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Weak solvability of the unconditionally stable difference scheme for the coupled sine-Gordon system*

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Abstract. In this paper, we study the existence and uniqueness of weak solution for the system of finite difference schemes for coupled sine-Gordon equations. A novel first order of accuracy unconditionally stable difference scheme is considered. The variational method also known as the energy method is applied to prove unique weak solvability. We also present a new unified numerical method for the approximate solution of this problem by combining the difference scheme and the fixed point iteration. A test problem is considered, and results of numerical experiments are presented with error analysis to verify the accuracy of the proposed numerical method.

Keywords: existence, uniqueness, weak solutions, finite difference, fixed point theory.

1 Introduction

Wave propagation problems are studied in several areas of engineering, physics, and applied mathematics including relativistic quantum mechanics, acoustics, biomedical engineering, and field theory problems (see, [1–4,6,9,11,14,22,25,27,31] and the references given therein). There have been extensive theoretical and numerical studies on nonlinear wave systems such as sine-Gordon, Klein–Gordon, and coupled sine-Gordon equations in the literature (see, [7, 10, 21, 24, 30] and the references given therein). Such type of problems attracted much attention in the last decades due to the presence of soliton solutions. Solitons are nonlinear waves, which occur in proteins, signal conduction between neurons, and deoxyribonucleic acid (DNA) [18, 34].

Due to low regularity of coefficients and source functions, unique solvability in the weak sense have drawn remarkable interest for many problems occurring in real world phenomena, including coupled sine-Gordon equations. In the weak solvability, solutions of complicated nonlinear systems and also linear or semilinear problems, which do not have a corresponding mild formulation, can be obtained even under less regularities of

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data. Solutions of these problems are obtained in the space of distributions by the energy method, also known as the variational method, which is a versatile tool in the theory of partial differential equations.

In this article, for the first time, the unique solvability of first order of accuracy unconditionally stable difference scheme for coupled sine-Gordon system, in the weak sense, is proved. Compared with other existing studies in the literature, the novelty of the present work is two fold: one is the generality of nonlinearity and damping effects in weak solvability via finite difference method, and the other is the unified numerical approach based on first order of accuracy unconditionally stable finite difference scheme and fixed point iteration.

The early investigations about the convergence of difference schemes for hyperbolic partial differential equations (PDEs) are contributed by Courant, Friedrichs, Lewy, von Neumann, Lax, and Richtmeyer et al. In studying these problems, a necessary condition for convergence of a finite difference scheme is Courant–Friedrichs–Lewy (CFL) condition. In the present study, the employed difference scheme provides good convergence and stability results without the need of a CFL condition. In numerical analysis, a unified numerical method, which combines the difference scheme and fixed point iteration with some error tolerances, is used. Combining with fixed point iteration, the numerical experiments for the solution of the difference scheme gives accurate results.

In this study, the nonlinear system of coupled sine-Gordon equations

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} - \beta_1 \Delta u + \gamma_1 \sin(\delta_{11}u + \delta_{12}v)\\ &\quad + \rho_{11}u + \rho_{12}v = f \quad \text{in } R,\\ &\frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} - \beta_2 \Delta v + \gamma_2 \sin(\delta_{21}u + \delta_{22}v)\\ &\quad + \rho_{21}u + \rho_{22}v = g \quad \text{in } R\end{aligned} \tag{1}$$

with boundary conditions

$$u = 0 \quad \text{and} \quad v = 0 \quad \text{on } S \tag{2}$$

and initial conditions

$$u(0,x) = \varphi_1(x) \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial t}(0,x) = \psi_1(x) \quad \text{in } \Omega, \tag{3}$$

$$v(0,x) = \varphi_2(x) \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial t}(0,x) = \psi_2(x) \quad \text{in } \Omega \tag{4}$$

is considered. Here, $\Omega \subset \mathbb{R}^n$ is a bounded open set with piecewise smooth boundary $\Gamma = \partial\Omega$, and Δ is Laplacian. The spaces R and S are defined as $R = (0,1) \times \Omega$ and $S = (0,1) \times \Gamma$, respectively. The constants are given as

$$\alpha_{ij}, \beta_i, \gamma_i, \delta_{ij}, \rho_{ij} \quad \text{– bounded nonzero real numbers for } i,j = 1,2. \tag{5}$$

Let us denote

$$\tilde{f}(t,x,u,v) = f(t,x) - \gamma_1 \sin(\delta_{11}u + \delta_{12}v),$$
$$\tilde{g}(t,x,u,v) = g(t,x) - \gamma_2 \sin(\delta_{21}u + \delta_{22}v).$$

Source functions $\tilde{f}$ and $\tilde{g}$ satisfy the Lipschitz conditions of the form

$$\left|\tilde{f}(t,x,u_1,v)-\tilde{f}(t,x,u_2,v)\right| \leqslant M_1|u_1-u_2|, \tag{6}$$

$$\left|\tilde{g}(t,x,u_1,v)-\tilde{g}(t,x,u_2,v)\right| \leqslant M_2|u_1-u_2|, \tag{7}$$

$$\left|\tilde{f}(t,x,u,v_1)-\tilde{f}(t,x,u,v_2)\right| \leqslant M_3|v_1-v_2|, \tag{8}$$

$$\left|\tilde{g}(t,x,u,v_1)-\tilde{g}(t,x,u,v_2)\right| \leqslant M_4|v_1-v_2| \tag{9}$$

on R, where M_i, $i=1,2,3,4$, are positive constants.

Let $A=-\Delta$ be an unbounded self-adjoint positive definite operator in a Hilbert space H. Then problem (1)–(4) can be written as

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}+\alpha_{11}\frac{\partial u}{\partial t}+\alpha_{12}\frac{\partial v}{\partial t}+\beta_1 Au+\gamma_1\sin(\delta_{11}u+\delta_{12}v)\\
&\quad+\rho_{11}u+\rho_{12}v=f, \quad 0<t<1,\\
&\frac{\partial^2 v}{\partial t^2}+\alpha_{21}\frac{\partial u}{\partial t}+\alpha_{22}\frac{\partial v}{\partial t}+\beta_2 Av+\gamma_2\sin(\delta_{21}u+\delta_{22}v)\\
&\quad+\rho_{21}u+\rho_{22}v=g, \quad 0<t<1,\\
&u(0)=u_0\in V, \qquad \frac{\mathrm{d}u}{\mathrm{d}t}(0)=u_0'\in H,\\
&v(0)=v_0\in V, \qquad \frac{\mathrm{d}v}{\mathrm{d}t}(0)=v_0'\in H.
\end{aligned} \tag{10}$$

Here, V is the Hilbert space satisfying the relation $V\subset H$. In the literature, a special case of the system in the form

$$u_{tt}-u_{xx}=-\delta^2\sin(u-v), \qquad v_{tt}-v_{xx}=\sin(u-v),$$

which describes the open states in DNA double helices, is studied by many researchers (see, [18,34] and the references given therein). Note that some applications and numerical results of the present study, without proof, are presented in [32, 33].

