Algebraic Properties of Parikh Matrices of Binary Picture Arrays

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Abstract

A word is a finite sequence of symbols. Parikh matrix of a word is an upper triangular matrix with ones in the main diagonal and non-negative integers above the main diagonal which are counts of certain scattered subwords in the word. On the other hand a picture array, which is a rectangular arrangement of symbols, is an extension of the notion of word to two dimensions. Parikh matrices associated with a picture array have been introduced and their properties have been studied. Here we obtain certain algebraic properties of Parikh matrices of binary picture arrays based on the notions of power, fairness and a restricted shuffle operator extending the corresponding notions studied in the case of words. We also obtain properties of Parikh matrices of arrays formed by certain geometric operations.

1 Introduction

"Combinatorics on words" [16] is a comparatively new branch of Discrete Mathematics with applications in many fields. The work [9] of the Norwegian mathematician Axel Thue (1863-1922) is considered to be the origin for the beginning of this new branch of Mathematics. A finite word or simply a word is a finite sequence of symbols in a finite set called an alphabet. The Parikh vector [20] of a finite word, which has played a significant role in the theory of formal languages [20], expresses a numerical property of the word by counting the number of occurrences of the different symbols in the word.

The recently introduced notion of Parikh matrix [19] of a word over an ordered alphabet is an extension of the Parikh vector. Parikh matrix of a word, which is based on subwords (also called scattered subwords) of the word, is a very interesting and effective tool in the study of certain numerical properties of the word. Intensive work (see, for example, [4, 6, 14, 18, 22, 24, 25]) has taken place in investigating properties of words based on associated Parikh matrices. Such theoretical studies have dealt with problems of great interest related to words such as inequalities on the numbers of occurrences of subwords, injectivity of the mapping involved in defining the Parikh matrix and other directions [21]. An application of Parikh matrix in message authentication is considered in [5].

On the other hand, a picture array or simply an array, having a rectangular arrangement of symbols in rows and columns, is an extension of a word to two-dimensions (2D) [20]. Several combinatorial properties of arrays have also been intensively investigated [1, 2, 3, 10, 11, 12, 15]. For instance, notions such as repetitions of subarrays in 2D arrays are studied in [2, 3, 10, 12] while periodicity in arrays are dealt with in [1, 11]. The notion of Parikh matrix of a word has been extended to row and column Parikh matrices of picture arrays in [23] and their properties have been studied. The problem of reconstruction of 2D binary images has been studied [17] based on Parikh matrices.

Here we consider binary picture arrays and establish properties of the Parikh matrices of power of an array, fairness of an array and a restricted shuffle operator on arrays, by extending the corresponding notions [7, 13, 15] investigated in the case of words. We also obtain properties of Parikh matrices of arrays formed by certain geometric operations. A preliminary version of this work was presented in the conference MICOPAM 2018 [8].

2 Preliminaries

For notions of formal string language theory and two-dimensional languages, not explained here, the reader is referred to [20]. We recall only some basic notions.

A set Σ, called an alphabet, is a finite set of symbols. A word w over Σ is a finite sequence of symbols over Σ. The set of all words over Σ is denoted by Σ^* and λ is the empty word with no symbols. An alphabet $\Sigma = \{a_1, a_2, \cdots, a_k\}$ with an order $a_1 < a_2 < \cdots < a_k$ defined on it, is called an ordered alphabet and we write $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$. A word u is said to be a scattered subword (or simply subword) of a word $w \in \Sigma^*$ if there exist words $x_1, x_2, \cdots, x_n, y_0, y_1, \cdots, y_n \in \Sigma^*$ (possibly empty) such that $u = x_1x_2 \cdots x_n$ and $w = y_0x_1y_1 \cdots y_{n-1}x_ny_n$. The length of a word $w \in \Sigma^*$, denoted by $|w|$, is the number of symbols present in w. The number of occurrences of a word u as a subword of w is denoted by $|w|_u$.

