THE CLASSIFICATION OF FUZZY SUBGROUPS OF FINITE CYCLIC GROUPS $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ AND $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ FOR DISTINCT PRIME NUMBERS $p, q, r, p_1, p_2, \cdots, p_n$ AND $n, m \in \mathbb{Z}^+$

by

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DECLARATION

This thesis is a presentation of my original research work. Except where acknowledged in the customary manner, the material presented in this thesis, to the best of my knowledge, has not been submitted in whole or part for a degree award in any other university or any other award.

Signed this $\underline{10^{th}}$ day of <u>April</u> 2018

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This is to confirm that this thesis work was carried out under my supervision and the submission has been done with my approval.

Signed this $\underline{10^{th}}$ day of <u>April</u> 2018

Prof. Babington B. Makamba

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DEDICATION

To my beloved wife Winfred and my daughter Patience.

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NOTATION

$\|A\|, \|x\|$	–	the order of a set A, the order of an element x
$\mathbb{N}$	–	the set of natural numbers
$\mathbb{Z},\ \mathbb{Z}^+$	–	the integers, the positive integers
$\mathbb{Q},\ \mathbb{Q}^+$	–	the rational numbers, the positive rational numbers
$\mathbb{R},\ \mathbb{R}^+$	–	the real numbers, the positive real numbers
$\mathbb{C},\ \mathbb{C}^\times$	–	the complex numbers, the non-zero complex numbers
$\mathbb{Z}/n\mathbb{Z}$	–	the integers modulo n
$(\mathbb{Z}/n\mathbb{Z})^\times$	–	the (multiplicative group of) invertible integers modulo n
$H \leq G$	–	H is a subgroup of G
$N \trianglelefteq G$	–	N is a normal subgroup of G
$\mathbb{Z}_n$	–	the cyclic group of order n
D_{2n}	–	the dihedral group of order $2n$
S_n	–	the symmetric group on n letters
A_n	–	the alternating group on n letters
Q_8	–	the quarternion group of order 8
$A \cong B$	–	A is isomorphic to B
$C_G(A)$	–	the centralizer of A in G
$N_G(A)$	–	the normalizer of A in G
$Z(G)$	–	the center of the group G
G_x	–	the stabilizer of x in G
$\langle A \rangle$	–	the group generated by the set A

$\langle x \rangle$	–	the group generated by the element x
$ker\ \varphi$	–	the kernel of the homomorphism φ
$im\ \varphi$	–	the image of the homomorphism φ
$\lvert G : H \rvert$	–	the index of the subgroup H in the group G
$\exists$	–	there exists
$\forall$	–	for all
$\lvert,\ :$	–	such that
(m, n)	–	the greatest common divisor of n and m

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ABSTRACT

Let G be the cyclic group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ where p, q, r are distinct primes and $n, m \in \mathbb{Z}^+$. Using the criss-cut method by Murali and Makamba, we determine in general the number of distinct fuzzy subgroups of G. This is achieved by using the maximal chains of subgroups of the respective groups, and the equivalence relation given in their research papers. For cases of m, the number of fuzzy subgroups is first given, from which the general pattern for G is achieved. Murali and Makamba discussed the number of fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ using the cross-cut method. A brief revisit of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is done using the criss-cut method. The formulae for finding the number of distinct fuzzy subgroups in each of the cases is given and proofs provided.

Furthermore, we classify the fuzzy subgroups of the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ for $p_1, p_2, \cdots, p_n$ distinct primes and $n \in \mathbb{Z}^+$ using the criss-cut method. An algorithm for counting the distinct fuzzy subgroups of this group is developed.

CHAPTER 1. GENERAL INTRODUCTION AND PRELIMINARIES

1.1 Introduction

In a crisp set, an element is either a member of the set or not. For example, a square matrix over the field of real numbers is either invertible or singular but not both. Most of our traditional tools for formal modelling, reasoning and computing are crisp, deterministic and precise in character. Crisp means dichotomous, that is, yes or no type rather than more or less type. In traditional dual logic, for instance, a statement can be true or false and nothing in between. In set theory, an element can either belong to a set or not; in optimization a solution can be feasible or not. Precision assumes that the parameters of a model represent exactly the real system that has been modeled. This, generally, also implies that the model is unequivocal, that is, it contains no ambiguities. Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known and that there are no doubts about their values or their occurrence. Unluckily these assumptions and beliefs are not justified if it is important, that the model describes well reality (which is neither crisp nor certain). In addition, the complete description of a real system would often require far more detailed data than a human being could ever recognize simultaneously, process and understand. This situation has already been recognized by thinkers in the past. In 1923, the philosopher Bertrand Russell referred to the first point when he wrote: “All traditional logic habitually assumes that precise symbols are being employed. It is, therefore, not applicable to this terrestrial life but only to an imagined celestial existence”.

For a long time, probability theory and statistics have been the predominant theories and tools to model uncertainties of reality. They are based, as all formal theories, on certain axiomatic assumptions, which are hardly ever tested when these theories are applied to reality. Formally speaking, fuzzy set theory was initially intended to be an extension of dual logic and/or classical set theory.

On the other hand, fuzzy[1] sets allow elements to be partially in a set. Each element is given a degree of membership in a set. This membership value can range from 0 (not an element of the set) to 1 (a member of the set). Clearly, if only the extreme membership values of 0 and 1 are allowed, then we get crisp sets. A membership function of a fuzzy set, often called the characteristic function, is the relationship between the values of an element and its degree of membership in a set. This is a function whose range is an ordered membership set containing more than two (often a continuum of) values (typically, the unit interval). The main objective of fuzzy sets is to capture the idea of this partial membership. Therefore a fuzzy set is a function. This definition drew criticism from mathematicians like Michael Arbib in 1977 since functions are already well-known, and the theory of functions already exists.

About hundred years ago, the American philosopher Charles Pierce, in 1931, was one of the first scholars in the modern age to point out, and to regret, that "Logicians have too much neglected the study of *vagueness*, not suspecting the important part it plays in mathematical thought".

In fuzzy set theory, the set union is performed by taking their pointwise maximum, their intersection by their pointwise minimum, their complementation by an order reversing automorphism of the membership scale, set inclusion by the pointwise inequality between functions. Such point of view had not been envisaged by mathematicians, except for some pioneers, mainly logicians. Fuzzy set theory is related to many-valued logics with degrees of membership understood as degrees of truth, intersection as conjunction, union as disjunction, complementation as negation and set inclusion as implication.

Since its advent, fuzzy set theory has seen wide applications in diverse fields like the fuzzy controlled house-hold appliances. Other applications are in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management

[1] From the oxford english dictionary, the word *fuzzy* means not clear in sound or shape. In the literature of fuzzy sets, the word often stands for the word *vague*.

science, quality control, operations research, decision analysis, pattern recognition, robotics, credit worthiness assessment and fraud detection. Some specific applications are in elevator control and scheduling, anti-lock braking system, train control, fuzzy traffic light control, where the controller is supposed to change the cycle time based on the densities of cars behind green and red lights and the current cycle time. In 1987, the first fuzzy logic-controlled subway was opened in Sendai in northern Japan. Here, fuzzy-logic controllers make subway journeys more comfortable with smooth braking and acceleration. Best of all, all the driver has to do is push the start button!

1.2 Literature Review

In artificial intelligence, the ultimate quest is for machines that think like humans. One man by the name Lotfi A. Zadeh recognized that the only way to truly think outside the box is to get rid of the boxes themselves. By abandoning the rigid idea of true or false, Zadeh [98] the Iranian-American, redefined how we think about logic. In his paper *"Fuzzy logic-a personal perspective"* [102], he says that binarization draws a sharp boundary between two classes and thus, it's a deeply entrenched Cartesian tradition. He adds that what people have not realized is how this tradition has outlived its usefulness. Even though he faced staunch opposition, he finally became the "Father of fuzzy logic". His work in fuzzy sets can be traced back in 1965, when then as an electrical engineering professor at University of California, Berkeley he published his first paper *Fuzzy Sets* in [98] on his new theory of fuzzy sets and systems. In this paper, Zadeh writes:

> The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the approach used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables.

Imprecision here means the sense of vagueness rather than the lack of knowledge about the value of a parameter. According to Zadeh in his original proposition, the novelty of fuzzy set theory is to treat functions as subsets of their domains, as such, functions are used to represent gradual categories. Therefore classical set theoretic notions like union, intersection, complement, inclusion, etc are extended so as to include functions ranging on an ordered membership set.

Fuzzy set theory provides a strict mathematical framework (there is nothing fuzzy about fuzzy set theory!), in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modelling language, well suited for situations in which fuzzy relations, criteria, and phenomena exist. Zadeh's tremendous contribution to the development of this field, saw him awarded the Benjamin Franklin medal in electrical engineering in 2009 and most recently in 2012, the BBVA (Banco Bilbao Vizcaya Argentaria) Foundation Frontiers of Knowledge Award for Information and Communication Technologies.

Zadeh noted that, more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. The observation emphasized on the gap that existed between mental representations of reality and usual mathematical representations thereof, which was based on binary logic, precise numbers, differential and similar concepts (Dubois, et-al). Zadeh's motivation in proposing fuzzy sets was centered explicitly on their potential application in the domain of pattern classification (Bellman, et-al 1966), processing and communication of information, abstraction and summarization [100], simple and effective uncertainty calculus called *Possibility Theory* by [101].

Even though claims that fuzzy sets were relevant in the said areas appeared unsustainable when they first emerged, namely in the sixties, the future development of information science and engineering proved that these intuitions were right beyond all expectations. When it is used as a tool to model reality better than traditional theories, then an empirical validation is very desirable.

Application of fuzzy sets in medicine was motivated by suggestions from Zadeh. Since symptoms and diseases are fuzzy in nature, fuzzy sets are feasible to represent these entity classes of knowledge. This application led to enhanced medical-decision making and development of computer-aided diagnosis in medicine.

After Zadeh, other authors came in to advance the field in the direction of a powerful fuzzy mathematics. Rosenfeld [77] introduced the theory of fuzzy subgroups of a group. He is known for proving that a homomorphic pre-image of a fuzzy subgroup is always a fuzzy subgroup, and a homomorphic image of a fuzzy subgroup that has supremum property is always a fuzzy subgroup. Makamba in [44] extended Rosenfeld work by proving that a homomorphic image of any fuzzy subgroup is always a fuzzy subgroup.

Classification of fuzzy subgroups of finite groups by their level subgroups was introduced by Das [19]. This introduction of level subgroups posed the problem of finding a fuzzy subgroup that is representative of all the level subgroups.

Bhattacharya [10] answered this question and proved that given any chain of subgroups of a finite group, there exists a fuzzy subgroup of that group whose level subgroups are precisely the members of that finite chain. Further, [10] showed that this fuzzy subgroup is not unique, that is, any two distinct fuzzy subgroups can have similar family of level subgroups. In [11], it was also proved that two fuzzy subgroups of finite groups with identical level subgroups are equal if and only if their image sets are equal.

Bhattacharya and Mukherjee in [55] introduced fuzzy normality. The same authors in [56] introduced a notion of a fuzzy normalizer of a fuzzy subgroup. The fuzzy normalizer is not a fuzzy set but a crisp group in which the fuzzy subgroup is fuzzy normal. They further introduced the concept of fuzzy solvable in [13] where, from their definition, they show a fuzzy solvable fuzzy subgroup is necessarily fuzzy normal. Since in the crisp case, a subgroup H of a group G can be solvable without being normal in G, there was a motivation to get a definition analogous to the fuzzy case. In [44], Makamba achieved this goal by giving a definition of fuzzy normality which ensured that a fuzzy solvable fuzzy subgroup need not be normal.

Makamba in [44] extended the work of [55] and developed more general notions of fuzzy normality and cosets. The author [44] also generalized the notion of a fuzzy coset given in [55] using the notion of fuzzy point given in [74] and then proved that if μ is a fuzzy normal subgroup, then the supremum of its fuzzy cosets is a fuzzy subgroup. Further in [44], Makamba continued the work of [56] by defining a fuzzy normalizer $N(\mu)$ of a fuzzy subgroup μ such that $N(\mu)$ is a fuzzy subgroup in which μ is fuzzy normal. In [44] the work of [13] was extended by giving a definition of fuzzy normality which guarantees that a fuzzy solvable fuzzy subgroup need not be fuzzy normal. The concepts of direct products and isomorphism of fuzzy subgroups were introduced in [45]; it was proved that the internal direct product of two fuzzy subgroups of a group is isomorphic to their external direct product. Makamba's notion of isomorphism was stronger than Ray's in [76] since he used it to prove the fuzzy version of the isomorphism theorems.

In [46, 62, 63, 64, 65], Murali and Makamba defined an equivalence relation on the class of all fuzzy sets of a set which they used to determine the number of equivalence classes of fuzzy subgroups of some p-groups and some cyclic groups. In [46], they proved that normal fuzzy subgroups and fuzzy congruence relations determine each other in a group theoretical sense. The two authors in [63] characterized fuzzy subgroups of some finite groups by use of keychains and introduced the notion of a pinned flag in order to study the operations of sum, union and intersection in relation to this natural equivalence. Other researchers have also studied this equivalence like in [72, 68]. Furthermore, fuzzy subgroups have also been studied by [9, 12, 52, 54, 61, 66, 70, 84].

1.3 Overview of the Thesis

Tables and Figures in this thesis are placed at the end of each chapter. In the remainder of Chapter 1, we cover some preliminary concepts in group theory necessary in later chapters in this thesis. In particular Section 1.4 covers some fundamental concepts in group theory.

We discuss the number of subgroups and maximal chains of finite cyclic groups in Chapter 2. Chapter 3 builds the fundamental concepts of fuzzy group theory. For distinct primes $p,\ q$, the number of distinct fuzzy subgroups for $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is discussed in Chapter 4. Our main research problems of determining the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ and $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, are discussed in Chapters 5 and 6. We conclude our work in Chapter

7 by giving a summary of our main research outputs. We also give recommendations on some possible problems that can be pursued in extending the work done in this thesis.

The results of Chapter 6 have been published in the peer reviewed journal [58] while a second paper from some of the results obtained in Chapter 5 has been submitted to a peer reviewed journal. A third paper from our main results in Chapter 5 is under preparation for onward submission.

1.4 Some Fundamentals of Group Theory

Getting into the history of group theory is like trying to sort through a closet full of favourite old possessions; we pick up an item, prepared to toss it if necessary, and suddenly a second and third look at the thing reminds us that this is fascinating stuff. According to Miller [53], the history of group theory can be divided into three periods. The first extends from the mathematical history to about 1770 A.D and is called the *implicit period* because the group concept was applied without explicitly being stated. The second period, called the *specialization period*, extends from 1770 to 1870. During this period, the theory of substitution groups was founded as an autonomous science and its usefulness in algebraic equations emphasized. The third period, known as the *generalization period*, extends from 1870 to date, is a period of increased generalizations by abstraction and the explicit use of groups in each of the large domains of mathematics.

One of the most fundamental property of elements of a group is that each group has one and only one element satisfying $xy = z$ whenever any two of these are replaced with equal or unequal elements of the group. This property is satisfied by the members of the number systems of the ancients, including the Babylolians, Egyptians and Greeks. But the zero in this system destroyed the group property of the entire system, since without it, these numbers ($\mathbb{R}$ and $\mathbb{C}$) formed a group under multiplication. The multiplication group of real numbers is one of the oldest groups in the history of mathematics. Later the addition group came in with the introduction of the zero. So the stone which the builders rejected later became the corner stone. Even though this impaired the multiplication group, it granted full number citizenship to the negative numbers.

The development of group theory was greatly motivated by the quest by mathematicians to get a general method for finding roots of algebraic equations. Al-Khwarizmi gave the first non-geometric general method for solving quadratic equations for positive roots, with Savasorda later giving the complete solution for quadratic equations. Al-Mahani and others worked on the cubic equations. The struggle on cubic and quartic equations continued among mathematicians, one of them Leonardo Fibonacci (1170-1250). This went on until the Italian mathematicians Scipione dal Ferro (1465-1526), N. Tartaglia (1500-1557) and G. Cardano (1501-1576) succeeded in finding a general method of solving general cubic equations. Later L. Ferrari (1522-1565) came up with a general method for finding solutions to the most general quartic equation. Most of the great mathematicians were working on a general method to find solutions to equations of all degrees until 19th century when the path took a sharp turn. This was after Niels Henrik Abel (1802-1829), inspired by Joseph-Louis Lagrange (1735-1813), published in 1824 his famous result *"The imposibility of solving the general quintic equation by means of radicals"*. A.L. Cauchy (1789-1857) was studying the group of permutations of roots of equations of higher orders and it was E. Galois (1811-1832) who gave the complete relationship between groups and algebraic equations.

According to D.J. Struik [89], Galois' unifying principle is one of the outstanding achievements of 19th century mathematics. In 1870, Galois gave the name group which would later be defined axiomatically by A. Cayley (1821-1895) and L. Kronecker (1823-1891). From this time, it took an explicit form and has since then played a fundamental role in all other fields of mathematics.

Definition 1.4.1. [23, p.16] A *group* is an ordered pair $(G, \star)$, where G is a set and $\star$ is a binary operation on G satisfying the following axioms:

(i) $(a \star b) \star c = a \star (b \star c)\ \forall\ a, b, c \in G$ i.e., $\star$ is *associative*,

(ii) $\exists$ an element $e \in G$ called an *identity* of G, such that $\forall\ a \in G$ we have $a \star e = e \star a = a$,

(iii) for each $a \in G$, $\exists\ a^{-1} \in G$ called an *inverse* of a such that $a \star a^{-1} = a^{-1} \star a = e$.

The group $(G, \star)$ is called *abelian(or commutative)* if $a \star b = b \star a$ for all $a, b \in G$.

Definition 1.4.2. [26, p.78] Let A be the finite set $A = \{1, 2, \cdots, n\}$. The group of all permutations of A is called the *symmetric group* on n *letters*, and is denoted by S_n.

The symmetric group S_n has order $n!$

Definition 1.4.3. The *dihedral group* denoted D_{2n}, $n \geq 3$, is the group of symmetries of a regular n-gon.

$D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle$ and $|D_{2n}| = 2n$.

Definition 1.4.4. A subset H of a group G is called a *subgroup* of G, denoted $H \leq G$, if H is nonempty and closed under products and inverses i.e., $x, y \in H$, $xy \in H$ and $x^{-1}y^{-1} \in H$.

Proposition 1.4.5. *[23, p.47] (The Subgroup Criterion) A subset H of a group G is a subgroup if and only if*

(i) $H \neq \emptyset$, and

(ii) for all $x, y \in H$, $xy^{-1} \in H$. Furthermore, if H is finite, then it suffices to check that H is nonempty and closed under multiplication.

Definition 1.4.6. A proper subgroup M of a group G is called a *maximal subgroup* if whenever $M \leq H \leq G$, then either $H = M$ or $H = G$.

Proposition 1.4.7. *Every subgroup of a finite group G is contained in a maximal subgroup.*

Definition 1.4.8. A partial order is a binary relation that is reflexive, antisymmetric, and transitive. A partially ordered set $(P, \leq)$, or in short poset, is a set P together with a partial order $\leq$ on P.

Definition 1.4.9. A total order T is a partial order in which, for each x and y in T, we have $x \leq y$ or $y \leq x$.

Definition 1.4.10. A chain is a totally ordered set or a totally ordered subset of a poset.

Definition 1.4.11. A chain of subgroups of a group is said to be a *maximal chain* if it cannot be properly contained in another chain.

Remark 1.4.12. Throughout this thesis, a maximal chain will be used to mean maximal chain of subgroups of a group unless otherwise stated.

Definition 1.4.13. The *lattice* of subgroups of a group is a directed graph that shows the relationship among its subgroups.

An example of a subgroup lattice diagram of S_4 is given in Figure 1.1 which can also be generated from [34].

Definition 1.4.14. [23, p.49] The set $\{C_G(A) = g \in G \mid ga = ag \ \forall \ a \in A\}$ of elements of G which commute with every element of A is called the *centralizer*[2] of A in G.

Definition 1.4.15. [23, p.50] The set $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$ of the elements commuting with all elements of G is called the *center* of G.

By Definition 1.4.14, we have that $Z(G) = C_G(G)$. Therefore $Z(G)$ is a subgroup of G.

Definition 1.4.16. [23, p.50] Let $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. The *normalizer* of A in G is the set $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$.

Remark 1.4.17. $N_G(A) \leq G$ and $C_G(A) \leq N_G(A)$.

Definition 1.4.18. [23, p.51] The set $G_x = \{g \in G \mid g \cdot x = x\}$ is called the *stabilizer* of x in G. Furthermore, $G_x \leq G$.

Definition 1.4.19. [23, p.77] For any $N \leq G$, and any $g \in G$. The sets $gN = \{gn \mid n \in N\}$ and $Ng = \{ng \mid n \in N\}$ are called respectively *left coset* and *right coset* of N in G.

Definition 1.4.20. [23, p.82] The element gng^{-1} is called the *conjugate* of $n \in N$ by g. The set $gNg^{-1} = \{gng^{-1} \mid n \in N\}$ is called the *conjugate* of N by g. The element g is said to *normalize* N if $gNg^{-1} = N$. A subgroup N of G is called *normal* written $N \trianglelefteq G$ if every element of G normalizes N, i.e., if $gNg^{-1} = N$ for all $g \in G$.

Theorem 1.4.21. *[23, p.82] Let N be a subgroup of G. The following are equivalent:*

(i) $N \trianglelefteq G$,

[2] is a subgroup of G with an elaborate proof given in [23, p.49].

(ii) $N_G(N) = G$,

(iii) $gN = Ng$ *for all* $g \in G$,

(iv) $gNg^{-1} \subseteq N$ *for all* $g \in G$.

Definition 1.4.22. [23, p.93] Let H and K be subgroups of a group and define

$$HK = \{hk \mid h \in H,\ k \in K\}.$$

Proposition 1.4.23. *If H and K are finite subgroups of a group then*

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Proposition 1.4.24. *[23, p.94] If H and K are subgroups of a group, HK is a group if and only if $HK = KH$.*

Theorem 1.4.25. *Suppose G is a group with subgroups H and K such that*

(i) H *and* K *are normal in* G, *and*

(ii) $H \cap K = 1$

Then $HK \cong H \times K$.

Remark 1.4.26. Let G be a group with subgroups H and K.

(i) $H \cap K$ is the maximal subgroup contained in both H and K.

(ii) $H \cup K$ is the minimal subgroup containing both H and K.

Corollary 1.4.27. *[23, p.94] If H and K are subgroups of a group G and $H \leq N_G(K)$, then HK is a subgroup of G. In particular, if $K \trianglelefteq G$ then $HK \leq G$ for any $H \leq G$.*

Theorem 1.4.28. *[23] (*Lagrange's Theorem*) Let G be a group and $H \leq G$, then the order of H divides then order of G (i.e., $|H|||G|$) and the number of cosets of H in G equals $\frac{|G|}{|H|}$.*

Definition 1.4.29. [23, p.90] If G is a group (possibly infinite) and $H \leq G$, the number of left cosets of H in G is called the *index* of H in G and is denoted by $|G : H|$.

Corollary 1.4.30. *[23, p.90] If G is a group and $x \in G$, then the order of x divides the order of G. In particular $x^{|G|} = 1$ for all $x \in G$.*

Definition 1.4.31. [23, p.54] A group G is *cyclic* if G can be generated by a single element, i.e., there exists some $x \in G$ such that $G = \{x^n | n \in \mathbb{Z}\}$ (where as usual the operation is multiplicative). In additive notation G is cyclic if $G = \{nx | n \in \mathbb{Z}\}$.

In both cases we write $G = \langle x \rangle$ and say G is *generated* by x (and x is a *generator* of G). A cyclic group can have more than one generator for example if $G = \langle x \rangle$, then also $G = \langle x^{-1} \rangle$ since $(x^{-1})^n = x^{-n}$.

Example 1.4.32. $\mathbb{Z}$ is cyclic since $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$. $\mathbb{Z}_n$, the cyclic group of order n, is, up to isomorphism, the unique cyclic group of order n.

Definition 1.4.33. [23, p.36] Let $(G, \star)$ and $(H, \diamond)$ be groups. A map $\varphi : G \to H$ is called a *homomorphism* if

$$\varphi(x \star y) = \varphi(x) \diamond \varphi(y), \text{ for all } x, y \in G.$$

Theorem 1.4.34. *[23, p.37] The map $\varphi : G \to H$ is called an isomorphism and G and H are said to be isomorphic, written $G \cong H$, if*

(i) φ is a homomorphism, and

(ii) φ is a bijection.

Corollary 1.4.35. *[23, p.90] If G is a group of prime order p, then G is cyclic, hence $G \cong \mathbb{Z}_p$.*

Definition 1.4.36. A (finite or infinite) group G is called *simple* if $|G| > 1$ and the only normal subgroups of G are 1 and G.

Theorem 1.4.37. *[23, p.104] (Feit-Thompson) If G is a simple group of odd order, then $G \cong \mathbb{Z}_p$ for some prime p.*

Theorem 1.4.38. *[88, p.274] (Cauchy's Theorem) If G is a finite group and p is a prime dividing $|G|$, then G contains an element of order p.*

Proposition 1.4.39. *[23, p.102] If G is a finite abelian group and p is a prime dividing $|G|$, then G contains an element of order p.*

Definition 1.4.40. [23, p.139]

(i) A group of order p^α for some $\alpha \geq 0$ is called a *p-group*. Subgroups of G which are p-groups are called *p-subgroups*.

(ii) If G is a group of order $p^\alpha m$, where $p \nmid m$, then a subgroup of order p^α is called a *Sylow p-subgroup* of G.

(iii) The set of Sylow p-subgroups of G is denoted by $Syl_p(G)$.

Theorem 1.4.41. *[23, 88, p.139, p.275] (Sylow's Theorem) Let G be a group of order $p^\alpha m$ where $p \nmid m$.*

(i) Sylow p-subgroups of G exist, i.e., $Syl_p(G) \neq \emptyset$.

(ii) If P is a Sylow p-subgroup of G and Q is any p-subgroup of G, then $\exists\ g \in G$ such that $Q \leq gPg^{-1}$, i.e., Q is contained in some conjugate of P. Therefore any two Sylow p-subgroups of G are conjugate in G.

(iii) $|Syl_p(G)| \equiv 1 \ mod\ p$.

Theorem 1.4.42. *Let H and K be two subgroups of a group G. Then intersection $H \cap K$ is a subgroup of G.*

Proposition 1.4.43. *[23, p.55]. Let G be any arbitrary group, $x \in G$ and suppose $m, n \in \mathbb{Z}$. If $x^n = 1$ and $x^m = 1$, then $x^d = 1$, where $d = (m, n)$. Specifically, if $x^m = 1$, $m \in \mathbb{Z}$, then $|x|$ divides m.*

Theorem 1.4.44. *[23, p.56] Any two cyclic groups of the same order are isomorphic. More specifically,*

(i) if $n \in \mathbb{Z}^+$ and $\langle x \rangle$ and $\langle y \rangle$ are both cyclic groups of order n, then the map

$$\varphi : \langle x \rangle \to \langle y \rangle$$
$$x^k \mapsto y^k$$

is well defined and an isomorphism,

(ii) *if $\langle x \rangle$ is an infinite cyclic group, the map*

$$\varphi : \mathbb{Z} \to \langle x \rangle$$
$$k \mapsto x^k$$

is well defined and an isomorphism.

We have that $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ and whenever additive notation is advantageous, we use the group $\mathbb{Z}/n\mathbb{Z}$ as a representative of isomorphism classes of order n.

Theorem 1.4.45. *[32, p.193] Every infinite cyclic group is isomorphic to $\mathbb{Z}$ and every finite cyclic group is isomorphic to $\mathbb{Z}_n$.*

Theorem 1.4.46. *[23, p.120] (*Cayley's Theorem*) Every group is isomorphic to some symmetric group. If G is a finite group of order n, then G is isomorphic to a subgroup of S_n.*

Proposition 1.4.47. *Suppose $G = \langle x \rangle$.*

(i) *If $|x| = \infty$, then $G = \langle x^a \rangle$ if and only if $a = \pm 1$.*

(ii) *If $|x| = n < \infty$, then $G = \langle x^a \rangle$ if and only if $(a, n) = 1$. Particularly, the number of generators of G is $\varphi(n)$ (where φ is the Euler's function).*

Theorem 1.4.48. *[23, p.58] Let G be a cyclic group.*

(i) *Every subgroup of G is cyclic. In particular, if $H \leq G$, then either $H = \{1\}$ or $H = \langle x^d \rangle$ where d is the smallest positive integer such that $x^d \in H$.*

(ii) *If $|G| = \infty$, then for any distinct nonnegative integers a and b, $\langle x^a \rangle \neq \langle x^b \rangle$. Furthermore, for every integer m, $\langle x^m \rangle = \langle x^{|m|} \rangle$, where $|m|$ is the absolute value of m, so that the nontrivial subgroups of G correspond bijectively with the integers $1, 2, 3, \cdots$.*

(iii) *If $|G| = n < \infty$, then for each positive integer l dividing n, there is a unique subgroup of G of order l. This subgroup is the cyclic group $\langle x^d \rangle$, where $d = \frac{n}{l}$. Furthermore, for every integer m, $\langle x^m \rangle = \langle x^{(n,m)} \rangle$, so that the subgroups of G correspond bijectively with the positive divisors of n.*

Theorem 1.4.49. *Every cyclic group G is abelian.*

Theorem 1.4.50. *If G is a cyclic group and N is a subgroup of G. Then G/N is cyclic.*

Theorem 1.4.51. *If G is an abelian group and N is a subgroup of G. Then G/N is abelian.*

Theorem 1.4.52. *Let G be a finite group and $Z(G)$ the center of G. If G is cyclic, then $G/Z(G)$ is abelian.*

Theorem 1.4.53. *(**The Fundamental Theorem of Finite Abelian Groups**) Every finite abelian group of order $n > 1$ is the direct product of cyclic groups, each of prime power order. That is if $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_s}$ with p_i, $i = 1, \cdots, k$, distinct primes and $r_i > 0$ for all i, then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$.*

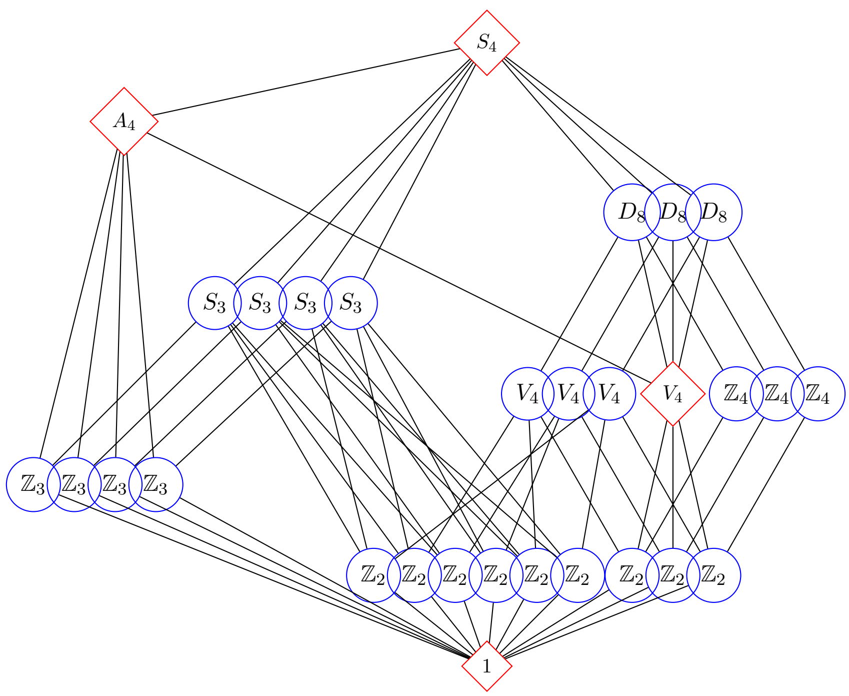

Figure 1.1: Subgroup Lattice of S_4

CHAPTER 2. COUNTING SUBGROUPS AND MAXIMAL CHAINS OF FINITE CYCLIC GROUPS

2.1 Introduction

In this chapter, we discuss the number of subgroups and maximal chains of the finite cyclic group $\mathbb{Z}_{{p_1}^{n_1}} \times \mathbb{Z}_{{p_2}^{n_2}} \times \cdots \times \mathbb{Z}_{{p_m}^{n_m}}$. It is the maximal chains that form the basis of our main work: counting the fuzzy subgroups of finite cyclic groups.

2.1.1 The Number of Subgroups in a Direct Product of Finite Abelian Groups

The motivation for this subsection is *How many subgroups?* as Joseph Petrillo puts it in his paper [75]. His main research area is on subgroup properties in finite groups.

As Joseph puts it, this question is not, in general, simple to answer. Different authors like [16], [17], [27] and [95] have done the counting in some finite groups. A finite abelian group is the direct sum of a finite number of cyclic groups of prime power orders.

2.1.2 Divisors of n

According to [75], let $L(G)$ and $|L(G)|$ denote respectively the subgroup lattice of G and the number of subgroups of G. Also let $d(n)$ denote the number of divisors of n.

Remark 2.1.1. $d(p^n) = n+1$. For example $d(3^2) = 3$ i.e., $1, 3, 9$ and $d(5^3) = 4$ i.e., $1, 5, 25, 125$.

Since the group $\mathbb{Z}_{p^n}$ has a unique subgroup of order d where $d|p^n$, then $|L(\mathbb{Z}_{p^n})| = n + 1$. We state this more generally in Proposition 2.1.2.

Proposition 2.1.2. *Let $m = p_1^{r_1} \cdots p_k^{r_k}$, where p_i, $i = 1, \cdots, k$, are distinct primes. Then by [75], we have*

$$d(m) = \prod_{i=1}^{k} (r_i + 1).$$

Example 2.1.3. When $m = 6$, we have $6 = 2 \times 3$ and

$$d(6) = \prod_{i=1}^{2} (r_i + 1) = \underset{r_1=1}{(1+1)}\underset{r_2=1}{(1+1)} = 4; \quad 1, 2, 3, 6.$$

Example 2.1.4. When $m = 60$, we have $60 = 2^2 \times 3 \times 5$ and

$$d(60) = \prod_{i=1}^{3} (r_i + 1) = \underset{r_1=2}{(2+1)}\underset{r_2=1}{(1+1)}\underset{r_3=1}{(1+1)} = 12; \quad 1, \cdots, 6, 10, 12, 15, 20, 30, 60.$$

Proposition 2.1.5. *[75] Let m and n be relatively prime with prime factorizations $m = p_1^{r_1} \cdots p_k^{r_k}$ and $n = q_1^{t_1} \cdots q_l^{t_l}$. Then the number of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ is*

$$|L(\mathbb{Z}_m \times \mathbb{Z}_n)| = d(m)d(n) = \prod_{i=1}^{k} (r_i + 1) \cdot \prod_{j=1}^{l} (t_j + 1).$$

Example 2.1.6. $|L(\mathbb{Z}_4 \times \mathbb{Z}_{27})| = \underset{4=2^2,\ 27=3^3}{d(4)d(27)} = \prod_{i=1}^{1} (r_i + 1) \cdot \prod_{j=1}^{l} (t_j + 1) = (2+1)(3+1) = 12.$

Remark 2.1.7. Generally, if m and n are relatively prime, then $d(mn) = d(m) \cdot d(n)$.

When m and n are prime powers i.e., $m = p_1^r$ and $n = p_2^s$, then we have a special case of Proposition 2.1.5 as seen in Example 2.1.6.

Corollary 2.1.8. *[75] If p and q are distinct primes, then the number of subgroups of $\mathbb{Z}_{p^r} \times \mathbb{Z}_{q^t}$ is $(r + 1)(t + 1)$.*

Theorem 2.1.9 below proved by Suzuki [91] in 1951, allows us to make a generalization of Proposition 2.1.5.

Theorem 2.1.9. *[91] Suppose G_1 and G_2 are finite groups. Then we have that $L(G_1 \times G_2) = L(G_1) \times L(G_2)$ and $|L(G_1 \times G_2)| = |L(G_1)| \cdot |L(G_2)|$ if and only if $|L(G_1)|$ and $|L(G_2)|$ are relatively prime.*

Proposition 2.1.10. *Let m and n be positive integers, and let $p_1, \cdots, p_k$ denote the list of distinct primes dividing the product mn so that $m = p_1^{r_1} \cdots p_k^{r_k}$ and $n = p_1^{s_1} \cdots p_k^{s_k}$. Then*

$$|L(\mathbb{Z}_m \times \mathbb{Z}_n)| = \prod_{i=1}^{k} |L(\mathbb{Z}_{p_i^{r_i}} \times \mathbb{Z}_{p_i^{s_i}})|.$$

Example 2.1.11. [75] Let $m = 18 = 2 \cdot 3^2$ and $n = 30 = 2 \cdot 3 \cdot 5$. Then we have $|L(\mathbb{Z}_{18} \times \mathbb{Z}_{30})| = |L(\mathbb{Z}_2 \times \mathbb{Z}_2)| \cdot |L(\mathbb{Z}_3 \times \mathbb{Z}_{3^2})| \cdot |L(\mathbb{Z}_1 \times \mathbb{Z}_5)| = 5 \cdot 10 \cdot 2 = 100.$

Corollary 2.1.12. *If $p_1^{n_1}, p_2^{n_2}, \cdots, p_m^{n_m}$ are powers of distinct primes p_i, $i = 1, 2, \cdots m$, then the number of subgroups of $\mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}}$ is*

$$(n_1 + 1)(n_2 + 1) \cdots (n_m + 1).$$

Proof. By Proposition 2.1.2, $|L(\mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}})| = (n_1 + 1)(n_2 + 1) \cdots (n_m + 1)$. □

Example 2.1.13. $|L(\underbrace{\mathbb{Z}_{2^4} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5}_{\cong \, \mathbb{Z}_{720}})| = (4 + 1)(2 + 1)(1 + 1) = 30$

As a check for the above example, it is clear that 720 has 30 divisors i.e., $1, 2, 3, 4, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 30, 36, 40, 45, 48, 60, 72, 80, 90, 120, 144, 180, 240, 360, 720$.

We now look at the maximal chains of a finite cyclic group. This is a crucial step in the process of characterizing fuzzy subgroups of a group. We determine a formula for the number of maximal chains for the group $\mathbb{Z}_{p_1{}^{n_1}} \times \mathbb{Z}_{p_2{}^{n_2}} \times \cdots \mathbb{Z}_{p_m{}^{n_m}}$, $p_i's$ distinct primes, $n_i's \in \mathbb{Z}^+$. This gives a good background for our main task of counting their fuzzy subgroups later in Chapter 5.

2.2 Counting Maximal Chains of some Finite Groups

Before getting into maximal chains of cyclic groups as discussed by [68], we first look at examples of some non-cyclic groups.