Unique solvability of problem (10) is presented as the limit of first order of accuracy unconditionally stable difference scheme

$$\begin{aligned}
&\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\alpha_{11}(2\tau)^{-1}(u_{k+1}-u_{k-1})\\
&\quad+\alpha_{12}(2\tau)^{-1}(v_{k+1}-v_{k-1})+\beta_1 Au_{k+1}\\
&\quad+\gamma_1\sin(\delta_{11}u_k+\delta_{12}v_k)+\rho_{11}u_k+\rho_{12}v_k=f_k,\\
&f_k=f(t_k), \quad t_k=k\tau, \quad 1\leqslant k\leqslant N-1,\ N\tau=1,\\
&\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+\alpha_{21}(2\tau)^{-1}(u_{k+1}-u_{k-1})\\
&\quad+\alpha_{22}(2\tau)^{-1}(v_{k+1}-v_{k-1})+\beta_2 Av_{k+1}\\
&\quad+\gamma_2\sin(\delta_{21}u_k+\delta_{22}v_k)+\rho_{21}u_k+\rho_{22}v_k=g_k,\\
&g_k=g(t_k), \quad t_k=k\tau, \quad 1\leqslant k\leqslant N-1,\ N\tau=1,
\end{aligned} \tag{11_1}$$

$$\begin{aligned} u_0 &= \varphi_1, \qquad u_0' = \frac{u_1 - u_0}{\tau} = \psi_1, \\ v_0 &= \varphi_2, \qquad v_0' = \frac{v_1 - v_0}{\tau} = \psi_2 \end{aligned} \tag{11$_2$}$$

with a modification for damped nonlinear system. The set of a family of grid points

$$\begin{aligned} \Omega_h = [0,1]_\tau \times [0,\pi]_h = \big\{(t_k, x_n)\colon\ &t_k = k\tau,\ 0 \leqslant k \leqslant N,\ N\tau = 1, \\ &x_n = nh,\ 0 \leqslant n \leqslant M,\ Mh = \pi\big\} \end{aligned} \tag{12}$$

with parameters τ and h is considered for the approximate solution of (10). Here, f_k, g_k, φ_1, φ_2, ψ_1, and ψ_2 are given nonzero functions. Convergence and unconditional stability of undamped linear form of difference scheme (11) is presented in [2,4].

The weak solvability of nonlinear systems are widely investigated in the literature (see [8,12,13,15–17,19,20,22,23,26–29,35,36] and the references given therein). In [8], endemic equilibrium for the PDE model of Zika virus, which leads to a major global public health emergency, is studied. In [15], approximate solution of coupled sine-Gordon equations with periodic boundary conditions is investigated. Also, in [22], the global weak solvability of coupled damped sine-Gordon equation in abstract form is proved, and the finite element method is used. The weak solutions for nongradient coupled sine-Gordon equations are studied in [27]. Regularity criteria of weak solutions for 3D incompressible viscous magnetohydrodynamics (MHD) equations are discussed in [28]. Several types of prey–predator models are investigated in [35].

2 Preliminaries

In this section, we present some preliminaries, which will be used in the theoretical statements of this paper. Let us define the Hilbert spaces H and V as $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, respectively. These spaces are equipped with the inner products and norms

$$(\psi, \phi) = \int_\Omega \psi(x)\phi(x)\,\mathrm{d}x, \quad |\psi| = (\psi,\psi)^{1/2}, \quad \forall \phi, \psi \in L^2(\Omega),$$

$$((\psi, \phi)) = \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\psi(x)\frac{\partial}{\partial x_i}\phi(x)\,\mathrm{d}x, \quad \|\psi\| = ((\psi,\psi))^{1/2}, \quad \forall \phi, \psi \in H_0^1(\Omega).$$

Let us define the dual spaces of V and H as V' and H', respectively. Here, the pair (V, H) is a Gelfand triple space with notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, where $V' = H^{-1}(\Omega)$. The embeddings $V \subset H$ and $H \subset V'$ are continuous, dense, and compact. The unique solvability results are presented in the setting of the triple space. The bilinear form

$$a(\phi, \varphi) = \int_\Omega \nabla\phi \cdot \nabla\varphi\,\mathrm{d}x = ((\phi, \varphi)) \quad \forall \phi, \varphi \in V = H_0^1(\Omega)$$

will be used in variational formulation. This form is bounded, symmetric on $V \times V = H_0^1(\Omega)^2$, and coercive, that is,

$$a(\phi, \phi) \geqslant \|\phi\|^2 \quad \forall \phi \in V.$$

The form is associated with the operator $A = -\Delta$ defined by

$$A(\phi, \varphi) = a(\phi, \varphi),$$

where A is an isomorphism from V onto V'. It is an unbounded self-adjoint operator in H with dense domain $D(A) = \{\phi \in V \mid A\phi \in H\}$ in V and in H. We consider system (10) in the following vector form:

$$\begin{aligned} &\mathbf{w}'' + \boldsymbol{\alpha}\mathbf{w}' + \boldsymbol{\beta}\mathbf{A}\mathbf{w} + \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{w} + \boldsymbol{\rho}\mathbf{w} = \mathbf{f}, \quad 0 < t < T, \\ &\mathbf{w}(\mathbf{0}) = \mathbf{w}_0, \qquad \mathbf{w}'(\mathbf{0}) = \mathbf{w}_0' \end{aligned} \tag{13}$$

with

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \mathbf{w}' = \begin{bmatrix} \frac{\mathrm{d}u}{\mathrm{d}t} \\ \frac{\mathrm{d}v}{\mathrm{d}t} \end{bmatrix}, \qquad \mathbf{w}'' = \begin{bmatrix} \frac{\mathrm{d}^2u}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2v}{\mathrm{d}t^2} \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} f \\ g \end{bmatrix}, \qquad \sin\mathbf{w} = \begin{bmatrix} \sin u \\ \sin v \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}, \qquad \boldsymbol{\delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix},$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \qquad \boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \qquad \mathbf{w}_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \qquad \mathbf{w}_0' = \begin{bmatrix} u_0' \\ v_0' \end{bmatrix}.$$

