclusion is that the optimum portfolio for an investor with log-utility would be to choose the optimum investment as if the jumps followed Poisson processes with constant intensities — except that those 'constant' intensities are changing all the time and so, theoretically, the portfolio should be continuously rebalanced (with no transaction costs).

Bian et al (2019) consider a simple portfolio of a risky asset and a risk-free asset over a period $(0,T)$ and optimize the portfolio by maximizing the expectation of a constant relative risk aversion (CRAA) utility function of terminal wealth $U(X_T)$, where $U(x) = x^p / p$. The exciting process is a simple exponential Hawkes process with dynamics

$$d\lambda_t = \alpha(\lambda_\infty - \lambda_t) + \beta dN_t. \tag{18}$$

Note that with β>0 the intensity jumps at the same time as N. The somewhat unusual jump-diffusion of the risky asset incorporates the jumps into a Geometric Brownian motion with stochastic volatility having dynamics

$$\frac{dS_t}{S_t} = \left[\mu(\lambda_t) + r\right]dt + \sigma(\lambda_t)dW_t + Y_t dN_t, \tag{19}$$

where the jump size $Y_t = e^{Z_t} - 1$ and Z_t is a Gaussian random variable. Note that the risk premium μ and the volatility σ are functions of the intensity λ_t , and therefore jump at the same time as the price of the risky asset. If π_t is the proportion of wealth invested in the risky asset at time t, an investment strategy is an adapted stochastic process $\{\pi_t\}_{t\in[0,T]}$. A proof of existence and uniqueness of an optimal strategy is given under general conditions on the functions $\sigma(\bullet), \mu(\bullet)$,

The paper seems to be essentially an exercise in detailed, complicated mathematical analysis. There is no attempt to confront the unusual model with actual financial data.

5.4 Bayesian analysis

5.4.1 Bayesian learning

Fulop *et al* (2014) propose an asset pricing model that takes into account cojumps between prices and diffusive volatility. Unusually, price jumps are excited only by the negative jumps: thus the jump intensity (for both positive and negative jumps) satisfies

$$d\lambda_t = k_2(\theta_2 - \lambda_t)dt - \beta_2 dJ_t^-, \tag{20}$$

where J_t^+, J_t^- are cumulative magnitudes of positive and negative jumps respectively. The negative jumps also contribute to a CIR process for the diffusive volatility

$$dV_t = k_1(\theta_1 - V_t)dt + \sigma\sqrt{V_t}dW_t - \beta_1 dJ_t^-. \tag{21}$$

A Bayesian learning approach is used to implement real time sequential analysis. This is quite complex as it involves simultaneously updating beliefs both about many state variables and about model parameters — also about model beliefs if a number of alternative models are contemplated. The analysis was based on daily returns of S&P 500 from January 2, 1980 to December 31, 2012.

The model has closed-form conditional expectations of the volatility components, making it convenient for risk management, volatility forecasting and option pricing. The learning process easily enables study of how the process changes over time: for example the self-exciting jump intensity clearly becomes more important since the 2008 financial crisis, leading to high levels of volatility during periods of financial crisis. This is particularly important in pricing short maturity out-of-money options.

The speed of learning for the diffusion parameters is considerably faster than that for the jump parameters, because of the low arrival rate of extreme events. There is strong asymmetry in the amount of learning over the tails of the return distribution: the left tail is soon well established but there is much uncertainty about the right tail behavior throughout the sample.

The full model clearly outperforms three sub-models obtained by omitting aspects of the jump process. However, from the point of view of predicting volatility, it cannot beat the simple GARCH(1,1) process.

The paper contains many impressive and interesting figures.

5.4.2 Bayes analysis of daily data together with some intra-day measures

Maneesoothorn *et al* (2017) discuss a Heston volatility diffusion with Hawkes jumps in both log-price and volatility. Thus the basic equations are

$$\begin{aligned} dp_t &= \left(\mu + \gamma V_t\right) + \sqrt{V_t}dB_t^p + dJ_t^p, \\ dV_t &= \kappa(\theta - V_t) + \sigma_v\sqrt{V_t}dB_t^v + dJ_t^v, \end{aligned} \tag{22}$$

where p_t is log-price; B_t^p, B_t^v are standard Brownian motions for log-price and volatility, respectively, with $corr(dB_t^p, dB_t^\upsilon) = \rho dt$; log-price and volatility jumps are denoted $dJ_t^i = Z_t^i dN_t^i$, for $i = (p, \upsilon)$, respectively. The additional term γV_t in the first of these equations allows for a volatility feedback (i.e. the impact of volatility on future returns).