A picture array (or simply an array) A over Σ of size $m \times n$, $m, n \geq 1$ is a rectangular arrangement of symbols in Σ in m rows and n columns. For example, $\begin{matrix} a & b & a \\ b & a & b \end{matrix}$ is a 2×3 binary array over the binary alphabet $\Sigma = \{a, b\}$. We denote the set of all $m \times n$ arrays over Σ by $\Sigma^{m \times n}$. If $X \in \Sigma^{m \times n}$, we denote by $|X_i|_x$, the number of symbol x in the i^{th} row (or in the i^{th} column) X_i of array X and by $|X|_x$, the sum $\Sigma_{i=1}^{m} X_i$. For two arrays X and Y with the same number of rows (respy. columns), the column (respy. row) catenation $X \circ Y$ (respy. $X \diamond Y$) is the array obtained by juxtaposing the array Y on the right (respy. below) the array X.

Throughout the rest of the paper we consider only a binary ordered alphabet Σ and binary arrays over Σ unless specified otherwise. We now recall the definition of Parikh matrix mapping [19] restricting it to a binary alphabet. Let $\mathcal{M}_3$ be the monoid of 3×3 upper triangular matrices with non-negative integer entries and unit diagonal with respect to multiplication of matrices. The unit 3×3 matrix is denoted by I_3. For a matrix $M \in \mathcal{M}_3$, the $(i, j)th$ entry is denoted by M_{ij}.

Definition 2.1 *[19] Let $\Sigma = \{a_1 < a_2\}$ be an ordered alphabet. The Parikh matrix mapping, denoted by ψ_3, is the morphism: $\psi_3 : \Sigma^* \longrightarrow \mathcal{M}_3$ defined as follows: $\psi_3(\lambda) = I_3$ and for $1 \leq k \leq 2$, $\psi_3(a_k) = (M_{ij})_{1 \leq i,j \leq 3}$ where $M_{ii} = 1$ for $1 \leq i \leq 3$, $M_{k(k+1)} = 1$ and all other entries are zero. For a word $w = w_1w_2 \cdots w_n$ with $w_i \in \Sigma$, the Parikh matrix of w is given by $\psi_3(w) = \psi_3(w_1)\psi_3(w_2) \cdots \psi_3(w_n)$.*

If $M_1, M_2 \in \mathcal{M}_3$ are two matrices, then the partial sum $M = M_1 \oplus M_2$ is defined [18] as the usual sum of matrices M_1 and M_2 except that the diagonal entries of M by definition have the value 1.

3 Row and Column Parikh Matrices of a Binary Picture Array

The notion of Parikh matrix of a word has been extended to a picture array in [23] by introducing row Parikh matrix and column Parikh matrix of an array, which we recall now again restricting to a binary alphabet.

Definition 3.1 *Let $\Sigma = \{a_1 < a_2\}$ and the array $A \in \Sigma^{m\times n}$. Let the word in the i^{th} row of A be x_i, $1 \le i \le m$ and the vertical word in the j^{th} column of A be y_j, $1 \le j \le n$. Let the Parikh matrices of x_i and y_j be respectively $M(x_i)$, $1 \le i \le m$ and $M(y_j)$, $1 \le j \le n$. Then the row Parikh matrix $M_r(A)$ of A is defined as $M_r(A) = M(x_1) \oplus \cdots \oplus M(x_m)$ and the column Parikh matrix $M_c(A)$ of A is defined as $M_c(A) = M(y_1) \oplus \cdots \oplus M(y_n)$.*

As an illustration, consider the array $A = \begin{matrix} a & b & a \\ b & a & b \end{matrix}$. Denoting the words in the rows as $u = aba$ and $v = bab$, the row Parikh matrix of A is $M_r(A) = M(u) \oplus M(v) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

We first obtain a property of the row (respy. column) Parikh matrix of a binary picture array, extending a corresponding property [18] of the Parikh matrix of a binary word.

