DOI https://doi.org/10.24297/jam.v19i.8885

On the solvability of a nonlinear functional integral equations via measure of noncompactness in $L^p(\mathbb{R}^N)$

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Abstract

Using the technique of a suitable measure of non-compactness and the Darbo fixed point theorem, we investigate the existence of a nonlinear functional integral equation of Urysohn type in the space of Lebesgue integrable functions $L^p(\mathbb{R}^N)$. In this space, we show that our functional-integral equation has at least one solution. Finally an example is also discussed to indicate the natural realizations of our abstract result.

Keywords: functional integral equation; measure of noncompactnes; existence; Darbo's fixed point theorem; fixed point.

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1 Introduction

Integral equations appear in many applications in describing numerous real world problems (see, for instance,

([30], [31], [5], [18]), and references therein).

Also many useful applications in describing problems of the real world and numerous events, which appear in physics, engineering, mechanics, biology, etc. See for example [1, 4, 8, 13, 15] can be depicted and demonstrated by methods of non-linear functional integral equations (for example, we refer to [25, 26, 28]). Apart from that, integral equations are often investigated in research papers and monographs (cf. [6, 12, 16, 18, 29, 32]) and the references cited therein.

2 Preliminaries

We will collect in this section some definitions and basic results which will be used further on throughout the paper. First, we denote $L^p(U)$ $(U \in \mathbb{R}^N)$ as the space of Lebesgue integrable functions on U with the standard norm $\| x \|_{L^p(U)} = (\int_U$

$| x(t) |^p \, dt^{\frac{1}{p}}$.

Theorem 2.1 *[1, 8, 9]*

Let F be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$. The closure of F in $L^p(\mathbb{R}^N)$ is compact if and only if

$$\lim_{h \to 0} \| \tau_h f - f \|_{L^p(\mathbb{R}^N)} = 0 \qquad uniformly \ in \ \ f \in F,$$

where $\tau_h f(x) = f(x+h)$ for all $x, h \in \mathbb{R}^N$. Also for $\epsilon > 0$ there is a bounded and measurable subset $\Omega \subset (\mathbb{R}^N)$ such that

$$\| f \|_{(\mathbb{R}^N \setminus \Omega)} < \epsilon \qquad forall \ \ f \in F.$$

Next, we recall the concept of measure of noncompactness, let E be an infinite dimensional Banach space with norm $\|.\|$ and zero element θ. Denote by$\mathcal{M}_E$ the family of all nonempty and bounded subsets of E , $\mathcal{N}_E$ and $\mathcal{N}_E^W$

the family of all nonempty relatively compact

and weakly relatively compact sets, respectively. The symbols $\bar{X}$ and ConvX stand for the closure and closed convex hull of a subset X of E, respectively. We use the standard notation $X + Y$ and λX for algebraic operations on sets,

while,

we denote $B_r = B(\theta, r)$ the closed ball centered at θ and with radius r.

Definition 2.1 *(Measure of noncompactness)*

[12]

A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following

conditions:

(1) *the family* $\ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ *is nonempty and* $ker\mu \subset \mathcal{N}_E$*, where* $ker\mu$ *is called the kernel of the measure* μ.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(ConvX) = \mu(X) = \mu(\overline{X})$.

(4) $\mu[\lambda X + (1-\lambda)Y] \leq \lambda\mu(X) + (1-\lambda)\mu(Y),\ \lambda \in [0,1]$.

(5) *If* $X_n \in \mathcal{M}_E,\ X_n = \bar{X_n}$ *and* $X_{n+1} \subset X_n$ *for* $n = 1, 2, \ldots$ *and if*

$\lim_{n\to\infty} \mu(X_n) = 0,$ *then*
$X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi$.

Theorem 2.2 *[1]*

Suppose $1 \leq p < \infty$ *and* X *is a bounded subset of* $(\mathbb{R}^N)$*. For* $x \in X$ *and* $\epsilon > 0$

$w^T(x,\epsilon) = \sup\{\| \tau_h x - x \|_{L^p(B_T)} \colon \|h\|_{\mathbb{R}^N} < \epsilon\},$

$w^T(X,\epsilon) = \sup\{w^T(x,\epsilon) : x \in X\},$

$w^T(X) = \lim_{\epsilon\to 0} w^T(X,\epsilon),$

w(X)$= \lim_{T\to\infty} w^T(X),$

d(X)$= \lim_{T\to\infty} \sup\{\|x\|_{L^p(\mathbb{R}^N\setminus B_T)} : x \in X\},$

where $B_T = \{a \in \mathbb{R}^{\mathbb{N}} : \|a\|_{\mathbb{R}^{\mathbb{N}}} \leq T\}$*. Then*

$$\mu(X) = w(X) + d(X)$$

is a measure of non compactness on $L^p(\mathbb{R}^N)$.