The self-exciting price jumps, have intensity satisfying that occur at times given by the counting process N_t^p,

$$d\lambda_t^p = \alpha_p(\lambda_\infty^p - \lambda_t^p) + \beta_{pp} dN_t^p. \tag{23}$$

Volatility jumps are self-exciting but are also generated differentially by positive and negative price jumps: thus, if a subset of price jumps with negative sizes is counted by the process whose jumps are denoted by $dN_t^{p(-)} = dN_t^p \mathbf{1}(Z_t^p < 0)$, then jump times of volatility are given by the counting process N_t^υ whose jump intensity satisfies

$$d\lambda_t^\upsilon = \alpha\ (\lambda_\infty^\upsilon - \lambda_t^\upsilon)dt + \beta_{\upsilon\upsilon} dN_t^\upsilon + \beta_{\upsilon p} dN_t^p + \beta_{\upsilon p}^{(-)} dN_t^{p(-)}. \tag{24}$$

The sizes of volatility jumps, Z_t^υ, are exponentially distributed. The absolute magnitudes of price jumps have a log-normal distribution with mean proportional to volatility; the jump signs (+1 or -1) are determined independently by a simple Bernoulli probability.

A discretized version of the model, having many parameters, is applied to daily open-to-close log-return of S&P 500 from 1996 to 2014. It is supplemented by intra-day measures based on 5-minute data, calculating volatility as measured by Bipower Variation (BV) and identifying at most one jump per day in terms of BV, realized variation (RV) and tripower quarticity as described in Barndorff-Nielsen and Shephard (2004, 2006). A Bayes analysis is calibrated by a hybrid of Gibbs and Hasting-Metropolis MCMC algorithms.

There is significant self-excitation of both return jumps and volatility jumps, but the cross-excitement from return jumps to volatility jumps is small. Volatility jumps tend to occur more frequently in periods of high volatility, notably in well-known crisis periods.

Using Bayes factors to compare the overall fit of the full model with 9 sub-models of the full model, together with two GARCH models, strongly supports the inclusion of Hawkes jumps. However, omitting the effect of price jumps on the intensity of volatility jumps gives a slightly better overall fit.

In terms of 1-step-ahead prediction of returns, jumps and volatility (log-bipower variation), models including Hawkes jumps predominate. However a simple log-RGARCH model is overall best at predicting returns, but not volatility, and is unable to predict anything at all about jumps.

The authors have completed a major exercise in data analysis, and the broad conclusions seem reasonable. However, the use of BV to estimate volatility and identify simply 0 or 1 jumps each day is very dubious in data, like S&P 500, which has contagious jumps that not infrequently lead to having several price jumps in a day (see Corsi et al, 2010).

5.5 Default risk

Errais, Giesecke and Goldberg (2010) is different from other work discussed in this paper: they start with a jump process and then add diffusion. The Hawkes process satisfies the equation

$$\lambda_t = c + e^{-\alpha t}(\lambda_0 - c) + \beta \int_{s=0}^{t} e^{-\alpha(t-s)}\, dL_s, \tag{25}$$

or

$$d\lambda_t = \alpha(c - \lambda_t)dt + \beta dL_t. \tag{26}$$

L_s is the cumulative loss from defaulting companies in a large portfolio of companies, therefore it jumps at the same time as the jump count N_s, which has intensity λ_s — so this is actually a marked Hawkes process with exponential kernel. The main part of the paper obtains various distributions associated with this process and then discusses pricing of index swaps and tranche swaps.

Then it generalizes by adding some diffusion. It is simplest to see this from the differential form of the above equation, which now becomes

$$d\lambda_t = \alpha(c-\lambda_t)dt + \beta dL_t. + \sigma\sqrt{\lambda_t}dW_t. \tag{27}$$

The diffusion represents some macroeconomic fluctuation. So in this model it is the jumps, representing defaults, that are important; the diffusion, instead of representing some observable such as log-price, contributes to the variation in the intensity of defaults. In that sense it is just a particular example of equation (6).