Theorem 3.2 *For integers $m, n(\ge 1)$, suppose $M = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_3$. If M is the row (respy. column) Parikh matrix of an $m \times n$ binary array A, then $r + s = mn$ and $t \le nr - \sum_{i=1}^{m} r_i^2$ (respy. $t \le mr - \sum_{i=1}^{n} c_i^2$), where $|A_i|_a = r_i$ $(1 \le i \le m)$ (respy. $|A_i|_a = c_i$ $(1 \le i \le n)$) with A_i being the i^{th} row (respy. column) of A.*

Proof.
We prove the result only for row Parikh matrix as the result for column Parikh matrix can be proved in a similar manner. Let M be the row Parikh matrix of an $m \times n$ binary array A. Then A has mn symbols, r a's and s b's, so that $r + s = mn$. Let $|A_i|_a = r_i$ with A_i being the i^{th} $(1 \le i \le m)$ row of A. Then $\sum_{i=1}^{m} r_i = r$, and the number of b's in the i^{th} row is $(n - r_i)$. Therefore the maximum number of ab's in the i^{th} row is $r_i(n-r_i)$. Thus the maximum number of ab's in the row Parikh matrix of A is $\sum_{i=1}^{m} r_i(n - r_i)$ so that $t \le nr - \sum_{i=1}^{m} r_i^2$.

Corollary 3.3 *Let M be as in Theorem 3.2. If M is the row (respy. column) Parikh matrix of an $m \times n$ array, then $r + s = mn$ and $t \le nr - \frac{r^2}{m}$ (respy. $t \le mr - \frac{r^2}{n}$).*

This result follows from Theorem 3.3 by the Cauchy Schwarz inequality $\sum_{i=1}^{m} r_i^2 \geq \frac{1}{m}(\sum_{i=1}^{m} r_i)^2$.

4 Parikh Matrix of Power of an Array

Parikh matrix of a word w raised to an arbitrary power, denoted as w^p, for an integer $p \geq 1$ has been studied in [7]. Here we consider power of an array which has been introduced in [15].

Definition 4.1 *Let A be an $m \times n$ array. Then $p \times q$ power of A, denoted by $A^{(p\times q)}$, is the $pm \times qn$ picture array such that $A_{ij}^{(p\times q)} = A_{(i \ mod \ m)(j \ mod \ n)}$, for all $1 \leq i \leq pm$ and $1 \leq j \leq qn$.*

Example 4.2 *Let $A = \begin{matrix} a & a & b \\ b & a & b \end{matrix}$ be an 2×3 two dimensional array. The 2×4 power of A is given by $A^{(2\times 4)} = \begin{matrix} a & a & b & a & a & b & a & a & b & a & a & b \\ b & a & b & b & a & b & b & a & b & b & a & b \\ a & a & b & a & a & b & a & a & b & a & a & b \\ b & a & b & b & a & b & b & a & b & b & a & b \end{matrix}$.*

Theorem 4.3 *Let $M = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ be the row Parikh matrix of a binary $m\times n$ array A over $\{a < b\}$. Then the row Parikh matrix of the power $A^{(p\times q)}$ is given by $\begin{pmatrix} 1 & pqr & pqt + \frac{pq(q-1)}{2}\sum_{i=1}^{m} r_i \cdot s_i \\ 0 & 1 & pqs \\ 0 & 0 & 1 \end{pmatrix}$, where $|A_i|_a = r_i$ and $|A_i|_b = s_i$, with A_i being the i^{th} row of A.*

Proof.
We have $A^{(p\times q)} = (A^{(1\times q)})^{(p\times 1)}$. Now $A^{(1\times q)}$ is the column catenation $A \circ \cdots \circ A$ of A with itself, q times. Let r_i, s_i and t_i denote the number of a's, b's and ab's in the i^{th} row x_i, $(1 \leq i \leq m)$ of A. Then the i^{th} row of $A^{(1\times q)}$ is x_i^q. Using the formula in ([7], Theorem 3.1), the Parikh matrix of x_i^q is given by $\psi_3(x_i^q) = \begin{pmatrix} 1 & qr_i & qt_i + \frac{q(q-1)}{2} r_i \cdot s_i \\ 0 & 1 & qs_i \\ 0 & 0 & 1 \end{pmatrix}$. Therefore the row Parikh matrix of $A^{(1\times q)}$ is $= \begin{pmatrix} 1 & qr & qt + \frac{q(q-1)}{2}\sum_{i=1}^{m} r_i \cdot s_i \\ 0 & 1 & qs \\ 0 & 0 & 1 \end{pmatrix}$. Since the array $A^{(p\times q)}$ is the row catenation $A^{(1\times q)} \diamond \cdots \diamond A^{(1\times q)}$ of the array $A^{(1\times q)}$ with itself p times, each of the rows of the array $A^{(1\times q)}$ is repeated p times in the same order in $A^{(p\times q)}$.