At the end of this section, we recall the fixed point theorem due to Darbo which enables us to prove the existence theorem for solutions of a several integral equations considered in nonlinear analysis. To quote this theorem we need the following definitions.

Definition 2.2 *[12]*

The function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition if it satisfies the following two conditions:

(1) f is measurable in $t \in I$ for any $x \in \mathbb{R}$.

(2) f is continuous in $x \in \mathbb{R}$ for almost all $t \in I$.

Definition 2.3 *(Darbo condition)[11]*

Let Ω be a nonempty subset of a Banach space E and let $A : \Omega \to E$ be a continuous operator which transforms bounded sets onto bounded ones. We say that A satisfies the Darbo condition (with a constant $k \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset X of

Ω, we have $\mu(AX) \leq k\mu(X)$.

Note that, if A satisfies the Darbo condition with $k < 1$, then it is called a contraction operator with respect to μ.

Theorem 2.3 *(Darbo fixed point theorem)[7]*

Let Ω be a nonempty, bounded, closed and convex subset of E and let $f : \Omega \to \Omega$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists a constant $k \in [0, 1)$ such that

$\mu(fX) \leq k\mu(X)$,

for any nonempty subset X of Ω. Then f has at least one fixed point in the set Ω.

3 Main result

This section is devoted to discuss the solvability of the following nonlinear functional integral equation

$$u(x) = f(x) + g_1(x, u(x)) + h_1\left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy\right). \tag{1}$$

Now, we will try to assume some assumptions under which we can prove our existence theorem.
Assume the following conditions are satisfied:

(i) $f \in L^p(\mathbb{R}^N)$;

(ii) $g_i : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory condition
(i.e. measurable in t for all $x \in \mathbb{R}^N$, and continuous in
x for all $t \in \mathbb{R}$) and there exists a constant $l \in [0, 1)$ and $a_i \in L^p(\mathbb{R}^N)$
such that

$$| g_i(x, u) - g_i(y, v) | \leq |a_i(x) - a_i(y) | + l \; |u - v|,$$

for any $u, v \in \mathbb{R}$ and almost all $x, y \in \mathbb{R}^N$ where $i = 1, 2$.

(iii) $h_1 : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$| h_1(x, y, z) | \leq a(x, y) + b_1|z|,$$

for all $x, y \in \mathbb{R}^N$, $a \in L^q(\mathbb{R}^N)$, where $| a(x, y) | \leq a_3(x) + b_2 | y |$ where $b_1, b_2 \geq 0$ are constant and $a_3 \in L^q(\mathbb{R}^N)$.

(iv) $| h_2(x, y, u) | \leq k(x, y)\{a_4(y) + b | u |\}$, where $h_2 : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$, $b > 0$, $a_4 \in L^p(\mathbb{R}^N)$ and $k(x, y)$ satisfies Carathéodory condition $k : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ and there exist f

$_1, f_2 \in L^p(\mathbb{R}^N)$ and $f^* \in L^q(\mathbb{R}^N)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that $| k(x, y) | \leq f^*(y)f_1(x)$, for all $x, y \in \mathbb{R}^N$ and

$$| k(x_1, y) - k(x_2, y) | \leq f^*(y)|f_2(x_1) - f_2(x_2)|.$$

(v) The operator Q is bounded linear operator and continuously maps the space $L^p(\mathbb{R}^N)$ into itself. Moreover, there exists a nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\| Qu \|_{L^p(\mathbb{R}^N)} \leq \psi(\| u \|_{L^p(\mathbb{R}^N)})$$

for any $u \in L^p(\mathbb{R}^N)$.

(vi) there exists a positive constant r_0 such that

$$\begin{aligned} \| & f \|_{L^p(\mathbb{R}^N)} + lr_0 + \| g_1(x, 0) \|_{L^p(\mathbb{R}^N)} + \| a_3 \|_{L^p(\mathbb{R}^N)} + b_2 lr_0 \\ + & b_2 \| g_2(x, 0) \|_{L^p(\mathbb{R}^N)} + b_1 \| K \|_1 \| a_4 \|_{L^p(\mathbb{R}^N)} + bb_1 \| K \|_1 \psi(r_0) \end{aligned}$$

$\leq r_0$,where

$$(Ku)(t) = \int_{\mathbb{R}^N} k(x, y)u(y)dy$$

and

$$\| K \|_1 = \{Sup \| Ku \|_{L^p(\mathbb{R}^N)} \quad : \| u \| \leq r$$

$_0\}$.

Remark 3.1 *The linear fredholm integral operator $K : L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is a continuous operator and* $\| K \|_1 \leq \infty$.