The norm $|\boldsymbol{\delta}|$ of the 2×2 matrix is defined by

$$\sum_{i,j=1,2} |\delta_{ij}|. \tag{14}$$

Let us introduce the product spaces $\mathcal{V} = V \times V$ and $\mathcal{H} = H \times H$ equipped with the inner products

$$\begin{aligned} ((\boldsymbol{\phi}, \boldsymbol{\psi})) &= ((\phi_1, \psi_1)) + ((\phi_2, \psi_2)), \quad \boldsymbol{\phi} = [\phi_1, \phi_2]^{\mathrm{T}}, \quad \boldsymbol{\psi} = [\psi_1, \psi_2]^{\mathrm{T}} \in \mathcal{V}, \\ (\boldsymbol{\phi}, \boldsymbol{\psi}) &= (\phi_1, \psi_1) + (\phi_2, \psi_2), \quad \boldsymbol{\phi} = [\phi_1, \phi_2]^{\mathrm{T}}, \quad \boldsymbol{\psi} = [\psi_1, \psi_2]^{\mathrm{T}} \in \mathcal{H}, \end{aligned} \tag{15}$$

respectively. Here, $[\cdot, \cdot]^{\mathrm{T}}$ is the transpose of $[\cdot, \cdot]$. Then the dual space $\mathcal{V}' = V' \times V'$ and the dual pairing between $\mathcal{V}'$ and $\mathcal{V}$ are

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \phi_1, \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle, \quad \boldsymbol{\phi} = [\phi_1, \phi_2]^{\mathrm{T}} \in \mathcal{V}', \quad \boldsymbol{\psi} = [\psi_1, \psi_2]^{\mathrm{T}} \in \mathcal{V}.$$

Let the operator A has a square root B such that $B = \sqrt{-\Delta}$. The self-adjoint positive definite operator B generates a C_0 semigroup in problem (13). Thus, the operator matrix $\mathbf{A}$

with operator entries A in (13) is a self-adjoint positive definite operator with a dense domain $\mathcal{D}(\mathcal{A}) = D(A) \times D(A)$ in $\mathcal{V}$ and in $\mathcal{H}$. It can be easily verified that $\mathbf{A}$ generates a C_0 semigroup [1, 3, 27].

By the embeddings $V \hookrightarrow H \hookrightarrow V'$, the pair $(\mathcal{V},\mathcal{H})$ is a Gelfand triple space with notation $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. The norms of $\mathcal{V}$ and $\mathcal{H}$ are denoted by $\|\psi\|$ and $|\psi|$, respectively. The weak solvability of (13) is stated in the following form.

Definition 1. (See [22].) A function $\mathbf{w}$ is said to be a weak solution of (13) if

$$\mathbf{w} \in \mathbf{W}(0,T) = W(0,T) \times W(0,T)$$

and $\mathbf{w}$ satisfies the variational (weak) formulation

$$\begin{aligned} &\langle \mathbf{w}''(\cdot), \boldsymbol{\phi} \rangle + \big(\boldsymbol{\alpha}\mathbf{w}'(\cdot), \boldsymbol{\phi}\big) + \big(\big(\boldsymbol{\beta}\mathbf{w}(\cdot), \boldsymbol{\phi}\big)\big) \\ &\quad + \big(\boldsymbol{\gamma} \sin \boldsymbol{\delta}\mathbf{w}(\cdot), \boldsymbol{\phi}\big) + \big(\boldsymbol{\rho}\mathbf{w}(\cdot), \boldsymbol{\phi}\big) = \big(\mathbf{f}(\cdot), \boldsymbol{\phi}\big), \\ &\mathbf{w}(\mathbf{0}) = \mathbf{w}_0, \qquad \mathbf{w}'(\mathbf{0}) = \mathbf{w}_0' \end{aligned}$$

for all $\boldsymbol{\phi} \in \mathcal{V}$. Here, the solution space is

$$W(0,T) = \big\{\varphi \mid \varphi \in L^2(0,T;V),\ \varphi' \in L^2(0,T;H),\ \varphi'' \in L^2(0,T;V')\big\}.$$

The following theorem states the continuous dependence of weak solutions for (10), and this theorem will be used in the proof of uniqueness.

Theorem 1. (*See* [22].) *Suppose that assumptions* (5)–(9) *hold. Let* $\mathbf{w}_A = [u_A, v_A]^{\mathrm{T}}$ (*resp.,* $\mathbf{w}_B = [u_B, v_B]^{\mathrm{T}}$) *be a weak solution of* (13) *with initial values* $(\mathbf{w}_{A0}, \mathbf{w}_{A1}) \in \mathcal{V} \times \mathcal{H}$ (*resp.,* $(\mathbf{w}_{B0}, \mathbf{w}_{B1}) \in \mathcal{V} \times \mathcal{H}$) *and* $\mathbf{f}_A \in L^2(0,T;\mathcal{H})$ (*resp.,* $\mathbf{f}_B \in L^2(0,T;\mathcal{H})$). *Then there exists a constant* $C > 0$ *depending only on* $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}$ *and* T *such that, for each* $t \in [0,T]$,

$$\begin{aligned} &\big\|\mathbf{w}_A(t) - \mathbf{w}_B(t)\big\|^2 + \big|\mathbf{w}_A'(t) - \mathbf{w}_B'(t)\big|^2 \\ &\quad \leqslant C\left(\|\mathbf{w}_{A0} - \mathbf{w}_{B0}\|^2 + |\mathbf{w}_{A1} - \mathbf{w}_{B1}|^2 + \int_0^t \big|\mathbf{f}_A(\sigma) - \mathbf{f}_B(\sigma)\big|^2 \,\mathrm{d}\sigma \right). \end{aligned}$$

By the properties of the operator matrix $\mathbf{A}$ considered in problem (13), and using the family of grid points Ω_h defined in (12), we can consider system (13) in difference form as

$$\begin{aligned} &\tau^{-2}(\mathbf{w}_{k+1} - 2\mathbf{w}_k + \mathbf{w}_{k-1}) + \alpha(2\tau)^{-1}(\mathbf{w}_{k+1} - \mathbf{w}_{k-1}) \\ &\quad + \boldsymbol{\beta}\mathbf{A}\mathbf{w}_k + \boldsymbol{\gamma} \sin \boldsymbol{\delta}\mathbf{w}_k + \boldsymbol{\rho}\mathbf{w}_k = \mathbf{f}_k, \quad 0 < t < T, \\ &\mathbf{w}_0 = \boldsymbol{\varphi}, \qquad \tau^{-1}(\mathbf{w}_1 - \mathbf{w}_0) = \boldsymbol{\psi}, \end{aligned} \tag{16}$$

where

$$\mathbf{w}_k = \begin{bmatrix} u_k \\ v_k \end{bmatrix}, \qquad \sin \mathbf{w}_k = \begin{bmatrix} \sin u_k \\ \sin v_k \end{bmatrix}, \qquad \mathbf{f}_k = \begin{bmatrix} f_k \\ g_k \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}, \qquad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix},$$

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}, \qquad \boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \qquad \boldsymbol{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \qquad \boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

and

$$f_k \to f \quad \text{and} \quad g_k \to g \quad \text{weakly in } L^2(0,T;\mathcal{V}). \tag{17}$$

We will obtain unique solvability results in the weak sense by constructing variational formulation for system of difference equation (16). We use some strong convergence properties of the sequences, which are obtained by compactness theorems. The existence and uniqueness of weak solutions are presented in the next section.