This means that $|A^{(p\times q)}|_a$ is p times $|A^{(1\times q)}|_a$. *i.e.* $|A^{(p\times q)}|_a$ is pqr. Likewise for $b's$ and $ab's$. This proves the required result.

The notion of M-ambiguity of words has been extended to two dimensional picture arrays in [23]. We now recall this.

Definition 4.4 *The arrays $A, B \in \Sigma^{m\times n}$ are said to be (i) M-row equivalent if $M_r(A) = M_r(B)$ and (ii) M-column equivalent if $M_c(A) = M_c(B)$. The arrays A and B are said to be M-equivalent, denoted by $A \equiv_M B$, if they are both M-row equivalent and M-column equivalent. An array $A \in \Sigma^{m\times n}$ is M-ambiguous (or simply ambiguous) if it is M-equivalent to another distinct array; otherwise it is unambiguous.*

In [7], it is shown that for any two words $v, w \in \Sigma^*$, $|\Sigma| \geq 2$, either of the following statements $(i), (ii)$ holds: (i) $v^k \equiv_M w^k$, for all positive integers k, (ii) $v^k \not\equiv_M w^k$, for all positive integers k. In the case of binary picture arrays the situation is different as **seen from** the following proposition.

Proposition 4.5 *There are M-row equivalent picture arrays whose powers are not M-row equivalent and conversely.*

This proposition is illustrated in the following example.

Example 4.6 *We consider binary arrays* $A = \begin{matrix} a & a & b \\ b & a & a \end{matrix}$ *and* $B = \begin{matrix} a & b & b \\ a & a & a \end{matrix}$.
Then $A^{(1\times 2)} = \begin{matrix} a & a & b & a & a & b \\ b & a & a & b & a & a \end{matrix}$, $B^{(1\times 2)} = \begin{matrix} a & b & b & a & b & b \\ a & a & a & a & a & a \end{matrix}$ *Now* $M_r(A) = M_r(B) = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ *so that the binary arrays A and B are $M-$equivalent.*
But $M_r(A^{(1\times 2)}) = \begin{pmatrix} 1 & 8 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $M_r(B^{(1\times 2)}) = \begin{pmatrix} 1 & 8 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ *so that* $M_r(A^{(1\times 2)})$ *and* $M_r(B^{(1\times 2)})$ *are not $M-$equivalent.*

We next consider binary arrays $C = \begin{matrix} a & a & b & b \\ b & b & a & a \end{matrix}$, $D = \begin{matrix} a & a & b & a \\ a & b & b & b \end{matrix}$. *Then* $C^{(1\times 2)} = \begin{matrix} a & a & b & b & a & a & b & b \\ b & b & a & a & b & b & a & a \end{matrix}$, $D^{(1\times 2)} = \begin{matrix} a & a & b & a & a & a & b & a \\ a & b & b & b & a & b & b & b \end{matrix}$.
We have $M_r(C) = \begin{pmatrix} 1 & 4 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $M_r(D) = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ *so that the binary arrays C and D are not $M-$equivalent.*
But $M_r(C^{(1\times 2)}) = \begin{pmatrix} 1 & 8 & 16 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix} = M_r(D^{(1\times 2)})$ *so that* $M_r(C^{(1\times 2)})$ *and* $M_r(D^{(1\times 2)})$ *are $M-$equivalent.*

The next result gives a sufficient condition for two M-row equivalent binary picture arrays to have their powers also M-row equivalent.

Theorem 4.7 *Let A and B be two $m \times n$ M-row equivalent binary arrays over $\Sigma = \{a < b\}$. Then their powers $A^{(p\times q)}$ and $B^{(p\times q)}$ are M-row equivalent if $\sum_{i=1}^{m} r_i^2 = \sum_{i=1}^{m} u_i^2$, where $|A_i|_a = r_i$ and $|B_i|_a = u_i$, $1 \le i \le m$, with A_i and B_i being the i^{th} rows of A and B respectively.*