5.6 *Interest rate models*

In this section we discuss two papers on interest rate models.

Hainaut (2016b) introduces a model to reproduce the clustering of shocks on the Euro overnight index average (EONIA) between 2004 and 2014, so data are observed at a daily frequency. These jumps are mainly caused by successive adjustments of rates for deposit and marginal lending facilities offered by the central banks. The clustering of these adjustments is partly explained by the emergency of such decisions in periods of economic crisis.

The short-term interest rate r_t is assumed to satisfy

$$dr_t = \alpha(\theta(t) - r_t)dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} J_i\right), \tag{28}$$

where $\theta(t)$ is the mean level to which interest rates tend to revert; W_t is a Wiener process with volatility σ; N_t is a Hawkes process of jumps whose jump sizes J_i are assumed to be i.i.d. with a double exponential distribution. The intensity of the Hawkes jumps is assumed to be

$$\lambda_t = c + e^{-\kappa t}(\lambda_0 - c) + \delta\int e^{-\kappa(t-s)}\, dL_s, \tag{29}$$

where $L_t = \sum_{i=1}^{N_t}|J_i|$ is the sum of the absolute jump sizes up to time t.

Theory is developed for a class of measures under which the features of the process are preserved. In particular the prices of bonds and their dynamics under a risk neutral measure are obtained and derivative pricing is obtained under a forward measure.

The model is fitted to EONIA data using a peaks-over-threshold analysis, which enables filtration of jumps and the intensity of their arrival process: the threshold is obtained by fitting the original reverting diffusion without jumps, i.e. a Vasicek model. Then the Brownian process, the jump intensity process and the jump distribution were fitted by three separate log-likelihood maximisations.

The clustering of jumps was clearly illustrated, showing the importance of including self-exciting jumps in the model. A variety of other conclusions for the EONIA data include the reduction of EONIA and the reversion level $\theta(t)$ since 2010; The average amplitude of jumps has also decreased over time, reducing the overall volatility of rates under the pricing measure.

Sun et al (2019) approached the problem with a slightly different model and a different mathematical approach. The basic model equations were

$$dr_t = a_1(t,r_t)dt + b_1(t,r_t)dW_1(t) + \int c_1(t,r_t,x)N(dt,dx)$$
$$d\lambda_t = a_2(t,\lambda_t)dt + b_2(t,\lambda_t)dW_2(t) + \int c_2(t,\lambda_t,x)N(dt,dx), \tag{30}$$

where $W_1(t)$, $W_1(t)$ are independent standard Brownian motions and the model coefficient functions a_1, b_1, c_1, a_2, b_2, c_2 are fairly general functions. $N(t,A)$ is the number of jumps whose jump times are less than or equal to t and the jump sizes are in the region A. This differs from equations (28), (29) in a having more general set of coefficients; in having a Brownian component to the dynamic of jump intensity; also the self-exciting effect of jumps on jump intensity is proportional to jump size rather then absolute value of jump size.

Girsanov's Theorem is used to obtain the dynamics of r_t, λ_t under an equivalent martingale measure $\mathbb{Q}$. Then the price at time t of a zero-coupon bond expiring at time T is given by

$$P(t,T) = E^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s\, ds\right) | \mathcal{F}_t\right], \tag{31}$$

where expectation is taken under the equivalent martingale measure $\mathbb{Q}$. It is shown that this has a unique solution

$$P(t,T) = \exp\left[A(t,T) + B(t,T)r_t + C(t,T)\lambda_t\right],$$

where A, B, C are determined by a set of three ordinary differential equations. These usually have no analytic solution but are solved by standard numerical methods for differential equations.

The paper contains no data, therefore no test of whether the model is a reasonable representation of any particular process, but ends with a numerical study that illustrates some properties of the model.