Example 2.2.1. The group $S_3 = \{e, (12), (13), (23), (123), (132)\}$ has the following 6 crisp subgroups: $\{e\}, \{e, (12)\}, \{e, (13)\}, \{e, (23)\}, \{e, (123), (132)\}$. These yield the following 4 maximal chains:

$$S_3 \supseteq \{e, (123), (132)\} \supseteq e$$

$$S_3 \supseteq \{e, (12)\} \supseteq e$$

$$S_3 \supseteq \{e, (13)\} \supseteq e$$

$$S_3 \supseteq \{e, (23)\} \supseteq e$$

Example 2.2.2. We look at the maximal chains of the dihedral group D_8. From the definition given in Section 1.4, $D_8 = \langle s, r | r^4 = s^2 = e,\ sr = r^{-1}s\rangle$. Thus we have $D_8 = \{e, s, r, r^2, r^3, sr, sr^2, sr^3\}$ which has the following 10 crisp subgroups:

$$e, \{e, s\}, \{e, r^2\}, \{e, sr\}, \{e, sr^2\}, \{e, sr^3\}, \{e, s, r^2, sr^2\}, \{e, r^2, sr, sr^3\}, \{e, r, r^2, r^3\}, D_8.$$

So D_8 has 7 maximal chains:

$$D_8 \supseteq \{e, s, r^2, sr^2\} \supseteq \{e, s\} \supseteq e$$

$$D_8 \supseteq \{e, s, r^2, sr^2\} \supseteq \{e, r^2\} \supseteq e$$

$$D_8 \supseteq \{e, s, r^2, sr^2\} \supseteq \{e, sr^2\} \supseteq e$$

$$D_8 \supseteq \{e, r^2, sr, sr^3\} \supseteq \{e, r^2\} \supseteq e$$

$$D_8 \supseteq \{e, r^2, sr, sr^3\} \supseteq \{e, sr\} \supseteq e$$

$$D_8 \supseteq \{e, r^2, sr, sr^3\} \supseteq \{e, sr^3\} \supseteq e$$

$$D_8 \supseteq \{e, r, r^2, r^3\} \supseteq \{e, r^2\} \supseteq e$$

Example 2.2.3. The quaternion group $Q_8 = \langle i, j, k | i^2 = j^2 = k^2 = ijk = -1,\ ij = k, jk = i, ki = j\rangle$ has 5 crisp subgroups $1, \{-1, 1\}, \{-1, 1 - i, i\}, \{-1, 1, -j, j\}, \{-1, 1, -k, k\}$ yielding 3 maximal chains:

$$Q_8 \supseteq \{-1, 1 - i, i\} \supseteq \{-1, 1\} \supseteq 1$$

$$Q_8 \supseteq \{-1, 1 - j, j\} \supseteq \{-1, 1\} \supseteq 1$$

$$Q_8 \supseteq \{-1, 1 - k, k\} \supseteq \{-1, 1\} \supseteq 1$$

Ngcibi 2001 [71] in his thesis also computed the number of maximal chains for $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$ for $n, m \in \mathbb{Z}^+$. We state without proof (the proofs are available in his thesis), the Lemmas 2.2.4–2.2.9 from his work.

Lemma 2.2.4. *$\mathbb{Z}_p \times \mathbb{Z}_p$ has $p + 1$ maximal chains.*

Lemma 2.2.5. $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ *has* $2p + 1$ *maximal chains.*

Lemma 2.2.6. $\mathbb{Z}_{p^3} \times \mathbb{Z}_p$ *has* $3p + 1$ *maximal chains.*

Lemma 2.2.7. $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$ *has* $np + 1$ *maximal chains.*

Lemma 2.2.8. $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^2}$ *has* $\left[\binom{n}{2} + \binom{n-1}{1}\right] p^2 + \left[\binom{n+1}{1}\right] p + 1$ *maximal chains.*

Lemma 2.2.9. $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^3}$ *has* $\left[\binom{n}{3} + 2\binom{n-1}{2} + 2\binom{n-2}{1}\right] p^3 + \left[\binom{n+1}{2} + \binom{n}{1}\right] p^2 + \binom{n+2}{1} p + 1$ *maximal chains.*

Once it's clear, we shall, throughout this thesis, use $p_1^{n_1} p_2^{n_2} \cdots p_n^{n_m}$ to refer to the group $\mathbb{Z}_{{p_1}^{n_1}} \times \mathbb{Z}_{{p_2}^{n_2}} \times \cdots \times \mathbb{Z}_{{p_m}^{n_m}}$.

2.3 The Number of Maximal Chains of Finite Cyclic Groups

2.3.1 Maximal Chains of $\mathbb{Z}_{p^n}$

Here we discuss the maximal chains of $\mathbb{Z}_{p^n}$, p a prime, $n \in \mathbb{Z}^+$. The cyclic group $\mathbb{Z}_{p^n}$ has only 1 maximal chain $\mathbb{Z}_{p^n} \supseteq \mathbb{Z}_{p^{n-1}} \supseteq \cdots \supseteq \mathbb{Z}_{p^2} \supseteq \mathbb{Z}_p \supseteq 0$.

For example for $n = 2$, we have the maximal chain $\mathbb{Z}_{p^2} \supseteq \mathbb{Z}_p \supseteq 0$ and in particular $\mathbb{Z}_{2^2} \supseteq \mathbb{Z}_2 \supseteq 0$ for $p = 2$.

2.3.2 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$

In this subsection, we look at the maximal chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ for p, q distinct primes, $n, m \in \mathbb{Z}^+$. To achieve this, we systematically break down this subsection into Subsubsections 2.3.2.1–2.3.2.3.

2.3.2.1 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$

Here we proceed case by case for increasing values of $n \in \mathbb{Z}^+$ with fixed q.

For $n = 1, 2$, the groups $\mathbb{Z}_p \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ have respectively 2 and 3 maximal chains:

$$pq \supseteq p \supseteq 0$$
$$pq \supseteq q \supseteq 0$$

$$p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^2q \supseteq p^2 \supseteq p \supseteq 0$$

When $n = 3, 4$, then $\mathbb{Z}_{p^3} \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^4} \times \mathbb{Z}_q$ have respectively 4 and 5 maximal chains:

$$p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$
$$p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

From the above cases, we generate Table 2.1 which shows that the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ has $\frac{(n+1)!}{n!1!}$ maximal chains.

Proposition 2.3.1. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ has $\frac{(n+1)!}{n!1!}$ maximal chains.*

2.3.2.2 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$

We fix $m = 2$ and observe the pattern for increasing values of n.

For $n = 1, 2$, the groups $\mathbb{Z}_p \times \mathbb{Z}_{q^2}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ have 3 and 6 maximal chains respectively:

$$pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$pq^2 \supseteq q^2 \supseteq q \supseteq 0$$

$$p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$

When $n = 3$, the group $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2}$ has 10 maximal chains:

$$p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

$$\underset{Ctd}{\Rightarrow}$$

$$p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

When $n = 4$, the group $\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^2}$ has 15 maximal chains:

$$p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$\underset{Ctd}{\Rightarrow}$$ [1]

$$p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$
$$p^4q^2 \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

The number of maximal chains for $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$ when $n = 1, 2, 3, 4$ are summarized in Table 2.2. From this table, we deduce that the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$ has $\frac{(n+2)!}{n!2!}$ maximal chains.

Proposition 2.3.2. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$ has $\frac{(n+2)!}{n!2!}$ maximal chains.*

2.3.2.3 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$

Similarly, we look at different cases of n and observe the general pattern in the number of maximal chains.

For $n = 1, 2$, the groups $\mathbb{Z}_p \times \mathbb{Z}_{q^3}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^3}$ have respectively 4 and 10 maximal chains:

[1] This means that the list of maximal chains of the group continues to the right.

$$p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^3 \supseteq p^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 \qquad p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 \qquad p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 \qquad p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 \qquad p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$

For $n = 3$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^3}$ has 20 maximal chains:

$$p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 \quad \underset{Ctd}{\Rightarrow} \quad p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 \qquad p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

From these cases, we generate Table 2.3 and deduce that $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$ has $\frac{(n+3)!}{n!3!}$ maximal chains.

Proposition 2.3.3. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$ has $\frac{(n+3)!}{n!3!}$ maximal chains.*

The formulas in Subsubsections 2.3.2.1– 2.3.2.3 are used to generate Table 2.4 of the general cases. From this table, we deduce that the number of maximal chains for $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is given by

$$\frac{(n+m)!}{n!m!}.$$

Theorem 2.3.4. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ has $\frac{(n+m)!}{n!m!}$ maximal chains.*

2.3.3 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$

Here we shall consider the maximal chains for $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ for distinct primes p, q, r and $n, m \in \mathbb{Z}^+$. We follow the same technique used in $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$.

2.3.3.1 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$

For $n = 1, 2$, the groups $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ have respectively 6 and 12 maximal chains listed below.

$$pqr \supseteq pq \supseteq p \supseteq 0$$
$$pqr \supseteq pq \supseteq q \supseteq 0$$
$$pqr \supseteq pr \supseteq p \supseteq 0$$
$$pqr \supseteq pr \supseteq r \supseteq 0$$
$$pqr \supseteq qr \supseteq q \supseteq 0$$
$$pqr \supseteq qr \supseteq r \supseteq 0$$

$$p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$
$$p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0$$
$$p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0$$
$$p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0$$
$$p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0$$
$$p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0$$
$$p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$
$$p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0$$
$$p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0$$

When $n = 3, 4$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and $\mathbb{Z}_{p^4} \times \mathbb{Z}_q \times \mathbb{Z}_r$ have respectively 20 and 30 maximal chains:

$$\begin{array}{rcr}
p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 & & p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 & & p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 & & p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 & & p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 & & p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 & \underset{Ctd}{\Rightarrow} & p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 & & p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 & & p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 & & p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 \\
p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 & & p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}$$

$$\begin{array}{rcr}
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 & & p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 & & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 & & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 & \underset{Ctd}{\Rightarrow} & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 & & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 & & p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 & & p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0
\end{array}$$

These cases are used to generate Table 2.5, from which we deduce that the number of maximal chains for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$ is a polynomial in n:

$$n^2 + 3n + 2 = (n+2)(n+1) = \frac{(n+2)!}{n!} = \frac{(n+1+1)!}{n!1!}.$$

Proposition 2.3.5. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $\frac{(n+1+1)!}{n!1!}$ maximal chains.*

2.3.3.2 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

For $n = 1$, the group $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has 12 maximal chains:

$$pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$
$$pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0$$
$$pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0$$
$$pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0$$
$$pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0$$
$$pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0$$

$$\underset{Ctd}{\Rightarrow}$$

$$pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0$$
$$pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0$$
$$pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0$$

For $n = 2$, the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has 30 maximal chains:

$$p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$\underset{Ctd}{\Rightarrow}$$

$$p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0$$

$\underset{Ctd}{\Rightarrow}$

$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0$$
$$p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$

$\underset{Ctd}{\Rightarrow}$

$$p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$

For $n = 3$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has 60 maximal chains:

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0$$

$\underset{Ctd}{\Rightarrow}$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0$$
$$p^3q^2r \supseteq^2 q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0$$

$\underset{Ctd}{\Rightarrow}$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 \quad \underset{Ctd}{\Rightarrow} \quad p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0$$
$$p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 \qquad p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

The above cases give us Table 2.6 from which $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $\frac{(n+2+1)!}{n!2!1!}$ maximal chains.

Proposition 2.3.6. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $\frac{(n+2+1)!}{n!2!1!}$ maximal chains.*

2.3.3.3 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

By a similar computation, the number of maximal chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ is $\frac{(n+3+1)!}{n!3!1!}$.

Proposition 2.3.7. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ has $\frac{(n+3+1)!}{n!3!1!}$ maximal chains.*

From the formulas in Subsubsections 2.3.3.1–2.3.3.3 , we generalize that the number of maximal chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ is given by $\frac{(n+m+1)!}{n!m!}$.

Proposition 2.3.8. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ has $\frac{(n+m+1)!}{n!m!}$ maximal chains.*

2.3.4 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_{r^s}$

More generally, from Proposition 2.3.8, we can deduce that the number of maximal chains of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_{r^s}$, p, q, r distinct primes, $n, m, s \in \mathbb{Z}^+$ is $\frac{(n+m+s)!}{n!m!s!}$.

Theorem 2.3.9. *[68] The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_{r^s}$ has $\frac{(n+m+s)!}{n!m!s!}$ maximal chains.*

2.3.5 Maximal Chains of $\mathbb{Z}_{p_1{}^{n_1}} \times \mathbb{Z}_{p_2{}^{n_2}} \times \cdots \times \mathbb{Z}_{p_m{}^{n_m}}$

The pattern from Theorem 2.3.9 can be extended to get Theorem 2.3.10, the most general formula for the number of maximal chains of any finite cyclic group.

Theorem 2.3.10. *[68] The group $\mathbb{Z}_{p_1{}^{n_1}} \times \mathbb{Z}_{p_2{}^{n_2}} \times \cdots \times \mathbb{Z}_{p_m{}^{n_m}}$, $p_i's$ distinct primes, $n_i's \in \mathbb{Z}^+$ has* $\dfrac{(n_1 + n_2 + \cdots + n_m)!}{n_1!n_2!\cdots n_m!} = \dfrac{\left(\sum_{i=1}^{m} n_i\right)!}{\prod_{i=1}^{m} n_i!}$ *maximal chains.*

Table 2.1: Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$

n	$p^n q$	Number of max chains
1	pq	$2 = \frac{(1+1)!}{1!1!}$
2	p^2q	$3 = \frac{(2+1)!}{2!1!}$
3	p^3q	$4 = \frac{(3+1)!}{3!1!}$
4	p^4q	$5 = \frac{(4+1)!}{4!1!}$
⋮	⋮	⋮
k	p^kq	$\frac{(k+1)!}{k!1!}$

Table 2.2: Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$

n	$p^n q^2$	Number of max chains
1	pq^2	$3 = \frac{(1+2)!}{1!2!}$
2	p^2q^2	$6 = \frac{(2+2)!}{2!2!}$
3	p^3q^2	$10 = \frac{(3+2)!}{3!2!}$
4	p^4q^2	$15 = \frac{(4+2)!}{4!2!}$
⋮	⋮	⋮
k	p^kq^2	$\frac{(k+2)!}{k!2!}$

Table 2.3: Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$

n	$p^n q^3$	Number of max chains
1	pq^3	$4 = \frac{(1+3)!}{1!3!}$
2	p^2q^3	$10 = \frac{(2+3)!}{2!3!}$
3	p^3q^3	$20 = \frac{(3+3)!}{3!3!}$
⋮	⋮	⋮
k	p^kq^3	$\frac{(k+3)!}{k!3!}$

Table 2.4: Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$

m	Group	Number of max chains
1	p^nq	$\frac{(n+1)!}{n!1!}$
2	p^nq^2	$\frac{(n+2)!}{n!2!}$
3	p^nq^3	$\frac{(n+3)!}{n!3!}$
⋮	⋮	⋮
k	p^nq^k	$\frac{(n+k)!}{n!k!}$

Table 2.5: Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$

n	$p^n qr$	Number of max chains
1	pqr	$6 = \frac{(1+1+1)!}{1!1!1!}$
2	p^2qr	$12 = \frac{(2+1+1)!}{2!1!1!}$
3	p^3qr	$20 = \frac{(3+1+1)!}{3!1!1!}$
4	p^4qr	$30 = \frac{(4+1+1)!}{4!1!1!}$
$\vdots$	$\vdots$	$\vdots$
k	p^kqr	$\frac{(k+1+1)!}{k!1!1!}$

Table 2.6: Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

n	$p^n q^2 r$	Number of max chains
1	pq^2r	$12 = \frac{(1+2+1)!}{1!2!1!}$
2	p^2q^2r	$30 = \frac{(2+2+1)!}{2!2!1!}$
3	p^3q^2r	$60 = \frac{(3+2+1)!}{3!2!1!}$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^2r	$\frac{(k+2+1)!}{k!2!1!}$

CHAPTER 3. FUZZY GROUP THEORY

3.1 Introduction

Just as its name implies, the theory of fuzzy sets is basically a theory of graded concepts in which everything is a matter of degree or, to put it figuratively, everything has elasticity. Earlier, it was established that fuzzy sets have a lot of application in different areas ranging from control of small devices such as cameras to control of large systems such as cement kilns, subways, and nuclear power plants. Therefore their study, through counting the fuzzy subgroups of a finite group in order to classify them is of great importance in fuzzy group theory. In this chapter, we discuss the theory of fuzzy sets, fuzzy subgroups, fuzzy equivalence relation and fuzzy isomorphism.

3.2 Fuzzy Set Theory

The motivation behind the development of fuzzy set theory was the quest to provide a formal setting for incomplete and gradual information, as expressed by people in natural language. A set can be described by enumerating its elements using a characteristic function. This is a function that assigns a value 0 and 1 to each element in the universe of discourse based on membership or non-membership of each element.

Definition 3.2.1. The *characteristic function or indicator function* of a subset A of a set X is the function $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Let X be a non-empty set. Crisp subsets of X are characterized by $\chi_A \in \{0,1\}^X$ ($\{0,1\}^X$ is the set of functions from X to $\{0,1\}$) where $A \subseteq X$.

Example 3.2.2. Let $X = \{0,1,2,3,4\}$ be the universal set. The crisp subset $A = \{0,1,2\}$, can be described by the function $f : X \to \{0,1\}$ defined by $\{(0,1),(1,1),(2,1),(3,0),(4,0)\}$.

If X is a collection of objects such that there is no defined conditions on the elements of $A \subseteq X$, then we cannot tell with certainty the truth value of the statement $x \in A$. It is this concept that motivated the development of fuzzy sets by [98].

Definition 3.2.3. If X is a non-empty set, the power set of X denoted $\mathcal{P}(X)$ is the set of all subsets of X.

Suppose 2 denotes the 2-element set $\{0,1\}$ with the ordering $0 \leq 1$, then each subset A of X can be identified with the characteristic function $\chi_A(x)$. Therefore the set of all characteristic functions on X can be identified with $\mathcal{P}(X)$ and both sets are sometimes denoted by 2^X [59].

Unlike in crisp sets where there is total membership, say $x \in A$, fuzzy sets allow partial membership, i.e., elements can partially belong to a set.

Example 3.2.4. Let $X = \{0,1,2,3,4,5\}$ be the universal set. We can define a fuzzy subset of "small numbers" of X, as a function $f : X \to [0,1]$ defined by $\{(0,1),(1,0.6),(2,0.4),(3,0.06),$ $(4,0),(5,0)\}$.

This means 0 is definitely small, 1 is pretty small, 2 is sort of small, 3 is not very small, and 4 and 5 are not at all small. Fuzzy logic allows us to use the "fuzzy" terminology which is very useful in ordinary human discourse.

Definition 3.2.5. [59] Let $I = [0,1]$ be the unit interval of real numbers with the usual ordering and let X be a non-empty set. A fuzzy subset of X is characterized by a function $\mu : X \to I$.

μ is called the *membership function* and $\mu(x)$ is the *degree of membership* of the element x to the fuzzy subset of X defined by μ. A fuzzy subset is therefore identified with its membership function μ.

We denote the family of all fuzzy subsets of X by I^X. This family I^X is a lattice [15, p.1], with the point-wise ordering induced by the ordering of I. Throughout this thesis, a fuzzy subset shall simply be called a fuzzy set. We use the interval $[0,1]$ with the usual ordering where $\vee$ means *supremum* (*or union*) and $\wedge$ means *infimum* (*or intersection*).

Remark 3.2.6. [59] Crisps (ordinary) subsets of X are fuzzy subsets of X when identified with characteristic functions of these subsets.

3.2.1 Operations on Fuzzy Sets

1. *Union*

   The union of two fuzzy sets μ_A and μ_B, also called the maximum criterion is defined as $\mu_{A\cup B} = max\{\mu_A, \mu_B\} = \mu_A \vee \mu_B$. Analogous to groups, this is the "smallest" fuzzy set containing both μ_A and μ_B.

2. *Intersection*

   The intersection of two fuzzy sets μ_A and μ_B, called the minimum criterion, is defined as $\mu_{A\cap B} = min\{\mu_A, \mu_B\} = \mu_A \wedge \mu_B$. Similarly this is the"largest" fuzzy set contained in both μ_A and μ_B.

3. *Complement*

   The complement of μ_A is defined as $\mu_A^C(x) = 1 - \mu_A(x)$.

4. *Inclusion*

   Let X be a non-empty set and suppose μ and ν are two fuzzy sets. Then $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$.

5. *Equality*

   $\mu = \nu$ if and only if $\mu(x) = \nu(x)$ for all $x \in X$.

6. *Null set*

   This is defined by the membership function $\mu_\emptyset(x) = 0$, $\forall\ x \in X$.

7. *Whole set*

   The whole fuzzy set of X is defined as $\mu_X(x) = 1$, $\forall\ x \in X$.

Remark 3.2.7. It is important for the reader to note the following:

(i) unlike in the conventional set theory, the intersection of a fuzzy set and its complement is not the empty fuzzy set unless the fuzzy set is a crisp subset,

(ii) the union of a fuzzy set and its complement is not the universal set X,

(iii) the membership grades of fuzzy sets on a finite universal set are not probabilities even though they take values in the unit interval $[0, 1]$. This is clear from the fact that the summation of these membership grades, unlike in probability theory, do not necessarily add up to 1. Moreover the definition of membership functions is very subjective since different people can define the membership functions for the same concept differently. This subjectivity should not be confused with randomness like in probability but rather differences in the way different people analyze and present abstract concepts. This subjectivity and non-randomness accounts for the primary difference between fuzzy set theory and probability theory.

Definition 3.2.8. A fuzzy set $\emptyset$ of X is called an empty fuzzy set if for each $x \in X$, we have $\emptyset(x) = 0$.

Definition 3.2.9. The core of a fuzzy set μ, denoted $core(\mu)$, is the set of all points $x \in X$ such that $\mu(x) = 1$.

Example 3.2.10. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and suppose we have a fuzzy set $\mu = \{(x_1, 0), (x_2, 1), (x_3, 0.8), (x_4, 0.4), (x_5, 0.6)\}$. Then $core(\mu) = \{(x_2, 1)\}$.

Definition 3.2.11. A fuzzy subset $\mu : X \to I$ is called a *fuzzy point* if $\mu(x) = 0$ for all except only one $x \in X$.

Definition 3.2.12. The *support* of μ, denoted by *supp* (μ), is defined as $supp(\mu) = \{x \in X : \mu(x) > 0\}$.

Example 3.2.13. Let $X = \{65, 70, 82, 94, 76, 88\}$ be the set of Calculus I test scores for six University of Fort Hare students. Define a fuzzy set μ on X as $\{(65, 0), (70, 0.52), (82, 0), (94, 0.68), (76, 0.76), (88, 0.9)\}$. Then $supp(\mu) = \{70, 94, 76, 88\}$.

Definition 3.2.14. The *co-support* of μ, denoted *co-supp*(μ), is a non-fuzzy set consisting of all elements that are completely outside a given fuzzy set.

Example 3.2.15. From Example 3.2.13 we have that *co-supp*$(\mu) = \{65, 82\}$.

Definition 3.2.16. A fuzzy set whose support is a singleton point in X is called a *fuzzy singleton*.

3.2.2 Images and Preimages of Fuzzy Sets

Let X and Y be two non-empty sets and $f : X \to Y$ a function from X to Y. Suppose $\mu : X \to I$ is a fuzzy subset of X. The fuzzy subset of Y, $f(\mu)$ is defined by

$$f(\mu)(y) = \begin{cases} sup\{\mu(x) : y = f(x)\} \\ 0 \quad \text{if } y \neq f(x) \ \forall x \in X. \end{cases}$$

Therefore for all x such that $f(x) = y$, the degree to which y belongs to $f(\mu)$ is at least as much as the degree to which x belongs to μ.

Definition 3.2.17. Suppose $f : X \to Y$ is a function. If μ is a fuzzy subset of Y, then $f^{-1}(\mu)$ is a fuzzy subset of X defined by $f^{-1}(\mu)(x) = \mu(f(x))$, $x \in X$.

In [3], the solution of the problem of showing a one-to-one correspondence between the family of fuzzy subgroups of a group, containing the kernel of a given homomorphism, and the family of fuzzy subgroups of the homomorphic image of the given group was provided. It was further shown by [3], that an ordinary kernel gives rise to the notion of fuzzy quotient group in a natural way. Consequently, the fundamental theorem of homomorphisms is established for fuzzy subgroups. Moreover, [3] proved that the homomorphic image of a fuzzy subgroup is always a fuzzy subgroup.

3.3 Fuzzy Subgroups

Definition 3.3.1. [77] Suppose G is a group. A fuzzy subset $\mu : G \to I$ of G is said to be a fuzzy subgroup of G if

(i) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in G$,

(ii) $\mu(x^{-1}) = \mu(x)$ for all $x \in G$.

Letting $y = x^{-1}$ in (i), we obtain $\mu(e) \geq \mu(x) \wedge \mu(x^{-1}) = \mu(x)$ by (ii). Therefore $\mu(x) \leq \mu(e)$ for all $x \in G$.

Clearly the *Supp* μ is a subgroup of G whenever μ is a fuzzy subgroup of G. In this thesis we assume $\mu(e) = 1$.

Proposition 3.3.2. *[4] Let G be a group. If μ is a fuzzy subgroup of G, then $\mu(xy) = min\{\mu(x), \mu(y)\}$ for each x, y in G with $\mu(x) \neq \mu(y)$.*

Proof. Assume $\mu(x) > \mu(y)$, then

$$\begin{aligned} \mu(y) &= \mu(x^{-1}xy) \geq min\{\mu(x^{-1}), \mu(xy)\} \\ &= min\{\mu(x), \mu(xy)\} = \mu(xy) \geq min\{\mu(x), \mu(y)\} = \mu(y) \end{aligned}$$

Therefore $\mu(xy) = \mu(y) = min\{\mu(x), \mu(y)\}$. □

Definition 3.3.3. A fuzzy subgroup μ of a group G is said to be proper if $Im\ \mu$ has at least two elements i.e., μ is not constant.

3.3.1 T-Norms

The triangular norm (t-norm) and triangular conorm (t-conorm) trace their roots in studies of probabilistic metric spaces [29]. Later, courtesy of [30, 5], the concepts would see their way in fuzzy set theory. Anthony and Sherwood in [6, 7] redefined fuzzy subgroups in terms of a t-norm. Membership values of intersection, union and complement of fuzzy sets can be calculated using t-norm, t-co-norm and negation functions, respectively.

Definition 3.3.4. A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions for all $x, y, z \in [0,1]$:

(i) $T(x,y) = T(y,x)$: *commutativity,*

(ii) $T(x,y) \leq T(x,z)$, if $y \leq z$: *monotonicity,*

(iii) $T(x, T(y,z)) = T(T(x,y),z)$: *associativity,*

(iv) $T(x,1) = x$.

Definition 3.3.5. A t-conorm is a function $T^* : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions for all $x, y, z \in [0,1]$:

(i) $T^*(x,y) = T^*(y,x)$: *commutativity,*

(ii) $T^*(x,y) \leq T^*(x,z)$, if $y \leq z$: *monotonicity,*

(iii) $T^*(x, T(y,z)) = T^*(T^*(x,y),z)$: *associativity,*

(iv) $T^*(x,0) = x$.

Definition 3.3.6. Let $N : [0,1] \to [0,1]$. N is a negation function if it satisfies the following conditions:

(i) $N(0) = 1$,

(ii) $N(1) = 0$,

(iii) $N(x) \leq N(y)$, if $x \leq y$: *monotonicity.*

A fuzzy subgroup can also be defined using t-norms and Zadeh's T-operators:

(i) $T(x,y) = min(x,y)$,

(ii) $T^*(x,y) = max(x,y)$,

(iii) $N(x) = 1 - x$,

where

$$min(x,y) = \begin{cases} x \text{ if } x \leq y \\ y \text{ if } y < x. \end{cases}$$

Definition 3.3.7. Let G be a group. A function $\mu : G \to [0,1]$ is a fuzzy subgroup of G with respect to t-norm T if and only if for all $x, y \in G$

(i) $\mu(x,y) \geq T(\mu(x), \mu(y))$,

(ii) $\mu(x^{-1}) = \mu(x)$,

(iii) $\mu(0) = 1$.

This definition leads to the following theorem by Osman.

Theorem 3.3.8. *[73] A fuzzy set $\mu : G \to [0,1]$ is a fuzzy subgroup of a group G with respect to t-norm T, if and only if $\mu(0) = 1$ and $\mu(x-y) \geq T(\mu(x), \mu(y))$ for all $x, y \in G$.*

Theorem 3.3.9. *[83] Let μ_i be a fuzzy subgroup of the group G_i for each $i = 1, 2, \cdots, n$. Then $\mu = \mu_1 \times \mu_2 \times \cdots \mu_n$ is a fuzzy subgroup of $G = G_1 \times G_2 \times \cdots \times G_n$ defined by*

$$\mu(x_1, x_2, \cdots, x_n) = min\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\}.$$

Proposition 3.3.10. *(**J.M. Anthony and H. Sherwood**) Let G be a group and suppose $\mu : G \to [0,1]$ satisfies*

(i) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) = \mu(x)$,

(iii) $\mu(0) > 0$.

Then the function φ defined by $\varphi(x) = \mu(x)/\mu(0)$ for each $x \in G$ is a fuzzy subgroup of G with respect to min such that $\varphi(0) = 1$.

Proof. [72]. □

Definition 3.3.11. [4] Let μ be a fuzzy subset of G and $I = [0, 1]$. For any $t \in I$, the set $G_t = \{x \in G | \mu(x) \geq t\}$ is called a *level subset* of the fuzzy subset μ.

Remark 3.3.12. If μ is a fuzzy subgroup of G, then G_t is a fuzzy subgroup of G for any $t \in I$.

Definition 3.3.13. The subgroups G_t are called *level subgroups* of G for any $t \in I$.

3.3.2 Operations on Fuzzy Subgroups

1. *Union*

   The union of two fuzzy subgroups μ and ν denoted $\mu \vee \nu$ is defined as

   $$(\mu \vee \nu)(x) = max\{\mu(x), \nu(x)\}.$$

   Proposition 3.3.14. *[77] A group cannot be the union of two proper fuzzy subgroups.*

2. *Intersection*

   The intersection of two fuzzy subgroups μ and ν denoted $\mu \wedge \nu$ is defined as

   $$(\mu \wedge \nu)(x) = min\{\mu(x), \nu(x)\}.$$

   Proposition 3.3.15. *If μ and ν are fuzzy subgroups of a group G, then their intersection $\mu \wedge \nu$ is a fuzzy subgroup of G.*

   Proof. [72]. □

3. *Sum*

   Definition 3.3.16. The sum of two fuzzy subgroups μ and ν of a group G is defined by

   $$(\mu + \nu)(x) = sup\{\mu(x_1) \wedge \nu(x_2) : x_1 + x_2 = x\}, \ x \in G.$$

3.3.3 Group Theory and Fuzzy Group Theory Analogs

Here, we look at some group theoretic concepts in a fuzzy group theoretic sense as also discussed by [72] . We discuss the terms *conjugacy*, *normal*, *normalizer*, *commutator*[1], *solvable*[2] and *Lagrange's Theorem*.

Definition 3.3.17. Let μ and ν be two fuzzy subgroups of a group G. Then μ is said to be *conjugate* to ν if $\exists$ some $g \in G$ such that $\mu(x) = \nu(g^{-1}xg)$ $\forall$ $x \in G$.

Theorem 3.3.18. *[4] If μ is a fuzzy subgroup of G, then the following conditions are equivalent:*

(i) $\mu(xy) = \mu(yx)$ *for all* $x, y \in G$,

(ii) $\mu(xyx^{-1}) = \mu(y)$ *for all* $x, y \in G$.

Proof. Let $x, y \in G$.
$(i) \Rightarrow (ii)$: If $\mu(xy) = \mu(yx)$, $\mu(xyx^{-1}) = \mu((xy)x^{-1}) = \mu(x^{-1}(xy)) = \mu(x^{-1}xy) = \mu(y)$.
$(ii) \Rightarrow (i)$: Let $\mu(xyx^{-1}) = \mu(y)$. Since $xy = x(yx)x^{-1}$, $\mu(xy) = \mu(x(yx)x^{-1}) = \mu(yx)$. □

Definition 3.3.19. A fuzzy subgroup μ of G which satisfies the equivalent conditions of Theorem 3.3.18 is said to be a *normal fuzzy subgroup* of G. We also say that μ is a *fuzzy normal subgroup* of G.

Definition 3.3.20. [103] Let μ be a fuzzy subgroup of G. For any $g \in G$, $g\mu$ and μg are fuzzy subsets of G, defined by

$$(g\mu)(x) = \mu(g^{-1}x), \ \forall g \in G,$$

$$(\mu g)(x) = \mu(xg^{-1}), \ \forall g \in G.$$

Proposition 3.3.21. *[55] Let μ be a fuzzy subgroup of G, then μ is a normal subgroup of G if and only if $g\mu = \mu g$ for all $g \in G$.*

Definition 3.3.22. Let μ be a fuzzy subgroup of G. The set

$$N_\mu(G) = \{g \in G | \mu(gx^{-1}g) = \mu(x) \ \forall \ x \in G\}$$

[1] For a group G and $x, y \in G$, the element $x^{-1}y^{-1}xy$ denoted by $[x, y]$ is called the commutator of x and y.

[2] A group G is solvable if there is a chain of subgroups $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G$ such that G_{i+1}/G_i, $0 \leq i \leq s-1$ is abelian.

is called the *normalizer* of μ in G.

Proposition 3.3.23. *If μ is a fuzzy subgroup of G. Then*

(i) $N_\mu(G)$ is a subgroup of G,

(ii) μ is fuzzy normal if and only if $N_\mu(G) = G$,

(ii) μ is a normal fuzzy subgroup of $N_\mu(G)$.

Proof. [72]. □

Definition 3.3.24. Let μ be a fuzzy subset of a group G. The fuzzy subgroup ν of G generated by μ is the smallest fuzzy subgroup of G containing μ and is also given by

$$\nu(x) = \bigvee_{x=a_1a_2\cdots a_n} \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n).$$

Definition 3.3.25. [72] Let μ and ν be fuzzy subsets of a group G. A *commutator* of μ, ν is a fuzzy subgroup $[\mu, \nu]$ of G which is generated by the fuzzy set (μ, ν) of G defined as follows

$$(\mu, \nu)(x) = \begin{cases} \bigvee\limits_{x=[a,b]} \{\mu(a) \wedge \nu(b)\} & \text{if x is a commutator of } G \\ 0 \quad \text{otherwise.} \end{cases}$$

Remark 3.3.26. The following remarks follow from the definition of a commutator of fuzzy subgroups:

(i) for any two fuzzy subsets μ, ν of G, $[\mu, \nu] = [\nu, \mu]$,

(ii) if μ and ν are normal fuzzy subgroups of G then $[\mu, \nu]$ is a normal fuzzy subgroup of G.

Definition 3.3.27. Let μ be a fuzzy subgroup of G. Then the set

$$C_\mu(G) = \{x \in G | [x, y] = \mu(e), \ \forall \ y \in G\}$$

is called the *centralizer* of μ in G, where $[x, y] = x^{-1}y^{-1}xy$.

Proposition 3.3.28. *Let μ be a fuzzy subgroup of a group G, then* *(i)* $C_\mu(G)$ *is a subgroup of* G*,*

(iii) $C_\mu(G)$ *is a normal subgroup of* $N_\mu(G)$*.*

Proposition 3.3.29. ***(Yunjie Zhand and Dong Yu)*** *Let* μ *be a fuzzy subgroup of a group* G *and suppose* $n \in \mathbb{N}$*. Then* $\mu((xy)^n) = \mu(x^n y^n),\ \forall\ x, y \in C_\mu(G)$*.*

Proof. From Propositions 3.3.23 (iii) and 3.3.28(ii), we have μ is a normal fuzzy subgroup of $C_\mu(G)$. Therefore for all $x, y \in C_\mu(G)$, we have

$$\begin{aligned}
\mu((xy)^n) &= \mu(xy\cdots xyxyxy) = \mu(xy\cdots xyxy^2x[x,y]) \\
&\geq min\{\mu(xy\cdots xyxy^2x), \mu([x,y])\} \\
&= \mu(xy\cdots xyxy^2x) = \mu(x^2y\cdots xyxy^2) \\
&= \mu(x^2y\cdots xy^3x[x,y]) \geq \mu(x^3y\cdots xy^3) \\
\geq \cdots &\geq \mu(x^{n-1}yxy^{n-1}) \\
&= \mu(x^{n-1}y^n x[x, y^{n-1}]) \\
&\geq \mu(x^{n-1}y^n x) = \mu(x^n y^n)
\end{aligned}$$

Thus we have

$$\mu((xy)^n) \geq \mu(x^n y^n). \tag{3.3.1}$$

$$\begin{aligned}
\mu(x^n y^n) &= \mu(x^{n-1}y^n x) = \mu(x^{n-1}yxy^{n-1}[y^{n-1}, x]) \\
&\geq \mu(x^{n-1}yxy^{n-1}) \\
\geq \cdots &\geq \mu(xy\cdots xyxy^2x) \\
&= \mu(xy\cdots xyxyxy[x,y]) \geq \mu((xy)^n).
\end{aligned}$$

Thus we have

$$\mu(x^n y^n) \geq \mu((xy)^n). \tag{3.3.2}$$

Therefore from Equations 3.3.1 and 3.3.2 we have $\mu((xy)^n) = \mu(x^n y^n)$. □

Definition 3.3.30. [44] Let μ be a fuzzy subgroup of G and ν a normal fuzzy subgroup of G such that $\nu \leq \mu$. The *quotient* μ/ν is a fuzzy subset of $\mathcal{F}_\nu = \{x\nu : x \in G\}$ defined by $\mu/\nu(x\nu) = \mu(x)$ for all $x \in G$.

Remark 3.3.31. The quotient μ/ν is a fuzzy subgroup of $\mathcal{F}_\nu$.

Definition 3.3.32. A fuzzy subgroup μ of a group G is *solvable* if there exists a finite chain $\{\mu_i\}$ of fuzzy subgroups of G

$$0 = \mu_0 \trianglelefteq \mu_1 \trianglelefteq \cdots \trianglelefteq \mu_s = \mu$$

such that μ_{i+1}/μ_i, $0 \leq i \leq s-1$ is abelian.

Proposition 3.3.33. *[44] Let μ be a solvable fuzzy subgroup of a group G. Then the Supp μ is solvable in G.*

Definition 3.3.34. Let μ be a fuzzy subgroup of group G. Let $H = \{x \in G | \mu(x) = \mu(e)\}$. Then μ is fuzzy abelian if H is an abelian subgroup of G.

Proposition 3.3.35. *[44] If μ is fuzzy abelian, then μ is solvable.*

Proposition 3.3.36. *A fuzzy subgroup of a solvable fuzzy subgroup is solvable.*

Proof. See Makamba [44]. □

Mukherjee and Bhattacharya in [55] gave the fuzzy version of the famous *Lagrange's Theorem*[3]. We discuss some of their results below.

Definition 3.3.37. [35] Let μ be a fuzzy subgroup of a group G. For $x \in G$, the least positive integer n, such that $\mu(x^n) = \mu(e)$ is called the fuzzy order of x with respect to μ and denoted by $F\mathcal{O}_\mu(x)$. If no such n exists, then x is said to have infinite fuzzy order with respect to μ.

Definition 3.3.38. [35] Let μ be a fuzzy subgroup of a group G. The least positive integer n, such that $\mu(x^n) = \mu(e)$ for all $x \in G$ is called the order of μ denoted by $\mathcal{O}(\mu)$. If no such n exists, then μ is said to have infinite order.

Proposition 3.3.39. *Let μ be a fuzzy subgroup of a group G. If $\mu(x^m) = \mu(e)$ for some integer m, then $F\mathcal{O}_\mu(x) | m$.*

Let H be a subgroup of a group G and μ a fuzzy subgroup of G. Then μ restricted to H, denoted by μ_H, is a fuzzy subgroup of H.

[3] If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$

Proposition 3.3.40. *If H is a subgroup of G and μ is a fuzzy subgroup of G, then $\mathcal{O}(\mu_H) \leq \mathcal{O}(\mu)$.*

Definition 3.3.41. [97] Let μ be a subgroup of G. Let $G/\mu = \{x\mu | x \in G\}$. Then the cardinal number of G/μ (finite or infinite), denoted by $[G : \mu]$, is called the *index* of μ in G. In the notation of 3.3.30, $|\mathcal{F}_\nu|$ is the index of μ in ν.