3 Unique solvability of the difference scheme

In the present section, theoretical statements on weak approximate solution to (16) is established by the unconditionally stable difference scheme (11). Applying variational formulation, it will be shown that difference problem (16) converges to a unique weak solution.

Let us consider the variational formulation of (11)

$$\begin{aligned}
&(u_{k+1},\bar u) + (u_{k-1},\bar u) + (\alpha_{11}\tau u_{k+1},\bar u) + (\alpha_{12}\tau v_{k+1},\bar u) + \big(\tau^2\rho_{11}u_k,\bar u\big) \\
&\qquad + \big(\tau^2\beta_1\nabla u_{k+1},\nabla\bar u\big) + \big(\tau^2\gamma_1\sin(\delta_{11}u_k+\delta_{12}v_k),\bar u\big) + \big(\tau^2\rho_{12}v_k,\bar u\big) \\
&\quad = (2u_k,\bar u) + (\alpha_{11}\tau u_{k-1},\bar u) + \big(\alpha_{12}\tau v_{k-1},\bar u\big) + \big(\tau^2 f_k,\bar u\big), \\
&(v_{k+1},\bar v) + (v_{k-1},\bar v) + (\alpha_{21}\tau u_{k+1},\bar v) + (\alpha_{22}\tau v_{k+1},\bar v) + \big(\tau^2\rho_{21}u_k,\bar v\big) \\
&\qquad + \big(\beta_2\tau^2\nabla v_{k+1},\nabla\bar v\big) + \big(\gamma_2\tau^2\sin(\delta_{21}u_k+\delta_{22}v_k),\bar v\big) + \big(\tau^2\rho_{22}v_k,\bar v\big) \\
&\quad = (2v_k,\bar v) + (\alpha_{21}\tau u_{k-1},\bar v) + (\alpha_{22}\tau v_{k-1},\bar v) + \big(\tau^2 g_k,\bar v\big), \\
&f_k = f(t_k), \quad g_k = g(t_k), \quad t_k = k\tau,\ 1 \leqslant k \leqslant N-1,\ N\tau = T, \\
&(u_0,\bar u) = (\varphi_1,\bar u), \qquad (v_0,\bar v) = (\varphi_2,\bar v), \\
&(u_1,\bar u) = (\psi_1\tau,\bar u) + (u_0,\bar u), \qquad (v_1,\bar v) = (\psi_2\tau,\bar v) + (v_0,\bar v),
\end{aligned} \tag{18}$$

where $\bar u$ and $\bar v$ are test functions in V. Throughout this paper, K represents a generic constant having possibly different values at different places.

Definition 2. The mesh functions $\{u_k\}$ and $\{v_k\}$ are said to be the approximate weak solutions of (11) if $u_k, v_k \in V$ satisfies (18).

Using the family of grid points (12), we introduce the Hilbert space

$$L_{2h}(\Omega) = L_2(\Omega_h)$$

equipped with the norms of grid functions

$$\|u_k\|_{L_{2h}(\Omega)} = \left(\sum_{j=1}^{N} \left|u_k^j\right|^2 h\right)^{1/2}, \qquad \|v_k\|_{L_{2h}(\Omega)} = \left(\sum_{j=1}^{N} |v_k^j|^2 h\right)^{1/2}.$$

Theorem 2. *Suppose that assumptions* (5)–(9), (17) *are satisfied. Then there exists a positive constant* K *such that*

$$\|\mathbf{w}_{k-1}\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_k\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)} \leqslant K, \tag{19}$$

where K *is independent of the grid parameters* τ *and* h *for all* $k \in \mathbb{N}$.

Proof. Setting $\bar{u} = u_{k+1}$, $\bar{v} = v_{k+1}$ in (18), system

$$\begin{aligned}
&(u_{k+1}, u_{k+1}) + (u_{k-1}, u_{k+1}) + \alpha_{11}\tau(u_k, u_{k+1}) + \alpha_{12}\tau(v_k, u_{k+1}) \\
&\quad + \tau^2\beta_1(Au_{k+1}, u_{k+1}) + \tau^2\gamma_1\bigl(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}\bigr) \\
&\quad + \tau^2\rho_{11}(u_k, u_{k+1}) + \tau^2\rho_{12}(v_k, u_{k+1}) - 2(u_k, u_{k+1}) \\
&\quad - \alpha_{11}\tau(u_{k-1}, u_{k+1}) - \alpha_{12}\tau(v_{k-1}, u_{k+1}) \\
&= \tau^2(f_k, u_{k+1}), \\
&(v_{k+1}, v_{k+1}) + (v_{k-1}, v_{k+1}) + \alpha_{21}\tau(u_k, v_{k+1}) + \alpha_{22}\tau(v_k, v_{k+1}) \\
&\quad + \beta_2\tau^2(Av_{k+1}, v_{k+1}) + \gamma_2\tau^2\bigl(\sin(\delta_{21}u_k + \delta_{22}v_k), v_{k+1}\bigr) \\
&\quad + \rho_{21}\tau^2(u_k, v_{k+1}) + \rho_{22}\tau^2(v_k, v_{k+1}) - 2(v_k, v_{k+1}) \\
&\quad - \alpha_{21}\tau(u_{k-1}, v_{k+1}) - \alpha_{22}\tau(v_{k-1}, v_{k+1}) \\
&= \tau^2(g_k, v_{k+1})
\end{aligned} \tag{20}$$

is obtained. A priori estimate will be presented by showing nonnegativity and boundedness for the components of system (20). By coercivity, the system can be written as

$$\begin{aligned}
&c_1(u_{k+1}, u_{k+1}) - 2\tau(u_k, u_{k+1}) + c_2(u_{k-1}, u_{k+1}) \\
&\quad + c_3(v_{k+1}, u_{k+1}) + c_4(v_{k-1}, u_{k+1}) \\
&\quad \leqslant \tau^2(f_k, u_{k+1}) - \tau^2\gamma_1\bigl(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}\bigr),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&d_1(v_{k+1}, v_{k+1}) - 2\tau(v_k, v_{k+1}) + d_2(v_{k-1}, v_{k+1}) \\
&\quad + d_3(u_{k+1}, v_{k+1}) + d_4(u_{k-1}, v_{k+1}) \\
&\quad \leqslant \tau^2(g_k, v_{k+1}) - \tau^2\gamma_2\bigl(\sin(\delta_{21}u_k + \delta_{22}v_k), v_{k+1}\bigr)
\end{aligned} \tag{22}$$

with

$$c_1 = 1 + \frac{\tau}{2}\alpha_{11} + \tau^2\rho_{11} + \tau^2\beta_1, \qquad c_2 = 1 - \frac{\tau}{2}\alpha_{11},$$
$$c_3 = \frac{\tau}{2}\alpha_{12} + \tau^2\rho_{12}, \qquad c_4 = -\frac{\tau}{2}\alpha_{12},$$

$$d_1 = 1 + \frac{\tau}{2}\alpha_{22} + \tau^2\rho_{22} + \tau^2\beta_2, \qquad d_2 = 1 - \frac{\tau}{2}\alpha_{22},$$
$$d_3 = \frac{\tau}{2}\alpha_{21} + \tau^2\rho_{21}, \qquad d_4 = -\frac{\tau}{2}\alpha_{21}.$$