Theorem 3.1 *If the above assumptions (i)-(vi) are satisfied then the functional integral equation 1 has at least one solution in $L^p(\mathbb{R}^N)$.*

Proof: *First of all, we define the operator $F : L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ by*

$$(Fu)(x) = f(x) + g_1(x, u(x)) + h_1 \left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy \right),$$

and $(GU)(x) = h_1 \left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy \right)$. Now Fu is measurable for any$u \in L^p(\mathbb{R}^N)$, we will prove that $Fu \in L^p(\mathbb{R}^N)$ for any $u \in L^p(\mathbb{R}^N)$ as $G : L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ using the above conditions, we have the following inequality

$| (Gu)(x) | = \left| h_1 \left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy \right) \right|$

$\leq a(x, g_2(x, u(x))) + b_1 \left|\int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy\right|$

$\leq a_3(x) + b_2 \mid g_2(x, u(x)) \mid +b_1 \int_{\mathbb{R}^N} \mid h_2(x, y, (Qu)(y)) \mid dy$

$\leq a_3(x) + b_2 \mid g_2(x, u(x)) - g_2(x, 0) \mid +b_2 \mid g_2(x, 0) \mid$

$+b_1 \int_{\mathbb{R}^N} k(x, y)[a_4(y) + b \mid (Qu)(y) \mid]dy$

$\leq a_3(x) + b_2 \mid a_2(x) - a_2(x) \mid +b_2 l \mid u \mid +b_2 \mid g_2(x, 0) \mid$

$+b_1 \int_{\mathbb{R}^N} k(x, y)a_4(y)dy + bb_1 \int_{\mathbb{R}^N} k(x, y) \mid (Qu)(y) \mid dy$

$\leq a_3(x) + b_2 l \mid u \mid +b_2 \mid g_2(x, 0) \mid +b_1 \int_{\mathbb{R}^N} k(x, y)a_4(y)dy$

$+ \ b \ b_1 \int_{\mathbb{R}^N} k(x, y) \mid (Qu)(y) \mid dy,$

$\| Gu \|_{L^p(\mathbb{R}^N)} \leq \| a_3 \|_{L^p(\mathbb{R}^N)} +b_2 l \| u \|_{L^p(\mathbb{R}^N)} +b_2 \| g_2(x, 0) \|_{L^p(\mathbb{R}^N)}$

$+b_1 \| K \|_1 \| a_4 \|_{L^p(\mathbb{R}^N)} +bb_1 \| K \|_1 \| Qu \|_{L^p(\mathbb{R}^N)}$

$< \infty,$

then from assumptions(i), (ii), $F(u) \in L^p(\mathbb{R}^N)$ *and* F *is will defined*

$\mid (Fu)(x) \mid \leq \mid f(x) \mid +$

$\mid g_1(x, u(x)) \mid + \mid Gx \mid$

$\leq \mid f(x) \mid +l \mid u \mid + \mid g_1(x, 0) \mid + \mid Gx \mid$

$\| Fu \|_{L^p(\mathbb{R}^N)} \leq \| f \|_{L^p(\mathbb{R}^N)} +l \| u \|_{L^p(\mathbb{R}^N)} + \| g_1(x, 0) \|_{L^p(\mathbb{R}^N)} + \| G \|_{L^p(\mathbb{R}^N)}$

$\leq \| f \|_{L^p(\mathbb{R}^N)} +l \| u \|_{L^p(\mathbb{R}^N)} + \| g_1(x, 0) \|_{L^p(\mathbb{R}^N)} + \| a_3 \|_{L^p(\mathbb{R}^N)}$

$+ \ b_2 l \| u \|_{L^p(\mathbb{R}^N)} +b_2 \| g_2(x, 0) \|_{L^p(\mathbb{R}^N)}$

$+b_1 \| K \|_1 \| a_4 \|_{L^p(\mathbb{R}^N)} +bb_1 \| K \|_1 \| Qu \|_{L^p(\mathbb{R}^N)}$

$< \infty.$

Next, we show that

$F : B_{r_0} \to B_{r_0}$ *where*

B_{r_0} *is closed ball of radius* r_0 *is constant, let* $u \in B_{r_0}$ *where* $(\| u \| \leq r_0)$

$$\begin{aligned} \| Fu \|_{L^p(\mathbb{R}^N)} &\leq \| f \|_{L^p(\mathbb{R}^N)} + lr_0 + \| g_1(x,0) \|_{L^p(\mathbb{R}^N)} + \| a_3 \|_{L^p(\mathbb{R}^N)} + b_2 lr_0 \\ &+ b_2 \| g_2(x,0) \|_{L^p(\mathbb{R}^N)} + b_1 \| K \|_1 \| a_4 \|_{L^p(\mathbb{R}^N)} \\ &+ bb_1 \| K \|_1 \psi(r_0) \end{aligned}$$