Theorem 3.3.42. *[55] Let μ be a fuzzy subgroup of a finite group G. Then the index of μ divides the order of G.*

Theorem 3.3.43. *[35] Let H be a subgroup of G of finite order and μ be a fuzzy subgroup of G. Then $\mathcal{O}(\mu_H) | \mathcal{O}(\mu)$.*

Proof. Let $\mathcal{O}(\mu) = n$, then $\mu(x^n) = \mu(e)$ for all $x \in G$. By Proposition 3.3.40, $\mathcal{O}(\mu_H) \leq n$. If $\mathcal{O}(\mu_H) = n$, then we are done. Now, assume $\mathcal{O}(\mu_H) < n$, say $\mathcal{O}(\mu_H) = m$, then we have $(\mu_H)(x^m) = \mu(e)$, for all $x \in H$ i.e., $\mu(x^m) = \mu(e)$, for all $x \in H$. Then we have that m is the fuzzy order of at least one element x in H. Therefore $F\mathcal{O}_\mu(x) = m$ and $\mu(x^n) = \mu(e)$ for all $x \in G$ and by Proposition 3.3.39, $m | n$. Hence $\mathcal{O}(\mu_H) | \mathcal{O}(\mu)$. □

Corollary 3.3.44. *If H is a subgroup of G, $x \in G$ and μ is a fuzzy subgroup of G of finite order, then $F\mathcal{O}_\mu(x) | \mathcal{O}(\mu)$.*

For more details on $F\mathcal{O}_\mu(x)$ and $\mathcal{O}_\mu(x)$, see [35].

3.4 Fuzzy Equivalence Relations and Fuzzy Isomorphism

3.4.1 Fuzzy Equivalence Relation

In this subsection, we discuss different equivalence relations as defined by different authors. Later, we focus on the equivalence relation defined by Murali and Makamba in [62, 63, 67].

Definition 3.4.1. A *relation* between two sets X and Y is a subset $\mathcal{R}$ of $X \times Y$. We read $(x, y) \in \mathcal{R}$ as *"x is related to y"* and write $x\mathcal{R}y$.

Example 3.4.2. The *equality relation* " $=$ " on a set X is the subset $\{(x,x) : x \in X\}$ of $X \times X$.

So for any $x \in X$, we have $x = x$, but for different elements x and y in X, $(x,y) \notin =$ and we write $x \neq y$.

Example 3.4.3. The graph of the function f for $f(x) = x^3$ for all $x \in \mathbb{R}$ is the subset $\{(x,x^3)|x \in \mathbb{R}\}$ of $\mathbb{R} \times \mathbb{R}$. Therefore it is a relation on $\mathbb{R}$.

Definition 3.4.4. A *partition* of a set X is a collection of nonempty subsets of X such that every element of X is in exactly one of the subsets. We denote by $\bar{x}$, the cell containing the element x of X.

Definition 3.4.5. A relation $\mathcal{R}$ on a set X is an *equivalence relation* if it satisfies the following three properties for all $x, y, x \in X$:

(i) $x\mathcal{R}x$: *Reflexive property*,

(ii) if $x\mathcal{R}y$, then $y\mathcal{R}x$: *Symmetric property*,

(iii) if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$: *Transitive property*.

Example 3.4.6. Each partition on a set X yields an equivalence relation on X in a natural way defined as follows: For $x, y \in X$, let $x\mathcal{R}y$ if and only if x and y are in the same cell of partition. Clearly, $x\mathcal{R}x$ and if x is in the same cell as y, i.e., $(x\mathcal{R}y)$, then y is in the same cell as x, i.e., $(y\mathcal{R}x)$. Transitive property can be verified similarly.

Example 3.4.7. The equality relation discussed in Example 3.4.2 is an equivalence relation on X.

Example 3.4.8. The relation $\mathcal{R}$ on the set $\mathbb{Z}$ defined by $n\mathcal{R}m$ if and only if $nm \geq 0$ is not an equivalence relation. This is because $-3\mathcal{R}0$ and $0\mathcal{R}5$, but $-3\mathcal{R}5$. Therefore the transitive property fails.

Definition 3.4.9. [60] A *fuzzy relation* μ on the sets X and Y is a fuzzy subset of $X \times Y$ given by $\mu : X \times Y \to \mu(x,y)$. We say $\mu(x,y)$ is the degree to which x is related to y.

Remark 3.4.10. We have that μ is a binary relation on the set X.

Definition 3.4.11. [60] A fuzzy relation μ is said to be:

(i) reflexive if $\forall\, x \in X,\ \mu(x,x) = 1$,

(ii) symmetric if $\forall\, x, y \in X,\ \mu(x,y) = \mu(y,x)$,

(iii) transitive if $\mu \circ \mu \leq \mu$, where $(\mu \circ \mu)(x,y) = \sup_{z \in X}(\mu(x,y) \wedge \mu(y,z))$.

Tărnăuceanu in [93], defined an equivalence relation on fuzzy subgroups as follows:

Definition 3.4.12. [93] Two fuzzy subgroups μ and ν of G are equivalent if and only if $\mu(x) \geq \nu(y) \iff \nu(x) \geq \nu(y)$ for all $x, y \in G$.

Murali and Makamba in [62, 63, 67] gave a different equivalence relation from [93] as follows:

Definition 3.4.13. [63] Two fuzzy subgroups μ and ν of G are said to be *equivalent*, denoted $\mu \sim \nu$, if and only if

(i) for all $x, y \in X$, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$,

(ii) $\mu(x) = 0$ if and only if $\nu(x) = 0$.

Clearly this relation is an equivalence relation on I^X and it coincides with equality of sets when restricted to 2^X. We can replace the strict inequality with $\geq$ in the above definition of our equivalence relation and the same equivalence classes of fuzzy sets will be determined.

The two equivalence relations by [93] and [62] are different. The equivalence relation by [93] lacks the support property, a condition added by [62]. As shall be seen later, this difference in defining an equivalence relation, accounts for the difference in the way different authors have classified fuzzy subgroups.

Definition 3.4.14. We say two fuzzy subgroups μ and ν are *distinct* if they are non-equivalent.

Remark 3.4.15. The condition $\mu(x) = 0 \iff \nu(x) = 0$ implies that the supports of μ and ν are equal.

Example 3.4.16. [62] Let $S_3 = \{e, a, a^2, b, ab, a^2b\}$ and define fuzzy subsets μ and ν as follows:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{1}{2} & \text{if} \quad x = a, a^2 \\ \frac{1}{3} & \text{otherwise} \end{cases} \qquad \nu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{1}{2} & \text{if} \quad x = a, a^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$ for $x, y \in S_3$ but $Supp\ \mu = [\frac{1}{3}, 1] \neq Supp\ \nu = [0, 1]$, therefore, $\mu \not\sim \nu$.

Proposition 3.4.17. *[62] If $\mu \sim \nu$, then $|Im(\mu)| = Im(\nu)|$.*

The converse of Proposition 3.4.17 is not true as demonstrated in Example 3.4.18 below.

Example 3.4.18. [62] Let $S_3 = \{e, a, a^2, b, ab, a^2b\}$ and define fuzzy subsets μ and ν as follows:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{1}{2} & \text{if} \quad x = b \\ \frac{1}{3} & \text{otherwise} \end{cases} \qquad \nu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{1}{2} & \text{if} \quad x = ab \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

Then we have $Im(\mu) = Im(\nu)$ and $Supp\ \mu = Supp\ \nu$. However, $\mu(b) > \mu(ab)$ but $\nu(b) \not> \nu(ab)$. Hence μ is not equivalent to ν.

Is it possible or not for a proper fuzzy subgroup to be realized as a union of two proper non-equivalent fuzzy subgroups?. This question, asked by Dixit et al in [21], was answered by Dixit et-al in [20] through Example 3.4.19.

Example 3.4.19. Let G be the cyclic group $\mathbb{Z}_{p^n}$, $n \geq 1$. Then G has a sequence of subgroups H_i's of order p^i, $i = 0, 1, 2, \cdots, n$. Define fuzzy subgroups μ and ν as follows:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{1}{2m} & \text{if} \quad x \in H_{2m} \backslash H_{2m-2} \end{cases} \qquad \nu(x) = \begin{cases} \frac{2}{3} & \text{if} \quad x \in H_1 \backslash H_0 \\ \frac{1}{2m+1} & \text{if} \quad x \in H_{2m+1} \backslash H_{2m-1}. \end{cases}$$

As defined, μ and ν are fuzzy subgroups of G such that neither $\mu \leq \nu$ nor $\nu \leq \mu$. (For let $x \in H_2$ such that $x \notin H_1$. So $x \notin H_0$ and $x \in H_3$ and thus $\mu(x) = \frac{1}{2}$ and $\nu(x) = \frac{1}{3}$ and therefore $\mu(x) > \nu(x)$. Let $y \in H_1$, $y \notin H_0$, thus $y \in H_2$. So $\mu(x) = \frac{1}{2}$ and $\nu(x) = \frac{2}{3}$ implying

$\nu(y) > \mu(y)$). The fuzzy subgroups μ and ν constructed here are totally non-equivalent. This is because no member of the family of level subgroups of μ is a member of the family of level subgroups of ν and vice-versa. Moreover, the union $\mu \cup \nu$ is given by

$$(\mu \cup \nu)(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{2}{3} & \text{if} \quad x \in H_1 \backslash H_0 \\ \frac{1}{2} & \text{if} \quad x \in H_2 \backslash H_1 \\ \frac{1}{3} & \text{if} \quad x \in H_3 \backslash H_2 \\ \frac{1}{n} & \text{if} \quad x \in H_n \backslash H_{n-1}. \end{cases}$$

The union $\mu \cup \nu$ as defined above is a fuzzy subgroup of G. Therefore we have two non-equivalent fuzzy subgroups whose union is a fuzzy subgroup.

Remark 3.4.20. The operations of intersection (infimum), union (supremum) and sum of fuzzy subgroups, do not necessarily preserve the equivalence classes of fuzzy subgroups. This is demonstrated in the Examples 3.4.21 and 3.4.22 below by Murali and Makamba in [64].

Example 3.4.21. Suppose $\mu \sim \nu$ and $\mu' \sim \nu'$. Then it is not necessarily that $(\mu \wedge \mu') \sim (\nu \wedge \nu')$. Let G be the group of integers under addition and let

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x \in 2\mathbb{Z} \\ \frac{1}{3} & \text{otherwise} \end{cases} \qquad \nu(x) = \begin{cases} 1 & \text{if} \quad x \in 3\mathbb{Z} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$\mu'(x) = \begin{cases} 1 & \text{if} \quad x \in 2\mathbb{Z} \\ \frac{1}{10} & \text{otherwise} \end{cases} \qquad \nu'(x) = \begin{cases} 1 & \text{if} \quad x \in 3\mathbb{Z} \\ \frac{1}{20} & \text{otherwise.} \end{cases}$$

Then

$$(\mu \wedge \nu)(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{1}{2} & \text{if} \quad x \in 2\mathbb{Z} \backslash 6\mathbb{Z} \\ \frac{1}{3} & \text{otherwise} \end{cases} \qquad (\mu' \wedge \nu')(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{1}{10} & \text{if} \quad x \in 3\mathbb{Z} \backslash 6\mathbb{Z} \\ \frac{1}{20} & \text{otherwise.} \end{cases}$$

We have that $\mu \sim \mu'$ and $\nu \sim \nu'$ but $(\mu \wedge \nu) \not\sim (\mu' \wedge \nu')$ since $(\mu \wedge \nu)(2) = \frac{1}{2} > (\mu \wedge \nu)(3) = \frac{1}{3}$ but $(\mu' \wedge \nu')(2) = \frac{1}{20} < (\mu' \wedge \nu')(3) = \frac{1}{10}$.

Similarly, by an example, it can be shown that $\mu \sim \mu'$ and $\nu \sim \nu'$ do not in general imply $(\mu \vee \nu) \sim (\mu' \vee \nu')$.

Example 3.4.22. Suppose $\mu \sim \nu$ and $\mu' \sim \nu'$. Then it is not necessarily that $(\mu+\mu') \sim (\nu+\nu')$. Suppose G is the group of integers under addition and let

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{1}{2} & \text{if} \quad x \in 2\mathbb{Z}\backslash 6\mathbb{Z} \\ \frac{1}{4} & \text{otherwise} \end{cases} \qquad \nu(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{2}{3} & \text{if} \quad x \in 3\mathbb{Z}\backslash 6\mathbb{Z} \\ \frac{1}{6} & \text{otherwise} \end{cases}$$

$$\mu'(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{3}{4} & \text{if} \quad x \in 2\mathbb{Z}\backslash 6\mathbb{Z} \\ \frac{1}{10} & \text{otherwise} \end{cases} \qquad \nu'(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{1}{2} & \text{if} \quad x \in 3\mathbb{Z}\backslash 6\mathbb{Z} \\ \frac{1}{6} & \text{otherwise.} \end{cases}$$

Then

$$(\mu+\nu)(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{2}{3} & \text{if} \quad x \in 3\mathbb{Z}\backslash 6\mathbb{Z} \\ \frac{1}{2} & \text{otherwise} \end{cases} \qquad (\mu'+\nu')(x) = \begin{cases} 1 & \text{if} \quad x \in 6\mathbb{Z} \\ \frac{3}{4} & \text{if} \quad x \in 2\mathbb{Z}\backslash 6\mathbb{Z} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Clearly $\mu \sim \mu$ and $\nu \sim \nu'$ but $(\mu + \nu) \not\sim (\mu' + \nu')$ since $(\mu + \nu)(3) = \frac{2}{3} > (\mu + \nu)(2) = \frac{1}{2}$ but $(\mu' + \nu')(3) = \frac{1}{2} < (\mu' + \nu')(2) = \frac{3}{4}$.

Proposition 3.4.23. *[64] Let μ and ν be fuzzy subgroups of a group G. If $\mu \sim \nu$ then $(\mu \wedge \nu) \sim \mu$. Consequently $(\mu \wedge \nu) \sim \nu$.*

Proposition 3.4.24. *[64] Let μ and ν be fuzzy subgroups of a group G. If $\mu \sim \nu$ then $(\mu \vee \nu) \sim \mu$. Consequently $(\mu \vee \nu) \sim \nu$.*

Proposition 3.4.25. *[64] Let μ and ν be fuzzy subgroups of a finite group G and $\mu \sim \nu$. Then $(\mu + \nu) \sim \mu$ and consequently $(\mu + \nu) \sim \nu$.*

Remark 3.4.26. By Zadeh's complement [98], if $\mu \sim \nu$, then $(1 - \mu) \sim (1 - \nu)$.

3.4.2 Fuzzy Homomorphism and Fuzzy Isomorphism

The concept of fuzzy isomorphism has been studied by different researchers among them Makamba [44] and Murali and Makamba [62]. These two researchers defined an equivalence relation as stated in Subsection 3.4.1. They studied distinct classes of fuzzy subgroups using their equivalence relation, and compared this equivalence relation to the notion of fuzzy isomorphism.

The concept of group homomorphism and group isomorphism was discussed in Chapter 1 and here we shall extend this definition to fuzzy subgroups.

Remark 3.4.27. If two subgroups are isomorphic, then they are equivalent. An illustration is given in Figure 5.5.

Theorem 3.4.28. *Isomorphism is an equivalence relation on any set of groups.*

Definition 3.4.29. [72] Let G and G' be groups, $f : G \to G'$ a homomorphism and μ a fuzzy subgroups of G. By $f(\mu)$ we mean fuzzy subset of $f(G)$ defined by

$$f(\mu)(f(x)) = sup\{\mu(y) : f(y) = f(x)\}.$$

Define $f(\mu)(y) = 0$ if $y \notin f(G)$. Then $f(\mu)$ is a fuzzy subgroup of G' [45, 46].

Akgŭl [4] also showed that the preimage of a fuzzy subgroup under the homomorphism f above is also a fuzzy subgroup.

Theorem 3.4.30. *[4] Let ν be a fuzzy subgroup of a group G'. Then $f^{-1}(\nu)$ is a fuzzy subgroup of G.*

Theorem 3.4.31. *[72] Suppose $f : G \to G'$ is a homomorphism and let μ and ν be fuzzy subgroups of G. If $\mu \sim \nu$ in G, then $f(\mu) \sim f(\nu)$ in G'.*

What about the pre-image under f?

Theorem 3.4.32. *[72] Suppose $f : G \to G'$ is a homomorphism and let μ' and ν' be fuzzy subgroups of G'. If $\mu' \sim \nu'$ in G', then $f^{-1}(\mu') \sim f^{-1}(\nu')$ in G.*

Definition 3.4.33. [18] A *fuzzy map* $f : X \to Y$ is an ordinary map from X to the set of all fuzzy subsets of Y satisfying the following conditions:

(i) $\forall\, x \in X,\, \exists\, y_x \in Y$ such that $(f(x))(y_x) = 1$,

(ii) $\forall\, x \in X,\, f(x)(y_1) = f(x)(y_2)$ implies that $y_1 = y_2$.

Remark 3.4.34. A fuzzy map $f : X \to Y$ gives rise to a unique ordinary map $\mu_f : X \times Y \to I$, given by $\mu_f(x, y) = f(x)(y)$. Furthermore, a fuzzy map from X to Y gives a unique ordinary map $f_1 : X \to Y$ defined by $f_1(x) = y_x$

Definition 3.4.35. [18] Let G and G' be two groups and $f : G \to G'$ a fuzzy map. Then f is said to be a *fuzzy homomorphism* if and only if $\mu_f(x_1x_2, y) = \vee_{y_1y_2=y}(\mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2))$, $\forall\, x_1, x_2 \in G$ and $y \in G'$,

Remark 3.4.36. If f is an ordinary map, then the above definition reduces to an ordinary homomorphism. One also observes that if a fuzzy map f is a fuzzy homomorphism, then the induced ordinary map f_1 is an ordinary homomorphism.

Definition 3.4.37. Let μ and ν be fuzzy subgroups of a group G. Then μ is said to be *fuzzy isomorphic* to ν denoted by $\mu \approx \nu$, if there exists an isomorphism $f : Supp\ \mu \to Supp\ \nu$ such that

$$\mu(x) > \mu(y) \iff \nu(f(x)) > \nu(f(y)),\ x, y \in Supp\ \mu.$$

Murali and Makamba in [63] introduced the notion of keychains in the study of fuzzy subgroups of finite groups. The concept of index of a keychain was explored in determining the number of fuzzy subgroups of a group. Although we do not use this keychains approach in this thesis, we briefly discuss a few concepts in the approach.

Definition 3.4.38. [63] A *finite n-chain* is a collection of numbers on the interval $[0, 1]$ of the form $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n$, where the last entry may or may not be zero. This is simply written $1\lambda_1\lambda_2 \cdots \lambda_{n-1}\lambda_n$ in descending order.

The numbers $1, \lambda_1, \lambda_2, \cdots \lambda_{n-1}, \lambda_n$ are called *pins* [63]. We say 1 occupies the first position and λ_i occupies the (i+1)st position for $i = 1, 2, \cdots, n$.

Remark 3.4.39. Since the length of an n-chain is $n+1$, then there are $n+1$ positions available in the n-chain.

Definition 3.4.40. [64] An n-chain is called a *keychain* if $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0$ where $\lambda_i' s$ are not necessarily distinct.

Remark 3.4.41. We note that in Definition 3.4.38, the $\lambda_i' s$ are distinct while in Definition 3.4.40, the $\lambda_i' s$ need not be distinct.

Definition 3.4.42. [64] A *flag* of G is a maximal chain of subgroups of G.

Definition 3.4.43. [64] A *pinned flag* is a pair (ζ, ℓ) of a flag ζ and a keychain ℓ, written as $0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \subset \cdots \subset G_n^{\lambda_n}$. We say $G_i^{\lambda_i}$, $i = 1, 2, \cdots, n$ is the $(i+1)st$ component of the pinned flag.

A fuzzy subgroup μ of a group G, is associated with the pinned flag (ζ, ℓ) as follows:

$$\mu(x) = \begin{cases} 1, & x = e \\ \lambda_1, & x \in G_1 \backslash \{e\} \\ \lambda_2, & x \in G_2 \backslash G_1 \\ \vdots & \\ \lambda_n, & x \in G_n \backslash G_{n-1} \end{cases}$$

where $G_n = G$. It can be checked from the definition, that μ is a fuzzy subgroup of G.

Remark 3.4.44. We say that μ is represented by the pinned flag $0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \subset \cdots \subset G_n^{\lambda_n}$ and conversely, every fuzzy subgroup μ can be decomposed into a pinned flag as above. More details are available in [62, 63].

CHAPTER 4. DISTINCT FUZZY SUBGROUPS OF FINITE CYCLIC GROUPS $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ for distinct primes p, q and $n, m \in \mathbb{Z}^+$

4.1 Introduction

To count the fuzzy subgroups of a finite group, an equivalence relation needs to be defined. As discussed in subsection 3.4.1, different authors have defined equivalence relation on fuzzy sets differently. This accounts for the differences in the ways in which the characterization of fuzzy subgroups of groups has evolved over time. In this thesis, we use the equivalence relation defined by Murali and Makamba in [62]. In their papers, [62, 65], Murali and Makamba worked on the number of fuzzy subgroups of the groups $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$, and $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$. Ngcibi [71] in his Masters thesis classified fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$. He further, in his PhD thesis [72] extended his masters work to classification of fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$ for $3 \leq m \leq 5$.

In Section 4.2, two techniques for counting fuzzy subgroups of finite groups are discussed. Murali and Makamba in [65] worked on the fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ using the cross-cut method. We revisit their work with a view of applying the criss-cut counting technique, which gives simpler formulas (in polynomial form) of finding the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$.

4.2 Counting Techniques

In this section, we discuss the two counting techniques introduced by Murali and Makamba in [67] and discussed by Ndiweni in [69]. The criss-cut counting technique is used in this thesis.

4.2.1 Cross-Cut Counting Technique

In this technique, the notion of keychain of a chain is used to determine the number of distinct fuzzy subgroups of a group G. From the length of the flags (maximal chains), we find the number of pins (or the levels) in each keychain. The number of fuzzy subgroups is a function of the number of maximal chains of G and the type of the keychains. Furthermore, a maximal chain with n levels can be associated with $2^n - 1$ keychains.

Once all maximal chains of G are listed, a keychain is picked and "passed" through each maximal chain. This gives the number of distinct fuzzy subgroups this first keychain contributes to G. A second keychain is picked and this process repeated. This process is performed on all keychains and the sum of distinct fuzzy subgroups from all the keychains is the number of distinct fuzzy subgroups for G. Example 4.2.1 illustrates this counting technique with $G = D_6$.

Example 4.2.1. Let $D_6 = \{e, s, r, r^2sr, sr^2\}$ be the dihedral group of order 6. The subgroups of D_6 are: $H_1 = \{e, r, r^2\}$, $H_2 = \{e, s\}$, $H_3 = \{e, sr\}$, $H_4 = \{e, r^2\}$, $\{e\}$ and D_6. From these subgroups, we have four flags (maximal subgroup chains):

$$\{e \subseteq H_1 \subseteq D_6\} \tag{4.2.1}$$

$$\{e \subseteq H_2 \subseteq D_6\} \tag{4.2.2}$$

$$\{e \subseteq H_3 \subseteq D_6\} \tag{4.2.3}$$

$$\{e \subseteq H_4 \subseteq D_6\} \tag{4.2.4}$$

We can represent each fuzzy subgroup using a three pin keychain since the flags are of length three. We have $2^3 - 1 = 7$ keychains namely 111 11λ 110 $1\lambda\lambda$ $1\lambda\beta$ $1\lambda0$ 100. The keychain $1\lambda\beta$ may be associated with a fuzzy subgroup μ on chain 4.2.1 as follows:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \lambda & \text{if} \quad x \in H_1 \backslash \{e\} \\ \beta & \text{if} \quad x \in D_6 \backslash H_1. \end{cases}$$

On chain 4.2.2,

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \lambda & \text{if} \quad x \in H_2 \backslash \{e\} \\ \beta & \text{if} \quad x \in D_6 \backslash H_2. \end{cases}$$

Similarly $1\lambda\beta$ can be associated with μ on chains 4.2.3 and 4.2.4. So $1\lambda\beta$ yields 4 distinct fuzzy subgroups. Similarly, the keychain $1\lambda 0$ yields 4 distinct fuzzy subgroups. The keychain $1\lambda\lambda$ yields

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \lambda & \text{if} \quad x \in D_6 \backslash \{e\} \end{cases}$$

on the chains 4.2.1, 4.2.2, 4.2.3 and 4.2.4, which is the same on all 4 flags. Therefore $1\lambda\lambda$ yields 1 fuzzy subgroup up to equivalence. Similarly, each of the keychain 100, 111 yields 1 fuzzy subgroup. The keychain 11λ yields

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x \in H_1 \\ \lambda & \text{if} \quad x \in D_6 \backslash H_1 \end{cases}$$

on chain 4.2.1. On chains 4.2.2, 4.2.3 and 4.2.4, we replace H_1 by H_2, H_3 an H_4 respectively. So 11λ; 110 each yields 4 fuzzy subgroups up to equivalence. Therefore D_6 has $4+4+1+1+1+1+4+4 = 19$ distinct fuzzy subgroups.

We shall later see that the cross-cut method is more cumbersome than the criss-cut method.

4.2.2 Criss-Cut Counting Technique

In criss-cut technique of counting, we first list all the maximal chains of each group. The number of distinct fuzzy subgroups is then computed as explained by [67, 69]. We give a summarized discussion of this technique below.

Remark 4.2.2. The order of listing our maximal subgroup chains does not matter and so does not alter the number of distinct fuzzy subgroups. Thus we can start the counting from any chain in the list and proceed in any order. Therefore the position of a chain here refers to the order in which we consider the chains in our counting.

Now let G be a group. From the list of the maximal subgroup chains, suppose our first chain is

$$0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G. \tag{4.2.5}$$

By [67], the first chain 4.2.5 always contributes $2^{n+1} - 1$ distinct fuzzy subgroups of G. Let our next maximal chain be

$$0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = G, \tag{4.2.6}$$

such that for some i, $J_i \neq H_i$ where $i \in \{1, 2, \cdots, n-1\}$. This new subgroup J_i is called a *distinguishing factor* of the maximal chain. The number of distinct fuzzy subgroups of G (which we state in Proposition 4.2.3) contributed by the chain 4.2.6 was introduced by [67] and later used in [69].

Proposition 4.2.3. *[69] The number of distinct fuzzy subgroups of G contributed by a maximal subgroup chain with a distinguishing factor is equal to $\frac{2^{n+1}}{2} = 2^n$ for $n \geq 2$.*

Suppose in our counting process, we encounter a maximal subgroup chain $0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = G$, such that $\{K_i, K_j\}$, $i \neq j$ is a pair of subgroups in this chain that have not appeared in any previous chain. We say such a chain has a *new pair* or *a distinguishing pair*. Here we assume that all the subgroups in the chain have appeared and have been used in the previous chains as distinguishing factors.

Proposition 4.2.4. *[69] In the process of our counting distinct fuzzy subgroups, a maximal subgroup chain that has no single distinguishing factor but has a distinguishing pair, contributes $\frac{2^{n+1}}{2^2} = 2^{n-1}$ new distinct fuzzy subgroups of G for $n \geq 4$.*

A new triple of subgroups in a maximal chain is called a *distinguishing triple* and such a chain contributes $\frac{2^{n+1}}{2^3}$ new distinct fuzzy subgroups. This counting argument continues inductively and can be generalized in Proposition 4.2.5.

Proposition 4.2.5. *In the process of counting distinct fuzzy subgroups, if a maximal subgroup chain of length $n+1$, other than the first chain, has no distinguishing $(m-1)$-tuple, but has a new m-tuple of subgroups that has not been used as a distinguishing m-tuple previously, then that chain contributes $\frac{2^{n+1}}{2^m}$ new distinct fuzzy subgroups of G, $m \geq 1$.*

For further details of the counting technique, see [67, 69]. Examples 4.2.7–4.2.10 illustrate the criss-cut counting technique using the groups S_3, D_8 and Q_8.

The remaining part of this subsection is devoted to some examples to illustrate the criss-cut counting technique. We first begin with some examples in non-cyclic groups.

The powers of 2 on the extreme right of each maximal chain (with the exception of the first chain, which is a power of 2 less 1), gives the number of distinct fuzzy subgroups contributed by that maximal chain.

Remark 4.2.6. In this thesis, a distinguishing factor is indicated by $*$, a distinguishing pair by $\{*, **\}$ and a distinguishing triple by $\{*, **, ***\}$. This indication can be extended similarly to a distinguishing quadruple and beyond. When our counting is clear from the examples 4.2.7–4.2.10, we will drop the use of $*$ and its derivatives (except where necessary in our proofs) in the subsequent chains. Furthermore, the number of fuzzy subgroups contributed by each maximal chain is indicated to its extreme right.

Example 4.2.7. From the maximal chains discussed earlier, S_3 has the following distinct fuzzy subgroups enumerated down the maximal chains tree diagram:

$$\{e, (123), (132)\} - \{e\} \to \; 2^3 - 1$$

$$\{e, (12)\}^* - \{e\} \to \; 2^2$$

$$S_3$$

$$\{e, (13)\}^* - \{e\} \to \; 2^2$$

$$\{e, (23)\}^* - \{e\} \to \; 2^2$$

So we have 3 maximal chains with a distinguishing factor in S_3. Therefore the group S_3 has a total of $1 \cdot (2^3 - 1) + 3 \cdot 2^2 = 19$ distinct fuzzy subgroups.

Remark 4.2.8. Tree diagrams for maximal subgroup chains of a group can be cumbersome and take a lot of space. Therefore in this thesis, our maximal chains are presented as a list.

Example 4.2.9. For the group D_8 we have the following fuzzy subgroups:

$$D_8 \supseteq \{e, s, r^2, sr^2\} \supseteq \{e, s\}^* \supseteq e : 2^4 - 1$$

$$D_8 \supseteq \{e, s, r^2, sr^2\} \supseteq \{e, r^2\}^* \supseteq e : 2^3$$

$$D_8 \supseteq \{e, s, r^2, sr^2\} \supseteq \{e, sr^2\}^* \supseteq e : 2^3$$

$$D_8 \supseteq \{e, r^2, sr, sr^3\}^* \supseteq \{e, r^2\} \supseteq e : 2^3$$

$$D_8 \supseteq \{e, r^2, sr, sr^3\} \supseteq \{e, sr\}^* \supseteq e : 2^3$$

$$D_8 \supseteq \{e, r^2, sr, sr^3\} \supseteq \{e, sr^3\}^* \supseteq e : 2^3$$

$$D_8 \supseteq \{e, r, r^2, r^3\}^* \supseteq \{e, r^2\} \supseteq e : 2^3$$

Thus D_8 has $1 \cdot (2^4 - 1) + 6 \cdot 2^3 = 63$ distinct fuzzy subgroups. As a check, $1 + 6 = 7$ maximal chains.

Example 4.2.10. The group Q_8 we has the following fuzzy subgroups:

$$Q_8 \supseteq \{-1, 1 - i, i\} \supseteq \{-1, 1\} \supseteq 1 : 2^4 - 1$$

$$Q_8 \supseteq \{-1, 1 - j, j\}^* \supseteq \{-1, 1\} \supseteq 1 : 2^3$$

$$Q_8 \supseteq \{-1, 1 - k, k\}^* \supseteq \{-1, 1\} \supseteq 1 : 2^3$$

So Q_8 has $1 \cdot (2^4 - 1) + 2 \cdot 2^3 = 31$ distinct fuzzy subgroups.

4.3 The Number of Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$

We first briefly look at the number of fuzzy subgroups of $\mathbb{Z}_{p^n}$ after which we shift our focus to $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ which shall, in turn, be extended to $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ in Chapter 5.

4.3.1 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n}$

We consider the following *trivial*[1] cases for $p = 2$, $n \geq 1$:

$$n = 1 : \quad \mathbb{Z}_2 : \quad \mathbb{Z}_2 \supseteq 0 : 2^{1+1} - 1$$

$$n = 2 : \quad \mathbb{Z}_{2^2} : \quad \mathbb{Z}_{2^2} \supseteq \mathbb{Z}_2 \supseteq 0 : 2^{2+1} - 1$$

[1] Since $\mathbb{Z}_{p^n}$ has only one maximal subgroup for all $n \geq 1$

$$n = 3: \quad \mathbb{Z}_{2^3}: \quad \mathbb{Z}_{2^3} \supseteq \mathbb{Z}_{2^2} \supseteq \mathbb{Z}_2 \supseteq 0 : 2^{3+1} - 1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$n = k: \quad \mathbb{Z}_{2^k}: \quad \mathbb{Z}_{2^k} \supseteq \cdots \supseteq \mathbb{Z}_{2^2} \supseteq \mathbb{Z}_2 \supseteq 0 : 2^{k+1} - 1$$

Theorem 4.3.1. *[62] The group $\mathbb{Z}_{p^n}$ has $2^{n+1} - 1$ distinct fuzzy subgroups.*

From the maximal chains obtained in Chapter 2, we now look at the distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$. Using the cross-cut technique, [65] worked on the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ and gave the results in Propositions 4.3.2–4.3.5.

Proposition 4.3.2. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is*

$$\left[2^{n+1+1} \sum_{r=0}^{1} 2^{-r} \binom{n}{r} \binom{1}{r}\right] - 1, \ n \geq 1.$$

Proposition 4.3.3. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$ is*

$$\left[2^{n+2+1} \sum_{r=0}^{2} 2^{-r} \binom{n}{r} \binom{2}{r}\right] - 1, \ n \geq 2.$$

Proposition 4.3.4. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$ is*

$$\left[2^{n+3+1} \sum_{r=0}^{3} 2^{-r} \binom{n}{r} \binom{3}{r}\right] - 1, \ n \geq 3.$$

Proposition 4.3.5. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4}$ is*

$$\left[2^{n+4+1} \sum_{r=0}^{4} 2^{-r} \binom{n}{r} \binom{4}{r}\right] - 1, \ n \geq 4.$$

From these cases, [65] gave the general formula for the number of distinct fuzzy subgroups in Theorem 4.3.6.

Theorem 4.3.6. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is*

$$\left[2^{n+m+1} \sum_{r=0}^{m} 2^{-r} \binom{n}{r} \binom{m}{r}\right] - 1, \ n \geq m.$$

We now apply the criss-cut counting technique to determine the number of fuzzy subgroups for the $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$. We break this into the Subsections 4.3.2–4.3.4.

4.3.2 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$

We count the distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ for increasing values of n.

When $n = 1, 2$, we have the respective number of distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ in the chains below.

$$pq \supseteq p \supseteq 0 : 2^3 - 1$$
$$pq \supseteq q^* \supseteq 0 : 2^2$$

$$p^2q \supseteq pq \supseteq p \supseteq 0 : 2^4 - 1$$
$$p^2q \supseteq pq \supseteq q^* \supseteq 0 : 2^3$$
$$p^2q \supseteq p^{2^*} \supseteq p \supseteq 0 : 2^3$$

Thus $\mathbb{Z}_p \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ have respectively $1 \cdot (2^3 - 1) + 1 \cdot 2^2 = 11$ and $1 \cdot (2^4 - 1) + 2 \cdot 2^3 = 31$ distinct fuzzy subgroups.

For $n = 3, 4$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_q$ and $\mathbb{Z}_{p^4} \times \mathbb{Z}_q$ have respectively $1 \cdot (2^5 - 1) + 3 \cdot 2^4 = 79$ and $1 \cdot (2^6 - 1) + 4 \cdot 2^5 = 191$ distinct fuzzy subgroups as shown below.

$$p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1$$
$$p^3q \supseteq p^2q \supseteq pq \supseteq q^* \supseteq 0 : 2^4$$
$$p^3q \supseteq p^2q \supseteq p^{2^*} \supseteq p \supseteq 0 : 2^4$$
$$p^3q \supseteq p^{3^*} \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$

$$p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6 - 1$$
$$p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q^* \supseteq 0 : 2^5$$
$$p^4q \supseteq p^3q \supseteq p^2q \supseteq p^{2^*} \supseteq p \supseteq 0 : 2^5$$
$$p^4q \supseteq p^3q \supseteq p^{3^*} \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^4q \supseteq p^{4^*} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

This counting can be extended to $n = 5$ as is summarized in Table 4.1. From the pattern obtained, the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is $2^{n+2} - 1 + n \cdot 2^{n+1}$.

Proposition 4.3.7. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is*

$$2^{n+2} - 1 + n \cdot 2^{n+1}, \ n \geq 1.$$

Proof. We proceed by induction on n. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_q$ has 2 maximal chains as seen earlier in this subsection. The first chain has $2^3 - 1$ distinct fuzzy subgroups while the second one yields 2^2. These add up to $2^3 - 1 + 2^2$ distinct fuzzy subgroups which can be obtained from Table 4.1 when $n = 1$, or Proposition 4.3.7 when $n = 1$. Therefore the proposition is true for $n = 1$.

Suppose the proposition is true for $n = k$ i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_q$ has $2^{k+2} - 1 + k \cdot 2^{k+1}$ distinct fuzzy subgroups. We show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q$ has $2^{k+3} - 1 + (k+1) \cdot 2^{k+2}$ distinct fuzzy subgroups. The group G has $(k+2)$ maximal chains. Furthermore, G has two maximal subgroups $H_1 = p^k q$ and $H_2 = p^{k+1}$ from which all the maximal chains of G extend as shown below.

$$p^{k+1}q \supseteq p^k q \supseteq \begin{cases} p^{k-1}q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ \\ p^k \supseteq \cdots \end{cases} \quad \text{and} \quad p^{k+1}q \supseteq p^{k+1} \supseteq p^k \supseteq \cdots$$

So we look at the number of distinct fuzzy subgroups contributed by the maximal chains extending from H_1 and H_2.

Case (i): $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_q$

By the inductive hypothesis, H_1 contributes $2^{k+3} - 1 + k \cdot 2^{k+2}$ (since each maximal chain of G is of length $k+3$).

Case (ii): $H_2 = \mathbb{Z}_{p^{k+1}}$

The maximal chain $p^{k+1}q \supseteq p^{k+1} \supseteq p^k \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$ has a distinguishing factor p^{k+1} hence, contributes 2^{k+2} distinct fuzzy subgroups of G.

The total contributions from case (i) and case (ii) give $2^{k+3} - 1 + (k+1) \cdot 2^{k+2}$ distinct fuzzy subgroups of G. This result can also be obtained from Proposition 4.3.7 when $n = k+1$. □

4.3.3 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$

We discuss the counting for $n = 1, 2, 3$, with more cases of n summarized in Table 4.1.

When $n = 1, 2$, $\mathbb{Z}_p \times \mathbb{Z}_{q^2}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ have respectively $1 \cdot (2^4 - 1) + 2 \cdot 2^3 = 31$ and $1 \cdot (2^5 - 1) + 4 \cdot 2^4 + 1 \cdot 2^3 = 103$ distinct fuzzy subgroups as shown below.

$$\begin{array}{ccc} pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^4 - 1 & p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1 & p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^4 \\ pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^3 & p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4 \underset{Ctd}{\Rightarrow} & p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^3 \\ pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^3 & p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4 & p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4 \end{array}$$

For $n = 3$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2}$ has $1 \cdot (2^6 - 1) + 6 \cdot 2^5 + 3 \cdot 2^4 = 303$ distinct fuzzy subgroups as discussed in the immediate chains.

$$\begin{array}{rcr} p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6 - 1 & & p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\ p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5 & & p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5 \\ p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5 & \underset{Ctd}{\Rightarrow} & p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4 \\ p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5 & & p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4 \\ p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4 & & p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \end{array}$$

From the pattern obtained and summarized in Table 4.1, the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$ is $2^{n+3} - 1 + 2n \cdot 2^{n+2} + \frac{n(n-1)}{2!} \cdot 2^{n+1}$.

Proposition 4.3.8. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$ is*

$$2^{n+3} - 1 + 2n \cdot 2^{n+2} + \frac{n(n-1)}{2!} \cdot 2^{n+1}, \; n \geq 2.$$

Proof. We prove by induction on n. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ has 3 maximal chains and $2^4 - 1 + 2 \cdot 2^3$ distinct fuzzy subgroups by Proposition 4.3.7 with $n = 2$, or $n = 1$ in Table4.1. This can also be achieved by substituting $n = 1$ in Proposition 4.3.8.