Proof.
Let A and B be two $m \times n$ M-row equivalent binary arrays over $\Sigma = \{a < b\}$ and $M_r(A) = M_r(B) = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$. Then $\sum_{i=1}^{m} r_i = \sum_{i=1}^{m} u_i = r$ where r_i and u_i, $1 \le i \le m$ are the number of a's in the i^{th} row of A and B respectively. Also the number of b's in the i^{th} row of A and B respectively are $(n - r_i)$ and $(n - u_i)$. Suppose $\sum_{i=1}^{m} r_i^2 = \sum_{i=1}^{m} u_i^2$. Now using Theorem 4.3, we have

$$M_r(A^{(p\times q)}) = \begin{pmatrix} 1 & pqr & \alpha \\ 0 & 1 & pqs \\ 0 & 0 & 1 \end{pmatrix} \text{ and } M_r(B^{p\times q}) = \begin{pmatrix} 1 & pqr & \beta \\ 0 & 1 & pqs \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha = pqt + \frac{pq(q-1)}{2} \sum_{i=1}^{m} r_i \cdot (n - r_i)$ and $\beta = pqt + \frac{pq(q-1)}{2} \sum_{i=1}^{m} u_i \cdot (n - u_i)$.
We now prove that $\alpha = \beta$ which will complete the proof.
We have
$\alpha = pqt + \frac{pq(q-1)}{2} \sum_{i=1}^{m} r_i \cdot (n - r_i) = pqt + \frac{pq(q-1)}{2} \left(n \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} r_i^2 \right)$
$= pqt + \frac{pq(q-1)}{2} \left(n \sum_{i=1}^{m} u_i - \sum_{i=1}^{m} u_i^2 \right)$
$= pqt + \frac{pq(q-1)}{2} \sum_{i=1}^{m} u_i \cdot (n - u_i) = \beta.$
This proves that $A^{(p\times q)}$ and $B^{(p\times q)}$ are M-row equivalent.

Remark 4.8 *The sufficient condition in Theorem 4.7 is not vacuous as can be seen from the following illustration.*
Consider the binary arrays $A = \begin{matrix} a & a & b & b \\ a & b & a & a \end{matrix}$ and $B = \begin{matrix} a & a & a & b \\ b & a & a & b \end{matrix}$ which are $M-$equivalent with the row Parikh matrix $\begin{pmatrix} 1 & 5 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. If the number of subword a in the rows of A (respy. B) are r_1 and r_2 (respy. u_1 and u_2),

then $r_1^2 + r_2^2 = u_1^2 + u_2^2 = 13$. Now $A^{(3\times 2)} = \begin{matrix} a & a & b & b & a & a & b & b \\ a & b & a & a & a & b & a & a \\ a & a & b & b & a & a & b & b \\ a & b & a & a & a & b & a & a \\ a & a & b & b & a & a & b & b \\ a & b & a & a & a & b & a & a \end{matrix}$ and

$B^{(3\times 2)} = \begin{matrix} a & a & a & b & a & a & a & b \\ a & a & a & b & a & a & a & b \\ a & a & a & b & a & a & a & b \\ a & a & a & b & a & a & a & b \\ a & a & a & b & a & a & a & b \\ a & a & a & b & a & a & a & b \end{matrix}$ and $M_r(A) = M_r(B) = \begin{pmatrix} 1 & 30 & 51 \\ 0 & 1 & 18 \\ 0 & 0 & 1 \end{pmatrix}$

so that the binary arrays $A^{(3\times 2)}$ *and* $B^{(3\times 2)}$ *are* $M-$*equivalent.*

5 Fair picture arrays

Fair words and their properties have been studied in [13]. A weak ratio property for an array is introduced in [23]. We now extend the notion of fair words to two dimensional arrays. We also recall the notion of weak ratio property restricting it to binary arrays.

Definition 5.1 *(i) A binary array* $A \in \Sigma^{m\times n}$ *is called fair if the total number of subwords ab in the rows (respectively columns) of A is equal to the total numbers of subwords ba in the rows (respectively columns) of A.*
(ii) Let A and B be two binary arrays over $\Sigma = \{a < b\}$*. The arrays A and B are said to satisfy a weak ratio property if* $\frac{|A|_a}{|B|_a} = \frac{|A|_b}{|B|_b} = k$ *where k, is a non-zero constant.*