5.7 *A C.I.R. process with exogenous and endogenous jumps*

Dassios et. al. (2019) introduced a generalised C.I.R. process that might be suitable for describing aggregate losses for insurance companies or interest rates in the fixed-income markets. The dynamics are supposed to follow

$$S_t = a + (S_0 - a)e^{-\delta t} + \sigma \int_0^t e^{-\delta(t-s)}\sqrt{S_s}\, dW_s + \sum_{0 \le T_i^{(X)} < t} X_i e^{-\delta(t-T_i^{(X)})} + \sum_{0 \le T_j^{(Y)} < t} Y_j e^{-\delta(t-T_j^{(Y)})}, \tag{32}$$

where the symbols in the diffusive part are those used throughout this review; $\{X_i\}_{i=1,2,\ldots}$ is a sequence of i.i.d. positive sizes of exogenous jumps, with distribution function $H(x)$, $x > 0$, that occur at random times $\{T_i^{(X)}\}_{i=1,2,\ldots}$ following a Poisson process $N_t^{(X)}$ of constant rate ρ; $\{Y_j\}_{j=1,2,\ldots}$ is a sequence of i.i.d. positive jump sizes, with distribution function $G(y)$, $y > 0$, that occur at random times $\{T_j^{(Y)}\}$ of a point process N_t that has a stochastic intensity

$$\lambda_t = S_t. \tag{33}$$

From equation (32) we see that the intensity of N_t is driven by the diffusion and the marked point processes $N_t^{(X)}$ and N_t itself, so it is self-exciting and is also excited by the exogenous jumps.

By applying standard martingale theory, the joint Laplace transform of S_t and its integrated process $Z_t = \int_0^t S_u \, du$ is obtained. This enables calculation of means and variances of losses, and hence premiums, for a variety of insurance models that are special cases of the general model. An application to finance is to calculate prices for default-free zero-coupon bonds. Numerical examples with/without jumps and with/without diffusion are presented. It is recognized that estimating parameters of the models is difficult, so that no applications to real data are presented.

5.8 Variance swaps

Liu and Zhu (2019) added Hawkes jumps to a Heston model

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= (r - m\lambda_t)dt + \sqrt{V_t}\,dB_t^S + J_t dN_t \\ dV_t &= \kappa(\theta - V_t) + \xi\sqrt{V_t}\,dB_t^V \\ d\lambda_t &= \beta(\lambda_\infty - \lambda_t)dt + \alpha dN_t; \end{aligned} \tag{34}$$

where S_t is the price of an asset and V_t its instantaneous variance. The Brownian terms for price and variance may be correlated. The jump occurrences form a simple exponential Hawkes process and $m = E(J)$ is the mean jump size. They further assume that $J_t = e^{Z_t} - 1$, where $Z_t \sim N(\mu_z, \sigma_z^2)$ are *i.i.d.* so that $m = e^{\mu_z + \frac{1}{2}\sigma_z^2} - 1$.

The main object of the paper is to obtain a relatively simple formula for pricing a variance swap using a discretized version of the model. The idea is to balance a lump sum payment at a fixed future time and accumulated payments at regular intervals that are a given multiple of the realized volatility over each interval.

Equivalent results are easily obtained as special cases of the above model, including for classical models such as Black-Scholes; Merton process with Poison jumps, Heston process with or without Poisson jumps. Jump clustering that occurs with Hawkes processes naturally leads to higher prices for variance swaps, emphasising the need to identify the best model for your data in order to obtain realistic prices.

5.9 *Exchange options*

Ma et al (2020) discuss exchange options. These offer the holder the right, but not the obligation, to exchange one risky asset for another. This basic idea is incorporated in some way in many financial arrangements. For example, in mergers and acquisitions the bidding firm often offers to exchange some of its own stocks for those of the target firm. In the case of a possible exchange between a stock and a market index, the exchange option price will include information about the correlation between them — hence a measure of beta.

The prices of the two assets are assumed to follow geometric Brownian motions with Hawkes process jumps, so the dynamics are given, in the usual notation, by

$$
\begin{aligned}
&S_{j,t} = S_{j,0}\exp(\mu_j t + X_{j,t}),\\
&dX_{j,t} = \left(-\tfrac{1}{2}\sigma_j^2 - \xi_j\lambda_{j,t}\right)dt + \sigma_j dW_{j,t} + dJ_{j,t},\ \ j = 1,2,
\end{aligned}
\tag{35}
$$

where $dJ_{j,t} = \varepsilon_{j,t}dN_{j,t}$, $N_{j,t}$ are counting processes for number of jumps and $\varepsilon_{j,t}$ are random jump sizes from a distribution function $F_j(y)$; $\xi_j = \int_{-\infty}^{\infty}(e^y - 1)dF_j(y)$ are the mean price jump sizes. The two Brownian terms are assumed correlated.