Suppose this result is true for $n = k$ i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^2}$ has $2^{k+3} - 1 + 2k \cdot 2^{k+2} + \frac{k(k-1)}{2!}$ distinct fuzzy subgroups. We need to show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2}$ has $2^{k+4} - 1 + 2(k+1) \cdot 2^{k+3} + \frac{(k+1)k}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups. All the $\frac{(k+3)(k+2)}{2!}$ maximal chains of G branch from its 2 maximal subgroups $H_1 = p^kq^2$ and $H_2 = p^{k+1}q$ as shown below.

$$p^{k+1}q^2 \supseteq p^kq^2 \supseteq \begin{cases} p^{k-1}q^2 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^kq \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \end{cases} \quad \text{and} \quad p^{k+1}q^2 \supseteq p^{k+1}q \supseteq \begin{cases} p^kq \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^{k+1} \supseteq \cdots \end{cases}$$

We therefore, proceed with the proof along these two cases, H_1 and H_2.

Case (i): $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^2}$

By the inductive hypothesis, H_1 contributes $2^{k+4} - 1 + 2k \cdot 2^{k+3} + \frac{k(k-1)}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups (since each maximal chain of G is of length $k+4$).

Case (ii): $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q$

Along H_2, the maximal chains $p^{k+1}q^2 \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$ and

$p^{k+1}q^2 \supseteq p^{k+1}q \supseteq p^{{k+1}^*} \supseteq p^k \supseteq \cdots \supseteq p \supseteq 0$ have distinguishing factors, and thus contribute $2 \cdot 2^{k+3}$ distinct fuzzy subgroups. Each of the remaining k maximal chains extending from H_2, as shown below, has a distinguishing pair. These k chains contribute $k \cdot 2^{k+2}$ distinct fuzzy subgroups.

$$p^{k+1}q^{2^*} \supseteq p^{k+1}q \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq p^{k-3}q \supseteq \cdots \supseteq p^2q \supseteq pq \supseteq q^{**} \supseteq 0$$
$$p^{k+1}q^{2^*} \supseteq p^{k+1}q \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq p^{k-3}q \supseteq \cdots \supseteq p^2q \supseteq p^{2^{**}} \supseteq p \supseteq 0$$
$$p^{k+1}q^{2^*} \supseteq p^{k+1}q \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq p^{k-3}q \supseteq \cdots \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$p^{k+1}q^{2^*} \supseteq p^{k+1}q \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{{k-1}^{**}} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$
$$p^{k+1}q^{2^*} \supseteq p^{k+1}q \supseteq p^kq \supseteq p^{k^{**}} \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

From case (i) and case (ii), $\left(2^{k+4} - 1 + 2k \cdot 2^{k+3} + \frac{k(k-1)}{2!} \cdot 2^{k+2}\right) + \left(2 \cdot 2^{k+3} + k \cdot 2^{k+2}\right)$ gives $2^{k+4} - 1 + 2(k+1) \cdot 2^{k+3} + \frac{(k+1)k}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups. The same result can be obtained from Proposition 4.3.8 when $n = k+1$. □

4.3.4 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$

When $n = 1, 2$, $\mathbb{Z}_p \times \mathbb{Z}_{q^3}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^3}$ have respectively the following fuzzy subgroups:

$$pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1$$
$$p^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$
$$pq^3 \supseteq q^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$

$$p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6 - 1$$
$$p^2q^3 \supseteq p^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$
$$p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$
$$p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$
$$p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

So $\mathbb{Z}_p \times \mathbb{Z}_{q^3}$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^3}$ have respectively $1 \cdot (2^5 - 1) + 3 \cdot 2^4 = 79$ and $1 \cdot (2^6 - 1) + 6 \cdot 2^5 + 3 \cdot 2^4 = 303$ distinct fuzzy subgroups.

Similarly, for $n = 3$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^3}$ has the following fuzzy subgroups:

$$p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7 - 1$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$
$$p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$
$$p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$\underset{Ctd}{\Rightarrow}$$

$$p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

Thus $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^3}$ has $1 \cdot (2^7 - 1) + 9 \cdot 2^6 + 9 \cdot 2^5 + 1 \cdot 2^4 = 1007$ distinct fuzzy subgroups.

Furthermore, when $n = 4$, $\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^3}$ yields the following fuzzy subgroups:

$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8 - 1$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^7$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$
$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \subseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \subseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \subseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \subseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \subseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \subseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \subseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \subseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \subseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \subseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \subseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \subseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^4 \subseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

Therefore $\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^3}$ has $1 \cdot (2^8 - 1) + 12 \cdot 2^7 + 18 \cdot 2^6 + 4 \cdot 2^5 = 3071$ distinct fuzzy subgroups. From the pattern obtained and summarized in Table 4.1, the number of distinct fuzzy subgroups of

$\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$ is $2^{n+4} - 1 + 3n \cdot 2^{n+3} + \frac{3n(n-1)}{2!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+1}$.

Proposition 4.3.9. *The number of distinct fuzzy subgroups for the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$

$$2^{n+4} - 1 + 3n \cdot 2^{n+3} + \frac{3n(n-1)}{2!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+1}, \; n \geq 3.$$

Proof. As in Propositions 4.3.7 and 4.3.8, we proceed by induction on n. For $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^3} \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_q$ has 4 maximal chains and $2^5 - 1 + 3 \cdot 2^4$ distinct fuzzy subgroups by Proposition 4.3.7 when $n = 3$, or $n = 1$ in Table 4.1. This can be obtained from Proposition 4.3.9 when $n = 1$.

Suppose the result holds for $n = k$, i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3}$ has $2^{k+4} - 1 + 3k \cdot 2^{k+3} + 3 \cdot \frac{k(k-1)}{2!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+1}$ distinct fuzzy subgroups. We show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^3}$ has $2^{k+5} - 1 + 3(k+1) \cdot 2^{k+4} + 3 \cdot \frac{(k+1)k}{2!} \cdot 2^{k+3} + \frac{(k+1)k(k-1)}{3!} \cdot 2^{k+2}$ distinct fuzzy subgroups. The group G has 2 maximal subgroups $H_1 = p^k q^3$ and $H_2 = p^{k+1} q^2$ as illustrated below.

$$p^{k+1}q^3 \supseteq p^k q^3 \supseteq \begin{cases} p^{k-1}q^3 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k q^2 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \end{cases} \quad \text{and} \quad p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq \begin{cases} p^k q^2 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^{k+1} q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \end{cases}$$

Therefore we proceed with our counting across the maximal chains through H_1 and H_2.

<u>*Case (i):*</u> $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3}$

By the inductive hypothesis, H_1 contributes $2^{k+5} - 1 + 3k \cdot 2^{k+4} + 3 \cdot \frac{k(k-1)}{3!} \cdot 2^{k+3} + \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+2}$ distinct fuzzy subgroups (since each maximal chain of G is of length $k + 5$).

<u>*Case (ii):*</u> $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2}$

We have 3 chains along $p^{k+1}q^2$ with a distinguishing factor as listed below.

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^k q^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq p^{k+1}q^* \supseteq p^k q \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq \cdots \supseteq p^2 q \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq p^k \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$$

These 3 maximal chains therefore, contribute $3 \cdot 2^{k+4}$ distinct fuzzy subgroups of G.

Each of the maximal chains in the 3 clusters discussed next, consists of a distinguishing pair.

First cluster: This cluster has $(k+1)$ maximal chains listed below.

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-3}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q^{**} \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-3}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq q^{2^{**}} \supseteq q \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-3}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq p^2q^{**} \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-3}q^2 \supseteq \cdots \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-2}q^{**} \supseteq p^{k-3}q \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-1}q^{**} \supseteq p^{k-2}q \supseteq p^{k-3}q \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^kq^{**} \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq p^{k-3}q \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$$

Second cluster: The second cluster has $(k-1)$ maximal chains listed below.

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq p^2q \supseteq p^{2^{**}} \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^3q^2 \supseteq p^3q \supseteq p^{3^{**}} \supseteq p^2 \supseteq p^{**} \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-2}q \supseteq p^{k-2^{**}} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-1}q \supseteq p^{k-1^{**}} \supseteq p^{k-2} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^kq \supseteq p^{k^{**}} \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$$

Third cluster: The third cluster has k maximal chains listed below.

$$p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq \cdots \supseteq p^2q \supseteq pq \supseteq q^{**} \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq \cdots \supseteq p^2q \supseteq p^{2^{**}} \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-2}q \supseteq \cdots \supseteq p^3q \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-1^{**}} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^2 \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k^{**}} \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

So H_2 has $(k+1)+(k-1)+k = 3k$ maximal chains with a distinguishing pair and therefore, contributes $3k \cdot 2^{k+3}$ distinct fuzzy subgroups of G.

The remaining chains along H_2 have a distinguishing triple as discussed below. The first of these clusters has the following $(k-1)$ maximal chains:

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq p^2q^{**} \supseteq pq \supseteq q^{***} \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{**} \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq p^{k-2}q^{**} \supseteq p^{k-3}q \supseteq p^{k-3^{***}} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-1}q^{**} \supseteq p^{k-2}q \supseteq p^{k-2^{***}} \supseteq p^{k-3} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^3 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^kq^{**} \supseteq p^{k-1}q \supseteq p^{k-1^{***}} \supseteq p^{k-2} \supseteq p^{k-3} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$$

Similarly, the second cluster has $(k-2)$ maximal chains, the third, $(k-3)$, in that order up to the last cluster with 1 maximal chain. This gives a total of $(k-1)$ such clusters. The sum of the maximal chains from the $(k-1)$ clusters is $\frac{k-1}{2}\{2(k-1)+(-1)(k-2)\} = \frac{k(k-1)}{2}$. Therefore H_2 contributes $3 \cdot 2^{k+4} + 3k \cdot 2^{k+3} + \frac{k(k-1)}{2} \cdot 2^{k+2}$ distinct fuzzy subgroups.

The sum of the contributions from case (i) and case (ii) yields

$$\begin{aligned} & 2^{k+5} - 1 + 3k \cdot 2^{k+4} + 3 \cdot \frac{k(k-1)}{2!} \cdot 2^{k+3} + \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+2} + \\ & \qquad 3 \cdot 2^{k+4} \quad + \quad 3k \cdot 2^{k+3} \quad + \quad \frac{k(k-1)}{2} \cdot 2^{k+2} \\ = {} & 2^{k+5} - 1 + 3(k+1) \cdot 2^{k+4} + 3 \cdot \frac{(k+1)k}{2!} \cdot 2^{k+3} + \frac{(k+3)k(k-1)}{3!} \cdot 2^{k+2} \end{aligned}$$

distinct fuzzy subgroups. This result can be obtained from Proposition 4.3.9 when $n = k+1$. □

4.3.5 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4}$

Counting the distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4}$ follows a similar technique as discussed in Subsections 4.3.2–4.3.4. The number of distinct fuzzy subgroups for $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4}$ obtained is $2^{n+5} - 1 + 4n \cdot 2^{n+4} + \frac{6n(n-1)}{2!} \cdot 2^{n+3} + \frac{4n(n-1)(n-2)}{3!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+1}$, which we state as Proposition 4.3.10.

Proposition 4.3.10. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4}$, $n \geq 4$,is*

$$2^{n+5} - 1 + 4n \cdot 2^{n+4} + \frac{6n(n-1)}{2!} \cdot 2^{n+3} + \frac{4n(n-1)(n-2)}{3!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+1}.$$

Proof. We take a similar inductive approach as used in Propositions 4.3.7–4.3.9. For $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^4} \cong \mathbb{Z}_{p^4} \times \mathbb{Z}_q$ has 5 maximal chains and $2^6 - 1 + 4 \cdot 2^5$ distinct fuzzy subgroups by Proposition 4.3.7 when $n = 4$. This can be obtained from Proposition 4.3.10 when $n = 1$.

Suppose the result holds for $n = k$, i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^4}$ has $2^{k+5} - 1 + 4k \cdot 2^{k+4} + 6 \cdot \frac{k(k-1)}{2!} \cdot 2^{k+3} + 4 \cdot \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)(k-3)}{4!} \cdot 2^{k+1}$ distinct fuzzy subgroups. We show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^4}$ has $2^{k+6} - 1 + 4(k+1) \cdot 2^{k+5} + 6 \cdot \frac{(k+1)k}{2!} \cdot 2^{k+4} + 4 \cdot \frac{(k+1)k(k-1)}{3!} \cdot 2^{k+3} + \frac{(k+1)k(k-1)(k-2)}{4!} \cdot 2^{k+2}$ distinct fuzzy subgroups. The proof continues along the maximal subgroups $H_1 = p^k q^4$ and $H_2 = p^{k+1}q^3$ of G, shown in the maximal chains below.

$$p^{k+1}q^4 \supseteq p^k q^4 \supseteq \begin{cases} p^{k-1}q^4 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k q^3 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \end{cases} \quad \text{and} \quad p^{k+1}q^4 \supseteq p^{k+1}q^3 \supseteq \begin{cases} p^k q^3 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^{k+1}q^2 \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \end{cases}$$

<u>*Case (i):*</u> $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^4}$

By the inductive hypothesis, H_1 contributes $2^{k+6} - 1 + 4k \cdot 2^{k+5} + 6 \cdot \frac{k(k-1)}{2!} \cdot 2^{k+4} + 4 \cdot \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+3} + \frac{k(k-1)(k-2)(k-3)}{4!} \cdot 2^{k+2}$ distinct fuzzy subgroups (since each maximal chain of G has length $k+6$).

<u>*Case (ii):*</u> $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^3}$

This case yields 4 distinguishing factors, $6k$ distinguishing pairs, $4 \cdot \frac{k(k-1)}{2!}$ distinguishing triples and $\frac{k(k-1)(k-2)}{3!}$ distinguishing quadruples. Therefore H_2 contributes $4 \cdot 2^{k+5} + 6k \cdot 2^{k+4} + 4 \cdot \frac{k(k-1)}{2!} \cdot 2^{k+3} + \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+2}$ distinct fuzzy subgroups.

Summing up the contributions from case (i) and case (ii), we have $[2^{k+6} - 1 + 4k \cdot 2^{k+5} + 6 \cdot \frac{k(k-1)}{2!} \cdot 2^{k+4} + 4 \cdot \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+3} + \frac{k(k-1)(k-2)(k-3)}{4!} \cdot 2^{k+2}] + [4 \cdot 2^{k+5} + 6k \cdot 2^{k+4} + 4 \cdot \frac{k(k-1)}{2!} \cdot 2^{k+3} + \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+2}]$. This sum gives our desired result $2^{k+6} - 1 + 4(k+1) \cdot 2^{k+5} + 6 \cdot \frac{(k+1)k}{2!} \cdot 2^{k+4} + 4 \cdot \frac{(k+1)k(k-1)}{3!} \cdot 2^{k+3} + \frac{(k+1)k(k-1)(k-2)}{4!} \cdot 2^{k+2}$. The result can also be obtained by substituting $n = k+1$ in Proposition 4.3.10. □

The general pattern observed in Propositions 4.3.7–4.3.10 is extended in Table 4.2, from which we deduce the most general case as stated in Theorem 4.3.11.

Theorem 4.3.11. *The number of distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is given by* $2^{n+m+1}-1+\binom{m}{1}\cdot n\cdot 2^{n+m}+\binom{m}{2}\cdot\frac{n(n-1)}{2!}\cdot 2^{n+m-1}+\binom{m}{3}\cdot\frac{n(n-1)(n-2)}{3!}\cdot 2^{n+m-2}+\binom{m}{4}\cdot\frac{n(n-1)(n-2)(n-3)}{4!}\cdot 2^{n+m-3}+\binom{m}{5}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cdot 2^{n+m-4}+\cdots+\binom{m}{r}\cdot\frac{n(n-1)(n-2)\cdots(n-(r-1))}{r!}\cdot 2^{n+m-(r-1)}+\cdots+\frac{n(n-1)(n-2)\cdots(n-(m-1))}{m!}\cdot 2^{n+1},\ r\le m\le n.$

Proof. We prove this theorem by induction on n, and in a similar way to Propositions 4.3.7–4.3.10, but in a more general sense. For $n=1$, $\mathbb{Z}_p\times\mathbb{Z}_{q^m}\cong\mathbb{Z}_{p^m}\times\mathbb{Z}_q$ has $2^{m+2}-1+m\cdot 2^{m+1}$ distinct fuzzy subgroups by Proposition 4.3.7 when $n=m$.

Suppose for $n=k$, $\mathbb{Z}_{p^k}\times\mathbb{Z}_{q^m}$ has $2^{k+m+1}-1+\binom{m}{1}\cdot k\cdot 2^{k+m}+\binom{m}{2}\cdot\frac{k(k-1)}{2!}\cdot 2^{k+m-1}+\binom{m}{3}\cdot\frac{k(k-1)(k-2)}{3!}\cdot 2^{k+m-2}+\binom{m}{4}\cdot\frac{k(k-1)(k-2)(k-3)}{4!}\cdot 2^{k+m-3}+\binom{m}{5}\cdot\frac{k(k-1)(k-2)(k-3)(k-4)}{5!}\cdot 2^{k+m-4}+\cdots+\binom{m}{r}\cdot\frac{k(k-1)(k-2)\cdots(k-(r-1))}{r!}\cdot 2^{k+m-(r-1)}+\cdots+\frac{k(k-1)(k-2)\cdots(k-(m-1))}{m!}\cdot 2^{k+1}$ distinct fuzzy subgroups. We need to show that the group $G=\mathbb{Z}_{p^{k+1}}\times\mathbb{Z}_{q^m}$ has $2^{k+m+2}-1+\binom{m}{1}\cdot(k+1)\cdot 2^{k+m+1}+\binom{m}{2}\cdot\frac{(k+1)k}{2!}\cdot 2^{k+m}+\binom{m}{3}\cdot\frac{(k+1)k(k-1)}{3!}\cdot 2^{k+m-1}+\binom{m}{4}\cdot\frac{(k+1)k(k-1)(k-2)}{4!}\cdot 2^{k+m-2}+\binom{m}{5}\cdot\frac{(k+1)k(k-1)(k-2)(k-3)}{5!}\cdot 2^{k+m-3}+\cdots+\binom{m}{r}\cdot\frac{(k+1)k(k-1)(k-2)\cdots(k-(r-2))}{r!}\cdot 2^{k+m-(r-2)}+\cdots+\frac{(k+1)k(k-1)(k-2)\cdots(k-(m-2))}{m!}\cdot 2^{k+2}$ distinct fuzzy subgroups. The group G has 2 maximal subgroups $H_1=\mathbb{Z}_{p^k}\times\mathbb{Z}_{q^m}$ and $H_2=\mathbb{Z}_{p^{k+1}}\times\mathbb{Z}_{q^{m-1}}$ as shown below, from which all the $\frac{(k+m+1)!}{(k+1)!m!}$ maximal chains extend.

$$p^{k+1}q^m \supseteq p^kq^m \supseteq \begin{cases} p^{k-1}q^m \supseteq \begin{cases}\cdots\\ \cdots\end{cases} \\ p^kq^{m-1} \supseteq \begin{cases}\cdots\\ \cdots\end{cases}\end{cases} \quad \text{and} \quad p^{k+1}q^m \supseteq p^{k+1}q^{m-1} \supseteq \begin{cases} p^kq^{m-1} \supseteq \begin{cases}\cdots\\ \cdots\end{cases} \\ p^{k+1}q^{m-2} \supseteq \begin{cases}\cdots\\ \cdots\end{cases}\end{cases}$$

Therefore we proceed along the two maximal subgroups H_1 and H_2 of G.

Case (i): $H_1=\mathbb{Z}_{p^k}\times\mathbb{Z}_{q^m}$

By the inductive hypothesis, H_1 contributes $2^{k+m+2}-1+\binom{m}{1}\cdot k\cdot 2^{k+m+1}+\binom{m}{2}\cdot\frac{k(k-1)}{2!}\cdot 2^{k+m}+\binom{m}{3}\cdot\frac{k(k-1)(k-2)}{3!}\cdot 2^{k+m-1}+\binom{m}{4}\cdot\frac{k(k-1)(k-2)(k-3)}{4!}\cdot 2^{k+m-2}+\binom{m}{5}\cdot\frac{k(k-1)(k-2)(k-3)(k-4)}{5!}\cdot 2^{k+m-3}+\cdots+\binom{m}{r}\cdot\frac{k(k-1)(k-2)\cdots(k-(r-1))}{r!}\cdot 2^{k+m-(r-1)}+\cdots+\frac{k(k-1)(k-2)\cdots(k-(m-1))}{m!}\cdot 2^{k+2}$ distinct fuzzy subgroups (since each maximal chain of G has length $k+m+2$).

Case (ii): $H_2=\mathbb{Z}_{p^{k+1}}\times\mathbb{Z}_{q^{m-1}}$

The maximal chains along H_2 contribute $\binom{m}{1}$ distinguishing factors, $\binom{m}{2}\cdot k$ distinguishing pairs,

$\binom{m}{3} \cdot \frac{k(k-1)}{2!}$ distinguishing triples, $\binom{m}{4} \cdot \frac{k(k-1)(k-2)}{3!}$ distinguishing quadruples, in that order up to $\binom{m}{r} \cdot \frac{k(k-1)(k-2)(k-3)\cdots(k-(r-2))}{(r-1)!}$ distinguishing r-tuples and finally $\frac{k(k-1)(k-2)\cdots(k-(m-2))}{(m-1)!}$ distinguishing m-tuples . Therefore H_2 contributes $\binom{m}{1} \cdot 2^{k+m+1} + \binom{m}{2} \cdot k \cdot 2^{k+m} + \binom{m}{3} \cdot \frac{k(k-1)}{2!} \cdot 2^{k+m-1} + \binom{m}{4} \cdot \frac{k(k-1)(k-2)}{3!} \cdot 2^{k+m-2} + \cdots + \binom{m}{r} \cdot \frac{k(k-1)(k-2)\cdots(k-(r-2))}{(r-1)!} \cdot 2^{k+m-(r-2)} + \cdots + \frac{k(k-1)(k-2)\cdots(k-(m-2))}{(m-1)!} \cdot 2^{k+2}$ distinct fuzzy subgroups. Summing up the contributions from case (i) and case (ii), we get $2^{k+m+2} - 1 + \binom{m}{1} \cdot (k+1) \cdot 2^{k+m+1} + \binom{m}{2} \cdot \frac{(k+1)k}{2!} \cdot 2^{k+m} + \binom{m}{3} \cdot \frac{(k+1)k(k-1)}{3!} \cdot 2^{k+m-1} + \binom{m}{4} \cdot \frac{(k+1)k(k-1)(k-2)}{4!} \cdot 2^{k+m-2} + \binom{m}{5} \cdot \frac{(k+1)k(k-1)(k-2)(k-3)}{5!} \cdot 2^{k+m-3} + \cdots + \binom{m}{r} \cdot \frac{(k+1)k(k-1)(k-2)\cdots(k-(r-2))}{r!} \cdot 2^{k+m-(r-2)} + \cdots + \frac{(k+1)k(k-1)(k-2)(k-3)\cdots(k-(m-2))}{m!} \cdot 2^{k+2}$ distinct fuzzy subgroups. This result can also be obtained by substituting $n = k+1$ in Theorem 4.3.11. □

Remark 4.3.12. The number in Theorem 4.3.11 can also be expressed in summation notation as

$$-1 + \sum_{r=0}^{m} \binom{m}{r} \binom{n}{r} 2^{n+m-(r-1)}.$$

Remark 4.3.13. The formula for $p^n q^m$ in Theorem 4.3.11 has $(m+1)$ terms. The coefficients of the powers of 2 from the second term onwards, are polynomials of degrees $1, 2, \cdots, m$.

Table 4.1: Fuzzy Subgroups of $\mathbb{Z}_{p^n},\ \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ for $m = 1, 2, 3$

n	p^n	$p^n q$	$p^n q^2$	$p^n q^3$
1	$1 \cdot (2^2 - 1)$	$1 \cdot (2^3 - 1) + 1 \cdot 2^2$	$1 \cdot (2^4 - 1) + 2 \cdot 2^3$	$1 \cdot (2^5 - 1) + 3 \cdot 2^4$
2	$1 \cdot (2^3 - 1)$	$1 \cdot (2^4 - 1) + 2 \cdot 2^3$	$1 \cdot (2^5 - 1) + 4 \cdot 2^4 + 1 \cdot 2^3$	$1 \cdot (2^6 - 1) + 6 \cdot 2^5 + 3 \cdot 2^4$
3	$1 \cdot (2^4 - 1)$	$1 \cdot (2^5 - 1) + 3 \cdot 2^4$	$1 \cdot (2^6 - 1) + 6 \cdot 2^5 + 3 \cdot 2^4$	$1 \cdot (2^7 - 1) + 9 \cdot 2^6 + 9 \cdot 2^5 + 1 \cdot 2^4$
4	$1 \cdot (2^5 - 1)$	$1 \cdot (2^6 - 1) + 4 \cdot 2^5$	$1 \cdot (2^7 - 1) + 8 \cdot 2^6 + 6 \cdot 2^5$	$1 \cdot (2^8 - 1) + 12 \cdot 2^7 + 18 \cdot 2^6 + 4 \cdot 2^5$
5	$1 \cdot (2^6 - 1)$	$1 \cdot (2^7 - 1) + 5 \cdot 2^6$	$1 \cdot (2^8 - 1) + 10 \cdot 2^7 + 10 \cdot 2^6$	$1 \cdot (2^9 - 1) + 15 \cdot 2^8 + 30 \cdot 2^7 + 10 \cdot 2^6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$2^{n+1} - 1$	$2^{n+2} - 1 + n \cdot 2^{n+1}$	$2^{n+3} - 1 + 2n \cdot 2^{n+2} + \frac{n(n-1)}{2!} \cdot 2^{n+1}$	$2^{n+4} - 1 + 3n \cdot 2^{n+3} + \frac{3n(n-1)}{2!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+1}$

Table 4.2: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$

$p^n q^m$	Number of fuzzy subgroups
p^n	$2^{n+1} - 1$
$p^n q$	$2^{n+2} - 1 + n \cdot 2^{n+1}$
$p^n q^2$	$2^{n+3} - 1 + 2 \cdot n \cdot 2^{n+2} + \frac{n(n-1)}{2!} \cdot 2^{n+1}$
$p^n q^3$	$2^{n+4} - 1 + 3 \cdot n \cdot 2^{n+3} + 3 \cdot \frac{n(n-1)}{2!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+1}$
$p^n q^4$	$2^{n+5} - 1 + 4 \cdot n \cdot 2^{n+4} + 6 \cdot \frac{n(n-1)}{2!} \cdot 2^{n+3} + 4 \cdot \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+1}$
$p^n q^5$	$2^{n+6} - 1 + 5 \cdot n \cdot 2^{n+5} + 10 \cdot \frac{n(n-1)}{2!} \cdot 2^{n+4} + 10 \cdot \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+3} + 5 \cdot \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+2} + \frac{n(n-1)\cdots(n-4)}{5!} \cdot 2^{n+1}$
$p^n q^6$	$2^{n+7} - 1 + 6 \cdot n \cdot 2^{n+6} + 15 \cdot \frac{n(n-1)}{2!} \cdot 2^{n+5} + 20 \cdot \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+4} + 15 \cdot \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+3} + 6 \cdot \frac{n(n-1)\cdots(n-4)}{5!} \cdot 2^{n+2} + \frac{n(n-1)\cdots(n-5)}{6!} \cdot 2^{n+1}$
$\vdots$	$\vdots$
$p^n q^k$	$2^{n+k+1} - 1 + \binom{k}{1} \cdot n \cdot 2^{n+k} + \binom{k}{2} \cdot \frac{n(n-1)}{2!} \cdot 2^{n+k-1} + \binom{k}{3} \cdot \frac{n(n-1)(n-2)}{3!} \cdot 2^{n+k-2} + \binom{k}{4} \cdot \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+k-3} + \binom{k}{5} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} \cdot 2^{n+k-4} + \cdots + \binom{k}{r} \cdot \frac{n(n-1)(n-2)(n-3)(n-(r-1))}{r!} \cdot 2^{n+k-(r-1)} + \cdots + \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!} \cdot 2^{n+1}, \quad r \leq k \leq n$

CHAPTER 5. DISTINCT FUZZY SUBGROUPS OF FINITE CYCLIC GROUPS $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ for distinct primes p, q, r and $n, m \in \mathbb{Z}^+$

5.1 Introduction

Using the cross-cut technique, Ndiweni in his masters thesis [68] looked at fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_{r^s}$ for some cases of r.

In this chapter, we use the criss-cut technique and extend the work of [68] by looking at the number of fuzzy subgroups of the finite cyclic groups $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ for $n, m \in \mathbb{Z}^+$. At the end of this chapter, we discuss isomorphism classes of (non-isomorphic) fuzzy subgroups of finite groups. We make a comparison between the number of equivalence classes (distinct fuzzy subgroups) and the non-isomorphic fuzzy subgroups.

5.2 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$

As done for $p^n q^m$, we similarly break $p^n q^m r$ into the subsections 5.2.1–5.2.10. In each of these subsections, for a fixed m, we observe the pattern in the number of distinct fuzzy subgroups as n increases. This general pattern gives the formula for the number of the distinct fuzzy subgroups in each case.

5.2.1 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$

When $n = 1, 2$, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$, have respectively $1 \cdot (2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2 = 51$ and $1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3 = 175$ distinct fuzzy subgroups as shown in the immediate maximal chains.

$$pqr \supseteq pq \supseteq p \supseteq 0 : 2^4 - 1$$
$$pqr \supseteq pq \supseteq q \supseteq 0 : 2^3$$
$$pqr \supseteq pr \supseteq p \supseteq 0 : 2^3$$
$$pqr \supseteq pr \supseteq r \supseteq 0 : 2^3$$
$$pqr \supseteq qr \supseteq q \supseteq 0 : 2^3$$
$$pqr \supseteq qr \supseteq r \supseteq 0 : 2^2$$

$$p^2qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1$$
$$p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^4$$
$$p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^4$$
$$p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^4$$
$$p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^3$$
$$p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^4$$
$$p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^3$$
$$p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^4$$
$$p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^3$$
$$p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^3$$

For $n = 3$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_r$ yields the following fuzzy subgroups:

$$p^3qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^6 - 1$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$\underset{Ctd}{\Rightarrow}$$

$$p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5$$
$$p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$

Therefore $\mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4 = 527$ distinct fuzzy subgroups.

When $n = 4$, $\mathbb{Z}_{p^4} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has the following fuzzy subgroups:

$$\begin{array}{rcr}
p^4qr \supseteq p^3qr \supseteq qpr \supseteq pq \supseteq p \supseteq 0 : 2^7 - 1 & & p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pq \supseteq q \supseteq 0 : 2^6 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq p \supseteq 0 : 2^6 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq pr \supseteq r \supseteq 0 : 2^6 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq q \supseteq 0 : 2^6 & & p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq qpr \supseteq qr \supseteq r \supseteq 0 : 2^5 & & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6 & & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5 & \underset{Ctd}{\Rightarrow} & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6 & & p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6 & & p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq p \supseteq 0 : 2^6 \\
p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq r \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5 & & p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5 \\
p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5 & & p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5
\end{array}$$

So $\mathbb{Z}_{p^4} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5 = 1471$ distinct fuzzy subgroups.

Similarly, for $m = 5, 6, \cdots, 10$ we have the following distinct fuzzy subgroups:

$$\mathbb{Z}_{p^5} \times \mathbb{Z}_q \times \mathbb{Z}_r : \ (2^8 - 1) + 16 \cdot 2^7 + 25 \cdot 2^6$$

$$\mathbb{Z}_{p^6} \times \mathbb{Z}_q \times \mathbb{Z}_r : \ (2^9 - 1) + 19 \cdot 2^7 + 36 \cdot 2^6$$

$$\mathbb{Z}_{p^7} \times \mathbb{Z}_q \times \mathbb{Z}_r : \ (2^{10} - 1) + 22 \cdot 2^7 + 49 \cdot 2^6$$

$$\mathbb{Z}_{p^8} \times \mathbb{Z}_q \times \mathbb{Z}_r : \ (2^{11} - 1) + 25 \cdot 2^7 + 64 \cdot 2^6$$

$$\mathbb{Z}_{p^9} \times \mathbb{Z}_q \times \mathbb{Z}_r : \ (2^{12} - 1) + 28 \cdot 2^{11} + 81 \cdot 2^{10}$$

$$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_q \times \mathbb{Z}_r : \ (2^{13} - 1) + 31 \cdot 2^{12} + 100 \cdot 2^{11}$$

The summary of the results of subsection 5.2.1 are presented in Table 5.1, from which a general pattern is deduced. Therefore the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$ is given by $2^{n+3} - 1 + (3n+1) \cdot 2^{n+2} + \frac{n \cdot n}{1!} \cdot 2^{n+1}$. This result is proved in Proposition 5.2.1.

Proposition 5.2.1. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$ *is*

$$2^{n+3} - 1 + (3n+1) \cdot 2^{n+2} + \frac{n \cdot n}{1!} \cdot 2^{n+1}.$$

Proof. We proceed by induction on n. For $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ has 6 maximal chains as discussed in 2.3.3.1. The first chain yields $2^4 - 1$ distinct fuzzy subgroups. Each of the next 4 chains has a distinguishing factor, and thus contributes 2^3 distinct fuzzy subgroups. The last chain has a new pair and therefore contributes 2^2 distinct fuzzy subgroups. Hence, $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^4 - 1 + 4 \cdot 2^3 + 2^2$ distinct fuzzy subgroups. Clearly this number is also obtainable by letting $n = 1$ in the formula of Proposition 5.2.1. Therefore the proposition is true for $n = 1$.

Now assume $\mathbb{Z}_{p^k} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^{k+3} - 1 + (3k+1) \cdot 2^{k+2} + k^2 \cdot 2^{k+1}$ distinct fuzzy subgroups. We want to show that $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^{k+4} - 1 + [3(k+1)+1] \cdot 2^{k+3} + (k+1)^2 \cdot 2^{k+2}$ distinct fuzzy subgroups. Let $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$.

The group G has 3 maximal subgroups $H_1 = p^k qr$, $H_2 = p^{k+1}q$ and $H_3 = p^{k+1}r$ from which all the maximal chains extend (see Figure 5.1). Thus we proceed along these three subgroups.

Case (i): $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_q \times \mathbb{Z}_r$

By Proposition 2.3.5, this subgroup has $(k+2)(k+1)$ maximal chains and the length of a maximal chain of G is $k+4$. By the inductive hypothesis, H_1 yields $2^{k+4} - 1 + (3k+1) \cdot 2^{k+3} + k^2 \cdot 2^{k+2}$ distinct fuzzy subgroups.

Case (ii): $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q$

The subgroup H_2 has $(k+2)$ maximal chains. Only 2 chains in H_2 have a distinguishing factor (in the language of [65]), viz. $p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^k q \supseteq p^{k-1}q \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$ and $p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq p^k \supseteq \cdots \supseteq pq \supseteq p \supseteq 0$. Thus the 2 chains contribute $2 \cdot 2^{k+3}$ distinct fuzzy subgroups of G. Each of the remaining k maximal chains along H_2 contributes a distinguishing pair. This accounts for $k \cdot 2^{k+2}$ distinct fuzzy subgroups. Therefore H_2 yields $2 \cdot 2^{k+3} + k \cdot 2^{k+2}$ distinct fuzzy subgroups.

Case (iii): $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_r$

The subgroup H_3 has $(k+2)$ maximal chains and only 1 chain has a distinguishing factor: $p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^k \supseteq p^{k-1} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$. This contributes 2^{k+3} distinct fuzzy subgroups. Each of the remaining $(k+1)$ maximal chains contributes a distinguishing pair, accounting for $(k+1) \cdot 2^{k+2}$ distinct fuzzy subgroups. Thus H_3 yields $2^{k+3} + (k+1) \cdot 2^{k+2}$ distinct fuzzy subgroups.

Summing up the contributions from *case(i)–case(iii)*, we have

$$\begin{aligned} 2^{k+4} - 1 + (3k+1) \cdot 2^{k+3} + k^2 \cdot 2^{k+2} &\quad + \\ 2 \cdot 2^{k+3} + k \cdot 2^{k+2} &\quad + \\ 2^{k+3} + (k+1) \cdot 2^{k+2} & \\ = 2^{k+4} - 1 + (3k+4) \cdot 2^{k+3} + (k^2 + 2k + 1) \cdot 2^{k+2} & \end{aligned}$$

Therefore the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$ is given by

$$2^{k+4} - 1 + [3(k+1) + 1] \cdot 2^{k+3} + (k+1)^2 \cdot 2^{k+2}.$$

This can also be obtained from the Proposition 5.2.1 with $n = k+1$. □

5.2.2 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

For $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ gives the following fuzzy subgroups:

$$\begin{array}{rcl} pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^5 - 1 & & pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^4 \\ pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^4 & & pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^3 \\ pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^4 & & pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4 \\ pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^4 & \underset{Ctd}{\Rightarrow} & pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^4 \\ pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^4 & & pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^3 \\ pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^3 & & pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^3 \end{array}$$

So $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3 = 175$ distinct fuzzy subgroups.

When $n = 2$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has the following fuzzy subgroups:

$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6 - 1$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4 \underset{Ctd}{\Rightarrow}$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^3$$
$$p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^4$$
$$p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^3$$
$$p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$

Therefore $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $1 \cdot (2^6 - 1) + 12 \cdot 2^5 + 15 \cdot 2^4 + 2 \cdot 2^3 = 703$ distinct fuzzy subgroups.

For $n = 3$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ yields the following fuzzy subgroups:

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7 - 1$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$
$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^4$$

$$p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

Thus $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $1 \cdot (2^7 - 1) + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4 = 2415$ distinct fuzzy subgroups.

Similarly, for $m = 4, 5, \cdots 10$ we have the following groups and their distinct fuzzy subgroups:

$$\begin{aligned}
\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r : &\quad 1 \cdot (2^8 - 1) + 22 \cdot 2^7 + 58 \cdot 2^6 + 24 \cdot 2^5 \\
\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r : &\quad 1 \cdot (2^9 - 1) + 27 \cdot 2^8 + 90 \cdot 2^7 + 50 \cdot 2^6 \\
\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{10} - 1) + 32 \cdot 2^9 + 129 \cdot 2^8 + 90 \cdot 2^7 \\
\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{11} - 1) + 37 \cdot 2^{10} + 175 \cdot 2^9 + 147 \cdot 2^8 \\
\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{12} - 1) + 42 \cdot 2^{11} + 228 \cdot 2^{10} + 224 \cdot 2^9 \\
\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{13} - 1) + 47 \cdot 2^{12} + 288 \cdot 2^{11} + 324 \cdot 2^{10} \\
\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{14} - 1) + 52 \cdot 2^{13} + 355 \cdot 2^{12} + 450 \cdot 2^{11}
\end{aligned}$$

More cyclic groups and their fuzzy subgroups are presented in Appendix A.

These results are presented in Table 5.2, from which we have the following number of distinct fuzzy subgroups: $2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + 2 \cdot \frac{n}{2} \frac{7n+1}{2!} \cdot 2^{n+2} + \frac{n(n-1)n}{2!} \cdot 2^{n+1}$. We give a proof of this result in Proposition 5.2.2.

Proposition 5.2.2. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ *is*

$$2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + 2 \cdot \frac{n}{2} \frac{(7n+1)}{2!} \cdot 2^{n+2} + \frac{n(n-1)n}{2!} \cdot 2^{n+1}.$$

Proof. We prove the simplified form $2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + \frac{7n^2+n}{2!} \cdot 2^{n+2} + \frac{n^2(n-1)}{2!} \cdot 2^{n+1}$ of the proposition. We proceed inductively on n. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^5 - 1 + 7 \cdot 2^4 + 4 \cdot 2^3$ distinct fuzzy subgroups by Proposition 5.2.1 with $n = 2$, or $n = 1$ in Table 5.2. This number can clearly be obtained from the substitution of $n = 1$ in the formula of Proposition 5.2.2, and therefore the proposition is true for $n = 1$.