Taking the sum of (21) and (22), we get

$$\begin{aligned} &c_1(u_{k+1}, u_{k+1}) + d_1(v_{k+1}, v_{k+1}) - 2\tau(u_k, u_{k+1}) - 2\tau(v_k, v_{k+1}) \\ &\quad + c_2(u_{k-1}, u_{k+1}) + d_2(v_{k-1}, v_{k+1}) + c_3(v_{k+1}, u_{k+1}) + d_3(u_{k+1}, v_{k+1}) \\ &\quad + c_4(v_{k-1}, u_{k+1}) + d_4(u_{k-1}, v_{k+1}) \\ &\leqslant \tau^2(f_k, u_{k+1}) - \tau^2\gamma_1\bigl(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}\bigr) \\ &\quad + \tau^2(g_k, v_{k+1}) - \tau^2\gamma_2\bigl(\sin(\delta_{21}u_k + \delta_{22}v_k), v_{k+1}\bigr). \end{aligned} \tag{23}$$

With the help of inner product defined in (15), conditions (6)–(9), (17), and the matrix norm in (14), system (23) can be written in the vector form

$$\begin{aligned} &(\mathbf{A}_1\mathbf{w}_{k+1}, \mathbf{w}_{k+1}) - 2\tau(\mathbf{w}_k, \mathbf{w}_{k+1}) + (\mathbf{A}_2\mathbf{w}_{k-1}, \mathbf{w}_{k+1}) \\ &\quad + (\mathbf{A}_3\mathbf{w}_{k+1}, \mathbf{w}_{k+1}) + (\mathbf{A}_4\mathbf{w}_{k-1}, \mathbf{w}_{k+1}) \\ &\leqslant \tau^2(\mathbf{F}_k - \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{w}_k,\ \mathbf{w}_{k+1}), \end{aligned} \tag{24}$$

where

$$\mathbf{w}_k = \begin{bmatrix} u_k \\ v_k \end{bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} c_1 & 0 \\ 0 & d_1 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} c_2 & 0 \\ 0 & d_2 \end{bmatrix},$$
$$\mathbf{A}_3 = \begin{bmatrix} 0 & c_3 \\ d_3 & 0 \end{bmatrix}, \qquad \mathbf{A}_4 = \begin{bmatrix} 0 & c_4 \\ d_4 & 0 \end{bmatrix}, \qquad \mathbf{F}_k = \begin{bmatrix} f_k & 0 \\ 0 & g_k \end{bmatrix}.$$

Let us denote

$$\begin{aligned} \Delta = \tau^2(\mathbf{F}_k - \boldsymbol{\gamma}\sin\boldsymbol{\delta}\mathbf{w}_k, \mathbf{w}_{k+1}) - (\mathbf{A}_1\mathbf{w}_{k+1}, \mathbf{w}_{k+1}) + 2\tau(\mathbf{w}_k, \mathbf{w}_{k+1}) \\ -(\mathbf{A}_2\mathbf{w}_{k-1}, \mathbf{w}_{k+1}) - (\mathbf{A}_3\mathbf{w}_{k+1}, \mathbf{w}_{k+1}) - (\mathbf{A}_4\mathbf{w}_{k-1}, \mathbf{w}_{k+1}). \end{aligned}$$

From (24), $\Delta \geqslant 0$. An upper bound should be constructed for Δ. Using triangle and Cauchy–Schwarz inequalities yields

$$\begin{aligned} \Delta &\leqslant \bigl(\|\mathbf{A}_1\|_{L_{2h}(\Omega)} + \|\mathbf{A}_3\|_{L_{2h}(\Omega)} + \tau\bigr)\|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)} + \tau\|\mathbf{w}_k\|^2_{L_{2h}(\Omega)} \\ &\quad + \frac{\tau^2}{2}|\boldsymbol{\gamma}||\boldsymbol{\delta}|\bigl(\|\mathbf{w}_k\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)}\bigr) + \frac{\tau^2}{2}\bigl(\|\mathbf{F}_k\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)}\bigr) \\ &\quad + \frac{1}{2}\bigl(\|\mathbf{A}_2\|_{L_{2h}(\Omega)} + \|\mathbf{A}_4\|_{L_{2h}(\Omega)}\bigr)\bigl(\|\mathbf{w}_{k-1}\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)}\bigr). \end{aligned}$$

Thus,

$$\Delta \leqslant C_1\|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)} + C_2\|\mathbf{w}_k\|^2_{L_{2h}(\Omega)} + C_3\|\mathbf{w}_{k-1}\|^2_{L_{2h}(\Omega)} + C_4\|\mathbf{F}_k\|^2_{L_{2h}(\Omega)}, \tag{25}$$

where

$$C_1 = \|\mathbf{A}_1\|_{L_{2h}(\Omega)} + \|\mathbf{A}_3\|_{L_{2h}(\Omega)} + \tau + \frac{\tau^2}{2}\bigl(1 + |\gamma||\boldsymbol{\delta}|\bigr) + \frac{1}{2}\bigl(\|\mathbf{A}_2\|_{L_{2h}(\Omega)} + \|\mathbf{A}_4\|_{L_{2h}(\Omega)}\bigr),$$

$$C_2 = \tau + \frac{\tau^2}{2}|\gamma||\boldsymbol{\delta}|, \qquad C_3 = \frac{1}{2}\bigl(\|\mathbf{A}_2\|_{L_{2h}(\Omega)} + \|\mathbf{A}_4\|_{L_{2h}(\Omega)}\bigr), \qquad C_4 = \frac{\tau^2}{2}.$$

Using assumptions (5)–(9) and (17), we conclude that $\mathbf{A}_i$, $i = 1, 2, 3, 4$, are invertible positive definite matrices and the function $\mathbf{F}_k$ is Lipschitz in Ω_h. Let us denote

$$M = \max\{C_1, C_2, C_3, C_4\}.$$

Then (25) can be written as

$$\Delta \leqslant M\bigl(\|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_k\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_{k-1}\|^2_{L_{2h}(\Omega)} + \|\mathbf{F}_k\|^2_{L_{2h}(\Omega)}\bigr).$$