Theorem 5.2 *Let A and B be two fair binary arrays over* $\Sigma = \{a < b\}$*, both having the same number of rows and satisfying weak ratio property. Then the arrays* $A \circ B$ *and* $B \circ A$ *are also fair. A corresponding result holds good for* $A \diamond B$ *and* $B \diamond A$*.*

Proof.
Let $A \in \Sigma^{m\times n}$ and $B \in \Sigma^{m\times l}$ be two fair words satisfying weak ratio property with ratio constant α. Denoting the total number of subword ab in the rows of a binary array by $|X|^r_{ab}$, we then have $|A|^r_{ab} = |A|^r_{ba}$ and $|B|^r_{ab} = |B|^r_{ba}$. Also we have $\frac{|A|_a}{|B|_a} = \frac{|A|_b}{|B|_b} = \alpha$. This implies that $mn = |A| = |A|_a + |A|_b = \alpha(|B|_a + |B|_b) = \alpha ml$, i.e., $n = \alpha l$.
Since $A \circ B$ is the column catenation of A and B, the column Parikh matrix of $A \circ B$ is $M_c(A \circ B) = M_c(A) \oplus M_c(B)$. Therefore the number of subword ab's column wise in $A \circ B$ is same as the number of ba's column wise in $A \circ B$.
Let x_i and u_i, $1 \le i \le m$ be the words in the i^{th} row of A and B respectively. Now the number of ab's row wise in $A \circ B$ is given by
$|A \circ B|^r_{ab} = \sum_{i=1}^{m}(|x_i|_{ab} + |y_i|_{ab} + |x_i|_a \cdot |y_i|_b) = |A|^r_{ab} + |B|^r_{ab} + \sum_{i=1}^{m} |x_i|_a \cdot |y_i|_b$.

We also have

$$|A \circ B|^r_{ba} = \sum_{i=1}^{m}(|x_i|_{ba} + |x_i|_{ba} + |x_i|_b \cdot |y_i|_a)$$

$$= |A|^r_{ba} + |B|^r_{ba} + \sum_{i=1}^{m} |x_i|_b \cdot |y_i|_a$$

$$= |A|^r_{ab} + |B|^r_{ab} + \sum_{i=1}^{m} |x_i|_b(l - |y_i|_b),$$

since $|y_i| = l$

$$= |A|^r_{ab} + |B|^r_{ab} + l\sum_{i=1}^{m} |x_i|_b - \sum_{i=1}^{m} |x_i|_b|y_i|_b$$

$$= |A|^r_{ab} + |B|^r_{ab} + \alpha l\sum_{i=1}^{m} |y_i|_b - \sum_{i=1}^{m} |x_i|_b|y_i|_b$$

$$= |A|^r_{ab} + |B|^r_{ab} + \sum_{i=1}^{m}(\alpha l - |x_i|_b)|y_i|_b$$

$$= |A|^r_{ab} + |B|^r_{ab} + \sum_{i=1}^{m}(n - |x_i|_b)|y_i|_b,$$

since $n = \alpha l$

$$= |A|^r_{ab} + |B|^r_{ab} + \sum_{i=1}^{m} |x_i|_a \cdot |y_i|_b = |A \circ B|^r_{ab}.$$

This proves that $A \circ B$ is a fair array. In a similar manner it can be shown that $B \circ A$ is also a fair array.

6 Restricted Shuffle operator on picture arrays

In [6], a restricted shuffle operator on two binary words, denoted as $SShuf$ is considered and properties of Parikh matrices of words under this operator are derived, especially over a binary alphabet. Here we extend this operator to picture arrays and obtain properties of Parikh matrices of arrays under this operator.

Definition 6.1 *Let $A, B \in \Sigma^{m\times n}$ be two picture arrays over $\Sigma = \{a < b\}$ such that* $A = \begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}$ *and* $B = \begin{matrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{matrix}$. *Then the restricted row shuffle operator on the pair of arrays A and B is defined by*

$$RSShuf(A,B) = \begin{array}{ccccccc} a_{11} & b_{11} & a_{12} & b_{12} & \cdots & a_{1n} & b_{1n} \\ a_{21} & b_{21} & a_{22} & b_{22} & \cdots & a_{2n} & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & b_{m1} & a_{m2} & b_{m2} & \cdots & a_{mn} & b_{mn} \end{array}$$

and similarly the restricted column shuffle operator is defined by

$$CSShuf(A,B) = \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ b_{11} & b_{12} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{array}.$$