Suppose the result holds for $n = k$, i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^{k+4} - 1 + (5k+2) \cdot 2^{k+3} + \frac{7k^2+k}{2!} \cdot 2^{k+2} + \frac{k^2(k-1)}{2!} \cdot 2^{k+1}$ distinct fuzzy subgroups. We need to show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^{k+5} - 1 + [(5(k+1)+2] \cdot 2^{k+4} + \frac{7(k+1)^2+(k+1)}{2!} \cdot 2^{k+3} + \frac{(k+1)^2 k}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups. The group G has 3 maximal subgroups $H_1 = p^k q^2 r$, $H_2 = p^{k+1} q r$ and $H_3 = p^{k+1} q^2$ through which all the maximal chains of G pass. These maximal chains are:

$$p^{k+1}q^2r \supseteq p^kq^2r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases}, \quad p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \text{ and } p^{k+1}q^2r \supseteq p^{k+1}q^2 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases}$$

Therefore the proof continues along these maximal subgroups H_1, H_2 and H_3.

Case (i): $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

By the inductive hypothesis, H_1 has $2^{k+5} - 1 + (5k+2) \cdot 2^{k+4} + \frac{7k^2+k}{2!} \cdot 2^{k+3} + \frac{k^2(k-1)}{2!} \cdot 2^{k+2}$ distinct fuzzy subgroups.

Case (ii): $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$

There are 4 maximal chains with a distinguishing factor along this subgroup:

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq p^{k-2}qr \supseteq \cdots \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k-1}r \supseteq \cdots \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$

Furthermore, there are 5 clusters of maximal chains through H_2 whose chains have a distinguishing pair.

First cluster: This cluster has k chains:

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq^{**} \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq p^2q^{**} \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^3q^{**} \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^kq^{**} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

<u>*Second cluster:*</u> This cluster has k chains:

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr^{**} \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq p^2r^{**} \supseteq pr \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^3r^{**} \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^kr^{**} \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$

<u>*Third cluster:*</u> This cluster has $(k+1)$ chains:

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr^{**} \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r^{**} \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^{2^{**}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^3qr \supseteq p^3q \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^kq \supseteq p^{k^{**}} \supseteq p^{k-1} \cdots \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

<u>*Fourth cluster:*</u> This cluster has k chains:

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q^{**} \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq p^{2^{**}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq p^kq \supseteq p^{k^{**}} \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

<u>*Fifth cluster:*</u> This cluster has $(k+1)$ chains:

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r^{**} \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq p^{2^{**}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k-1}r \supseteq \cdots \supseteq p^3r \supseteq p^{3^{**}} \supseteq p^2 \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq p^{k^{**}} \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^{k+1^{**}} \supseteq p^k \supseteq p^{k-1} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

From these five clusters, H_2 yields a total of $(5k+2)$ maximal chains with a distinguishing pair.

Similarly, H_2 has clusters of maximal chains, each of which has k chains with a distinguishing triple. The first cluster has chains ending with $pr \supseteq r \supseteq 0$, the second ends with $pq \supseteq q \supseteq 0$, the third, $p^2 \supseteq p \supseteq 0$, the fourth, $p^3 \supseteq p^2 \supseteq p \supseteq p \supseteq 0$, in that order with the last cluster comprising of chains ending with $p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^2 \supseteq p \supseteq 0$. This gives a total of k clusters, hence H_2 has k^2 maximal chains with a distinguishing triple.

<u>*Case (iii):*</u> $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2}$

There is only 1 maximal chain with a distinguishing factor through this subgroup:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0.$$

Extending through H_3, we have 2 clusters of maximal chains with a distinguishing pair:

<u>*First cluster:*</u> This first cluster has $(k+2)$ maximal chains:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q^{**} \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq q^{2^{**}} \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq p^{k-2}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq p^2q^{**} \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^kq^{**} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

<u>*Second cluster:*</u> This cluster has k maximal chains:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q^{**} \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{**} \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^kq^{**} \supseteq p^k \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^{k+1} \supseteq p^k \supseteq p^{k-1} \supseteq p^{k-2} \supseteq \cdots \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

Therefore the subgroup H_3 yields $(2k+2)$ maximal chains with a distinguishing pair.

Next we look at the maximal chains through H_3 contributing a distinguishing triple to the group G. Following similar criteria, we have two major clusters of maximal chains with specific patterns.

<u>*First cluster:*</u> This cluster has k maximal chains:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q^{***} \supseteq pq \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q^{***} \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{***} \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{***} \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^{k-1}q^2 \supseteq p^{k-1}q^{***} \supseteq \cdots \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^{2^{**}} \supseteq p^kq^{***} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq^{***} \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0$$

<u>*Second cluster:*</u>

The second cluster is broken down into subclusters of maximal chains ending with $p^2 \supseteq p \supseteq 0$.

<u>*First subcluster:*</u> This has 3 maximal chains:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q^{**} \supseteq p^2q \supseteq p^2 \supseteq p^{***} \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{**} \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^4q^2 \supseteq p^4q^{**} \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0$$

<u>*Second subcluster:*</u> This has 3 maximal chains:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{**} \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{**} \supseteq p^4q \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^5q^2 \supseteq p^5q^{**} \supseteq p^4q \supseteq p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

<u>*Third subcluster:*</u> This has 4 maximal chains:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^4q \supseteq p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^6q^2 \supseteq p^6q^{**} \supseteq p^5q \supseteq p^{5^{***}} \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

<u>*Fourth subcluster:*</u> This has 5 maximal chains:

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^4q \supseteq p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^5q \supseteq p^{5^{***}} \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq \cdots \supseteq p^7q^2 \supseteq p^7q^{**} \supseteq p^6q \supseteq p^{6^{***}} \supseteq p^5 \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

So far, the first, second, third and fourth subclusters have respectively $3,\ 3,\ 4$ and 5 maximal chains. Similarly, the fifth, sixth and seventh subclusters (which are not included here), have respectively $6,\ 7,$ and 8 maximal chains. There is a total of $(k-2)$ subclusters, and the last one has $(k-1)$ maximal chains which we discuss here and give a summary in Table 5.3.

<u>*$(k-2)$nd subcluster:*</u>

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supset p^3q \supseteq p^2q \supseteq p^{2^{***}} \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supset p^3q \supseteq p^{3^{***}} \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^4q \supset p^{4^{***}} \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq \cdots \supseteq p^5q \supseteq p^{5^{***}} \supseteq p^4 \supset p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k-1}q \supseteq p^{k-1^{***}} \supseteq \cdots \supseteq p^5 \supseteq p^4 \supset p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^{k+1}q^{**} \supseteq p^kq \supseteq p^{k^{***}} \supseteq p^{k-1} \supseteq \cdots \supseteq p^5 \supseteq p^4 \supset p^3 \supseteq p^2 \supseteq p \supseteq 0$$

From Table 5.3, S-cl is used as a shorthand for subcluster. When $m = k$, each of the first two subclusters have 3 chains. From the third subcluster onwards, the number of chains form an arithmetic sequence $4,\ 5,\ 6,\ 7,\ 8,\ \cdots,\ k-1$. This sequence has $k-4$ terms (subclusters) with the first term $a = 4$, common difference $d = 1$ and last term $l = k - 1$. Thus the number of maximal chains from the third subcluster is

$$\begin{aligned} S_n &= \frac{n}{2}(a+l) \\ &= \frac{k-4}{2}(4+k-1) \\ &= \frac{k-4}{2}(k+3) \\ &= \frac{(k-4)(k+3)}{2}. \end{aligned}$$

Therefore the total number of maximal chains with a distinguishing triple contributed by H_3 is

$$\begin{aligned} \frac{(k-4)(k+3)}{2!} + (6+k) &= \frac{k^2+k}{2!} \\ &= \frac{k(k+1)}{2!}. \end{aligned}$$

Summing up the contributions from case (i)-case (iii), we get

$$\begin{aligned} & 2^{k+5} - 1 + (5k+1)\cdot 2^{k+4} + \frac{7k^2+k}{2!}\cdot 2^{k+3} + \frac{k^2(k-1)}{2!}\cdot 2^{k+2} + \\ & \quad 4\cdot 2^{k+4} + (5k+2)\cdot 2^{k+3} + k^2\cdot 2^{k+2} + \\ & \quad 2^{k+4} + (2k+2)\cdot 2^{k+3} + \frac{k(k+1)}{2!}\cdot 2^{k+2} \\ &= 2^{k+5} - 1 + (5k+7)\cdot 2^{k+4} + \frac{7k^2+15k+8}{2!}\cdot 2^{k+3} + \frac{k(k^2+2k+1)}{2!}\cdot 2^{k+2} \end{aligned}$$

Hence, the group $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^{k+5} + [(5(k+1)+2)+2]\cdot 2^{k+4} + \frac{7(k+1)^2+(k+1)}{2!}\cdot 2^{k+3} + \frac{(k+1)^2 k}{2!}\cdot 2^{k+2}$ distinct fuzzy subgroups. This result can also be obtained by substituting $n = k+1$ in Proposition 5.2.2. □

5.2.3 Distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

A similar counting procedure is adopted as shown in Appendix B. This gives us the following distinct fuzzy subgroups for the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$.

$$\mathbb{Z}_p \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1\cdot(2^6-1) + 10\cdot 2^5 + 9\cdot 2^4$$

$$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1\cdot(2^7-1) + 17\cdot 2^6 + 33\cdot 2^5 + 9\cdot 2^4$$

$$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^8 - 1) + 24 \cdot 2^7 + 72 \cdot 2^6 + 40 \cdot 2^5 + 3 \cdot 2^4$$

$$\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^9 - 1) + 31 \cdot 2^8 + 126 \cdot 2^7 + 106 \cdot 2^6 + 16 \cdot 2^5$$

$$\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^{10} - 1) + 38 \cdot 2^9 + 195 \cdot 2^8 + 220 \cdot 2^7 + 50 \cdot 2^6$$

$$\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^{11} - 1) + 45 \cdot 2^{10} + 279 \cdot 2^9 + 395 \cdot 2^8 + 120 \cdot 2^7$$

$$\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^{12} - 1) + 52 \cdot 2^{11} + 378 \cdot 2^{10} + 644 \cdot 2^9 + 245 \cdot 2^8$$

$$\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^{13} - 1) + 59 \cdot 2^{12} + 492 \cdot 2^{11} + 980 \cdot 2^{10} + 448 \cdot 2^9$$

$$\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^{14} - 1) + 66 \cdot 2^{13} + 621 \cdot 2^{12} + 1416 \cdot 2^{11} + 756 \cdot 2^{10}$$

$$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r : \quad 1 \cdot (2^{15} - 1) + 73 \cdot 2^{13} + 765 \cdot 2^{31} + 1965 \cdot 2^{12} + 1200 \cdot 2^{11}$$

The above results are summarized in Table 5.4, from which the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ is given by $2^{n+5} - 1 + (7n+3) \cdot 2^{n+4} + 3 \cdot \frac{n}{2}\frac{(10n+2)}{2!} \cdot 2^{n+3} + \frac{3 \cdot 2}{2!} \cdot \frac{n(n-1)}{3}\frac{(13n+1)}{3!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)n}{3!} \cdot 2^{n+1}$. This is stated in Proposition 5.2.3.

Proposition 5.2.3. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ *is* $2^{n+5} - 1 + (7n+3) \cdot 2^{n+4} + 3 \cdot \frac{n}{2}\frac{(10n+2)}{2!} \cdot 2^{n+3} + \frac{3 \cdot 2}{2!} \cdot \frac{n(n-1)}{3}\frac{(13n+1)}{3!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)n}{3!} \cdot 2^{n+1}$.

Proof. We prove the simplified form $2^{n+5} - 1 + (7n+3) \cdot 2^{n+4} + \frac{(15n^2+3n)}{2!} \cdot 2^{n+3} + \frac{(13n^2+n)(n-1)}{3!} \cdot 2^{n+2} + \frac{n^2(n-1)(n-2)}{3!} \cdot 2^{n+1}$ of the proposition. The proof follows an induction on n. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_r$ has $2^6 - 1 + 10 \cdot 2^5 + 9 \cdot 2^4$ distinct fuzzy subgroups by Proposition 5.2.1 with $n = 3$, or $n = 1$ in Table 5.4. This can also be obtained by substituting $n = 1$ in Proposition 5.2.3. Similarly, when $n = 2$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ has $2^7 - 1 + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4$ distinct fuzzy subgroups by Proposition 5.2.2 when $n = 3$. Therefore the proposition is true for $n = 1, 2$.

Suppose the group $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ has $2^{k+5} - 1 + (7k+3) \cdot 2^{k+4} + \frac{(15k^2+3k)}{2!} \cdot 2^{k+3} + \frac{(13k^2+k)(k-1)}{3!} \cdot 2^{k+2} + \frac{k^2(k-1)(k-2)}{3!} \cdot 2^{k+1}$ distinct fuzzy subgroups. We need to show that the group $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ has $2^{k+6} - 1 + [7(k+1)+3] \cdot 2^{k+5} + \frac{15(k+1)^2+3(k+1)}{2!} \cdot 2^{k+4} + \frac{[13(k+1)^2+(k+1)]k}{3!} \cdot 2^{k+3} + \frac{(k+1)^2k(k-1)}{3!} \cdot 2^{k+2}$ distinct fuzzy subgroups.

The group G has 3 maximal subgroups $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$, $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$ and $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^3}$ through which all maximal chains of G pass:

$$p^{k+1}q^3r \supseteq p^kq^3r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases}, \; p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \text{ and } \; p^{k+1}q^3r \supseteq p^{k+1}q^3 \supseteq \begin{cases} \cdots \\ \cdots \end{cases}$$

By a similar argument done in Propositions 5.2.1–5.2.2, we first proceed along these three subgroups to count the number of maximal chains with a distinguishing factor, a distinguishing pair and a distinguishing triple. However, in the fourth case, we look at the number of maximal chains with a distinguishing quadruple in both H_2 and H_3.

Case (i): $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

By the inductive hypothesis, H_1 has $2^{k+6} - 1 + (7k+3)\cdot 2^{k+5} + \frac{(15k^2+3k)}{2!}\cdot 2^{k+4} + \frac{(13k^2+k)(k-1)}{3!}\cdot 2^{k+3} + \frac{k^2(k-1)(k-2)}{3!}\cdot 2^{k+2}$ distinct fuzzy subgroups since maximal chains of G have length $k+6$.

Case (ii): $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

The subgroup H_2 has 6 maximal chains with a distinguishing factor as listed below.

$$p^{k+1}q^3r \supseteq p^{k+1}q^2r^* \supseteq p^kq^2r \supseteq p^{k-1}q^2r \supseteq p^{k-2}q^2r \supseteq \cdots \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \cdots \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q \supseteq p^kq^* \supseteq \cdots \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq \cdots \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq p^kr \supseteq \cdots \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^2 \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq p^{k-1}q^2 \supseteq \cdots \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$

The subgroup H_2 has 7 clusters of maximal chains with distinguishing pairs. The process of enumerating these chains is similar to that used in the proof of Proposition 5.2.2. Therefore we simply state the representative of each cluster and the number of chains therein.

The first and second clusters consists respectively of chains ending with $pq \supseteq p \supseteq 0$ and $p^2 \supseteq p \supseteq 0$ and each of these clusters has $(5k-3)$ maximal chains. The third cluster has 4 chains ending with $pq \supseteq q \supseteq 0$. Each of the fourth and fifth clusters has 3 maximal chains ending respectively with $pr \supseteq r \supseteq 0$ and $qr \supseteq q \supseteq 0$. The sixth cluster has $2k$ chains all of which end with $pr \supseteq p \supseteq 0$ while the seventh cluster has 2 chains both of which end with

$q^2 \supseteq q \supseteq 0$. These 7 clusters give a total of $2(5k-3)+4+6+2 = (12k+6)$ maximal chains through H_2 with a distinguishing pair.

For the maximal chains with a distinguishing triple, we have that the subgroups p^2q^2r, p^3q^2r, p^4q^2r, p^5q^2r, p^6q^2r, $\cdots$ have $6,\ 22,\ 48,\ 84,\ 130,\ \cdots$ maximal chains respectively. Therefore as k increases, the number of chains for $p^{k+1}q^2r$ with a distinguishing triple exhibits a quadratic sequence. The n^{th} term of the sequence is given by $T_n = an^2+bn+c$, where $2a = \textit{first term}$, $3a+b = \textit{first term of the first difference row}$ and $2a = \textit{second term}$. This sequence is illustrated in Figure 5.2a.

We therefore have the system $2a = 10$, $3a+b = 16$ and $a+b+c = 6$, whose solution is $a = 5$, $b = 1$ and $c = 0$. For our sequence, $n = k$, implying that $T_k = 5k^2+k = k(5k+1)$ is the number of maximal chains in H_2 with a distinguishing triple.

Case (iii): $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q3}$

There is 1 maximal chain passing through H_3 with a distinguishing factor. This is the chain

$$p^{k+1}q^3r \supseteq p^{k+1}q^{3^*} \supseteq p^kq^3 \supseteq p^{k-1}q^3 \supseteq \cdots \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0.$$

The subgroup H_3 has 4 clusters of maximal chains with a distinguishing pair. The first cluster has all its maximal chains ending with $pq \supseteq p \supseteq 0$ and has $2k$ such chains. The second and third clusters' maximal chains end with $pq \supseteq q \supseteq 0$ and $q^2 \supseteq q \supseteq 0$ respectively and have respectively 1 and 2 maximal chain(s). The fourth cluster consists of k maximal chains all of which end with $p^2 \supseteq p \supseteq 0$. These 4 clusters contribute a total of $(3k+3)$ maximal chains with a distinguishing pair. Similarly, H_3 has $\frac{k(3k+3)}{2!}$ maximal chains with a distinguishing triple.

Case (iv): Quadruples in both H_2 and H_3

Lastly, we look at the number of quadruples contributed by H_2 and H_3 combined. This last category has four clusters. The first cluster has $(k-1)$ maximal chains all of which end with $qr \supseteq r \supseteq 0$. The second cluster consists of maximal chains ending with $pr \supseteq r \supseteq 0$. The number of chains with a distinguishing quadruple in this cluster for the groups $p^3q^3r,\ p^4q^3r,\ p^5q^3r,\ p^6q^3r,\ p^7q^3r,\ \cdots$, is $1,\ 3,\ ,6,\ 10,\ 15,\ \cdots$ respectively. This is a sequence of triangular numbers whose n-th term is given by $T_n = \frac{n(n+1)}{2!}$. In our case, $n = k-2$ and therefore this cluster has $T_{k-2} = \frac{(k-1)(k-2)}{2!}$ chains.

The third cluster has maximal chains ending with $pq \supseteq q \supseteq 0$. The number of chains in this cluster for the groups $p^3q^3r,\ p^4q^3r,\ p^5q^3r,\ p^6q^3r,\ \cdots$, is $2,\ 7,\ 15,\ 26, \cdots$ respectively. This is a sequence of quadratic numbers as shown in Figure 5.2b. In our case, $n = k-1$, thus we have $2a = 3$, $3a+b = 5$ and $a+b+c = 2$. Solving this system gives $a = \frac{3}{2}$, $b = \frac{1}{2}$ and $c = 0$. So $T_n = \frac{3}{2}n^2 + \frac{1}{2}n = \frac{n(3n+1)}{2!}$ and therefore this cluster has $T_{k-1} = \frac{(k-1)[3(k-1)+1]}{2!} = \frac{(k-1)(3k-2)}{2!}$ maximal chains. The fourth cluster consists of chains $\frac{(k-1)(k-2)(4k-3)}{3!}$ all of which end with $p^2 \supseteq p \supseteq 0$. Thus from case (iv) we get $(k-1) + \frac{(k-1)(k-2)}{2!} + \frac{(k-1)(3k-2)}{2!} + \frac{(k-1)(k-2)(4k-3)}{3!} = \frac{k(k-1)(4k+1)}{3!}$ maximal chains with a distinguishing quadruple.

Summing up the contributions from case(i)–case(iv), we have $[2^{k+6} - 1 + (7k+3) \cdot 2^{k+5} + \frac{(15k^2+3k)}{2!} \cdot 2^{k+4} + \frac{(13k^2+k)(k-1)}{3!} \cdot 2^{k+3} + \frac{k^2(k-1)(k-2)}{3!} \cdot 2^{k+2}] + [6+1] \cdot 2^{k+5} + [(12k+6) + (3k+3)] \cdot 2^{k+4} + [k(5k+1) + \frac{k(3k+3)}{2!}] \cdot 2^{k+3} + [\frac{k(k-1)(4k+1)}{3!}] \cdot 2^{k+2}$. Therefore G has $2^{k+6} - 1 + [7(k+1)+3] \cdot 2^{k+5} + \frac{15(k+1)^2+3(k+1)}{2!} \cdot 2^{k+4} + \frac{[13(k+1)^2+(k+1)]k}{3!} \cdot 2^{k+3} + \frac{(k+1)^2k(k-1)}{3!} \cdot 2^{k+2}$ distinct fuzzy subgroups. This can also be obtained by substituting $n = k+1$ in Proposition 5.2.3. □

5.2.4 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$

For the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$, we get the following distinct fuzzy subgroups.

$$\begin{aligned}
\mathbb{Z}_p \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5 \\
\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^8 - 1) + 22 \cdot 2^7 + 58 \cdot 2^6 + 24 \cdot 2^5 \\
\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^9 - 1) + 31 \cdot 2^8 + 126 \cdot 2^7 + 106 \cdot 2^6 + 16 \cdot 2^5 \\
\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{10} - 1) + 40 \cdot 2^9 + 220 \cdot 2^8 + 280 \cdot 2^7 + 85 \cdot 2^6 + 4 \cdot 2^5 \\
\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{11} - 1) + 49 \cdot 2^{10} + 340 \cdot 2^9 + 580 \cdot 2^8 + 265 \cdot 2^7 + 25 \cdot 2^6 \\
\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{12} - 1) + 58 \cdot 2^{11} + 486 \cdot 2^{10} + 1040 \cdot 2^9 + 635 \cdot 2^8 + 90 \cdot 2^7 \\
\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{13} - 1) + 67 \cdot 2^{12} + 658 \cdot 2^{11} + 1694 \cdot 2^{10} + 1295 \cdot 2^9 + 245 \cdot 2^8 \\
\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{14} - 1) + 76 \cdot 2^{13} + 856 \cdot 2^{12} + 2576 \cdot 2^{11} + 2366 \cdot 2^{10} + 560 \cdot 2^9 \\
\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{15} - 1) + 85 \cdot 2^{14} + 1080 \cdot 2^{13} + 3720 \cdot 2^{12} + 3990 \cdot 2^{11} + 1134 \cdot 2^{10} \\
\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r : &\quad 1 \cdot (2^{16} - 1) + 94 \cdot 2^{15} + 1330 \cdot 2^{14} + 5160 \cdot 2^{13} + 6330 \cdot 2^{12} + 2100 \cdot 2^{11}
\end{aligned}$$

Some counting maximal chains are presented in Appendix C. These results are summarized in Table 5.5 from which we have that the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ has $2^{n+6} - 1 + (9n+4) \cdot 2^{n+5} + 4 \cdot \frac{n}{2} \frac{(13n+3)}{2!} \cdot 2^{n+4} + \frac{4 \cdot 3}{2!} \frac{n(n-1)}{3} \frac{(17n+2)}{3!} \cdot 2^{n+3} + \frac{4 \cdot 3 \cdot 2}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(21n+1)}{4!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)n}{4!} \cdot 2^{n+1}$ distinct fuzzy subgroups. This result is stated and proved in Proposition 5.2.4.

Proposition 5.2.4. *The number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ is* $2^{n+6} - 1 + (9n+4) \cdot 2^{n+5} + 4 \cdot \frac{n}{2} \frac{(13n+3)}{2!} \cdot 2^{n+4} + \frac{4 \cdot 3}{2!} \frac{n(n-1)}{3} \frac{(17n+2)}{3!} \cdot 2^{n+3} + \frac{4 \cdot 3 \cdot 2}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(21n+1)}{4!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)n}{4!} \cdot 2^{n+1}$

Proof. We inductively prove the simplified form $2^{n+6} - 1 + (9n+4) \cdot 2^{n+5} + (13n^2+3n) \cdot 2^{n+4} + \frac{(34n^2+4n)(n-1)}{3!} \cdot 2^{n+3} + \frac{(21n^2+n)(n-1)(n-2)}{4!} \cdot 2^{n+2} + \frac{n^2(n-1)(n-2)(n-3)}{4!} \cdot 2^{n+1}$ of the proposition. We follow a similar approach as in Propositions 5.2.1–5.2.4. When $n = 1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^4} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and has $2^7 - 1 + 13 \cdot 2^6 + 16 \cdot 2^5$ distinct fuzzy subgroups by Proposition 5.2.1 with $n = 4$, or $n = 1$ in Table 5.5. This can also be obtained by substituting $n = 1$ in Proposition 5.2.4.

Suppose the group $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ has $2^{k+6} - 1 + (9k+4) \cdot 2^{k+5} + (13k^2+3k) \cdot 2^{k+4} + \frac{(34k^2+4k)(k-1)}{3!} \cdot 2^{k+3} + \frac{(21k^2+k)(k-1)(k-2)}{4!} \cdot 2^{k+2} + \frac{k^2(k-1)(k-2)(k-3)}{4!} \cdot 2^{k+1}$ distinct fuzzy subgroups. We need to show that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ has $2^{k+7} - 1 + [9(k+1)+4] \cdot 2^{k+6} + [13(k+1)^2 + 3(k+1)] \cdot 2^{k+5} + \frac{[34(k+1)^2+4(k+1)]k}{3!} \cdot 2^{k+4} + \frac{[21(k+1)^2+(k+1)]k(k-1)}{4!} \cdot 2^{k+3} + \frac{(k+1)^2 k(k-1)(k-2)}{4!} \cdot 2^{k+2}$ distinct fuzzy subgroups. The group G has 3 maximal subgroups $H_1 = p^k q^4 r$, $H_2 = p^{k+1} q^3 r$ and $H_3 = p^{k+1} q^4$ as illustrated below, and the proof continues along four different cases.

$$p^{k+1}q^4r \supseteq p^kq^4r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases}, \quad p^{k+1}q^4r \supseteq p^{k+1}q^3r \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \quad \text{and} \quad p^{k+1}q^4r \supseteq p^{k+1}q^4 \supseteq \begin{cases} \cdots \\ \cdots \end{cases}$$

<u>*Case (i):*</u> $H_1 = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$

By the inductive hypothesis, H_1 has $2^{k+7} - 1 + (9k+4) \cdot 2^{k+6} + (13k^2+3k) \cdot 2^{k+5} + \frac{(34k^2+4)(k-1)}{3!} \cdot 2^{k+4} + \frac{(21k^2+k)(k-1)(k-2)}{4!} \cdot 2^{k+3} + \frac{k^2(k-1)(k-2)(k-3)}{4!} \cdot 2^{k+2}$ distinct fuzzy subgroups since maximal chains of G have length $k+7$.

<u>*Case (ii):*</u> $H_2 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

We have 8 maximal chains through H_2 with a distinguishing factor as illustrated below.

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^kq^3r^* \supseteq p^{k-1}q^3r \supseteq \cdots \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^{k+1}q^2r^* \supseteq p^kq^2r \supseteq p^{k-1}q^2r \supseteq \cdots \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr^* \supseteq p^kqr \supseteq \cdots \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q^* \supseteq \cdots \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1^*} \supseteq \cdots \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}qr \supseteq p^{k+1}r^* \supseteq \cdots \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^{k+1}q^2r \supseteq p^{k+1}q^{2^*} \supseteq p^kq^2 \supseteq \cdots \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$

$$p^{k+1}q^3r \supseteq p^{k+1}q^3r \supseteq p^{k+1}q^{3^*} \supseteq p^kq^3 \supseteq p^{k-1}q^3 \supseteq \cdots \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$

The subgroup H_2 has 7 clusters of maximal chains with a distinguishing pair. The first, second and the third cluster consists of chains ending with $pq \supseteq q \supseteq 0$, $qr \supseteq q \supseteq 0$ and $q^2 \supseteq q \supseteq 0$ respectively. Each of these three clusters has 6 maximal chains. The fourth cluster has 4 maximal chains all of which end with $pr \supseteq r \supseteq 0$. The fifth and sixth clusters have maximal chains ending with $pq \supseteq p \supseteq 0$ and $pr \supseteq p \supseteq 0$ respectively. The sum of the chains in these two clusters is $15k - 6$. The seventh cluster has $7k - 4$ maximal chains all of which end with $p^2 \supseteq p \supseteq 0$. Therefore H_2 has $18 + 4 + (15k - 6) + (7k - 4) = 22k + 12$ maximal chains with a distinguishing pair. Similarly, H_3 has $14k^2 + 4k$ maximal chains with a distinguishing triple.

<u>*Case (iii):*</u> $H_3 = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^4}$

The subgroup H_3 has 1 maximal chain with a distinguishing factor:

$$p^{k+1}q^4r \supseteq p^{k+1}q^{4^*} \supseteq p^kq^4 \supseteq p^{k-1}q^4 \supseteq p^{k-2}q^4 \supseteq \cdots \supseteq p^2q^4 \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0$$

Furthermore, H_3 has 4 clusters of maximal chains with a distinguishing pair. The first cluster consists of chains ending with $pq \supseteq p \supseteq 0$. This cluster has $3k$ chains. The second cluster has 1 chain and ends with $pq \supseteq q \supseteq 0$. The third and fourth clusters consists of maximal chains ending with $p^2 \supseteq p \supseteq 0$ and $q^2 \supseteq q \supseteq 0$ respectively. These last two clusters have k and 3 maximal chains respectively. Therefore H_3 has $4k + 4$ maximal chains with a distinguishing pair.

In addition, we have 4 clusters of maximal chains with a distinguishing triple along H_3. Each of the first and second clusters has $3k$ maximal chains ending with $pq \supseteq q \supseteq 0$ and $q^2 \supseteq q \supseteq 0$ respectively. The third and fourth clusters end respectively with $p^2 \supseteq p \supseteq 0$ and $pq \supseteq p \supseteq 0$. Each of the third and fourth clusters has $\frac{3k(k-1)}{2}$ maximal chains. Thus from the 4 clusters we get $6k + 3k(k-1) = 3k(k+1)$ maximal chains along H_3 with a distinguishing triple.

Case (iv): Quadruples and Quintuples in both H_2 and H_3

Using a similar approach, the sum of maximal chains with a distinguishing quadruple from H_2 and H_3 is given by $\frac{k(k-1)(21k+6)}{3!}$. Furthermore, there are $\frac{k(k-1)(k-2)(5k+1)}{4!}$ maximal chains with a distinguishing quintuple in this category.

The sum of the contributions from case (i)–case (iv) gives us $[2^{k+7}-1+(9k+4)\cdot 2^{k+6}+(13k^2+3k)\cdot 2^{k+5}+\frac{(34k^2+4)(k-1)}{3!}\cdot 2^{k+4}+\frac{(21k^2+k)(k-1)(k-2)}{4!}\cdot 2^{k+3}+\frac{k^2(k-1)(k-2)(k-3)}{4!}\cdot 2^{k+2}]+[8+1]\cdot 2^{k+6}+[(22k+12)+(4k+4)]\cdot 2^{k+5}+[(14k^2+4k)+3k(k+1)]\cdot 2^{k+4}+\frac{k(k-1)(21k+6)}{3!}\cdot 2^{k+3}+\frac{k(k-1)(k-2)(5k+1)}{4!}\cdot 2^2$. Thus G has $2^{k+7}-1+[9(k+1)+4]\cdot 2^{k+6}+[13(k+1)^2+3(k+1)]\cdot 2^{k+5}+\frac{[34(k+1)^2+4(k+1)]k}{3!}\cdot 2^{k+4}+\frac{[21(k+1)^2+(k+1)]k(k-1)}{4!}\cdot 2^{k+3}+\frac{(k+1)^2k(k-1)(k-2)}{4!}\cdot 2^{k+2}$ distinct fuzzy subgroups. This result can also be obtained by substituting $n = k+1$ in Proposition 5.2.4. □

In Subsections 5.2.5–5.2.10, we determine the formulae for the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ for $m = 5, 6, \cdots, 10$. The results of these subsections are given in Propositions 5.2.5–5.2.10. The proofs follow a similar argument of Propositions 5.2.1–5.2.4; they are therefore left out in this thesis.

5.2.5 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$

For the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$, we get the following distinct fuzzy subgroups:

$\mathbb{Z}_p \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$: $1\cdot(2^8-1)+16\cdot 2^7+25\cdot 2^6$

$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$: $1\cdot(2^9-1)+27\cdot 2^8+90\cdot 2^7+50\cdot 2^6$

$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$: $1\cdot(2^{10}-1)+38\cdot 2^9+195\cdot 2^8+220\cdot 2^7+50\cdot 2^6$

$\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$: $1\cdot(2^{11}-1)+49\cdot 2^{10}+340\cdot 2^9+580\cdot 2^8+265\cdot 2^7+25\cdot 2^6$

$\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$: $1\cdot(2^{12}-1)+60\cdot 2^{11}+525\cdot 2^{10}+1200\cdot 2^9+825\cdot 2^8+156\cdot 2^7+5\cdot 2^6$

$\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{13} - 1) + 71 \cdot 2^{12} + 750 \cdot 2^{11} + 2150 \cdot 2^{10} + 1975 \cdot 2^9 + 561 \cdot 2^8 + 36 \cdot 2^7$

$\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{14} - 1) + 82 \cdot 2^{13} + 1015 \cdot 2^{12} + 3500 \cdot 2^{11} + 4025 \cdot 2^{10} + 1526 \cdot 2^9$
$+147 \cdot 2^8$

$\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{15} - 1) + 93 \cdot 2^{14} + 1320 \cdot 2^{13} + 5320 \cdot 2^{12} + 7350 \cdot 2^{11} + 3486 \cdot 2^{10}$
$+448 \cdot 2^9$

$\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{16} - 1) + 104 \cdot 2^{15} + 1665 \cdot 2^{14} + 7680 \cdot 2^{13} + 12390 \cdot 2^{12} + 7056 \cdot 2^{11}$
$+1134 \cdot 2^{10}$

$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{17} - 1) + 115 \cdot 2^{16} + 2050 \cdot 2^{15} + 10650 \cdot 2^{14} + 19650 \cdot 2^{13} + 13062 \cdot 2^{12}$
$+2520 \cdot 2^{11}$

Some counting maximal chains are presented in Appendix D. These results are summarized in Table 5.6 from which we have Proposition 5.2.5.

Proposition 5.2.5. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$ *is* $2^{n+7} - 1 + (11n+5) \cdot 2^{n+6} + 5 \cdot \frac{n}{2} \frac{(16n+4)}{2!} \cdot 2^{n+5} + \frac{5 \cdot 4}{2!} \cdot \frac{n(n-1)}{3} \frac{(21n+3)}{3!} \cdot 2^{n+4} + \frac{5 \cdot 4 \cdot 3}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(26n+2)}{4!} \cdot 2^{n+3} + \frac{5 \cdot 4 \cdot 3 \cdot 2}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(31n+1)}{5!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)(n-4)n}{5!} \cdot 2^{n+1}$

5.2.6 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r$

The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r$ gives us the following distinct fuzzy subgroups:

$\mathbb{Z}_{p} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{9} - 1) + 19 \cdot 2^{8} + 36 \cdot 2^7$

$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{10} - 1) + 32 \cdot 2^{9} + 129 \cdot 2^8 + 90 \cdot 2^7$

$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{11} - 1) + 45 \cdot 2^{10} + 279 \cdot 2^9 + 395 \cdot 2^8 + 120 \cdot 2^7$

$\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{12} - 1) + 58 \cdot 2^{11} + 486 \cdot 2^{10} + 1040 \cdot 2^9 + 635 \cdot 2^8 + 90 \cdot 2^7$

$\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{13} - 1) + 71 \cdot 2^{12} + 750 \cdot 2^{11} + 2150 \cdot 2^{10} + 1975 \cdot 2^9 + 561 \cdot 2^8 + 36 \cdot 2^7$

$\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{14} - 1) + 84 \cdot 2^{13} + 1071 \cdot 2^{12} + 3850 \cdot 2^{11} + 4725 \cdot 2^{10} + 2016 \cdot 2^9$
$+259 \cdot 2^8 + 6 \cdot 2^7$

$\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r:$ $\quad 1 \cdot (2^{15} - 1) + 97 \cdot 2^{14} + 1449 \cdot 2^{13} + 6265 \cdot 2^{12} + 9625 \cdot 2^{11} + 5481 \cdot 2^{10}$
$+1057 \cdot 2^9 + 49 \cdot 2^8$

$\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{16} - 1) + 110 \cdot 2^{15} + 1884 \cdot 2^{14} + 9520 \cdot 2^{13} + 17570 \cdot 2^{12} + 12516 \cdot 2^{11}$
$+3220 \cdot 2^{10} + 224 \cdot 2^{9}$

$\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{17} - 1) + 123 \cdot 2^{16} + 2376 \cdot 2^{15} + 13740 \cdot 2^{14} + 29610 \cdot 2^{13} + 25326 \cdot 2^{12}$
$+8148 \cdot 2^{11} + 756 \cdot 2^{10}$

$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{18} - 1) + 136 \cdot 2^{17} + 2925 \cdot 2^{16} + 19050 \cdot 2^{15} + 46950 \cdot 2^{14} + 468722 \cdot 2^{13}$
$+18102 \cdot 2^{12} + 2100 \cdot 2^{11}$

Again, the above results are summarized in Table 5.7 from which we have Proposition 5.2.6.

Proposition 5.2.6. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r$ *is* $2^{n+8} - 1 + (13n+6) \cdot 2^{n+7} + 6 \cdot \frac{n}{2} \frac{(19n+5)}{2!} \cdot 2^{n+6} + \frac{6 \cdot 5}{2!} \cdot \frac{n(n-1)}{3} \frac{(25n+4)}{3!} \cdot 2^{n+5} + \frac{6 \cdot 5 \cdot 4}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(31n+3)}{4!} \cdot 2^{n+4} + \frac{6 \cdot 5 \cdot 4 \cdot 3}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(37n+2)}{5!} \cdot 2^{n+3} + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(43n+1)}{6!} \cdot 2^{n+3} + \frac{n(n-1)(n-2) \cdots (n-5)n}{6!} \cdot 2^{n+1}$

5.2.7 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r$

Similarly, the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r$ gives us the following distinct fuzzy subgroups:

$\mathbb{Z}_{p} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{10} - 1) + 22 \cdot 2^{9} + 49 \cdot 2^{8}$

$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{11} - 1) + 37 \cdot 2^{10} + 175 \cdot 2^{9} + 147 \cdot 2^{8}$

$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{12} - 1) + 52 \cdot 2^{11} + 378 \cdot 2^{10} + 644 \cdot 2^{9} + 245 \cdot 2^{8}$

$\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{13} - 1) + 67 \cdot 2^{12} + 658 \cdot 2^{11} + 1964 \cdot 2^{10} + 1295 \cdot 2^{9} + 245 \cdot 2^{8}$

$\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{14} - 1) + 82 \cdot 2^{13} + 1015 \cdot 2^{12} + 3500 \cdot 2^{11} + 4025 \cdot 2^{10} + 1526 \cdot 2^{9}$
$+147 \cdot 2^{8}$

$\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{15} - 1) + 97 \cdot 2^{14} + 1449 \cdot 2^{13} + 6265 \cdot 2^{12} + 9625 \cdot 2^{11} + 5481 \cdot 2^{10}$
$+1057 \cdot 2^{9} + 49 \cdot 2^{8}$

$\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{16} - 1) + 112 \cdot 2^{15} + 1960 \cdot 2^{14} + 10192 \cdot 2^{13} + 19600 \cdot 2^{12} + 14896 \cdot 2^{11}$
$+4312 \cdot 2^{10} + 400 \cdot 2^{9} + 7 \cdot 2^{8}$

$\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{17} - 1) + 127 \cdot 2^{16} + 2548 \cdot 2^{15} + 15484 \cdot 2^{14} + 35770 \cdot 2^{13} + 34006 \cdot 2^{12}$

$+13132 \cdot 2^{11} + 1828 \cdot 2^{10} + 64 \cdot 2^{9}$

$\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r$: $1 \cdot (2^{18} - 1) + 142 \cdot 2^{17} + 3213 \cdot 2^{16} + 22344 \cdot 2^{15} + 60270 \cdot 2^{14} + 68796 \cdot 2^{13}$
$+33222 \cdot 2^{12} + 6168 \cdot 2^{11} + 324 \cdot 2^{10}$

$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r$: $1 \cdot (2^{19} - 1) + 157 \cdot 2^{18} + 3955 \cdot 2^{17} + 30975 \cdot 2^{16} + 95550 \cdot 2^{15} + 127302 \cdot 2^{14}$
$+73794 \cdot 2^{13} + 171302 \cdot 2^{12} + 1200 \cdot 2^{11}$

These results are summarized in Table 5.8. The number of fuzzy subgroups for $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r$ is given by the formula in Proposition 5.2.7.