Components of this system are nonnegative and bounded above. Thus, the following energy inequality holds:

$$0 \leqslant \Delta \leqslant \|\mathbf{w}_{k+1}\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_k\|^2_{L_{2h}(\Omega)} + \|\mathbf{w}_{k-1}\|^2_{L_{2h}(\Omega)} \leqslant K.$$

Hence, Theorem 2 is proved. □

Theorem 3. *Suppose that assumptions* (5)–(9) *and* (17) *hold. Then there exists a positive constant K, independent of grid parameters τ and h, such that for all $k \in \mathbb{N}$,*

$$\left(\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2_{L_{2h}(\Omega)}\right) \leqslant K. \tag{26}$$

Proof. We construct the weak formulation for the derivative terms by modifying (11). Multiplying by τ and taking the inner product for the first equation of (11) by $(u_{k+1} - u_k)/\tau$ and $(u_k - u_{k-1})/\tau$, we get

$$\begin{aligned}
&\left(\frac{u_{k+1}-u_k}{\tau} - \frac{u_k-u_{k-1}}{\tau}, \frac{u_{k+1}-u_k}{\tau}\right) \\
&\quad + \tau\alpha_{11}\left(\frac{u_{k+1}-u_{k-1}}{2\tau}, \frac{u_{k+1}-u_k}{\tau}\right) + \tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau}, \frac{u_{k+1}-u_k}{\tau}\right) \\
&\quad + \tau\beta_1\left(Au_{k+1}, \frac{u_{k+1}-u_k}{\tau}\right) + \tau\gamma_1\left(\sin(\delta_{11}u_k + \delta_{12}v_k), \frac{u_{k+1}-u_k}{\tau}\right) \\
&\quad + \tau\rho_{11}\left(u_{k+1}, \frac{u_{k+1}-u_k}{\tau}\right) + \tau\rho_{12}\left(v_{k+1}, \frac{u_{k+1}-u_k}{\tau}\right) \\
&= \tau\left(f_k, \frac{u_{k+1}-u_k}{\tau}\right)
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\left(\frac{u_{k+1}-u_k}{\tau}-\frac{u_k-u_{k-1}}{\tau},\frac{u_k-u_{k-1}}{\tau}\right)\\
&\quad+\tau\alpha_{11}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{u_k-u_{k-1}}{\tau}\right)\\
&\quad+\tau\beta_1\left(Au_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\gamma_1\left(\sin(\delta_{11}u_k+\delta_{12}v_k),\frac{u_k-u_{k-1}}{\tau}\right)\\
&\quad+\tau\rho_{11}\left(u_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\rho_{12}\left(v_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)\\
&=\tau\left(f_k,\frac{u_k-u_{k-1}}{\tau}\right),
\end{aligned}\tag{28}$$

respectively. Taking the sum of (27) and (28),

$$\begin{aligned}
&\left\{\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2_{L_{2h}(\Omega)}-\left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2_{L_{2h}(\Omega)}\right\}\\
&\quad+2\tau\alpha_{11}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2_{L_{2h}(\Omega)}+\tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{u_{k+1}-u_{k-1}}{\tau}\right)\\
&\quad+\tau\beta_1\left(Au_{k+1},\frac{u_{k+1}-u_{k-1}}{\tau}\right)+\tau\gamma_1\left(\sin(\delta_{11}u_k+\delta_{12}v_k),\frac{u_{k+1}-u_{k-1}}{\tau}\right)\\
&\quad+\tau\rho_{11}\left(u_{k+1},\frac{u_{k+1}-u_{k-1}}{\tau}\right)+\tau\rho_{12}\left(v_{k+1},\frac{u_{k+1}-u_{k-1}}{\tau}\right)\\
&=\tau\left(f_k,\frac{u_{k+1}-u_{k-1}}{\tau}\right)
\end{aligned}\tag{29}$$

is obtained. Applying the same procedure for v_k,

$$\begin{aligned}
&\left\{\left\|\frac{v_{k+1}-v_k}{\tau}\|^2_{L_{2h}(\Omega)}-\right\|\frac{v_k-v_{k-1}}{\tau}\right\|^2_{L_{2h}(\Omega)}\right\}\\
&\quad+2\tau\alpha_{22}\left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2_{L_{2h}(\Omega)}+\tau\alpha_{21}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{v_{k+1}-v_{k-1}}{\tau}\right)\\
&\quad+\tau\beta_2\left(Av_{k+1},\frac{v_{k+1}-v_{k-1}}{\tau}\right)+\tau\gamma_2\left(\sin(\delta_{21}u_k+\delta_{22}v_k),\frac{v_{k+1}-v_{k-1}}{\tau}\right)\\
&\quad+\tau\rho_{21}\left(u_{k+1},\frac{v_{k+1}-v_{k-1}}{\tau}\right)+\tau\rho_{22}\left(v_{k+1},\frac{v_{k+1}-v_{k-1}}{\tau}\right)\\
&=\tau\left(g_k,\frac{v_{k+1}-v_{k-1}}{\tau}\right)
\end{aligned}\tag{30}$$

is obtained. Multiplying (29) by α_{21} and (30) by α_{12} and subtracting these terms, we get

$$\begin{aligned}
\Delta_1 = \alpha_{21}\biggl\{\biggl\|\frac{u_{k+1}-u_k}{\tau}\biggr\|^2_{L_{2h}(\Omega)} - \biggl\|\frac{u_k-u_{k-1}}{\tau}\biggr\|^2_{L_{2h}(\Omega)}\biggr\} \\
+ 2\tau\alpha_{11}\alpha_{21}\biggl\|\frac{u_{k+1}-u_{k-1}}{2\tau}\biggr\|^2_{L_{2h}(\Omega)} \\
- \alpha_{12}\biggl\{\biggl\|\frac{v_{k+1}-v_k}{\tau}\biggr\|^2_{L_{2h}(\Omega)} - \biggl\|\frac{v_k-v_{k-1}}{\tau}\biggr\|^2_{L_{2h}(\Omega)}\biggr\} \\
+ 2\tau\alpha_{12}\alpha_{22}\biggl\|\frac{v_{k+1}-v_{k-1}}{2\tau}\biggr\|^2_{L_{2h}(\Omega)} + s_2(u_{k+1}, v_{k-1}) \\
+ 2\tau\bigl(\|u_{k+1}\|^2_{L_{2h}(\Omega)} + \|v_{k+1}\|^2_{L_{2h}(\Omega)}\bigr) + s_1(u_{k+1}, v_{k+1}) \\
+ s_3(u_{k+1}, u_{k-1}) + s_4(v_{k+1}, v_{k-1}) + s_5(v_{k+1}, u_{k-1}),
\end{aligned} \tag{31}$$

where

$$s_1 = \alpha_{21}\rho_{12} - \alpha_{12}\rho_{21}, \qquad s_2 = \alpha_{21}\beta_1 - \alpha_{21}\rho_{11},$$
$$s_3 = \alpha_{12}\beta_2 - \alpha_{12}\rho_{22}, \qquad s_4 = \alpha_{12}\rho_{21}\tau, \qquad s_5 = -\alpha_{21}\rho_{12}.$$