Example 6.2 *Let* $A, B \in \Sigma^{3\times 3}$ *over the binary alphabet* $\Sigma = \{a < b\}$ *be given by* $A = \begin{array}{ccc} a & a & b \\ b & a & a \end{array}$, $B = \begin{array}{ccc} b & a & a \\ a & b & a \end{array}$. *Then* $RSShuf(A,B) = \begin{array}{cccccc} a & b & a & a & b & a \\ b & a & a & b & a & a \end{array}$ *and*

$$CSShuf(A,B) = \begin{array}{ccc} a & a & b \\ b & a & a \\ b & a & a \\ a & b & a \end{array}.$$

We observe a few facts which are immediate from the definition :
(*i*) $M_r(CSShuf(A,B)) = M_r(A) \oplus M_r(B)$,
(*ii*) $M_c(RSShuf(A,B)) = M_c(A) \oplus M_c(B)$.

In [14], the authors introduced a notion of the positions of letters in a word and using this notion characterized the M-equivalent words over binary alphabet. The sum of positions of a letter a in a word w of length n over an alphabet Σ_k, denoted by $S_a(w)$ is defined by $S_a(w) = \sum\limits_{w[i]=a, 1\le i\le n} i$.

Here we introduce the sum of positions of a letter in a binary array over $\Sigma = \{a < b\}$ as follows.

Definition 6.3 *Let A be a binary $m \times n$ array over $\{a < b\}$, then the row wise sum of positions of a letter a in A is defined by* $S_a^r(A) = \sum\limits_{1\le i\le m} S_a(x_i)$, *where x_i is the i^{th} row of the array A.*

Similarly, the column wise sum of positions of a letter a in A is defined by $S_a^c(A) = \sum\limits_{1\le i\le n} S_a(y_i)$, *where y_i is the i^{th} column of the array A.*

Theorem 6.4 *Two arrays A and B over $\{a < b\}$ is M-row equivalent (column equivalent) to each other if each row (column) of A and B have the same number of b's and $S_b^r(A) = S_b^r(B)$ ($S_b^c(A) = S_b^c(B)$ respectively).*

Proof.
Let x_i and y_i be the i^{th} row of the arrays A and B respectively. Also let $|x_i|_b = |y_i|_b$, for all $1 \le i \le m$ and $S_b^r(A) = S_b^r(B)$. then the number of ab's in the row Parikh matrix of A is equal to $\sum\limits_{1\le i\le m} |x_i|_{ab}$. Now,

$$\sum_{1\le i\le m} |x_i|_{ab} = \sum_{1\le i\le m} [S_b(x_i) - \tfrac{|x_i|_b(|x_i|_b+1)}{2}] = \sum_{1\le i\le m} S_b(x_i) - \sum_{1\le i\le m} \tfrac{|x_i|_b(|x_i|_b+1)}{2}$$
$$= \sum_{1\le i\le m} S_b(y_i) - \sum_{1\le i\le m} \tfrac{|y_i|_b(|y_i|_b+1)}{2} = \sum_{1\le i\le m} [S_b(y_i) - \tfrac{|y_i|_b(|y_i|_b+1)}{2}] = \sum_{1\le i\le m} |y_i|_{ab}$$

which is the number of ab's in the row Parikh matrix of B. Hence the binary arrays A and B are M-row equivalent.

Similarly the other case of M-column equivalence can be proved.

Lemma 6.5 *Let $A, B \in \Sigma^{m\times n}$ where $\Sigma = \{a < b\}$, then (i) $S_b^r(RSShuf(A,B)) = 2(S_b^r(A) + S_b^r(B)) - |A|_b$ and (ii) $S_b^r(CSShuf(A,B)) = 2(S_b^c(A) + S_b^c(B)) - |A|_b$ where $|A|_b$ is the number of b's in the array A.*

Proof.
Let x_i and y_i be the i^{th} row of the arrays A and B respectively. Then we have,

$$S_b^r(RSShuf(A,B)) = \sum_{i=1}^{m} S_b(SShuf(x_i,y_i)) = \sum_{i=1}^{m} [2\{S_b(x_i) + S_b(y_i)\} - |x_i|_b]$$
$$= 2[\sum_{i=1}^{m} S_b(x_i) + \sum_{i=1}^{m} S_b(y_i)] - \sum_{i=1}^{m} |x_i|_b = 2(S_b^r(A) + S_b^r(B)) - |A|_b$$

Similarly we can prove the statement (ii). A sufficient condition for the row shuffle operator of two binary arrays is given as follows.