Proposition 5.2.7. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r$ *is* $2^{n+9} - 1 + (15n+7) \cdot 2^{n+8} + 7 \cdot \frac{n}{2} \frac{(22n+6)}{2!} \cdot 2^{n+7} + \frac{7 \cdot 6}{2!} \cdot \frac{n(n-1)}{3} \frac{(29n+5)}{3!} \cdot 2^{n+6} + \frac{7 \cdot 6 \cdot 5}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(36n+4)}{4!} \cdot 2^{n+5} + \frac{7 \cdot 6 \cdot 5 \cdot 4}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(43n+3)}{5!} \cdot 2^{n+4} + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(50n+2)}{6!} \cdot 2^{n+3} + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6!} \cdot \frac{n(n-1)(n-2) \cdots (n-5)}{7} \frac{(57n+1)}{7!} \cdot 2^{n+2} + \frac{n(n-1)(n-2) \cdots (n-6) n}{7!} \cdot 2^{n+1}$

5.2.8 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$

By a similar approach, the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$ gives us the following distinct fuzzy subgroups:

$\mathbb{Z}_{p} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{11} - 1) + 25 \cdot 2^{10} + 64 \cdot 2^{9}$

$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{12} - 1) + 42 \cdot 2^{11} + 228 \cdot 2^{10} + 224 \cdot 2^{9}$

$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{13} - 1) + 59 \cdot 2^{12} + 492 \cdot 2^{11} + 980 \cdot 2^{10} + 448 \cdot 2^{9}$

$\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{14} - 1) + 76 \cdot 2^{13} + 856 \cdot 2^{12} + 2576 \cdot 2^{11} + 2366 \cdot 2^{10} + 560 \cdot 2^{9}$

$\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{15} - 1) + 93 \cdot 2^{14} + 1320 \cdot 2^{13} + 5320 \cdot 2^{12} + 7350 \cdot 2^{11} + 3486 \cdot 2^{10}$
$+448 \cdot 2^{9}$

$\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{16} - 1) + 110 \cdot 2^{15} + 1884 \cdot 2^{14} + 9520 \cdot 2^{13} + 17570 \cdot 2^{12} + 12516 \cdot 2^{11}$
$+3220 \cdot 2^{10} + 224 \cdot 2^{9}$

$\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{17} - 1) + 127 \cdot 2^{16} + 2548 \cdot 2^{15} + 15484 \cdot 2^{14} + 35770 \cdot 2^{13} +$
$34006 \cdot 2^{12} + 13132 \cdot 2^{11} + 1828 \cdot 2^{10} + 64 \cdot 2^{9}$

$\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$: $1 \cdot (2^{18} - 1) + 144 \cdot 2^{17} + 3312 \cdot 2^{16} + 23520 \cdot 2^{15} + 65268 \cdot 2^{14} +$

$$77616 \cdot 2^{13} + 39984 \cdot 2^{12} + 8352 \cdot 2^{11} + 585 \cdot 2^{10} + 8 \cdot 2^{9}$$

$\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{19} - 1) + 161 \cdot 2^{18} + 4176 \cdot 2^{17} + 33936 \cdot 2^{16} + 109956 \cdot 2^{15} +$

$$156996 \cdot 2^{14} + 101136 \cdot 2^{13} + 28176 \cdot 2^{12} + 2961 \cdot 2^{11} + 81 \cdot 2^{10}$$

$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{20} - 1) + 178 \cdot 2^{19} + 5140 \cdot 2^{18} + 47040 \cdot 2^{17} + 174300 \cdot 2^{16} +$

$$290472 \cdot 2^{15} + 224616 \cdot 2^{14} + 78240 \cdot 2^{13} + 10965 \cdot 2^{12} + 450 \cdot 2^{11}$$

The above results are summarized in Table 5.9 and the formula for the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$ is given in Proposition 5.2.8.

Proposition 5.2.8. *The number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$ is* $2^{n+10} - 1 + (17n+8) \cdot 2^{n+9} + 8 \cdot \frac{n}{2} \frac{(25n+7)}{2!} \cdot 2^{n+8} + \frac{8 \cdot 7}{2!} \cdot \frac{n(n-1)}{3} \frac{(33n+6)}{3!} \cdot 2^{n+7} + \frac{8 \cdot 7 \cdot 6}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(41n+5)}{4!} \cdot 2^{n+6} + \frac{8 \cdot 7 \cdot 6 \cdot 5}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(49n+4)}{5!} \cdot 2^{n+5} + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(57n+3)}{6!} \cdot 2^{n+3} + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6!} \cdot \frac{n(n-1)(n-2)\cdots(n-5)}{7} \frac{(65n+2)}{7!} \cdot 2^{n+2} + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{7!} \cdot \frac{n(n-1)(n-2)\cdots(n-6)}{8} \frac{(73n+1)}{8!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)\cdots(n-7)n}{8!} \cdot 2^{n+1}$

5.2.9 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r$

The group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r$ has the following distinct fuzzy subgroups:

$\mathbb{Z}_{p} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{12} - 1) + 28 \cdot 2^{11} + 81 \cdot 2^{10}$

$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{13} - 1) + 47 \cdot 2^{12} + 288 \cdot 2^{11} + 342 \cdot 2^{10}$

$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{14} - 1) + 66 \cdot 2^{13} + 621 \cdot 2^{12} + 1416 \cdot 2^{11} + 756 \cdot 2^{10}$

$\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{15} - 1) + 85 \cdot 2^{14} + 1080 \cdot 2^{13} + 3720 \cdot 2^{12} + 3990 \cdot 2^{11} + 1134 \cdot 2^{10}$

$\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{16} - 1) + 104 \cdot 2^{15} + 1665 \cdot 2^{14} + 7680 \cdot 2^{13} + 12390 \cdot 2^{12} + 7056 \cdot 2^{11}$

$$+134 \cdot 2^{10}$$

$\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{17} - 1) + 123 \cdot 2^{16} + 2376 \cdot 2^{15} + 13740 \cdot 2^{14} + 29610 \cdot 2^{13} + 25326 \cdot 2^{12}$

$$+8148 \cdot 2^{11} + 756 \cdot 2^{10}$$

$\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{18} - 1) + 142 \cdot 2^{17} + 3213 \cdot 2^{16} + 22344 \cdot 2^{15} + 60270 \cdot 2^{14} + 68796 \cdot 2^{13}$

$$+33222 \cdot 2^{12} + 6168 \cdot 2^{11} + 324 \cdot 2^{10}$$

$\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{19} - 1) + 161 \cdot 2^{18} + 4176 \cdot 2^{17} + 33936 \cdot 2^{16} + 109956 \cdot 2^{15} +$

$156996 \cdot 2^{14} + 101136 \cdot 2^{13} + 28176 \cdot 2^{12} + 2961 \cdot 2^{11} + 81 \cdot 2^{10}$

$\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r$: $1 \cdot (2^{20} - 1) + 180 \cdot 2^{19} + 5265 \cdot 2^{18} + 48960 \cdot 2^{17} + 185220 \cdot 2^{16} + 317520 \cdot 2^{15} + 255780 \cdot 2^{14} + 95040 \cdot 2^{13} + 14985 \cdot 2^{12} + 820 \cdot 2^{11} + 9 \cdot 2^{10}$

$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r$: $1 \cdot (2^{21} - 1) + 199 \cdot 2^{20} + 6480 \cdot 2^{19} + 67860 \cdot 2^{18} + 293580 \cdot 2^{17} + 587412 \cdot 2^{16} + 568008 \cdot 2^{15} + 263880 \cdot 2^{14} + 55485 \cdot 2^{13} + 4555 \cdot 2^{12} + 100 \cdot 2^{11}$

These results are summarized in Table 5.10 from which we get the generalization in Proposition 5.2.9.

Proposition 5.2.9. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r$ *is* $2^{n+11} - 1 + (19n+9) \cdot 2^{n+10} + 9 \cdot \frac{n}{2} \frac{(28n+8)}{2!} \cdot 2^{n+9} + \frac{9 \cdot 8}{2!} \cdot \frac{n(n-1)}{3} \frac{(37n+7)}{3!} \cdot 2^{n+8} + \frac{9 \cdot 8 \cdot 7}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(46n+6)}{4!} \cdot 2^{n+7} + \frac{9 \cdot 8 \cdot 7 \cdot 6}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(55n+5)}{5!} \cdot 2^{n+6} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(64n+4)}{6!} \cdot 2^{n+5} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{6!} \cdot \frac{n(n-1)(n-2)\cdots(n-5)}{7} \frac{(73n+3)}{7!} \cdot 2^{n+4} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7!} \cdot \frac{n(n-1)(n-2)\cdots(n-6)}{8} \frac{(82n+2)}{8!} \cdot 2^{n+3} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{8!} \cdot \frac{n(n-1)(n-2)\cdots(n-7)}{8} \frac{(91n+1)}{9!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)\cdots(n-8)n}{9!} \cdot 2^{n+1}$

5.2.10 Distinct Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$

For the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$, we have the following distinct fuzzy subgroups.

$\mathbb{Z}_{p} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$: $1 \cdot (2^{13} - 1) + 31 \cdot 2^{12} + 100 \cdot 2^{11}$

$\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$: $1 \cdot (2^{14} - 1) + 52 \cdot 2^{13} + 355 \cdot 2^{12} + 450 \cdot 2^{11}$

$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$: $1 \cdot (2^{15} - 1) + 73 \cdot 2^{14} + 765 \cdot 2^{13} + 1965 \cdot 2^{12} + 1200 \cdot 2^{11}$

$\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$: $1 \cdot (2^{16} - 1) + 94 \cdot 2^{15} + 1330 \cdot 2^{14} + 5160 \cdot 2^{13} + 6330 \cdot 2^{12} + 2100 \cdot 2^{11}$

$\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$: $1 \cdot (2^{17} - 1) + 115 \cdot 2^{16} + 2050 \cdot 2^{15} + 10650 \cdot 2^{14} + 19650 \cdot 2^{13} + 13062 \cdot 2^{12} + 2520 \cdot 2^{11}$

$\mathbb{Z}_{p^6} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$: $1 \cdot (2^{18} - 1) + 136 \cdot 2^{17} + 2925 \cdot 2^{16} + 19050 \cdot 2^{15} + 46950 \cdot 2^{14} + 468722 \cdot 2^{13} + 18102 \cdot 2^{12} + 2100 \cdot 2^{11}$

$\mathbb{Z}_{p^7} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$: $1 \cdot (2^{19} - 1) + 157 \cdot 2^{18} + 3955 \cdot 2^{17} + 30975 \cdot 2^{16} + 95550 \cdot 2^{15} +$

$$127302 \cdot 2^{14} + 73794 \cdot 2^{13} + 171302 \cdot 2^{12} + 1200 \cdot 2^{11}$$

$\mathbb{Z}_{p^8} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{20} - 1) + 178 \cdot 2^{19} + 5140 \cdot 2^{18} + 47040 \cdot 2^{17} + 174300 \cdot 2^{16} + 290472 \cdot 2^{15} + 224616 \cdot 2^{14} + 78240 \cdot 2^{13} + 10965 \cdot 2^{12} + 450 \cdot 2^{11}$

$\mathbb{Z}_{p^9} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{21} - 1) + 199 \cdot 2^{20} + 6480 \cdot 2^{19} + 67860 \cdot 2^{18} + 293580 \cdot 2^{17} + 587412 \cdot 2^{16} + 568008 \cdot 2^{15} + 263880 \cdot 2^{14} + 55485 \cdot 2^{13} + 4555 \cdot 2^{12} + 100 \cdot 2^{11}$

$\mathbb{Z}_{p^{10}} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r :$ $\quad 1 \cdot (2^{22} - 1) + 220 \cdot 2^{21} + 7975 \cdot 2^{20} + 94050 \cdot 2^{19} + 465300 \cdot 2^{18} + 1086624 \cdot 2^{17} + 121260 \cdot 2^{16} + 732600 \cdot 2^{15} + 205425 \cdot 2^{14} + 25300 \cdot 2^{13} + 1111 \cdot 2^{12} + 10 \cdot 2^{11}$

Similarly, we summarize these results in Table 5.11 from which we have the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$, as stated in Proposition 5.2.10.

Proposition 5.2.10. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$ *is* $2^{n+12} - 1 + (21n + 10) \cdot 2^{n+11} + 10 \cdot \frac{n}{2} \frac{(31n+9)}{2!} \cdot 2^{n+10} + \frac{10 \cdot 9}{2!} \cdot \frac{n(n-1)}{3} \frac{(41n+8)}{3!} \cdot 2^{n+9} + \frac{10 \cdot 9 \cdot 8}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(51n+7)}{4!} \cdot 2^{n+8} + \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(61n+6)}{5!} \cdot 2^{n+7} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(71n+5)}{6!} \cdot 2^{n+6} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6!} \cdot \frac{n(n-1)(n-2)\cdots(n-5)}{7} \frac{(81n+4)}{7!} \cdot 2^{n+5} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{7!} \cdot \frac{n(n-1)(n-2)\cdots(n-6)}{8} \frac{(91n+3)}{8!} \cdot 2^{n+4} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{8!} \cdot \frac{n(n-1)(n-2)\cdots(n-7)}{9} \frac{(101n+2)}{9!} \cdot 2^{n+3} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{9!} \cdot \frac{n(n-1)(n-2)\cdots(n-8)}{10} \frac{(111n+1)}{10!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)\cdots(n-9)n}{10!} \cdot 2^{n+1}$

Remark 5.2.11. From Table 5.12, it's observed that the number of terms stabilizes for $n \geq m$. Therefore the group $p^n q^m$ has $(m + 1)$ terms while $q^n q^m r$ has $(m + 2)$ terms for $n \geq m$.

The results of Propositions 5.2.1–5.2.10 can be summarized in Table 5.13. To get the general formula for counting the fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$, we look at the coefficients of the powers of 2 in $p^n q^m r$, for $m = 1, 2, \cdots, 10$ from Table 5.13. This is done by analyzing each coefficient (starting with the second ones, since the first terms can be deduced independently) over the increasing values of m. The objective is to establish a general pattern when $m = k$ for that particular coefficient. The powers of 2 can be easily generalized by observation. We seek to find a generalization for the coefficients in the formula for the number of distinct fuzzy subgroups of the group $p^n q^m r$. Together with the general powers of 2, we have the counting formula.

The first terms of $p^n q^m r$ are respectively $(2^{n+3}-1),\ (2^{n+4}-1),\ (2^{n+5}-1),\cdots,(2^{n+12}-1)$ for $m=1,2,\cdots,10$. From this, we see that for $m=k$, the first general term is

$$(2^{n+k+2}-1).$$

The second coefficients are $(3n+1),\ (5n+2),\ (7n+3),\ (9n+4),\cdots,(21n+10)$. The coefficients of n here are $3,5,7,9$. So when $m=k$, $(2k+1)$ is the coefficient of n. The constant term is equal to m. The powers of 2 are $2^{n+2},\ 2^{n+3},\ 2^{n+4},\cdots,2^{n+11}$ which generally gives 2^{n+k+1}. Thus the second term can generally be written as

$$[(2m+1)n+m]\cdot 2^{n+m+1}.$$

For the third term, we look at the pattern emanating from two cases: when m is odd and m is even.

Case (i): m odd

We have the coefficients $\frac{(2n^2)}{2},\ \frac{3(5n^2+n)}{2},\ \frac{5(8n^2+2n)}{2},\ \frac{7(11n^2+3n)}{2},\ \frac{9(14n^2+4n)}{2}$ when $m=1,3,5,7,9$ respectively. We now find the general cases for the constants in these coefficients. The constants $1,3,5,7,9$ in these terms are equal to m. The coefficients of n^2 are $2,5,8,11,14$ and form an arithmetic progression with first term 2, common difference 3 and the number of terms is $\frac{m+1}{2}$. Therefore the n^{th} term of this arithmetic sequence is

$$\begin{aligned} T_n &= 2+3\left(\frac{(m+1)}{2}-1\right) \\ &= 2+\frac{3(m+1)}{2}-3 \\ &= \frac{3m+1}{2}. \end{aligned}$$

So when $m=k$, the coefficient of n^2 is $\frac{3k+1}{2}$.

The coefficients of n in these terms are $0,1,2,3,4$ and are equal to $\frac{k-1}{2}$ for $m=k$. The powers of 2 are $2^{n+1},\ 2^{n+2},\ 2^{n+3},\ 2^{n+4},\cdots,2^{n+10}$, generally yielding 2^{n+k}. Therefore when $m=k$, the third term is generally given by

$$\frac{k\left[(\frac{3k+1}{2})\cdot n^2+(\frac{k-1}{2})\cdot n\right]}{2}=\frac{k\left[(3k+1)\cdot n^2+(k-1)\cdot n\right]}{4}.$$

Case (ii): m even

We have the coefficients $\frac{1(7n^2+n)}{2}$, $\frac{2(13n^2+3n)}{2}$, $\frac{3(19n^2+5n)}{2}$, $\frac{4(25n^2+7n)}{2}$, $\frac{5(31n^2+9n)}{2}$, when $m = 2, 4, 6, 8, 10$ respectively. The constants $1, 2, 3, 4, 5$ are equal to $\frac{m}{2}$. The coefficients $7, 13, 19, 25, 31$ of n^2 are in arithmetic progression whose first term is 7, common difference is 6 and $\frac{m}{2}$ the number of terms. Thus the n^{th} term for this sequence is

$$\begin{aligned} T_n &= 7 + 6\left(\frac{m}{2} - 1\right) \\ &= 3m + 1. \end{aligned}$$

Therefore when $m = k$, the coefficient of n^2 is $(3k + 1)$. The coefficients of n are $1, 3, 5, 7, 9$, which give $(k - 1)$ as the general case when $m = k$. Therefore when $m = k$, the general third term is given by

$$\frac{\frac{k}{2}\left[(3k+1)\cdot n^2 + (k-1)\cdot n\right]}{2} = \frac{k\left[(3k+1)\cdot n^2 + (k-1)\cdot n\right]}{4}.$$

From the two cases, we have that in general, the third coefficient is given by

$$\frac{m\left[(3m+1)\cdot n^2 + (m-1)\cdot n\right]}{4} = \binom{m}{1}\frac{n}{2}\frac{\left[(3m+1)n + (m-1)\right]}{2!}.$$

Similarly, we find that the coefficients of the powers of 2 in the 4^{th}, 5^{th}, 6^{th} , $\cdots$, $(r+2)^{nd}$, $\cdots$, $(m+1)^{st}$ (second last) and $(m+2)^{nd}$ (last) terms are respectively: $\binom{m}{2}\frac{n(n-1)}{3}\frac{[(4m+1)n+(m-2)]}{3!}$, $\binom{m}{3}\frac{n(n-1)(n-2)}{4}\frac{[(5m+1)n+(m-3)]}{4!}$, $\binom{m}{4}\frac{n(n-1)(n-2)(n-3)}{5}\frac{[(6m+1)n+(m-4)]}{5!}$, $\cdots$, $\binom{m}{r}\frac{n(n-1)\cdots(n-(r-1))}{(r+1)}\frac{[((r+2)m+1)n+(m-r)]}{(r+1)!}$, $\cdots$, $\binom{m}{m-1}\frac{n(n-1)\cdots(n-(m-2))}{m}\frac{[((m+1)m+1)n+1]}{m!}$ and $\frac{n(n-1)\cdots(n-(m-1))n}{m!}$.

From these coefficients, we get the general formula for finding the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$. Therefore our main result is Theorem 5.2.12.

Theorem 5.2.12. *The number of distinct fuzzy subgroups of the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ *is* $2^{n+m+2} - 1 + [(2m+1)n + m]\cdot 2^{n+m+1} + \binom{m}{1}\frac{n}{2}\frac{[(3m+1)n+(m-1)]}{2!}\cdot 2^{n+m} + \binom{m}{2}\frac{n(n-1)}{3}\frac{[(4m+1)n+(m-2)]}{3!}\cdot 2^{n+m-1} + \binom{m}{3}\frac{n(n-1)(n-2)}{4}\frac{[(5m+1)n+(m-3)]}{4!}\cdot 2^{n+m-2} + \binom{m}{4}\frac{n(n-1)(n-2)(n-3)}{5}\frac{[(6m+1)n+(m-4)]}{5!}\cdot 2^{n+m-3} + \binom{m}{5}\frac{n(n-1)\cdots(n-4)}{6}\frac{[(7m+1)n+(m-5)]}{6!}\cdot 2^{n+m-4} + \cdots + \binom{m}{r}\frac{n(n-1)\cdots(n-(r-1))}{(r+1)}\frac{[((r+2)m+1)n+(m-r)]}{(r+1)!}\cdot 2^{n+m-(r-1)} + \cdots + \binom{m}{m-1}\frac{n(n-1)\cdots(n-(m-2))}{m}\frac{[((m+1)m+1)n+1]}{m!}\cdot 2^2 + \frac{n(n-1)\cdots(n-(m-1))n}{m!}\cdot 2^{n+1}, \quad r \leq m \leq n.$

Proof. We prove the theorem by induction on n. When $n=1$, $\mathbb{Z}_p \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_q \times \mathbb{Z}_r$ and has $2^{m+3}-1+(3m+1)\cdot 2^{m+2}+\frac{m\cdot m}{1!}\cdot 2^{m+1}$ distinct fuzzy subgroups by Proposition 5.2.1 with $n=m$. This result can also be obtained by substituting $m=1$ in Theorem 5.2.12.

Suppose the result holds for $n=k$, i.e., $\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ has
$2^{k+m+2}-1+[(2m+1)k+m]\cdot 2^{k+m+1}+\binom{m}{1}\frac{k}{2}\frac{[(3m+1)k+(m-1)]}{2!}\cdot 2^{k+m}+\binom{m}{2}\frac{k(k-1)}{3}\frac{[(4m+1)k+(m-2)]}{3!}\cdot 2^{k+m-1}+\binom{m}{3}\frac{k(k-1)(k-2)}{4}\frac{[(5m+1)k+(m-3)]}{4!}\cdot 2^{k+m-2}+\binom{m}{4}\frac{k(k-1)(k-2)(k-3)}{5}\frac{[(6m+1)k+(m-4)]}{5!}\cdot 2^{k+m-3}+\binom{m}{5}\frac{k(k-1)\cdots(k-4)}{6}\frac{[(7m+1)k+(m-5)]}{6!}\cdot 2^{k+m-4}+\cdots+\binom{m}{r}\frac{k(k-1)\cdots(k-(r-1))}{(r+1)}\frac{[((r+2)m+1)k+(m-r)]}{(r+1)!}\cdot 2^{k+m-(r-1)}+\cdots+\binom{m}{m-1}\frac{k(k-1)\cdots(k-(m-2))}{m}\frac{[((m+1)m+1)k+1]}{m!}\cdot 2^{k+2}+\frac{k(k-1)\cdots(k-(m-1))k}{m!}\cdot 2^{k+1}$ distinct fuzzy subgroups.

We need to show that $G=\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ has
$2^{k+m+3}-1+[(2m+1)(k+1)+m]\cdot 2^{k+m+2}+\binom{m}{1}\frac{(k+1)}{2}\frac{[(3m+1)(k+1)+(m-1)]}{2!}\cdot 2^{k+m+1}+\binom{m}{2}\frac{(k+1)k}{3}\frac{[(4m+1)(k+1)+(m-2)]}{3!}\cdot 2^{k+m}+\binom{m}{3}\frac{(k+1)k(k-1)}{4}\frac{[(5m+1)(k+1)+(m-3)]}{4!}\cdot 2^{k+m-1}+\binom{m}{4}\frac{k(k+1)(k-1)(k-2)}{5}\frac{[(6m+1)(k+1)+(m-4)]}{5!}\cdot 2^{k+m-2}+\binom{m}{5}\frac{k(k+1)(k-1)(k-2)(k-3)}{6}\frac{[(7m+1)(k+1)+(m-5)]}{6!}\cdot 2^{k+m-3}+\cdots\cdots+\binom{m}{r}\frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)}\frac{[((r+2)m+1)(k+1)+(m-r)]}{(r+1)!}\cdot 2^{k+m-(r-2)}+\cdots+\binom{m}{m-1}\frac{(k+1)k(k-1)\cdots(k-(m-3))}{m}\frac{[((m+1)m+1)(k+1)+1]}{m!}\cdot 2^{k+3}+\frac{(k+1)k(k-1)\cdots((k-(m-2))(k+1)}{m!}\cdot 2^{k+2}$ distinct fuzzy subgroups.

The group G has 3 maximal subgroups $H_1=p^kq^mr$, $H_2=p^{k+1}q^{m-1}r$ and $H_3=p^{k+1}q^m$. The maximal chains of G are illustrated below.

$$p^{k+1}q^mr \supseteq p^kq^mr \supseteq \begin{cases}\cdots\\ \cdots\\ \cdots\end{cases}, \quad p^{k+1}q^mr \supseteq p^{k+1}q^{m-1}r \supseteq \begin{cases}\cdots\\ \cdots\\ \cdots\end{cases} \quad \text{and}$$

$$p^{k+1}q^mr \supseteq p^{k+1}q^m \supseteq \begin{cases}\cdots\\ \cdots\end{cases}$$

Case (i): $H_1=\mathbb{Z}_{p^k} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$

By the inductive hypothesis, the subgroup H_1 has
$2^{k+m+3}-1+[(2m+1)k+m]\cdot 2^{k+m+2}+\binom{m}{1}\frac{k}{2}\frac{[(3m+1)k+(m-1)]}{2!}\cdot 2^{k+m+1}+\binom{m}{2}\frac{k(k-1)}{3}\frac{[(4m+1)k+(m-2)]}{3!}\cdot 2^{k+m}+\binom{m}{3}\frac{k(k-1)(k-2)}{4}\frac{[(5m+1)k+(m-3)]}{4!}\cdot 2^{k+m-1}+\binom{m}{4}\frac{k(k-1)(k-2)(k-3)}{5}\frac{[(6m+1)k+(m-4)]}{5!}\cdot 2^{k+m-2}+\binom{m}{5}\frac{k(k-1)\cdots(k-4)}{6}\frac{[(7m+1)k+(m-5)]}{6!}\cdot 2^{k+m-3}+\cdots+\binom{m}{r}\frac{k(k-1)\cdots(k-(r-1))}{(r+1)}\frac{[((r+2)m+1)k+(m-r)]}{(r+1)!}\cdot 2^{k+m-(r-2)}$

$+ \cdots + \binom{m}{m-1} \frac{k(k-1)\cdots(k-(m-2))}{m} \frac{[((m+1)m+1)k+1]}{m!} \cdot 2^{k+3} + \frac{k(k-1)\cdots(k-(m-1))k}{m!} \cdot 2^{k+2}$ distinct fuzzy subgroups (since maximal chains of G have length $k+m+3$).

Case (ii): Contribution from H_2 and H_3

There are $2m$ maximal chains with a distinguishing factor through H_2 and 1 maximal chain with a distinguishing factor through H_3. This gives a sum of $(2m+1)$ maximal chains with a distinguishing factor. From case (i) and case (ii), we have a total of

$$[(2m+1)k+m] + (2m+1) = [(2m+1)(k+1)+m]$$

maximal chains with a distinguishing factor. This is clearly the coefficient of 2^{k+m+2} in the second term of the formula for the number of distinct fuzzy subgroups of G.

When $m = 1, 2, 3, 4, 5, 6$, we have respectively k, $5k+2$, $12k+6$, $22k+12$, $35k+20$ and $51k+30$ maximal chains with a distinguishing pair through the maximal subgroup H_2. This pattern yields $\frac{m(3m-1)k}{2!} + m(m-1)$ maximal chains along H_2 with a distinguishing pair. Along the subgroup H_3, for $m = 1, 2, 3, 4, 5, 6$, we have respectively $k+1$, $2k+2$, $3k+3$, $4k+4$, $5k+5$ and $6k+6$ maximal chains with a distinguishing pair. Therefore H_3 has $(mk+m)$ maximal chains with a distinguishing pair. From H_2 and H_3, we have

$$\begin{aligned}\frac{3m^2-mk}{2!} + (m^2-m) + (mk+m) &= \frac{3m^2k+mk+2m^2}{2!} \\ &= \frac{mk(3m+1)+2m^2}{2!} \\ &= \frac{m[(3m+1)k+2m]}{2!} \\ &= \binom{m}{1}\frac{[(3m+1)k+2m]}{2!}.\end{aligned}$$

Therefore case (ii) yields $\binom{m}{1}\frac{[(3m+1)k+2m]}{2!}$ maximal chains with a distinguishing pair.

Case (i) has $\binom{m}{1}\frac{k}{2}\frac{[(3m+1)k+(m-1)]}{2!} = \frac{m^2k^2+mk^2+m^2k-mk}{4}$ maximal chains with a distinguishing pair. Summing up the totals from case (i) and case (ii), we have

$$\begin{aligned}\frac{m^2k^2+mk^2+m^2k-mk}{4} + \frac{3m^2k+mk+2m^2}{2!} &= \frac{3m^2k^2+7m^2k+mk^2+mk+4m^2}{4} \\ &= \binom{m}{1}\frac{(k+1)}{2}\frac{[(3m+1)(k+1)+(m-1)]}{2!}\end{aligned}$$

Therefore G has $\binom{m}{1}\frac{(k+1)}{2}\frac{[(3m+1)(k+1)+(m-1)]}{2!}$ maximal chains with a distinguishing pair. This is the coefficient of 2^{k+m+1} in the third term of the formula for the number of distinct fuzzy subgroups of G.

Similarly, when $m = 1, 2, 3, 4, 5, 6$, the subgroup H_2 has respectively 0, k^2, $5k(k+1)$, $2k(7k+2)$, $5k(6k+2)$, $5k(13k+4)$ maximal chains with a distinguishing triple. From this pattern, the number of maximal chains with a distinguishing triple through H_2 is given by

$$\begin{aligned}\frac{2m^3k^2 - 3m^2k^2 + m^3k - 3m^2k + mk^2 + 2mk}{3!} &= \frac{mk \cdot (2m^2k - 3mk + m^2 - 3m + k + 2)}{3!} \\ &= \frac{mk \cdot [(2m^2 - 3m + 1)k + (m^2 - 3m + 2)]}{3!} \\ &= \frac{mk \cdot [(2m-1)(m-1)k + (m-2)(m-1)]}{3!} \\ &= \frac{m(m-1)k}{2!} \cdot \frac{[(2m-1)k + (m-2)]}{3}.\end{aligned}$$

Moreover, the subgroup H_3 has 0, $\frac{k(k+1)}{2!}$, $\frac{3k(k+1)}{2!}$, $\frac{6k(k+1)}{2!}$, $\frac{10k(k+1)}{2!}$, $\frac{15k(k+1)}{2!}$ maximal chains with a distinguishing triple when $m = 1, 2, 3, 4, 5, 6$ respectively. Thus in general, H_3 has $\frac{m(m-1)}{2!} \cdot \frac{k(k+1)}{2!}$ maximal chains with a distinguishing triple. Therefore case (ii) contributes

$$\begin{aligned}\frac{m(m-1)k}{2!}\frac{[(2m-1)k + (m-2)]}{3} + \frac{m(m-1)}{2!}\frac{k(k+1)}{2!} &= \frac{m(m-1)k(4mk + k + 2m - 1)}{12} \\ &= \binom{m}{2}\frac{k[(4m+1)k + (2m-1)]}{3!}.\end{aligned}$$

maximal chains with a distinguishing triple. Case (i) yields $\binom{m}{2}\frac{k(k-1)}{3}\frac{[(4m+1)k+(m-2)]}{3!}$ maximal chains with a distinguishing triple. Thus case (i) and case (ii) yield $\binom{m}{2}\frac{k(k-1)}{3}\frac{[(4m+1)k+(m-2)]}{3!} + \binom{m}{2}\frac{k[(4m+1)k+(2m-1)]}{3!} = \binom{m}{2}\frac{k(k+1)}{3}\frac{[(4m+1)(k+1)+(m-2)]}{3!}$ maximal chains with a distinguishing triple. This is the coefficient of 2^{k+m} in the fourth term of the formula for the number of distinct fuzzy subgroups of G.

By a similar approach, the number of maximal chains with a distinguishing quadruple contributed by H_2 and H_3 (case (ii)) is $\binom{m}{3}\frac{k(k-1)[(5m+1)k+(2m-2)]}{4!}$. Case (i) yields $\binom{m}{3}\frac{k(k-1)(k-2)}{4}\frac{[(5m+1)k+(m-3)]}{4!}$ maximal chains with a distinguishing quadruple. Therefore the number of maximal chains with a distinguishing quadruple from case (i) and case (ii) is $\binom{m}{3}\frac{k(k-1)(k-2)}{4}\frac{[(5m+1)k+(m-3)]}{4!} + \binom{m}{3}\frac{k(k-1)[(5m+1)k+(2m-2)]}{4!} = \binom{m}{3}\frac{k(k+1)(k-1)}{4}\frac{[(5m+1)(k+1)+(m-3)]}{4!}$. This again is the coefficient of 2^{k+m-1} in the fourth term of the formula for the number of distinct fuzzy subgroups of G.

We can extend the argument discussed above to get, the number of maximal chains with a distinguishing $5 - tuple$, $6 - tuple$, $\cdots$, $(r+1) - tuple$ for $r \leq m$, $\cdots$, $m - tuple$ (second last coefficient) and $(m+1) - tuple$ (last coefficient) contributed by case (ii). From this argument, the number of these maximal chains are respectively $\binom{m}{4}\frac{k(k-1)(k-2)[(6m+1)k+(2m-3)]}{5!}$, $\binom{m}{5}\frac{k(k-1)(k-2)(k-3)[(7m+1)k+(2m-4)]}{6!}$, $\cdots$, $\binom{m}{r}\frac{k(k-1)\cdots(k-(r-2))[((r+2)m+1)k+(2m-(r-1))]}{(r+1)!}$, $\cdots$, $\binom{m}{m-1}\frac{k(k-1)\cdots(k-(m-3))\{[(m+1)m+1)]k+[2m-((m-1)-1)]\}}{m!}$ and $\frac{k(k-1)\cdots(k-(m-2))[(m+1)k+1]}{m!}$.

The sum of the contributions by case (i) and case (ii) to maximal chains with a distinguishing $5 - tuple$, $6 - tuple$, $\cdots$, $(r+1) - tuple$ for $r \leq m$, $\cdots$, $m - tuple$ (second last coefficient) and $(m+1) - tuple$ (last coefficient) gives respectively $\binom{m}{4}\frac{k(k+1)(k-1)(k-2)}{5}\frac{[(6m+1)(k+1)+(m-4)]}{5!}$, $\binom{m}{5}\frac{k(k+1)(k-1)(k-2)(k-3)}{6}\frac{[(7m+1)(k+1)+(m-5)]}{6!}$, $\cdots$, $\binom{m}{r}\frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)}\frac{[((r+2)m+1)(k+1)+(m-r)]}{(r+1)!}$, $\cdots$, $\binom{m}{m-1}\frac{(k+1)k(k-1)\cdots(k-(m-3))}{m}\frac{\{[(m+1)m+1](k+1)+1\}}{m!}$ and $\frac{(k+1)k(k-1)\cdots(k-(m-2))(k+1)}{m!}$. These terms are the coefficient of 2^{k+m-2}, 2^{k+m-3}, $\cdots$, $2^{k+m-(r-2)}$, $\cdots$, 2^{k+3} and 2^{k+2} respectively in the 6^{th}, 7^{th}, ..., $(r+2)^{nd}$, $\cdots$, $(m+1)^{st}$ and the $(m+2)^{nd}$ (last) terms of the formula for the number of distinct fuzzy subgroups of G.

In particular, we use the $(r+1) - tuple$ and $(m+1) - tuple$ to demonstrate the sum from case (i) and case (ii). We label the number of maximal chains with $(r+1) - tuple$ from case (i) and case (ii) by the Expressions 5.2.1 and 5.2.2 respectively.

$$\binom{m}{r}\frac{k(k-1)\cdots(k-(r-1))}{(r+1)}\frac{[((r+2)m+1)k+(m-r)]}{(r+1)!} \tag{5.2.1}$$

$$\binom{m}{r}\frac{k(k-1)\cdots(k-(r-2))[((r+2)m+1)k+(2m-(r-1))]}{(r+1)!} \tag{5.2.2}$$

Suppose the sum of Expressions 5.2.1 and 5.2.2 is z. Then z can be simplified as follows:

$$\begin{aligned} z &= \binom{m}{r}\frac{k(k-1)\cdots(k-(r-2))(k+1)(rmk+rm+k+2mk+3m-r+1)}{(r+1)(r+1)!} \\ &= \binom{m}{r}\frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)}\frac{[(rmk+2mk+k+rm+2m+1)+(m-r)]}{(r+1)!} \\ &= \binom{m}{r}\frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)}\frac{[(rm+2m+1)(k+1)+(m-r)]}{(r+1)!} \\ &= \binom{m}{r}\frac{(k+1)k(k-1)\cdots(k-(r-2))}{(r+1)}\frac{[((r+2)m+1)(k+1)+(m-r)]}{(r+1)!}. \end{aligned}$$

When $r = m$ in z, we get the $(m+1) - tuple$, the last coefficient as discussed next.

$$\begin{aligned} z &= \binom{m}{m}\frac{(k+1)k(k-1)\cdots(k-(m-2))}{(m+1)}\frac{[((m+2)m+1)(k+1)+(m-m)]}{(m+1)!} \\ &= \frac{(k+1)k(k-1)\cdots(k-(m-2))}{(m+1)}\frac{(m^2+2m+1)(k+1)}{(m+1)!} \\ &= \frac{(k+1)k(k-1)\cdots(k-(m-2))}{(m+1)}\frac{(m+1)(m+1)(k+1)}{(m+1)!} \\ &= \frac{(k+1)k(k-1)\cdots(k-(m-2))(k+1)}{m!}. \end{aligned}$$

The summation for the other tuples can be done similarly. This result can also be obtained by substituting $n = k+1$ in Theorem 5.2.12. This completes the proof of the theorem. □

5.3 Isomorphism Classes of Fuzzy Subgroups

Generally, isomorphism in groups is a form of an equivalence relation. This is illustrated by the Venn diagram in Figure 5.5.

Proposition 5.3.1. *Isomorphism on groups is an equivalence relation.*

Proof. We show that isomorphism is reflexive, symmetric and transitive.