Using coercivity, we obtain

$$\begin{aligned}
\Delta_1 \leqslant \alpha_{21}(f_k, u_{k+1}) - \alpha_{21}(f_k, u_{k-1}) - \alpha_{12}(g_k, v_{k+1}) + \alpha_{21}(g_k, v_{k-1}) \\
+ s_6\bigl(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}\bigr) - s_6\bigl(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k-1}\bigr) \\
- s_7\bigl(\sin(\delta_{21}u_k + \delta_{22}v_k), v_{k+1}\bigr) + s_7\bigl(\sin(\delta_{11}u_k + \delta_{12}v_k), v_{k-1}\bigr),
\end{aligned} \tag{32}$$

where $s_6 = \alpha_{21}\gamma_1$, $s_7 = \alpha_{12}\gamma_2$. Here, the coefficients s_i, $i = 1, 2, \dots, 7$, in (31) and (32) are bounded constants by (5). Using the inequalities

$$\bigl(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}\bigr) \leqslant |\delta_{11}u_k + \delta_{12}v_k||u_{k+1}|,$$
$$(u_{k+1}, v_{k+1}) \leqslant \frac{1}{2}\bigl(|u_{k+1}|^2 + |v_{k+1}|^2\bigr),$$

(31) and (32) can be written as

$$\begin{aligned}
\alpha_{21}\biggl\{\biggl\|\frac{u_{k+1}-u_k}{\tau}\biggr\|^2_{L_{2h}(\Omega)} - \biggl\|\frac{u_k-u_{k-1}}{\tau}\biggr\|^2_{L_{2h}(\Omega)}\biggr\} \\
+ 2\tau\alpha_{11}\alpha_{21}\biggl\|\frac{u_{k+1}-u_{k-1}}{2\tau}\biggr\|^2_{L_{2h}(\Omega)} \\
+ \alpha_{12}\biggl\{\biggl\|\frac{v_{k+1}-v_k}{\tau}\biggr\|^2_{L_{2h}(\Omega)} - \biggl\|\frac{v_k-v_{k-1}}{\tau}\biggr\|^2_{L_{2h}(\Omega)}\biggr\} \\
+ 2\tau\alpha_{12}\alpha_{22}\biggl\|\frac{v_{k+1}-v_{k-1}}{2\tau}\biggr\|^2_{L_{2h}(\Omega)} + a_1\|u_{k+1}\|^2_{L_{2h}(\Omega)} + a_2\|u_k\|^2_{L_{2h}(\Omega)}
\end{aligned}$$

$$+ a_3\|u_{k-1}\|^2_{L_{2h}(\Omega)} + a_4\|f_k\|^2_{L_{2h}(\Omega)} + b_1\|v_{k+1}\|^2_{L_{2h}(\Omega)} + b_2\|v_k\|^2_{L_{2h}(\Omega)} + b_3\|v_{k-1}\|^2_{L_{2h}(\Omega)} + b_4\|g_k\|^2_{L_{2h}(\Omega)}, \tag{33}$$

where

$$a_1 = a_3 = \frac{\tau s_6}{2}\big(|\delta_{11}| + |\delta_{12}|\big), \qquad a_2 = \tau\big(s_6|\delta_{11}| + s_7|\delta_{12}|\big), \qquad a_4 = \alpha_{21},$$
$$b_1 = b_3 = \frac{\tau s_7}{2}\big(|\delta_{21}| + |\delta_{22}|\big), \qquad b_2 = \tau\big(s_6|\delta_{12}| + s_7|\delta_{22}|\big), \qquad b_4 = \alpha_{12}.$$

We denote

$$N = \max\{\alpha_{21}, 2\tau\alpha_{11}\alpha_{21}, 2\tau\alpha_{12}\alpha_{22}, \alpha_{12}\},$$

Then using Theorem 2, assumptions (5)–(9), and (17), the estimation for (33) can be written as

$$N\left\{\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2_{L_{2h}(\Omega)} - \left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2_{L_{2h}(\Omega)} + \left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2_{L_{2h}(\Omega)} - \left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2_{L_{2h}(\Omega)}\right\} \leqslant K.$$

Thus, Theorem 3 is proved. □

Next theorem states that the mesh functions $\{u_k\}$ and $\{v_k\}$ are compact in $L_{2h}(\Omega)$ topology.

Theorem 4. *Under the hypotheses of Theorems* 2 *and* 3, *there exist subsequences*

$$\{u_{k_m}\} \subset \{u_k\} \quad \textit{and} \quad \{v_{k_m}\} \subset \{v_k\},$$

which converge in V *to bounded measurable functions* u *and* v, *respectively. Moreover, the limit functions* u *and* v *are unique weak solutions satisfying* (19) *and* (26).

Proof. Estimates (19), (26) and discrete Gronwall lemma [23] imply that

$$\{u_k\} \quad \text{and} \quad \{v_k\} \quad \text{are bounded in } L^\infty(0,T;V).$$

Then by Rellich theorem [9] there exists a subsequence $\mathbf{w}_{k_m} = [u_{k_m}, v_{k_m}]^\mathrm{T}$ of $\mathbf{w}_k = [u_k, v_k]^\mathrm{T}$ and $\tilde{\mathbf{w}}_k \in L^\infty(0,T;\mathcal{V})$ such that

$$\tilde{\mathbf{w}}_k \in L^\infty(0,T;\mathcal{V}) \subset L^2(0,T;\mathcal{V})$$

and

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \quad \text{weak star in } L^\infty(0,T;\mathcal{V}) \text{ and weakly in } L^2(0,T;\mathcal{V}).$$

By the Aubin compactness theorem [5], the above convergence results imply

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \quad \text{strongly in } L^2(0,T;\mathcal{H}), \tag{34}$$

and by (34),

$$\sin\boldsymbol{\delta}\mathbf{w}_{k_m} \to \sin\boldsymbol{\delta}\tilde{\mathbf{w}}_k \quad \text{strongly in } L^2(0,T;\mathcal{H}),$$

which shows the existence of $\tilde{\mathbf{w}}_k$ a.e. in $\mathcal{H}$ and $\tilde{\mathbf{w}}_0 = \mathbf{w}_0$.