The jump occurrence times $N_{j,t}, j = 1,2,$ are assumed to be mutually-exciting Hawkes processes in which the intensities may take one of two possible forms, with or without jump sizes. Then

$$
\begin{aligned}
&d\lambda_{j,t} = \eta_j(\lambda_{j,\infty} - \lambda_{j,t})dt + \sum_{l=1}^{2}\theta_{j,l}dN_{l,t},\ \ \text{or}\\
&d\lambda_{j,t} = \eta_j(\lambda_{j,\infty} - \lambda_{j,t})dt + \sum_{l=1}^{2}\theta_{j,l}dJ_{l,t}.
\end{aligned}
\tag{36}
$$

In these equations all symbols not previously defined are constants. In general, under either assumption, when a jump occurs in one asset price the jump intensity of both processes increase immediately then decay exponentially at rate $\eta_j, j = 1,2.$

Theoretical expressions are obtained for the joint characteristic function of the processes $(X_{1,t}, X_{2,t})$ that depends on functions $A(t)$, $B(t)$ that satisfy a set of ODEs. These generally have no analytical solution but can be solved numerically using standard Runge-Kutta methods.

These functions also appear in a Fourier transform approach to pricing an exchange option under a risk neutral measure. The paper then obtains the optimal mean-variance hedging strategy and a set of Greeks (Delta, Theta, Vega, Gamma). Simulation is used to study the effect of model parameters on option prices.

Next an option-implied beta is obtained when the exchange option is between an asset and a market index. Again simulation is used to study the effect of model parameters on implied betas from the Hawkes jump diffusion, Poisson jump diffusion and Black-Scholes diffusion models.

Finally, data from stocks of eight large financial companies were compared with S&P 500 index. It was concluded that the prediction performance of the Poisson-diffusion implied beta was better than the beta of historical data and the implied beta of the Hawkes jump-diffusion model was better still. This was especially so during financial crisis of 2008-2009 when clustering of jumps was particularly obvious. It was concluded that superior prediction of implied beta under the Hawkes jump-diffusion model, especially during a financial crisis, would enable investors to manage portfolios and systematic risk more effectively.

6. Summary

This paper has discussed some examples of jump-diffusion processes in finance, in which the jumps are self-exciting or mutually-exciting Hawkes processes. They show that, compared to traditional jump-diffusions, this may improve model fit, forecasting, pricing, hedging or portfolio management. There are clearly many more financial problems that could be treated in this way.

The Hawkes processes used in the models in this review generally have exponential kernels, with or without marks. This is understandable in view of their relative simplicity. When using Hawkes process in finance it has often been found that power law kernels fit better than exponential: however, this may not be true in the diffusion environment and the extra mathematical complexity in an already complex problem is probably not worth it.

A variety of mathematical methods have been used to solve the problems in this review, including Laplace or Fourier transforms, martingale theory, changes of measure, numerical solution of ordinary differential equations, simulation, etc. Then there is the problem of estimating model parameters when you want to apply the model to actual data. Various possibilities include method of moments, maximum likelihood (usually difficult), simulation, peaks-over-threshold, particle filters etc. A systematic comparison of the efficiency of various mathematical and statistical methods would be interesting. Bayes methods, including statistical learning, are perhaps heavy on computation but offer a systematic way of following changes over time in the process dynamics that occur due to changes in economic conditions and market technology.

Incorporating Hawkes processes into diffusion models usually tends to make the resulting mathematics and computation considerably more difficult: Thus it is a good idea to check whether your particular data shows obvious signs of jump-clustering. If not, you may well be better off using simple Lévy or Poisson jumps, or perhaps no jumps at all. However, the papers discussed in this review clearly demonstrate that when jumps show clustering or contagion effects, including Hawkes processes into the model usually leads to significant improvements. It is hoped that research in this area will continue to advance and develop efficient computer programs that could be used on a routine commercial basis.

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