(i) *Reflexive:* Let $\underset{I(x)=x,\ \forall\ x\in G}{I : G \to G}$ be the identity map on G. The map I is an isomorphism and therefore G is isomorphic to itself via I.

(ii) *Symmetric:* Suppose $\theta : G \to H$ is an isomorphism between the groups G and H. Then, we know that θ is a homomorphism and a bijection. We need to show that $\theta^{-1} : H \to G$ is an isomorphism. The map θ is a bijection implies θ^{-1} is too. For homomorphism, we need to show that $\theta^{-1}(hh') = \theta^{-1}(h)\theta^{-1}(h')$, $h, h' \in H$. But θ^{-1} is a bijection implies that $\exists$ $g, g' \in G$ such that $\theta^{-1}(h) = g$ and $\theta^{-1}(h') = g'$. Further, $\theta(gg') = \theta(g)\theta(g') = hh'$ which implies $\theta^{-1}(hh') = gg'$. Similarly $\theta^{-1}(h)\theta^{-1}(h') = gg'$. Thus $\theta^{-1}(hh') = \theta^{-1}(h)\theta^{-1}(h')$.

(iii) *Transitive:* Suppose $G \cong H$ and $H \cong K$. Then $\exists$ two isomorphisms $f : G \to H$ and $g : H \to K$. We show that $g \circ f : G \to K$ is an isomorphism. Let $h, h' \in G$

$$(gf)(hh') = g(f(hh'))$$

$$\begin{aligned} &= g(f(h)f(h')), \text{ since f is a homomorphism} \\ &= g(f(h))g(f(h')), \text{ since g is a homomorphism} \\ &= (gf)(h)(gf)(h'). \end{aligned}$$

Since f and g are bijections, then clearly $f \circ g$ is a bijective map. Therefore $f \circ g$ is an isomorphism. This completes the proof.

□

Remark 5.3.2. The notion of fuzzy equivalence relation as defined in this thesis and in [76, 77, 62] is finer[1] than the notion of fuzzy isomorphism. Therefore if two fuzzy subgroups are equivalent, then they are fuzzy isomorphic and not vice versa as shown in Figure 5.6. This is illustrated in an example given in [62].

Definition 5.3.3. [69] Two or more maximal chains are said to be *isomorphic*, if their lengths are equal and their corresponding components are isomorphic subgroups.

In Examples 5.3.4 and 5.3.5, we discuss the number of non-isomorphic fuzzy subgroups of S_3 and D_8 respectively.

Example 5.3.4. From the maximal chains of S_3 discussed earlier in Example 2.2.1, $S_3 \supseteq \{e,(12)\} \supseteq e$, $S_3 \supseteq \{e,(13)\} \supseteq e$ and $S_3 \supseteq \{e,(23)\} \supseteq e$ are isomorphic. Furthermore, all the three chains are non-isomorphic to the chain $S_3 \supseteq \{e,(123),(132)\} \supseteq e$. We therefore have two non-isomorphic chains namely $S_3 \supseteq \{e,(123),(132)\} \supseteq e$ and one of the isomorphic cases, say $S_3 \supseteq \{e,(12)\} \supseteq e$. These two non-isomorphic chains give $2^3 - 1 + 2^2 = 11$ non-isomorphic fuzzy subgroups of S_3.

Example 5.3.5. Similarly the group D_8 has three isomorphism classes of maximal chains:

$$\begin{aligned} D_8 \supseteq \{e,s,r^2,sr^2\} \supseteq \{e,s\} \supseteq e \ &\cong\ D_8 \supseteq \{e,s,r^2,sr^2\} \supseteq \{e,r^2\} \supseteq e \\ &\cong\ D_8 \supseteq \{e,s,r^2,sr^2\} \supseteq \{e,sr^2\} \supseteq e \\ D_8 \supseteq \{e,r^2,sr,sr^3\} \supseteq \{e,sr\} \supseteq e \ &\cong\ D_8 \supseteq \{e,r^2,sr,sr^3\} \supseteq \{e,sr^3\} \supseteq e \end{aligned}$$

[1] Fuzzy equivalence is a special case of fuzzy isomorphism

$$D_8 \supseteq \{e, r^2, sr, sr^3\} \supseteq \{e, r^2\} \supseteq e \quad \cong \quad D_8 \supseteq \{e, r, r^2, r^3\} \supseteq \{e, r^2\} \supseteq e$$

From the three non isomorphic fuzzy subgroups, we get $2^4 - 1 + 2^3 + 2^3 = 31$ non-isomorphic fuzzy subgroups. We note that this is less than the 63 distinct fuzzy subgroups for D_8.

Proposition 5.3.6. *[8] The number of maximal subgroup chains of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \mathbb{Z}_p$, for p a prime, is*

$$(p+1) + (p^2 + p)\left[\frac{n(n+1)}{2}p + n\right].$$

Proposition 5.3.7. *[8] The number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \mathbb{Z}_p$, for p a prime, is*

$$2^{n+3} - 1 + 2^{n+2}\left[p(pn + n + 1)\right] + 2^{n+1}\left[p + 1 + (p^2 + p)\left(\frac{n(n+1)}{2}p + n\right) - p(2pn + n + 1) - 1\right].$$

Proposition 5.3.8. *[8] The number of maximal subgroup chains of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \mathbb{Z}_p$, for p a prime, is*

$$\frac{n(n+1)}{2}.$$

Proposition 5.3.9. *[8] The number of non-isomorphic fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_p \times \mathbb{Z}_p$, for p a prime, is*

$$2^{n+3} - 1 + 2(n-1)2^{n+2} + \frac{(n-1)(n-2)}{2}2^{n+1}.$$

Remark 5.3.10. The number of non-isomorphic maximal chains of a finite group G is less or equal to that of the total number of maximal chains of G. Therefore the number of non-isomorphic fuzzy subgroups of G is in general less or equal to the number of distinct fuzzy subgroups.

Proposition 5.3.11. *The maximal chains for the groups $\prod_i^k \mathbb{Z}_{p_i^{n_i}}$ and $\prod_i^k \mathbb{Z}_{p_i}$, for p_i's distinct primes, n_i's $\in \mathbb{Z}^+$ and $i = 1, 2, \cdots, k$, are non-isomorphic. Therefore the number of non-isomorphic fuzzy subgroups is equal to that of distinct fuzzy subgroups.*

Table 5.1: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q \times \mathbb{Z}_r$

n	$p^n qr$	Number of fuzzy subgroups
1	pqr	$1 \cdot (2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2$
2	p^2qr	$1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3$
3	p^3qr	$1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4$
4	p^4qr	$1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5$
5	p^5qr	$1 \cdot (2^8 - 1) + 16 \cdot 2^7 + 25 \cdot 2^6$
6	p^6qr	$1 \cdot (2^9 - 1) + 19 \cdot 2^8 + 36 \cdot 2^7$
7	p^7qr	$1 \cdot (2^{10} - 1) + 22 \cdot 2^9 + 49 \cdot 2^8$
8	p^8qr	$1 \cdot (2^{11} - 1) + 25 \cdot 2^{10} + 64 \cdot 2^9$
9	p^9qr	$1 \cdot (2^{12} - 1) + 28 \cdot 2^{11} + 81 \cdot 2^{10}$
10	$p^{10}qr$	$1 \cdot (2^{13} - 1) + 31 \cdot 2^{12} + 100 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	p^kqr	$2^{k+3} - 1 + (3k+1) \cdot 2^{k+2} + k \cdot k \cdot 2^{k+1}$

Table 5.2: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

n	$p^n q^2 r$	Number of fuzzy subgroups
1	pq^2r	$1 \cdot (2^5 - 1) + 7 \cdot 2^4 + 4 \cdot 2^3$
2	p^2q^2r	$1 \cdot (2^6 - 1) + 12 \cdot 2^5 + 15 \cdot 2^4 + 2 \cdot 2^3$
3	p^3q^2r	$1 \cdot (2^7 - 1) + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4$
4	p^4q^2r	$1 \cdot (2^8 - 1) + 22 \cdot 2^7 + 58 \cdot 2^6 + 24 \cdot 2^5$
5	p^5q^2r	$1 \cdot (2^9 - 1) + 27 \cdot 2^8 + 90 \cdot 2^7 + 50 \cdot 2^6$
6	p^6q^2r	$1 \cdot (2^{10} - 1) + 32 \cdot 2^9 + 129 \cdot 2^8 + 90 \cdot 2^7$
7	p^7q^2r	$1 \cdot (2^{11} - 1) + 37 \cdot 2^{10} + 175 \cdot 2^9 + 147 \cdot 2^8$
8	p^8q^2r	$1 \cdot (2^{12} - 1) + 42 \cdot 2^{11} + 228 \cdot 2^{10} + 224 \cdot 2^9$
9	p^9q^2r	$1 \cdot (2^{13} - 1) + 47 \cdot 2^{12} + 288 \cdot 2^{11} + 324 \cdot 2^{10}$
10	$p^{10}q^2r$	$1 \cdot (2^{14} - 1) + 52 \cdot 2^{13} + 355 \cdot 2^{12} + 450 \cdot 2^{11}$
⋮	⋮	⋮
k	p^kq^2r	$2^{k+4} - 1 + (5k+2) \cdot 2^{k+3} + 2 \cdot \frac{k}{2} \frac{(7k+1)}{2!} \cdot 2^{k+2} + \frac{k(k-1)k}{2!} \cdot 2^{k+1}$

Table 5.3: Maximal Chains of $H_3 = p^{m+1}q^2$ with a Triple in Subclusters of the Second Cluster

S-cl \ m	3	4	5	6	7	8	⋯	k
1	3	3	3	3	3	3	⋯	3
2		3	3	3	3	3	⋯	3
3			4	4	4	4	⋯	4
4				5	5	5	⋯	5
5					6	6	⋯	6
6						7	⋯	7
⋮							⋱	⋮
$k-2$								$k-1$

Table 5.4: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

n	p^nq^3r	Number of fuzzy subgroups
1	pq^3r	$1 \cdot (2^6 - 1) + 10 \cdot 2^5 + 9 \cdot 2^4$
2	p^2q^3r	$1 \cdot (2^7 - 1) + 17 \cdot 2^6 + 33 \cdot 2^5 + 9 \cdot 2^4$
3	p^3q^3r	$1 \cdot (2^8 - 1) + 24 \cdot 2^7 + 72 \cdot 2^6 + 40 \cdot 2^5 + 3 \cdot 2^4$
4	p^4q^3r	$1 \cdot (2^9 - 1) + 31 \cdot 2^8 + 126 \cdot 2^7 + 106 \cdot 2^6 + 16 \cdot 2^5$
5	p^5q^3r	$1 \cdot (2^{10} - 1) + 38 \cdot 2^9 + 195 \cdot 2^8 + 220 \cdot 2^7 + 50 \cdot 2^6$
6	p^6q^3r	$1 \cdot (2^{11} - 1) + 45 \cdot 2^{10} + 279 \cdot 2^9 + 395 \cdot 2^8 + 120 \cdot 2^7$
7	p^7q^3r	$1 \cdot (2^{12} - 1) + 52 \cdot 2^{11} + 378 \cdot 2^{10} + 644 \cdot 2^9 + 245 \cdot 2^8$
8	p^8q^3r	$1 \cdot (2^{13} - 1) + 59 \cdot 2^{12} + 492 \cdot 2^{11} + 980 \cdot 2^{10} + 448 \cdot 2^9$
9	p^9q^3r	$1 \cdot (2^{14} - 1) + 66 \cdot 2^{13} + 621 \cdot 2^{12} + 1416 \cdot 2^{11} + 756 \cdot 2^{10}$
10	$p^{10}q^3r$	$1 \cdot (2^{15} - 1) + 73 \cdot 2^{14} + 765 \cdot 2^{13} + 1965 \cdot 2^{12} + 1200 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^3r	$2^{k+5} - 1 + (7k+3) \cdot 2^{k+4} + 3 \cdot \frac{k}{2} \frac{(10k+2)}{2!} \cdot 2^{k+3} + \frac{3 \cdot 2}{2!} \cdot \frac{k(k-1)}{3} \frac{(13k+1)}{3!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)k}{3!} \cdot 2^{k+1}$

Table 5.5: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$

n	$p^n q^4 r$	Number of fuzzy subgroups
1	pq^4r	$1 \cdot (2^7 - 1) + 13 \cdot 2^6 + 16 \cdot 2^5$
2	p^2q^4r	$1 \cdot (2^8 - 1) + 22 \cdot 2^7 + 58 \cdot 2^6 + 24 \cdot 2^5$
3	p^3q^4r	$1 \cdot (2^9 - 1) + 31 \cdot 2^8 + 126 \cdot 2^7 + 106 \cdot 2^6 + 16 \cdot 2^5$
4	p^4q^4r	$1 \cdot (2^{10} - 1) + 40 \cdot 2^9 + 220 \cdot 2^8 + 280 \cdot 2^7 + 85 \cdot 2^6 + 4 \cdot 2^5$
5	p^5q^4r	$1 \cdot (2^{11} - 1) + 49 \cdot 2^{10} + 340 \cdot 2^9 + 580 \cdot 2^8 + 265 \cdot 2^7 + 25 \cdot 2^6$
6	p^6q^4r	$1 \cdot (2^{12} - 1) + 58 \cdot 2^{11} + 486 \cdot 2^{10} + 1040 \cdot 2^9 + 635 \cdot 2^8 + 90 \cdot 2^7$
7	p^7q^4r	$1 \cdot (2^{13} - 1) + 67 \cdot 2^{12} + 658 \cdot 2^{11} + 1694 \cdot 2^{10} + 1295 \cdot 2^9 + 245 \cdot 2^8$
8	p^8q^4r	$1 \cdot (2^{14} - 1) + 76 \cdot 2^{13} + 856 \cdot 2^{12} + 2576 \cdot 2^{11} + 2366 \cdot 2^{10} + 560 \cdot 2^9$
9	p^9q^4r	$1 \cdot (2^{15} - 1) + 85 \cdot 2^{14} + 1080 \cdot 2^{13} + 3720 \cdot 2^{12} + 3990 \cdot 2^{11} + 1134 \cdot 2^{10}$
10	$p^{10}q^4r$	$1 \cdot (2^{16} - 1) + 94 \cdot 2^{15} + 1330 \cdot 2^{14} + 5160 \cdot 2^{13} + 6330 \cdot 2^{12} + 2100 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^4r	$2^{k+6} - 1 + (9k+4) \cdot 2^{k+5} + 4 \cdot \frac{k}{2} \frac{(13k+3)}{2!} \cdot 2^{k+4} + \frac{4 \cdot 3}{2!} \frac{k(k-1)}{3} \frac{(17k+2)}{3!} \cdot 2^{k+3} + \frac{4 \cdot 3 \cdot 2}{3!} \cdot \frac{k(k-1)(k-2)}{4} \frac{(21k+1)}{4!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)(k-3)k}{4!} \cdot 2^{k+1}$

Table 5.6: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$

n	p^nq^5r	Number of fuzzy subgroups
1	pq^5r	$1 \cdot (2^8 - 1) + 16 \cdot 2^7 + 25 \cdot 2^6$
2	p^2q^5r	$1 \cdot (2^9 - 1) + 27 \cdot 2^8 + 90 \cdot 2^7 + 50 \cdot 2^6$
3	p^3q^5r	$1 \cdot (2^{10} - 1) + 38 \cdot 2^9 + 195 \cdot 2^8 + 220 \cdot 2^7 + 50 \cdot 2^6$
4	p^4q^5r	$1 \cdot (2^{11} - 1) + 49 \cdot 2^{10} + 340 \cdot 2^9 + 580 \cdot 2^8 + 265 \cdot 2^7 + 25 \cdot 2^6$
5	p^5q^5r	$1 \cdot (2^{12} - 1) + 60 \cdot 2^{11} + 525 \cdot 2^{10} + 1200 \cdot 2^9 + 825 \cdot 2^8 + 156 \cdot 2^7 + 5 \cdot 2^6$
6	p^6q^5r	$1 \cdot (2^{13} - 1) + 71 \cdot 2^{12} + 750 \cdot 2^{11} + 2150 \cdot 2^{10} + 1975 \cdot 2^9 + 561 \cdot 2^8 + 36 \cdot 2^7$
7	p^7q^5r	$1 \cdot (2^{14} - 1) + 82 \cdot 2^{13} + 1015 \cdot 2^{12} + 3500 \cdot 2^{11} + 4025 \cdot 2^{10} + 1526 \cdot 2^9 + 147 \cdot 2^8$
8	p^8q^5r	$1 \cdot (2^{15} - 1) + 93 \cdot 2^{14} + 1320 \cdot 2^{13} + 5320 \cdot 2^{12} + 7350 \cdot 2^{11} + 3486 \cdot 2^{10} + 448 \cdot 2^9$
9	p^9q^5r	$1 \cdot (2^{16} - 1) + 104 \cdot 2^{15} + 1665 \cdot 2^{14} + 7680 \cdot 2^{13} + 12390 \cdot 2^{12} + 7056 \cdot 2^{11} + 1134 \cdot 2^{10}$
10	$p^{10}q^5r$	$1 \cdot (2^{17} - 1) + 115 \cdot 2^{16} + 2050 \cdot 2^{15} + 10650 \cdot 2^{14} + 19650 \cdot 2^{13} + 13062 \cdot 2^{12} + 2520 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^5r	$2^{k+7} - 1 + (11k + 5) \cdot 2^{k+6} + 5 \cdot \frac{k}{2} \frac{(16k+4)}{2!} \cdot 2^{k+5} + \frac{5 \cdot 4}{2!} \cdot \frac{k(k-1)}{3} \frac{(21k+3)}{3!} \cdot 2^{k+4} + \frac{5 \cdot 4 \cdot 3}{3!} \cdot \frac{k(k-1)(k-2)}{4} \frac{(26k+2)}{4!} \cdot 2^{k+3} + \frac{5 \cdot 4 \cdot 3 \cdot 2}{4!} \cdot \frac{k(k-1)(k-2)(k-3)}{5} \frac{(31k+1)}{5!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)(k-3)(k-4)k}{5!} \cdot 2^{k+1}$

Table 5.7: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^6} \times \mathbb{Z}_r$

n	p^nq^6r	Number of fuzzy subgroups
1	pq^6r	$1 \cdot (2^9 - 1) + 19 \cdot 2^8 + 36 \cdot 2^7$
2	p^2q^6r	$1 \cdot (2^{10} - 1) + 32 \cdot 2^9 + 129 \cdot 2^8 + 90 \cdot 2^7$
3	p^3q^6r	$1 \cdot (2^{11} - 1) + 45 \cdot 2^{10} + 279 \cdot 2^9 + 395 \cdot 2^8 + 120 \cdot 2^7$
4	p^4q^6r	$1 \cdot (2^{12} - 1) + 58 \cdot 2^{11} + 486 \cdot 2^{10} + 1040 \cdot 2^9 + 635 \cdot 2^8 + 90 \cdot 2^7$
5	p^5q^6r	$1 \cdot (2^{13} - 1) + 71 \cdot 2^{12} + 750 \cdot 2^{11} + 2150 \cdot 2^{10} + 1975 \cdot 2^9 + 561 \cdot 2^8 + 36 \cdot 2^7$
6	p^6q^6r	$1 \cdot (2^{14} - 1) + 84 \cdot 2^{13} + 1071 \cdot 2^{12} + 3850 \cdot 2^{11} + 4725 \cdot 2^{10} + 2016 \cdot 2^9 + 259 \cdot 2^8 + 6 \cdot 2^7$
7	p^7q^6r	$1 \cdot (2^{15} - 1) + 97 \cdot 2^{14} + 1449 \cdot 2^{13} + 6265 \cdot 2^{12} + 9625 \cdot 2^{11} + 5481 \cdot 2^{10} + 1057 \cdot 2^9 + 49 \cdot 2^8$
8	p^8q^6r	$1 \cdot (2^{16} - 1) + 110 \cdot 2^{15} + 1884 \cdot 2^{14} + 9520 \cdot 2^{13} + 17570 \cdot 2^{12} + 12516 \cdot 2^{11} + 3220 \cdot 2^{10} + 224 \cdot 2^9$
9	p^9q^6r	$1 \cdot (2^{17} - 1) + 123 \cdot 2^{16} + 2376 \cdot 2^{15} + 13740 \cdot 2^{14} + 29610 \cdot 2^{13} + 25326 \cdot 2^{12} + 8148 \cdot 2^{11} + 756 \cdot 2^{10}$
10	$p^{10}q^6r$	$1 \cdot (2^{18} - 1) + 136 \cdot 2^{17} + 2925 \cdot 2^{16} + 19050 \cdot 2^{15} + 46950 \cdot 2^{14} + 468722 \cdot 2^{13} + 18102 \cdot 2^{12} + 2100 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^6r	$2^{k+8} - 1 + (13k+6) \cdot 2^{k+7} + 6 \cdot \frac{k}{2} \frac{(19k+5)}{2!} \cdot 2^{k+6} + \frac{6\cdot5}{2!} \cdot \frac{k(k-1)}{3} \frac{(25k+4)}{3!} \cdot 2^{k+5} + \frac{6\cdot5\cdot4}{3!} \cdot \frac{k(k-1)(k-2)}{4} \frac{(31k+3)}{4!} \cdot 2^{k+4} + \frac{6\cdot5\cdot4\cdot3}{4!} \cdot \frac{k(k-1)(k-2)(k-3)}{5} \frac{(37k+2)}{5!} \cdot 2^{k+3} + \frac{6\cdot5\cdot4\cdot3\cdot2}{5!} \cdot \frac{k(k-1)(k-2)(k-3)(k-4)}{6} \frac{(43k+1)}{6!} \cdot 2^{k+3} + \frac{k(k-1)(k-2)\cdots(k-5)k}{6!} \cdot 2^{k+1}$

Table 5.8: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^7} \times \mathbb{Z}_r$

n	$p^n q^7 r$	Number of fuzzy subgroups
1	pq^7r	$1 \cdot (2^{10}-1) + 22 \cdot 2^9 + 49 \cdot 2^8$
2	p^2q^7r	$1 \cdot (2^{11}-1) + 37 \cdot 2^{10} + 175 \cdot 2^9 + 147 \cdot 2^8$
3	p^3q^7r	$1 \cdot (2^{12}-1) + 52 \cdot 2^{11} + 378 \cdot 2^{10} + 644 \cdot 2^9 + 245 \cdot 2^8$
4	p^4q^7r	$1 \cdot (2^{13}-1) + 67 \cdot 2^{12} + 658 \cdot 2^{11} + 1964 \cdot 2^{10} + 1295 \cdot 2^9 + 245 \cdot 2^8$
5	p^5q^7r	$1 \cdot (2^{14}-1) + 82 \cdot 2^{13} + 1015 \cdot 2^{12} + 3500 \cdot 2^{11} + 4025 \cdot 2^{10} + 1526 \cdot 2^9 + 147 \cdot 2^8$
6	p^6q^7r	$1 \cdot (2^{15}-1) + 97 \cdot 2^{14} + 1449 \cdot 2^{13} + 6265 \cdot 2^{12} + 9625 \cdot 2^{11} + 5481 \cdot 2^{10} + 1057 \cdot 2^9 + 49 \cdot 2^8$
7	p^7q^7r	$1 \cdot (2^{16}-1) + 112 \cdot 2^{15} + 1960 \cdot 2^{14} + 10192 \cdot 2^{13} + 19600 \cdot 2^{12} + 14896 \cdot 2^{11} + 4312 \cdot 2^{10} + 400 \cdot 2^9 + 7 \cdot 2^8$
8	p^8q^7r	$1 \cdot (2^{17}-1) + 127 \cdot 2^{16} + 2548 \cdot 2^{15} + 15484 \cdot 2^{14} + 35770 \cdot 2^{13} + 34006 \cdot 2^{12} + 13132 \cdot 2^{11} + 1828 \cdot 2^{10} + 64 \cdot 2^9$
9	p^9q^7r	$1 \cdot (2^{18}-1) + 142 \cdot 2^{17} + 3213 \cdot 2^{16} + 22344 \cdot 2^{15} + 60270 \cdot 2^{14} + 68796 \cdot 2^{13} + 33222 \cdot 2^{12} + 6168 \cdot 2^{11} + 324 \cdot 2^{10}$
10	$p^{10}q^7r$	$1 \cdot (2^{19}-1) + 157 \cdot 2^{18} + 3955 \cdot 2^{17} + 30975 \cdot 2^{16} + 95550 \cdot 2^{15} + 127302 \cdot 2^{14} + 73794 \cdot 2^{13} + 171302 \cdot 2^{12} + 1200 \cdot 2^{11}$
⋮	⋮	⋮
k	p^kq^7r	$2^{k+9} - 1 + (15k+7) \cdot 2^{k+8} + 7 \cdot \frac{k}{2} \frac{(22k+6)}{2!} \cdot 2^{k+7} + \frac{7\cdot6}{2!} \cdot \frac{k(k-1)}{3} \frac{(29k+5)}{3!} \cdot 2^{k+6} + \frac{7\cdot6\cdot5}{3!} \cdot \frac{k(k-1)(k-2)}{4} \frac{(36k+4)}{4!} \cdot 2^{k+5} + \frac{7\cdot6\cdot5\cdot4}{4!} \cdot \frac{k(k-1)(k-2)(k-3)}{5} \frac{(43k+3)}{5!} \cdot 2^{k+4} + \frac{7\cdot6\cdot5\cdot4\cdot3}{5!} \cdot \frac{k(k-1)(k-2)(k-3)(k-4)}{6} \frac{(50k+2)}{6!} \cdot 2^{k+3} + \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{6!} \cdot \frac{k(k-1)(k-2)\cdots(k-5)}{7} \frac{(57k+1)}{7!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)\cdots(k-6)k}{7!} \cdot 2^{k+1}$

Table 5.9: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^8} \times \mathbb{Z}_r$

n	p^nq^8r	Number of fuzzy subgroups
1	pq^8r	$1 \cdot (2^{11} - 1) + 25 \cdot 2^{10} + 64 \cdot 2^9$
2	p^2q^8r	$1 \cdot (2^{12} - 1) + 42 \cdot 2^{11} + 228 \cdot 2^{10} + 224 \cdot 2^9$
3	p^3q^8r	$1 \cdot (2^{13} - 1) + 59 \cdot 2^{12} + 492 \cdot 2^{11} + 980 \cdot 2^{10} + 448 \cdot 2^9$
4	p^4q^8r	$1 \cdot (2^{14} - 1) + 76 \cdot 2^{13} + 856 \cdot 2^{12} + 2576 \cdot 2^{11} + 2366 \cdot 2^{10} + 560 \cdot 2^9$
5	p^5q^8r	$1 \cdot (2^{15} - 1) + 93 \cdot 2^{14} + 1320 \cdot 2^{13} + 5320 \cdot 2^{12} + 7350 \cdot 2^{11} + 3486 \cdot 2^{10} + 448 \cdot 2^9$
6	p^6q^8r	$1 \cdot (2^{16} - 1) + 110 \cdot 2^{15} + 1884 \cdot 2^{14} + 9520 \cdot 2^{13} + 17570 \cdot 2^{12} + 12516 \cdot 2^{11} + 3220 \cdot 2^{10} + 224 \cdot 2^9$
7	p^7q^8r	$1 \cdot (2^{17} - 1) + 127 \cdot 2^{16} + 2548 \cdot 2^{15} + 15484 \cdot 2^{14} + 35770 \cdot 2^{13} + 34006 \cdot 2^{12} + 13132 \cdot 2^{11} + 1828 \cdot 2^{10} + 64 \cdot 2^9$
8	p^8q^8r	$1 \cdot (2^{18} - 1) + 144 \cdot 2^{17} + 3312 \cdot 2^{16} + 23520 \cdot 2^{15} + 65268 \cdot 2^{14} + 77616 \cdot 2^{13} + 39984 \cdot 2^{12} + 8352 \cdot 2^{11} + 585 \cdot 2^{10} + 8 \cdot 2^9$
9	p^9q^8r	$1 \cdot (2^{19} - 1) + 161 \cdot 2^{18} + 4176 \cdot 2^{17} + 33936 \cdot 2^{16} + 109956 \cdot 2^{15} + 156996 \cdot 2^{14} + 101136 \cdot 2^{13} + 28176 \cdot 2^{12} + 2961 \cdot 2^{11} + 81 \cdot 2^{10}$
10	$p^{10}q^8r$	$1 \cdot (2^{20} - 1) + 178 \cdot 2^{19} + 5140 \cdot 2^{18} + 47040 \cdot 2^{17} + 174300 \cdot 2^{16} + 290472 \cdot 2^{15} + 224616 \cdot 2^{14} + 78240 \cdot 2^{13} + 10965 \cdot 2^{12} + 450 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^8r	$2^{k+10} - 1 + (17k+8) \cdot 2^{k+9} + 8 \cdot \frac{k}{2} \frac{(25k+7)}{2!} \cdot 2^{k+8} + \frac{8\cdot7}{2!} \cdot \frac{k(k-1)}{3} \frac{(33k+6)}{3!} \cdot 2^{k+7} + \frac{8\cdot7\cdot6}{3!} \cdot \frac{k(k-1)(k-2)}{4} \frac{(41k+5)}{4!} \cdot 2^{k+6} + \frac{8\cdot7\cdot6\cdot5}{4!} \cdot \frac{k(k-1)(k-2)(k-3)}{5} \frac{(49k+4)}{5!} \cdot 2^{k+5} + \frac{8\cdot7\cdot6\cdot5\cdot4}{5!} \cdot \frac{k(k-1)(k-2)(k-3)(k-4)}{6} \frac{(57k+3)}{6!} \cdot 2^{k+3} + \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3}{6!} \cdot \frac{k(k-1)(k-2)\cdots(k-5)}{7} \frac{(65k+2)}{7!} \cdot 2^{k+2} + \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}{7!} \cdot \frac{k(k-1)(k-2)\cdots(k-6)}{8} \frac{(73k+1)}{8!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)\cdots(k-7)k}{8!} \cdot 2^{k+1}$

Table 5.10: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^9} \times \mathbb{Z}_r$

n	p^nq^9r	Number of fuzzy subgroups
1	pq^9r	$1 \cdot (2^{12} - 1) + 28 \cdot 2^{11} + 81 \cdot 2^{10}$
2	p^2q^9r	$1 \cdot (2^{13} - 1) + 47 \cdot 2^{12} + 288 \cdot 2^{11} + 342 \cdot 2^{10}$
3	p^3q^9r	$1 \cdot (2^{14} - 1) + 66 \cdot 2^{13} + 621 \cdot 2^{12} + 1416 \cdot 2^{11} + 756 \cdot 2^{10}$
4	p^4q^9r	$1 \cdot (2^{15} - 1) + 85 \cdot 2^{14} + 1080 \cdot 2^{13} + 3720 \cdot 2^{12} + 3990 \cdot 2^{11} + 1134 \cdot 2^{10}$
5	p^5q^9r	$1 \cdot (2^{16} - 1) + 104 \cdot 2^{15} + 1665 \cdot 2^{14} + 7680 \cdot 2^{13} + 12390 \cdot 2^{12} + 7056 \cdot 2^{11} + 134 \cdot 2^{10}$
6	p^6q^9r	$1 \cdot (2^{17} - 1) + 123 \cdot 2^{16} + 2376 \cdot 2^{15} + 13740 \cdot 2^{14} + 29610 \cdot 2^{13} + 25326 \cdot 2^{12} + 8148 \cdot 2^{11} + 756 \cdot 2^{10}$
7	p^7q^9r	$1 \cdot (2^{18} - 1) + 142 \cdot 2^{17} + 3213 \cdot 2^{16} + 22344 \cdot 2^{15} + 60270 \cdot 2^{14} + 68796 \cdot 2^{13} + 33222 \cdot 2^{12} + 6168 \cdot 2^{11} + 324 \cdot 2^{10}$
8	p^8q^9r	$1 \cdot (2^{19} - 1) + 161 \cdot 2^{18} + 4176 \cdot 2^{17} + 33936 \cdot 2^{16} + 109956 \cdot 2^{15} + 156996 \cdot 2^{14} + 101136 \cdot 2^{13} + 28176 \cdot 2^{12} + 2961 \cdot 2^{11} + 81 \cdot 2^{10}$
9	p^9q^9r	$1 \cdot (2^{20} - 1) + 180 \cdot 2^{19} + 5265 \cdot 2^{18} + 48960 \cdot 2^{17} + 185220 \cdot 2^{16} + 317520 \cdot 2^{15} + 255780 \cdot 2^{14} + 95040 \cdot 2^{13} + 14985 \cdot 2^{12} + 820 \cdot 2^{11} + 9 \cdot 2^{10}$
10	$p^{10}q^9r$	$1 \cdot (2^{21} - 1) + 199 \cdot 2^{20} + 6480 \cdot 2^{19} + 67860 \cdot 2^{18} + 293580 \cdot 2^{17} + 587412 \cdot 2^{16} + 568008 \cdot 2^{15} + 263880 \cdot 2^{14} + 55485 \cdot 2^{13} + 4555 \cdot 2^{12} + 100 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	p^kq^9r	$2^{k+11} - 1 + (19k + 9) \cdot 2^{k+10} + 9 \cdot \frac{k}{2} \frac{(28k+8)}{2!} \cdot 2^{k+9} + \frac{9 \cdot 8}{2!} \cdot \frac{k(k-1)}{3} \frac{(37k+7)}{3!} \cdot 2^{k+8} + \frac{9 \cdot 8 \cdot 7}{3!} \cdot \frac{k(k-1)(k-2)}{4} \frac{(46k+6)}{4!} \cdot 2^{k+7} + \frac{9 \cdot 8 \cdot 7 \cdot 6}{4!} \cdot \frac{k(k-1)(k-2)(k-3)}{5} \frac{(55k+5)}{5!} \cdot 2^{k+6} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5!} \cdot \frac{k(k-1)(k-2)(k-3)(k-4)}{6} \frac{(64k+4)}{6!} \cdot 2^{k+5} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{6!} \cdot \frac{k(k-1)(k-2) \cdots (k-5)}{7} \frac{(73k+3)}{7!} \cdot 2^{k+4} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7!} \cdot \frac{k(k-1)(k-2) \cdots (k-6)}{8} \frac{(82k+2)}{8!} \cdot 2^{k+3} + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{8!} \cdot \frac{k(k-1)(k-2) \cdots (k-7)}{8} \frac{(91k+1)}{9!} \cdot 2^{k+2} + \frac{k(k-1)(k-2) \cdots (k-8)k}{9!} \cdot 2^{k+1}$

Table 5.11: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^{10}} \times \mathbb{Z}_r$

n	$p^n q^{10} r$	Number of fuzzy subgroups
1	$pq^{10}r$	$1 \cdot (2^{13}-1) + 31 \cdot 2^{12} + 100 \cdot 2^{11}$
2	$p^2q^{10}r$	$1 \cdot (2^{14}-1) + 52 \cdot 2^{13} + 355 \cdot 2^{12} + 450 \cdot 2^{11}$
3	$p^3q^{10}r$	$1 \cdot (2^{15}-1) + 73 \cdot 2^{14} + 765 \cdot 2^{13} + 1965 \cdot 2^{12} + 1200 \cdot 2^{11}$
4	$p^4q^{10}r$	$1 \cdot (2^{16}-1) + 94 \cdot 2^{15} + 1330 \cdot 2^{14} + 5160 \cdot 2^{13} + 6330 \cdot 2^{12} + 2100 \cdot 2^{11}$
5	$p^5q^{10}r$	$1 \cdot (2^{17}-1) + 115 \cdot 2^{16} + 2050 \cdot 2^{15} + 10650 \cdot 2^{14} + 19650 \cdot 2^{13} + 13062 \cdot 2^{12} + 2520 \cdot 2^{11}$
6	$p^6q^{10}r$	$1 \cdot (2^{18}-1) + 136 \cdot 2^{17} + 2925 \cdot 2^{16} + 19050 \cdot 2^{15} + 46950 \cdot 2^{14} + 468722 \cdot 2^{13} + 18102 \cdot 2^{12} + 2100 \cdot 2^{11}$
7	$p^7q^{10}r$	$1 \cdot (2^{19}-1) + 157 \cdot 2^{18} + 3955 \cdot 2^{17} + 30975 \cdot 2^{16} + 95550 \cdot 2^{15} + 127302 \cdot 2^{14} + 73794 \cdot 2^{13} + 171302 \cdot 2^{12} + 1200 \cdot 2^{11}$
8	$p^8q^{10}r$	$1 \cdot (2^{20}-1) + 178 \cdot 2^{19} + 5140 \cdot 2^{18} + 47040 \cdot 2^{17} + 174300 \cdot 2^{16} + 290472 \cdot 2^{15} + 224616 \cdot 2^{14} + 78240 \cdot 2^{13} + 10965 \cdot 2^{12} + 450 \cdot 2^{11}$
9	$p^9q^{10}r$	$1 \cdot (2^{21}-1) + 199 \cdot 2^{20} + 6480 \cdot 2^{19} + 67860 \cdot 2^{18} + 293580 \cdot 2^{17} + 587412 \cdot 2^{16} + 568008 \cdot 2^{15} + 263880 \cdot 2^{14} + 55485 \cdot 2^{13} + 4555 \cdot 2^{12} + 100 \cdot 2^{11}$
10	$p^{10}q^{10}r$	$1 \cdot (2^{22}-1) + 220 \cdot 2^{21} + 7975 \cdot 2^{20} + 94050 \cdot 2^{19} + 465300 \cdot 2^{18} + 1086624 \cdot 2^{17} + 121260 \cdot 2^{16} + 732600 \cdot 2^{15} + 205425 \cdot 2^{14} + 25300 \cdot 2^{13} + 1111 \cdot 2^{12} + 10 \cdot 2^{11}$
$\vdots$	$\vdots$	$\vdots$
k	$p^kq^{10}r$	$2^{k+12} - 1 + (21k+10) \cdot 2^{k+11} + 10 \cdot \frac{k}{2} \frac{(31k+9)}{2!} \cdot 2^{k+10} + \frac{10 \cdot 9}{2!} \cdot \frac{k(k-1)}{3} \frac{(41k+8)}{3!} \cdot 2^{k+9} + \frac{10 \cdot 9 \cdot 8}{3!} \cdot \frac{k(k-1)(k-2)}{4} \frac{(51k+7)}{4!} \cdot 2^{k+8} + \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} \cdot \frac{k(k-1)(k-2)(k-3)}{5} \frac{(61k+6)}{5!} \cdot 2^{k+7} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5!} \cdot \frac{k(k-1)(k-2)(k-3)(k-4)}{6} \frac{(71k+5)}{6!} \cdot 2^{k+6} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6!} \cdot \frac{k(k-1)(k-2)\cdots(k-5)}{7} \frac{(81k+4)}{7!} \cdot 2^{k+5} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{7!} \cdot \frac{k(k-1)(k-2)\cdots(k-6)}{8} \frac{(91k+3)}{8!} \cdot 2^{k+4} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{8!} \cdot \frac{k(k-1)(k-2)\cdots(k-7)}{9} \frac{(101k+2)}{9!} \cdot 2^{k+3} + \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{9!} \cdot \frac{k(k-1)(k-2)\cdots(k-8)}{10} \frac{(111k+1)}{10!} \cdot 2^{k+2} + \frac{k(k-1)(k-2)\cdots(k-9)k}{10!} \cdot 2^{k+1}$

Table 5.12: Number of Terms in the Formulae for Fuzzy Subgroups of p^n, p^nq, p^nq^m, p^nq^mr for $m = 1, 2 \cdots 10$

n	p^n	p^nq	p^nq^2	p^nq^3	p^nqr	p^nq^2r	p^nq^3r	p^nq^4r	p^nq^5r	p^nq^6r	p^nq^7r	p^nq^8r	p^nq^9r	$p^nq^{10}r$
1	1	2	2	2	3	3	3	3	3	3	3	3	3	3
2	1	2	3	3	3	4	4	4	4	4	4	4	4	4
3	1	2	3	4	3	4	5	5	5	5	5	5	5	5
4	1	2	3	4	3	4	5	6	6	6	6	6	6	6
5	1	2	3	4	3	4	5	6	7	7	7	7	7	7
6	1	2	3	4	3	4	5	6	7	8	8	8	8	8
7	1	2	3	4	3	4	5	6	7	8	9	9	9	9
8	1	2	3	4	3	4	5	6	7	8	9	10	10	10
9	1	2	3	4	3	4	5	6	7	8	9	10	11	11
10	1	2	3	4	3	4	5	6	7	8	9	10	11	12