Theorem 6.6 *Let $A, B \in \Sigma^{m\times n}$ where $\Sigma = \{a < b\}$, then $RSShuf(A,B) \equiv_M RSShuf(B,A)$ if $|A|_b = |B|_b$.*

This can be seen using the Lemma 6.5 and the fact that $M_c(RSShuf(A,B)) = M_c(A) \oplus M_c(B)$.

7 Geometric operations on picture arrays

Geometric operations on picture arrays such as reflection, rotation are now considered. Properties of Parikh matrices of the arrays resulting from the geometric operations are obtained.

Proposition 7.1 *Let A be a binary $m\times n$ picture array over $\{a < b\}$. Reflection of A about its rightmost vertical yields an array A_v with the properties*
(i) $|A_v|_a = |A|_a$ and $|A_v|_b = |A|_b$
(ii) The column Parikh matrices of A and A_v are the same
(iii) The number of ab's row wise in A_v is $|A_v|_{ab}^r = \sum\limits_{1\le i\le m} (|r_i|_a |r_i|_b - |r_i|_{ab})$, where r_i is the i^{th} row of A.

Similarly, reflection of A about its bottommost horizontal yields an array A_h with the properties

(i) $|A_h|_a = |A|_a$ *and* $|A_h|_b = |A|_b$
(ii) The row Parikh matrices of A *and* A_h *are the same*
(iii) The number of ab's column wise in A_h *is* $|A_h|^c_{ab} = \sum_{1\le i\le n} (|c_i|_a|c_i|_b - |c_i|_{ab})$, *where* c_i *is the* i^{th} *column of* A.

The following proposition is a consequence of Proposition 7.1.

Proposition 7.2 *If two arrays* A *and* B *of same sizes are* M*-equivalent, then their reflections about their rightmost verticals and their bottommost horizontals are also* M*-equivalent.*

Definition 7.3 *Let* $A \in \Sigma^{m\times n}$ *be a picture array over* $\Sigma = \{a < b\}$ *such that* $A = \begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}$. *A picture array obtained from* A *by rotating it by* 90° *clockwise, denoted by* A^{90° *is defined as* $A^{90^\circ} = \begin{matrix} a_{m1} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{mn} & \cdots & a_{1n} \end{matrix}$.

Note that A^{90° ia an array of size $n \times m$ such that the first row of A is the last column of A^{90°, the second row of A is the last but second column of A^{90° and so on, and the last row of A is the first column of A^{90°.

Similarly one can define A^{180° (which is same as $(A^{90^\circ})^{90^\circ}$), A^{270° and A^{360°. It is easy to see that $A^{360^\circ} = A$.

Now we state in the following proposition, the relations between the row and column Parikh matrices of the rotated arrays.

Proposition 7.4 *Let* $A \in \Sigma^{m\times n}$ *be a picture array where* $\Sigma = \{a < b\}$, *then (i)* $M_r(A^{90^\circ}) = M_c(A_h)$ *and* $M_c(A^{90^\circ}) = M_r(A)$, *(ii)* $M_r(A^{180^\circ}) = M_r(A_v)$ *and* $M_c(A^{180^\circ}) = M_c(A_v)$ *and (iii)* $M_r(A^{270^\circ}) = M_c(A)$ *and* $M_c(A^{270^\circ}) = M_r(A_v)$ *where* A_h *and* A_v *are the reflections of the array* A *about its bottommost horizontal and rightmost vertical.*

8 Concluding Remarks

Motivated by applications in areas such as pattern recognition, computer vision and others, several studies have been done on combinatorial properties of two-dimensional arrays [2] . The study done in this paper is a contribution to this area as well, and it extends notions and concepts well-studied in the context of strings. It will be of interest to consider picture arrays of three or more symbols and examine the applicability of the notions and results considered here.

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10 Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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