Uniqueness follows from convergence of difference scheme (11) and by Theorem 1. Hence, Theorem 4 is proved. □

4 Numerical analysis

In the present section, we verify the theoretical results of our study by numerical experiments. A composite numerical method based on finite difference method and fixed point iteration is employed. The fixed point iteration is applied for nonlinear part of the problem. We propose a unified numerical method to obtain more accurate results for the solution of an initial boundary value problem (IBVP) for one dimensional coupled sine-Gordon equations. We choose an exact solution

$$w(t,x) = \{u(t,x), v(t,x)\}$$

with

$$u(t,x) = \mathrm{e}^{-2t}\sin\pi x, \qquad v(t,x) = \mathrm{e}^{-t}\sin\pi x,$$

and we formulate a boundary value problem that leads to this solution. Let us consider the following IBVP:

$$\begin{aligned}
&u_{tt} - u_{xx} + u_t + u = -\sin(u-v) + (\pi^2+3)\mathrm{e}^{-2t}\sin\pi x \\
&\quad + \sin(\mathrm{e}^{-2t}\sin\pi x - \mathrm{e}^{-t}\sin\pi x), \quad 0<t<1,\ 0<x<1, \\
&v_{tt} - v_{xx} + v_t + v = \sin(u-v) + (\pi^2+1)\mathrm{e}^{-t}\sin\pi x \\
&\quad - \sin(\mathrm{e}^{-2t}\sin\pi x - \mathrm{e}^{-t}\sin\pi x), \quad 0<t<1,\ 0<x<1, \\
&u(0,x) = \sin\pi x, \quad u_t(0,x) = -2\sin\pi x, \quad 0\leqslant x\leqslant 1, \\
&v(0,x) = \sin\pi x, \quad v_t(0,x) = -\sin\pi x, \quad 0\leqslant x\leqslant 1, \\
&u(t,0) = u(t,1) = 0, \quad v(t,0) = v(t,1) = 0, \quad 0\leqslant t\leqslant 1.
\end{aligned} \tag{35}$$

System (35) is used for modelling the wave propagation on an infinite chain of elastically bound atoms lying over a fixed lower chain of similar atoms. The second-order derivative terms describe the elastic interaction energy between neighboring atoms and their kinetic energy, respectively. The nonlinear terms containing sine stand for the potential energy due to the fixed lower chain. The remaining terms are damping terms and source functions.

For the approximate solution of problem (35), the corresponding difference scheme (11) is considered. The modified Gauss elimination method is used for the solution of system (11). The set of a family of grid points

$$\begin{aligned}
\Omega_h &= [0,1]_\tau \times [0,1]_h \\
&= \{(t_k,x_n)\colon t_k = k\tau,\ 0\leqslant k\leqslant N,\ N\tau=1,\ x_n = nh,\ 0\leqslant n\leqslant M,\ Mh=1\}
\end{aligned}$$

is considered.

Table 1. Errors for the approximate solution of problem (35).

ε	$N = M$	Error of w	Rate of convergence	m	CPU times
10^{-15}	20	0.1564	1.012329	9	2.8901
	40	0.0785	1.003370	10	3.9512
	80	0.0393	1.005520	11	5.5020
	160	0.0114	0.996334	12	15.8911
10^{-20}	20	0.1564	1.012329	10	3.1602
	40	0.0785	1.003370	11	3.5925
	80	0.0393	1.005520	12	5.8238
	160	0.0114	0.996334	13	18.0502

Table 2. Errors for the approximate solution of problem (35).

ε	$N = M$	Error of w	Rate of convergence	m	CPU times
10^{-20}	20	0.1564	1.012329	10	3.7344
	40	0.0785	1.003370	11	4.8879
	80	0.0393	1.005520	11	8.8649
	160	0.0114	0.996334	12	17.1365

Errors, rate of convergence, number of iterations, and related CPU times are presented in tables for different values of N and M. The numerical implementations are carried out by MATLAB R2018b software package, by a PC System of 64 bit, Core i5 CPU, 1.80 GHz, 8 GB of RAM. Errors are computed by the following formula:

$$\max_{\substack{1\leqslant k\leqslant N-1\\ 1\leqslant n\leqslant M-1}} \left|w(t_k, x_n) - w_n^k\right|.$$

The numerical algorithm is performed for $m = 1, 2, \dots, p$, where p depends on a given error tolerance ε such that

$$\left|{}_p u_n - {}_{p-1} u_n\right| < \varepsilon \quad \text{and} \quad \left|{}_p v_n - {}_{p-1} v_n\right| < \varepsilon.$$

Here, m is the index representing the number of fixed point iteration. The exact solution is denoted by $w(t_k, x_n) = [u(t_k, x_n), v(t_k, x_n)]^{\mathrm{T}}$, and the numerical solution is denoted by $w_n^k = [u_n^k, v_n^k]^{\mathrm{T}}$ for the approximate solution of problem (35) at (t_k, x_n). The numerical results are presented in the following tables.

Table 1 shows the errors for the approximate solution of (35) with a stopping criteria $\varepsilon = 10^{-15}$ and 10^{-20}. In the iteration, the initials are taken as vectors of the form

$$ {}_0u_n^k = \mathrm{rand}(N+1,\ 1), \tag{36}$$
$$ {}_0v_n^k = 0(N+1,\ 1), \tag{37}$$

where (36) is a random vector with dimension $N + 1$.

Table 2 shows the errors for the approximate solution of (35) with $\varepsilon = 10^{-20}$. In the iteration, the initials are taken as the identity matrices of the form

$$ {}_0u_n^k = I(N+1,\ M+1), \tag{38}$$
$$ {}_0v_n^k = I(N+1,\ M+1). \tag{39}$$

The numerical solutions are obtained by using difference scheme (11) jointly with fixed point iteration. The difference scheme converges for different iteration numbers m, $N = M$ values, initial vectors ${}_0u_n^k$, ${}_0v_n^k$, and termination criteria ε. When the maximum difference at grid points of two successive results gets less than ε, the iterative process is stopped. Note that if the initials ${}_0u_n^k$, ${}_0v_n^k$ in (36)–(37), (38)–(39) and ε are changed, the number of iterations and the CPU times increase when the error and the rate of convergence become constant for a certain $N = M$ value.

As it is obvious from the tables that, if N and M are doubled, the value of errors decrease approximately by a factor of $1/2$ for difference scheme (11). The errors and the rate of convergence presented in tables indicate the convergence of the difference scheme and the accuracy of the results. It is observed that the difference scheme has first order of convergence as it is expected.

5 Conclusion

In this work, the unique solvability for the system of finite difference schemes for coupled sine-Gordon equations is proved by using the variational formulation. A novel unified numerical method, which combines the first order of accuracy unconditionally stable difference scheme with the fixed point iteration, is constructed. Numerical experiments are implemented to verify theoretical results and to show the efficiency of the unified method.

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