Table 5.13: Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$

m	Group	Number of fuzzy subgroups
1	p^nqr	$2^{n+3} - 1 + (3n+1) \cdot 2^{n+2} + \frac{n \cdot n}{1!} \cdot 2^{n+1}$
2	p^nq^2r	$2^{n+4} - 1 + (5n+2) \cdot 2^{n+3} + 2 \cdot \frac{n}{2} \frac{(7n+1)}{2!} \cdot 2^{n+2} + \frac{n(n-1)n}{2!} \cdot 2^{n+1}$
3	p^nq^3r	$2^{n+5} - 1 + (7n+3) \cdot 2^{n+4} + 3 \cdot \frac{n}{2} \frac{(10n+2)}{2!} \cdot 2^{n+3} + \frac{3 \cdot 2}{2!} \cdot \frac{n(n-1)}{3} \frac{(13n+1)}{3!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)n}{3!} \cdot 2^{n+1}$
4	p^nq^4r	$2^{n+6} - 1 + (9n+4) \cdot 2^{n+5} + 4 \cdot \frac{n}{2} \frac{(13n+3)}{2!} \cdot 2^{n+4} + \frac{4 \cdot 3}{2!} \frac{n(n-1)}{3} \frac{(17n+2)}{3!} \cdot 2^{n+3} + \frac{4 \cdot 3 \cdot 2}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(21n+1)}{4!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)n}{4!} \cdot 2^{n+1}$
5	p^nq^5r	$2^{n+7} - 1 + (11n+5) \cdot 2^{n+6} + 5 \cdot \frac{n}{2} \frac{(16n+4)}{2!} \cdot 2^{n+5} + \frac{5 \cdot 4}{2!} \cdot \frac{n(n-1)}{3} \frac{(21n+3)}{3!} \cdot 2^{n+4} + \frac{5 \cdot 4 \cdot 3}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(26n+2)}{4!} \cdot 2^{n+3} + \frac{5 \cdot 4 \cdot 3 \cdot 2}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(31n+1)}{5!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)(n-3)(n-4)n}{5!} \cdot 2^{n+1}$
6	p^nq^6r	$2^{n+8} - 1 + (13n+6) \cdot 2^{n+7} + 6 \cdot \frac{n}{2} \frac{(19n+5)}{2!} \cdot 2^{n+6} + \frac{6 \cdot 5}{2!} \cdot \frac{n(n-1)}{3} \frac{(25n+4)}{3!} \cdot 2^{n+5} + \frac{6 \cdot 5 \cdot 4}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(31n+3)}{4!} \cdot 2^{n+4} + \frac{6 \cdot 5 \cdot 4 \cdot 3}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(37n+2)}{5!} \cdot 2^{n+3} + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(43n+1)}{6!} \cdot 2^{n+3} + \frac{n(n-1)(n-2)\cdots(n-5)n}{6!} \cdot 2^{n+1}$
7	p^nq^7r	$2^{n+9} - 1 + (15n+7) \cdot 2^{n+8} + 7 \cdot \frac{n}{2} \frac{(22n+6)}{2!} \cdot 2^{n+7} + \frac{7 \cdot 6}{2!} \cdot \frac{n(n-1)}{3} \frac{(29n+5)}{3!} \cdot 2^{n+6} + \frac{7 \cdot 6 \cdot 5}{3!} \cdot \frac{n(n-1)(n-2)}{4} \frac{(36n+4)}{4!} \cdot 2^{n+5} + \frac{7 \cdot 6 \cdot 5 \cdot 4}{4!} \cdot \frac{n(n-1)(n-2)(n-3)}{5} \frac{(43n+3)}{5!} \cdot 2^{n+4} + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5!} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{6} \frac{(50n+2)}{6!} \cdot 2^{n+3} + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6!} \cdot \frac{n(n-1)(n-2)\cdots(n-5)}{7} \frac{(57n+1)}{7!} \cdot 2^{n+2} + \frac{n(n-1)(n-2)\cdots(n-6)n}{7!} \cdot 2^{n+1}$
$\vdots$	$\vdots$	$\vdots$
k	p^nq^kr	$2^{n+k+2} - 1 + [(2k+1)n+k] \cdot 2^{n+k+1} + \binom{k}{1} \frac{n}{2} \frac{[(3k+1)n+(k-1)\cdot]}{2!} \cdot 2^{n+k} + \binom{k}{2} \frac{n(n-1)}{3} \frac{[(4k+1)n+(k-2)\cdot]}{3!} \cdot 2^{n+k-1} + \binom{k}{3} \frac{n(n-1)(n-2)}{4} \frac{[(5k+1)n+(k-3)\cdot]}{4!} \cdot 2^{n+k-2} + \binom{k}{4} \frac{n(n-1)(n-2)(n-3)}{5} \frac{[(6k+1)n+(k-4)\cdot]}{5!} \cdot 2^{n+k-3} + \binom{k}{5} \frac{n(n-1)\cdots(n-4)}{6} \frac{[(7k+1)n+(k-5)\cdot]}{6!} \cdot 2^{n+k-4} + \binom{k}{6} \frac{n(n-1)\cdots(n-5)}{7} \frac{[(8k+1)n+(k-6)\cdot]}{7!} \cdot 2^{n+k-5} + \binom{k}{7} \frac{n(n-1)\cdots(n-6)}{8} \frac{[(9k+1)n+(k-7)\cdot]}{8!} \cdot 2^{n+k-6} + \binom{k}{8} \frac{n(n-1)\cdots(n-7)}{9} \frac{[(10k+1)n+(k-8)\cdot]}{9!} \cdot 2^{n+k-7} + \cdots + \binom{k}{r} \frac{n(n-1)\cdots(n-(r-1))}{(r+1)} \frac{[((r+2)k+1)n+(k-r)]}{(r+1)!} \cdot 2^{n+k-(r-1)} + \cdots + \binom{k}{k-1} \frac{n(n-1)\cdots[n-(k-2)]}{9} \frac{[((k+1)k+1)n+1]}{k!} \cdot 2^2 + \frac{n(n-1)\cdots(n-(k-1))n}{k!} \cdot 2^{n+1}, \quad r \le k \le n$

$$p^{k+1}qr \supseteq p^kqr \supseteq p^{k-1}qr \supseteq \begin{cases} p^{k-2}qr \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \\ p^{k-1}q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^{k-1}r \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \end{cases}$$

$$p^{k+1}qr \supseteq p^kqr \supseteq p^kq \supseteq \begin{cases} p^{k-1}q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$p^{k+1}qr \supseteq p^kqr \supseteq p^kr \supseteq \begin{cases} p^{k-1}r \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$\underset{Ctd}{\Rightarrow}$$

$$p^{k+1}qr \supseteq p^{k+1}q \supseteq p^kq \supseteq \begin{cases} p^{k-1}q \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$p^{k+1}qr \supseteq p^{k+1}q \supseteq p^{k+1} \supseteq p^k \supseteq p^{k-1} \supseteq \cdots \supseteq 0$$

$$p^{k+1}qr \supseteq p^{k+1}r \supseteq p^kr \supseteq \begin{cases} p^{k-1}r \supseteq \begin{cases} \cdots \\ \cdots \end{cases} \\ p^k \supseteq \cdots \end{cases}$$

$$p^{k+1}qr \supseteq p^{k+1}r \supseteq p^{k+1} \supseteq p^k \supseteq p^{k-1} \supseteq \cdots \supseteq 0$$

Figure 5.1: Maximal Chains of $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_q \times \mathbb{Z}_r$.

6 22 48 84 ⋯

16 16 36

10 10

(a) Triples for $H_2 = p^{k+1}q^2r$

2 7 15 26 ⋯

5 8 11

3 3

(b) Quadruples for $H_2 = p^{k+1}q^2r$ and $H_3 = p^{k+1}q^3$

Figure 5.3: Distinguishing Triples and Quadruples

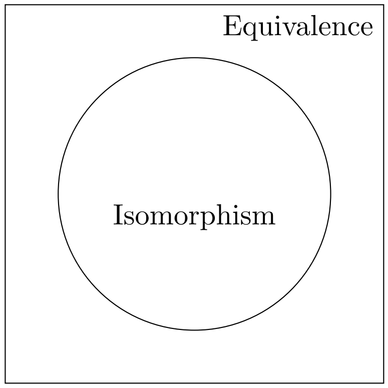

Figure 5.5 Equivalence and Isomorphism

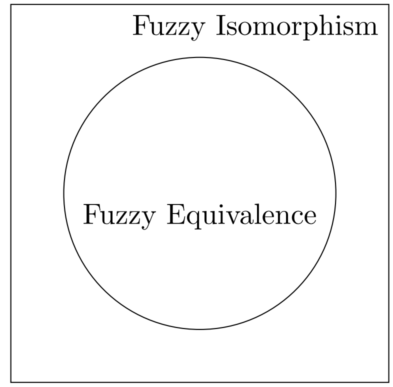

Figure 5.6 Fuzzy Isomorphism and Fuzzy Equivalence

CHAPTER 6. FUZZY SUBGROUPS OF THE FINITE CYCLIC GROUP $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ for $p_1, p_2, \cdots, p_n$ distinct primes

6.1 Introduction

Let G be the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ where $p_1, p_2, \cdots, p_n$ are distinct primes and $n \in \mathbb{Z}^+$. The group G is cyclic since it's isomorphic to the cyclic group $\mathbb{Z}_{p_1 p_2 \cdots p_n}$. We shall often use $p_1 p_2 \cdots p_n$ to denote the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. R. Sulaiman and A.G. Ahmad [90], worked on the fuzzy subgroups of the particular case $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5} \times \mathbb{Z}_{p_6}$ for distinct primes $p_1,\ p_2 \cdots,\ p_6$. Using the equivalence relation of [90], the authours [31, 33] generalized these results to the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. M. Tărnăuceanu in [93], using a different equivalence relation from [90], also worked on fuzzy subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. Other papers which have used the equivalence relation of [93] in counting fuzzy subgroups of finite cyclic groups are [81] and [80].

In [63], Murali and Makamba partially worked on the equivalence classes of fuzzy subgroups of G using the cross-cut technique. Using the equivalence relation defined by Murali and Makamba in [62], we classify the fuzzy subgroups of G. We achieve our goal in this chapter, by extending and completing the work of [63] using the criss-cut method. We develop an algorithm for finding the number of distinct fuzzy subgroups of the group G.

6.2 The Number of Distinct Fuzzy Subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

We investigate this group by making a deduction from the pattern obtained for the distinct primes p_i, $i = 1, 2, \cdots n$ as n increases.

For $i = 1, 2, 3$, the groups $\mathbb{Z}_{p_1}$, $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ and $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3}$, have respectively $2^2 - 1$, $2^3 - 1 + 2^2$ and $(2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2$ distinct fuzzy subgroups listed as follows:

$$p_1 \supseteq 0 : 2^2 - 1$$

$$p_1p_2 \supseteq p_1 \supseteq 0 : 2^3 - 1$$
$$p_1p_2 \supseteq p_2 \supseteq 0 : 2^2$$

$$p_1p_2p_3 \supseteq p_1p_2 \supseteq p_1 \supseteq 0 : 2^4 - 1$$
$$p_1p_2p_3 \supseteq p_1p_2 \supseteq p_2 \supseteq 0 : 2^3$$
$$p_1p_2p_3 \supseteq p_1p_3 \supseteq p_1 \supseteq 0 : 2^3$$
$$p_1p_2p_3 \supseteq p_1p_3 \supseteq p_3 \supseteq 0 : 2^3$$
$$p_1p_2p_3 \supseteq p_2p_3 \supseteq p_2 \supseteq 0 : 2^3$$
$$p_1p_2p_3 \supseteq p_2p_3 \supseteq p_3 \supseteq 0 : 2^2$$

When $i = 4$, $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}$ yields the following fuzzy subgroups:

$$p_1p_2p_3p_4 \supseteq p_1p_2p_3 \supseteq p_1p_2 \supseteq p_1 \supseteq 0 : 2^5 - 1$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_3 \supseteq p_1p_2 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_3 \supseteq p_1p_3 \supseteq p_1 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_3 \supseteq p_1p_3 \supseteq p_3 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_3 \supseteq p_2p_3 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_3 \supseteq p_2p_3 \supseteq p_3 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_4 \supseteq p_1p_2 \supseteq p_1 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_4 \supseteq p_1p_2 \supseteq p_2 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_4 \supseteq p_1p_4 \supseteq p_1 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_4 \supseteq p_1p_4 \supseteq p_4 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_4 \supseteq p_2p_4 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_2p_4 \supseteq p_2p_4 \supseteq p_4 \supseteq 0 : 2^3$$

$$\overset{\Rightarrow}{Ctd}$$

$$p_1p_2p_3p_4 \supseteq p_1p_3p_4 \supseteq p_1p_3 \supseteq p_1 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_3p_4 \supseteq p_1p_3 \supseteq p_3 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_1p_3p_4 \supseteq p_1p_4 \supseteq p_1 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_1p_3p_4 \supseteq p_1p_4 \supseteq p_4 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_1p_3p_4 \supseteq p_3p_4 \supseteq p_3 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_1p_3p_4 \supseteq p_3p_4 \supseteq p_4 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_2p_3p_4 \supseteq p_2p_3 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4 \supseteq p_2p_3p_4 \supseteq p_2p_3 \supseteq p_3 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_2p_3p_4 \supseteq p_2p_4 \supseteq p_2 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_2p_3p_4 \supseteq p_2p_4 \supseteq p_4 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_2p_3p_4 \supseteq p_3p_4 \supseteq p_3 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4 \supseteq p_2p_3p_4 \supseteq p_3p_4 \supseteq p_4 \supseteq 0 : 2^2$$

So $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}$ has $(2^5 - 1) + 11 \cdot 2^4 + 11 \cdot 2^3 + 2^2$ distinct fuzzy subgroups.

A similar approach gives us the following results for $i = 5, 6, 7, 8$:

When $i = 5$, $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_5}$ has $(2^6 - 1) + 26 \cdot 2^5 + 66 \cdot 2^4 + 26 \cdot 2^3 + 2^2$ distinct fuzzy subgroups.

When $i = 6$, $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_6}$ has $(2^7 - 1) + 57 \cdot 2^6 + 302 \cdot 2^5 + 302 \cdot 2^4 + 57 \cdot 2^3 + 2^2$ distinct fuzzy subgroups.

When $i = 7$, $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_7}$ has $(2^8 - 1) + 120 \cdot 2^7 + 1191 \cdot 2^6 + 2416 \cdot 2^5 + 1191 \cdot 2^4 + 120 \cdot 2^3 + 2^2$

distinct fuzzy subgroups.

When $i = 8$, $\mathbb{Z}_{p_1}\times\cdots\times\mathbb{Z}_{p_8}$ has $2^9-1+247\cdot 2^8+4293\cdot 2^7+15653\cdot 2^6+15653\cdot 2^5+4293\cdot 2^4+247\cdot 2^3+2^2$ distinct fuzzy subgroups.

The summary of these results is presented in Table 6.1 and extended to $n = 10$. Some more maximal chains used in counting the distinct fuzzy subgroups of G are presented in Appendix E.

6.3 Algorithm for the Number of Distinct Fuzzy Subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

We get a fascinating pattern from the respective coefficients of $2^{n+1} - 1, 2^n, 2^{n-2}, \cdots, 2^2$ for $n = 1, 2, \cdots, 10$ in the expression for the number of fuzzy subgroups of G. This pattern gives the Euler's number triangle, which we present in Figure 6.1. The sum of the i-th row of the triangle gives the number of maximal chains of the group $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_i}$. The terms of this triangle can be calculated from Table 6.2. This helps us to investigate a pattern of getting the coefficients in each column (each position).

In the remaining part of this section, we discuss the algorithm used to generate the triangle in Figure 6.1. The first and last terms of the triangle are each equal to 1, and indeed the triangle is symmetric about the vertical line consisting of the values $1, 4, 66, 2416, 156190, \cdots$. We use the group $G_6 = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_6}$ in explaining how the number of distinct fuzzy subgroups can be obtained using the previous level group $G_5 = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_5}$. The second term of G_6 in Figure 6.1 is given by $5\cdot 1+2\cdot 26$, where $5 = 6-1$, 1 and 26 are the first and second terms of G_5 respectively while 2 represents the position of the term. The second term of $G_7 = \mathbb{Z}_{p_1}\times\cdots\times\mathbb{Z}_{p_7}$ is $6 \cdot 1 + 2 \cdot 57$. Similarly $6 = 7 - 1$, 1 and 57 are the first and second terms of G_6 respectively while 2 is similarly the position of the term. This pattern can be seen in the other second terms for $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_i}$ as shown in Table 6.2. So generally, the second term of $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_n}$ can be expressed as $(n - 1)t_1 + 2t_2$, where $t_1 = 1$ and t_2 are the first and the second terms of $G_{n-1} = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{n-1}}$ respectively in Figure 6.1.

For the third term of G_6, we have $4 \cdot 26 + 3 \cdot 66$ where $4 = 6 - 2$, 26 and 66 are repectively the second and the third terms of G_5 in Figure 6.1 while 3 represents the position of the term (in this case, the third term). A similar trend is seen in G_7 with $5 \cdot 57 + 3 \cdot 302$ implying $5 = 7 - 2$, 57 and 302 are respectively the second and third terms of G_6 and 3 the position. Therefore the general formula for the third term of $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_n}$ is $(n-2)t_2 + 3t_3$ where t_2 and t_3 are the second and the third terms of G_{n-1} respectively. Continuing this pattern, we have that the k^{th} term of $G = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_n}$ is given by $[n - (k-1)]t_{k-1} + kt_k$ where t_{k-1} and t_k are the $(k-1)^{st}$ and k^{th} terms respectively of $G_{n-1} = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{n-1}}$.

Remark 6.3.1. By k^{th} term in the algorithm we mean the coefficient of $2^{n-(k-2)}$ in the formula for the number of distinct fuzzy subgroups of G.

Proposition 6.3.2. *Suppose from the triangle of Figure 6.1, the number of distinct fuzzy subgroups of the group* $G_{n-1} = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{n-1}}$ *is given in expanded form (with the decreasing powers of* 2 *retained). Let* t_i, *for* $i = 1, 2, \cdots n-1$, *be the coefficients of these powers of* 2. *Then the number of distinct fuzzy subgroups of* $G_n = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_n}$ *is given by* $2^{n+1} + [(n-1)t_1 + 2t_2] \cdot 2^n + [(n-2)t_2 + 3t_3] \cdot 2^{n-1} + [(n-3)t_3 + 4t_4] \cdot 2^{n-2} + \cdots + [(n-(k-1))t_{k-1} + kt_k] \cdot 2^{n-(k-2)} + \cdots + [2t_{n-2} + (n-1)t_{n-1}] \cdot 2^3 + 2^2 - 1$ *for* $k \leq n$.

The recursive algorithm discussed above and summarized in Proposition 6.3.2, can be used to determine the number of distinct fuzzy subgroups of $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_n}$ for $n \in \mathbb{Z}^+$.

Table 6.1: Fuzzy Subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

n	$p_1 p_2 \cdots p_n$	Number of fuzzy subgroups
1	p_1	$1 \cdot (2^2 - 1)$
2	$p_1 p_2$	$1 \cdot (2^3 - 1) + 2^2$
3	$p_1 p_2 p_3$	$1 \cdot (2^4 - 1) + 4 \cdot 2^3 + 2^2$
4	$p_1 p_2 p_3 p_4$	$1 \cdot (2^5 - 1) + 11 \cdot 2^4 + 11 \cdot 2^3 + 2^2$
5	$p_1 p_2 p_3 p_4 p_5$	$1 \cdot (2^6 - 1) + 26 \cdot 2^5 + 66 \cdot 2^4 + 26 \cdot 2^3 + 2^2$
6	$p_1 p_2 p_3 p_4 p_5 p_6$	$1 \cdot (2^7 - 1) + 57 \cdot 2^6 + 302 \cdot 2^5 + 302 \cdot 2^4 + 57 \cdot 2^3 + 2^2$
7	$p_1 p_2 p_3 p_4 p_5 p_6 p_7$	$1 \cdot (2^8 - 1) + 120 \cdot 2^7 + 1191 \cdot 2^6 + 2416 \cdot 2^5 + 1191 \cdot 2^4 + 120 \cdot 2^3 + 2^2$
8	$p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8$	$1 \cdot (2^9 - 1) + 247 \cdot 2^8 + 4293 \cdot 2^7 + 15653 \cdot 2^6 + 15653 \cdot 2^5 + 4293 \cdot 2^4 + 247 \cdot 2^3 + 2^2$
9	$p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8 p_9$	$1 \cdot (2^{10} - 1) + 502 \cdot 2^9 + 14608 \cdot 2^8 + 88234 \cdot 2^7 + 156190 \cdot 2^6 + 88234 \cdot 2^5 + 14608 \cdot 2^4 + 502 \cdot 2^3 + 2^2$
10	$p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8 p_9 p_{10}$	$1 \cdot (2^{11} - 1) + 1013 \cdot 2^{10} + 47840 \cdot 2^9 + 455192 \cdot 2^8 + 1310354 \cdot 2^7 + 1310354 \cdot 2^6 + 455192 \cdot 2^5 + 47840 \cdot 2^4 + 1013 \cdot 2^3 + 2^2$
$\vdots$	$\vdots$	$\vdots$
k	$p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8 p_9 p_{10} \cdots p_k$	$2^{k+1} + [(k-1)t_1 + 2t_2] \cdot 2^k + [(k-2)t_2 + 3t_3] \cdot 2^{k-1} + [(k-3)t_3 + 4t_4] \cdot 2^{k-2} + \cdots + [(k-(r-1))t_{r-1} + rt_r] \cdot 2^{k-(r-2)} + \cdots + [(2t_{k-2} + (k-1)t_{k-1}] \cdot 2^3 + 2^2 - 1, \quad r \leq k$

n										
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15653	15653	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 6.1: Euler's Number Triangle for Coefficients of the Terms in the Number of Fuzzy Subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

Table 6.2: Calculation of Coefficients of the Terms in the Number of Fuzzy Subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

n	2nd coeff	3rd coeff	4th coeff	5th coeff	6th coeff
3	$4 = 2 \cdot 1 + 2 \cdot 1$				
4	$11 = 3 \cdot 1 + 2 \cdot 4$	$11 = 2 \cdot 4 + 3 \cdot 1$			
5	$26 = 4 \cdot 1 + 2 \cdot 11$	$66 = 3 \cdot 11 + 3 \cdot 11$	$26 = 2 \cdot 11 + 4 \cdot 1$		
6	$57 = 5 \cdot 1 + 2 \cdot 26$	$302 = 4 \cdot 26 + 3 \cdot 66$	$302 = 3 \cdot 66 + 4 \cdot 26$	$57 = 2 \cdot 26 + 5 \cdot 1$	
7	$120 = 6 \cdot 1 + 2 \cdot 57$	$1191 = 5 \cdot 57 + 3 \cdot 302$	$2416 = 4 \cdot 302 + 4 \cdot 302$	$1191 = 3 \cdot 302 + 5 \cdot 57$	$120 = 2 \cdot 57 + 6 \cdot 1$
8	$247 = 7 \cdot 1 + 2 \cdot 120$	$4293 = 6 \cdot 120 + 3 \cdot 1191$	$15619 = 5 \cdot 1191 + 4 \cdot 2416$	$15619 = 4 \cdot 2416 + 5 \cdot 1191$	$4293 = 3 \cdot 1191 + 6 \cdot 120$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

CHAPTER 7. GENERAL CONCLUSIONS

7.1 General Discussion

Previously known results on fuzzy subgroups were presented in Chapter 1, Section 1.2. In Chapter 2, the number of subgroups and maximal chains of a finite cyclic group is discussed with formulae given and clear examples provided.

A background theory of fuzzy subgroups is given in Chapter 3. Previously, the criss-cut method by Murali and Makamba in counting fuzzy subgroups had not been applied on the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ for distinct primes p, q and $n, m \in \mathbb{Z}^+$. In Chapter 4, the criss-cut method is used to determine the number of fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ and simpler polynomial formulae with proofs provided. Chapter 5, extends our discussion to the number of fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m} \times \mathbb{Z}_r$ for p, q, r distinct primes and $n, m \in \mathbb{Z}^+$. A polynomial formula is obtained and a proof provided.

In Chapter 6, we discuss the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ where $p_1, p_2, \cdots, p_n$ are distinct primes. With the Eulers's triangle presented in Figure 6.1 and the recursive algorithm developed in Proposition 6.3.2, we can find the number of distinct fuzzy subgroups for a given cyclic group G.

From this research, we notice that, our equivalence gives more distinct fuzzy subgroups than the ones arising from the equivalence of [90, 93]. Therefore the equivalence relation by Murali and Makamba is finer than that of [90, 93].

7.2 Recommendations for Further Research

An immediate extension of this work would be to find the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m} \times \mathbb{Z}_{r^s}$ where $n, m, s \in \mathbb{Z}^+$, using the equivalence relation of Murali and Makamba. By the same equivalence relation, another ambitious undertaking would be to find the number of fuzzy subgroups for the group $\mathbb{Z}_{{p_1}^{n_1}} \times \mathbb{Z}_{{p_2}^{n_2}} \times \cdots \times \mathbb{Z}_{{p_m}^{n_m}}$ for $p_1, p_2, \cdots, p_m$ distinct primes and $n_1, n_2, \cdots, n_m \in \mathbb{Z}^+$.

As further research, one would want to write a computer program that automatically generates the triangle in Figure 6.1 for any given n, and add it to the existing computer algebra systems like MATLAB, MuPAD, Mathematica, GAP and Magma.

APPENDIX A. COUNTING FUZZY SUBGROUPS OF $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$, $n = 4, 5$

A.1 The Cyclic Group $\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8 - 1$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

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$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

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$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

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$$p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

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$$p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

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$$p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

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$$p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

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$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

A.2 The Cyclic Group $\mathbb{Z}_{p^5} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^9 - 1$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6 5$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5q \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5q \supseteq p^5 \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5r \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5r \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5r \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5r \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5r \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5qr \supseteq p^5r \supseteq p^5 \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^4q^2 \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^5q \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^5q \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^5q^2r \supseteq p^5q^2 \supseteq p^5q \supseteq p^5 \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

APPENDIX B. COUNTING FUZZY SUBGROUPS OF $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$, $n = 1, 2, 3, 4$

B.1 The Cyclic Group $\mathbb{Z}_p \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

$$pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6 - 1$$
$$pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^5$$
$$pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5$$
$$pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5$$
$$pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5$$
$$pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4$$
$$pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$
$$pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^5$$

$$\underset{Ctd}{\Rightarrow}$$

$$pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^4$$
$$pq^3r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$
$$pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5$$
$$pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$
$$pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$
$$pq^3r \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$
$$pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq p \supseteq 0 : 2^5$$
$$pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^4$$
$$pq^3r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq r \supseteq 0 : 2^4$$
$$pq^3r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$

B.2 The Cyclic Group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7 - 1$$
$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$
$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

B.3 The Cyclic Group $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8 - 1$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq pq^3r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^2q^3r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^4$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^4$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^3q^3r \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

B.4 The Cyclic Group $\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$

The first branch of the maximal chains in counting the fuzzy subgroups of $\mathbb{Z}_{p^4} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ is given in B.4.1.

$$p^4q^3r \supseteq p^3q^3r \supseteq \begin{cases} p^2q^3r \supseteq \begin{cases} \dots \\ \dots \\ \dots \end{cases} \\ p^3q^2r \supseteq \begin{cases} \dots \\ \dots \\ \dots \end{cases} \\ p^3q^3 \supseteq \begin{cases} \dots \\ \dots \end{cases} \end{cases} \tag{B.4.1}$$

The number of fuzzy subgroups corresponding to each chain is similar to that of $\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^3} \times \mathbb{Z}_r$ in Appendix B.3, with the powers of 2 increased by 1. This gives $1 \cdot (2^9 - 1) + 24 \cdot 2^8 + 72 \cdot 2^7 + 40 \cdot 2^6 + 3 \cdot 2^5$ fuzzy subgroups. The remaining chains in our counting, are listed below.

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^2q^2r \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2r \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3q \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^3q^2r \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^2qr \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^3qr \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^8$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq p \supseteq 0 : 2^8$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^2r \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^3r \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4qr \supseteq p^4r \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q^2 \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^2r \supseteq p^4q^2 \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^8$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^2q^3 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^3q^3 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^2q^2 \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^5$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^3q^2 \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^2q \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^3q \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^6$$

$$p^4q^3r \supseteq p^4q^3 \supseteq p^4q^2 \supseteq p^4q \supseteq p^4 \supseteq p^3 \supseteq p^2 \supseteq p \supseteq 0 : 2^7$$

APPENDIX C. COUNTING FUZZY SUBGROUPS OF $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$, $n = 1, 2$

C.1 The Cyclic Group $\mathbb{Z}_p \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^7 - 1$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^5$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5 \underset{Ctd}{\Rightarrow}$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$

$pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$

$pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^6$

$pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq pq^4 \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^6$

$pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^5$

$pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

$pq^4r \supseteq q^4r \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^5$

C.2 The Cyclic Group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$

The first branch of the maximal chains in counting the fuzzy subgroups of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ is given in C.2.1.

$$p^2q^4r \supseteq pq^4r \supseteq \begin{cases} pq^3r \supseteq \begin{cases} \dots \\ \dots \\ \dots \end{cases} \\ pq^4 \supseteq \begin{cases} \dots \\ \dots \end{cases} \\ q^4r \supseteq \begin{cases} \dots \\ \dots \end{cases} \end{cases} \tag{C.2.1}$$

The number of fuzzy subgroups corresponding to each chain is similar to that of $\mathbb{Z}_p \times \mathbb{Z}_{q^4} \times \mathbb{Z}_r$ in Appendix C.1, with the powers of 2 increased by 1. This gives $1 \cdot (2^8 - 1) + 13 \cdot 2^7 + 16 \cdot 2^6$ fuzzy subgroups. The remaining chains in our counting contribute $9 \cdot 2^7 + 42 \cdot 2^6 + 24 \cdot 2^5$ distinct fuzzy subgroups.

APPENDIX D. COUNTING FUZZY SUBGROUPS OF $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$, $n = 1, 2$

D.1 The Cyclic Group $\mathbb{Z}_p \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq p \supseteq 0 : 2^8 - 1$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pq \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq p \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq pr \supseteq r \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pqr \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq q^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^2r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^3r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^4 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq pq^4 \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq q^4r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^4r \supseteq q^4r \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^5 \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq p \supseteq 0 : 2^7$$

$$pq^5r \supseteq pq^5 \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq pq \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^5 \supseteq pq^4 \supseteq pq^3 \supseteq pq^2 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^5 \supseteq pq^4 \supseteq pq^3 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^5 \supseteq pq^4 \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq pq^5 \supseteq q^5 \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq q^5r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq q \supseteq 0 : 2^7$$

$$pq^5r \supseteq q^5r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq qr \supseteq r \supseteq 0 : 2^6$$

$$pq^5r \supseteq q^5r \supseteq q^4r \supseteq q^3r \supseteq q^2r \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq q^5r \supseteq q^4r \supseteq q^3r \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq q^5r \supseteq q^4r \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

$$pq^5r \supseteq q^5r \supseteq q^5 \supseteq q^4 \supseteq q^3 \supseteq q^2 \supseteq q \supseteq 0 : 2^6$$

D.2 The Cyclic Group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$

The first branch of the maximal chains in counting the fuzzy subgroups of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$ is given in D.2.1.

$$p^2q^5r \supseteq pq^5r \supseteq \begin{cases} pq^4r \supseteq \begin{cases} \dots \\ \dots \\ \dots \end{cases} \\ pq^5 \supseteq \begin{cases} \dots \\ \dots \end{cases} \\ q^5r \supseteq \begin{cases} \dots \\ \dots \end{cases} \end{cases} \tag{D.2.1}$$

The number of fuzzy subgroups corresponding to each chain is similar to that of $\mathbb{Z}_p \times \mathbb{Z}_{q^5} \times \mathbb{Z}_r$ in Appendix D.1, with the powers of 2 increased by 1. This gives $1 \cdot (2^9 - 1) + 16 \cdot 2^8 + 25 \cdot 2^7$ fuzzy subgroups. The remaining chains in our counting contribute $11 \cdot 2^8 + 65 \cdot 2^7 + 50 \cdot 2^6$ distinct fuzzy subgroups.

APPENDIX E. COUNTING FUZZY SUBGROUPS OF $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, $n = 5, 6$

E.1 The Cyclic Group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_5}$

The first branch of the maximal chains in counting the fuzzy subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_5}$ is given in E.1.1.

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_4 \supseteq \begin{cases} p_1p_2p_3 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \\ p_1p_2p_4 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \\ p_1p_3p_4 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \\ p_2p_3p_4 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases} \end{cases} \tag{E.1.1}$$

The number of fuzzy subgroups corresponding to each chain is similar to that of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_4}$, with the powers of 2 increased by 1. This gives $1 \cdot (2^6 - 1) + 11 \cdot 2^5 + 11 \cdot 2^4 + 2^3$

fuzzy subgroups. The remaining maximal chains contribute $15 \cdot 2^5 + 55 \cdot 2^4 + 25 \cdot 2^3 + 2^2$ distinct fuzzy subgroups as listed below.

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_3 \supseteq p_1p_2 \supseteq p_1 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_3 \supseteq p_1p_2 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_3 \supseteq p_1p_3 \supseteq p_1 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_3 \supseteq p_1p_3 \supseteq p_3 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_3 \supseteq p_2p_3 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_3 \supseteq p_2p_3 \supseteq p_3 \supseteq 0 : 2^3$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_5 \supseteq p_1p_2 \supseteq p_1 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_5 \supseteq p_1p_2 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_5 \supseteq p_1p_5 \supseteq p_1 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_5 \supseteq p_1p_5 \supseteq p_5 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_5 \supseteq p_2p_5 \supseteq p_2 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_2p_5 \supseteq p_2p_5 \supseteq p_5 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_3p_5 \supseteq p_1p_3 \supseteq p_1 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_3p_5 \supseteq p_1p_3 \supseteq p_3 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_3p_5 \supseteq p_1p_5 \supseteq p_1 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_3p_5 \supseteq p_1p_5 \supseteq p_5 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_3p_5 \supseteq p_3p_5 \supseteq p_3 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_1p_3p_5 \supseteq p_3p_5 \supseteq p_5 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_2p_3p_5 \supseteq p_2p_3 \supseteq p_2 \supseteq 0 : 2^5$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_2p_3p_5 \supseteq p_2p_3 \supseteq p_3 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_2p_3p_5 \supseteq p_2p_5 \supseteq p_2 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_2p_3p_5 \supseteq p_2p_5 \supseteq p_5 \supseteq 0 : 2^4$$
$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_2p_3p_5 \supseteq p_3p_5 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_3p_5 \supseteq p_2p_3p_5 \supseteq p_3p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_4 \supseteq p_1p_2 \supseteq p_1 \supseteq 0 : 2^5$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_4 \supseteq p_1p_2 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_4 \supseteq p_1p_4 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_4 \supseteq p_1p_4 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_4 \supseteq p_2p_4 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_4 \supseteq p_2p_4 \supseteq p_4 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_5 \supseteq p_1p_2 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_5 \supseteq p_1p_2 \supseteq p_2 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_5 \supseteq p_1p_5 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_5 \supseteq p_1p_5 \supseteq p_5 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_5 \supseteq p_2p_5 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_2p_5 \supseteq p_2p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_4 \supseteq p_1 \supseteq 0 : 2^5$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_4 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_5 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_5 \supseteq p_5 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_4p_5 \supseteq p_4p_5 \supseteq p_4 \supseteq 0 : 2^5$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_1p_4p_5 \supseteq p_4p_5 \supseteq p_5 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_4 \supseteq p_2 \supseteq 0 : 2^5$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_4 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_5 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_5 \supseteq p_5 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_2p_4p_5 \supseteq p_4p_5 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_2p_4p_5 \supseteq p_2p_4p_5 \supseteq p_4p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_4 \supseteq p_1p_3 \supseteq p_1 \supseteq 0 : 2^5$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_4 \supseteq p_1p_3 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_4 \supseteq p_1p_4 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_4 \supseteq p_1p_4 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_4 \supseteq p_3p_4 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_4 \supseteq p_3p_4 \supseteq p_4 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_5 \supseteq p_1p_3 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_5 \supseteq p_1p_3 \supseteq p_3 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_5 \supseteq p_1p_5 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_5 \supseteq p_1p_5 \supseteq p_5 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_5 \supseteq p_3p_5 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_3p_5 \supseteq p_3p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_4 \supseteq p_1 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_4 \supseteq p_4 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_5 \supseteq p_1 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_4p_5 \supseteq p_1p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_4p_5 \supseteq p_4p_5 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_1p_4p_5 \supseteq p_4p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_4 \supseteq p_3 \supseteq 0 : 2^5$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_4 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_5 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_5 \supseteq p_5 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_4p_5 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_1p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_4p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_4 \supseteq p_2p_3 \supseteq p_2 \supseteq 0 : 2^5$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_4 \supseteq p_2p_3 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_4 \supseteq p_2p_4 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_4 \supseteq p_2p_4 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_4 \supseteq p_3p_4 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_4 \supseteq p_3p_4 \supseteq p_4 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_5 \supseteq p_2p_3 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_5 \supseteq p_2p_3 \supseteq p_3 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_5 \supseteq p_2p_5 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_5 \supseteq p_2p_5 \supseteq p_5 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_5 \supseteq p_3p_5 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_3p_5 \supseteq p_3p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_4 \supseteq p_2 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_4 \supseteq p_4 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_5 \supseteq p_2 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_4p_5 \supseteq p_2p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_4p_5 \supseteq p_4p_5 \supseteq p_4 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_2p_4p_5 \supseteq p_4p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_4 \supseteq p_3 \supseteq 0 : 2^4$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_4 \supseteq p_4 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_5 \supseteq p_3 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_3p_5 \supseteq p_5 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_4p_5 \supseteq p_4 \supseteq 0 : 2^3$$

$$p_1p_2p_3p_4p_5 \supseteq p_2p_3p_4p_5 \supseteq p_3p_4p_5 \supseteq p_4p_5 \supseteq p_5 \supseteq 0 : 2^2$$

E.2 The Cyclic Group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_6}$

The first branch of the maximal chains in counting the fuzzy subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_6}$ is given in E.2.1.

$$p_1p_2p_3p_4p_6 \supseteq p_1p_2p_3p_4p_5 \supseteq \begin{cases} p_2p_3p_4p_5 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \\ p_2p_3p_4p_6 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \\ p_2p_3p_5p_6 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \\ p_2p_4p_5p_6 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \\ p_3p_4p_5p_6 \supseteq \begin{cases} \cdots \\ \cdots \\ \cdots \\ \cdots \end{cases} \end{cases} \tag{E.2.1}$$

The number of fuzzy subgroups corresponding to each chain is similar to that of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_5}$, with the powers of 2 increased by 1. This gives $(2^7 - 1) + 26 \cdot 2^6 + 66 \cdot 2^5 + 26 \cdot 2^4 + 2^3$ distinct fuzzy subgroups. The remaining chains contribute $31 \cdot 2^6 + 236 \cdot 2^5 + 276 \cdot 2^4 + 56 \cdot 2^3 + 2^2$ distinct fuzzy subgroups.

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