$\leq r_0.$

Now, we show that $w_0(FX) \leq l(b_2+1)w_0(X)$ *for any nonempty set* $X \subset B_{r_0}$. *To do this, we fix arbitrary* $T > 0$ *and* $\epsilon > 0$, *let us choose* $u \in X$ *and for* $x, h \in B_T$ *with* $\| h \|_{\mathbb{R}^N} \leq \epsilon$, *we have*

|(Gu)(x+h)-(Gu)(x)|

$= \Big| h_1\left(x+h, g_2(x+h, u(x+h)), \int_{\mathbb{R}^N} h_2(x+h, y, (Qu)(y))dy\right)$

$- h_1\left(x, g_2(x, u(x)), \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy\right) \Big|$

$\leq | a_3(x+h) + b_2 | g_2(x+h, u(x+h)) | - a_3(x) - b_2 | g_2(x, u(x)) ||$

$+ b_1(| \int_{\mathbb{R}^N} h_2(x+h, y, (Qu)(y))dy | - | \int_{\mathbb{R}^N} h_2(x, y, (Qu)(y))dy |)$

$\leq | a_3(x+h) - a_3(x) | + b_2 | g_2(x+h, u(x+h)) - g_2(x, u(x)) |$

$+ b_1 \Bigg(\int_{\mathbb{R}^N} k(x+h, y)[a_4(y) + b | (Qu)(y) |]dy$

$- \int_{\mathbb{R}^N} k(x, y)$

$\times [a_4(y) + b | (Qu)(y) |]dy \Bigg)$

$\leq | a_3(x+h) - a_3(x) | + b_2 | g_2(x+h, u(x+h)) - g_2(x, u(x)) |$

$+ b_1 \left(\int_{\mathbb{R}^N} | k(x+h, y) - k(x, y) | [a_4(y) + b | (Qu)(y) |]dy \right)$

$\leq | a_3(x+h) - a_3(x) | + b_2 | g_2(x+h, u(x+h)) - g_2(x+h, u(x)) |$

$+ b_2 | g_2(x+h, u(x)) - g_2(x, u(x)) | + b_1 \int_{\mathbb{R}^N} f^*(y)(| f_2(x+h) - f_2(x) |)$

$\times [a_4(y) + b | (Qu)(y) |]dy$

$\leq | a_3(x+h) - a_3(x) | + b_2 l | u(x+h) - u(x) | + b_2(| a_2(x+h) - a_2(x) |$

$+ \; b_2 l | u(x) - u(x) |) + b_1 \int_{\mathbb{R}^N} f^*(y) | f_2(x+h) - f_2(x) | a_4(y)dy$

$+ \; b \; b_1 \int_{\mathbb{R}^N} f^*(y) | f_2(x+h) - f_2(x) || (Qu)(y) | dy.$

$$\|\tau_h Gu \quad - \quad Gu\|_{L^p} = \left(\int_{B^T} |(Gu)(x+h) - (Gu)(x)|^p dx\right)^{\frac{1}{p}}$$

$\leq \left(\int_{B^T} \mid a_3(x+h) - a_3(x) \mid^p dx\right)^{\frac{1}{p}} + lb_2 \left(\int_{B^T} |u(x+h) - u(x)|^p dx\right)^{\frac{1}{p}}$

$+ \left(\int_{B^T} b_2 \mid a_2(x+h) - a_2(x) \mid^p dx\right)^{\frac{1}{p}}$

$+b_1$

$\left[\int_{B^T} \left(\int_{\mathbb{R}^N} \mid f^*(y) \mid^q a_4(y)|f_2(x+h) - f_2(x)|^q)|a_2(y)|^q dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$

$$+ \quad bb_1 \left[\int_{B^T} \left(\int_{\mathbb{R}^N} |f^*(y)|^q |f_2(x+h) - f_2(x)|^q |(Qu)(y)|^q dy\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}$$

$\|\tau_h Gu - Gu\|_{L^p}$

$\leq \|\tau_h a_3 - a_3\|_{L^p(B^T)} + lb_2\|\tau_h u - u\|_{L^p(B^T)} + b_2\|\tau_h a_2 - a_2\|_{L^p(B^T)}$

$+b_1\|f^*\|_{L^q(\mathbb{R}^N)}$

$\times\|\tau_h f_2 - f_2\|_{L^p(B^T)}\|a_4\|_{L^p(\mathbb{R}^N)}$

$+b\ \ b_1\|f^*\|$

$L^q\mathbb{R}^N)\|\tau_h f_2 - f_2\|_{L^p(B^T)}\|Qu\|_{L^p(\mathbb{R}^N)}$

$\leq w^T(a_3, \epsilon) + lb_2 w^T(u, \epsilon) + b_2 w^T(a_2, \epsilon)$

$+b_1 w^T(f_2, \epsilon)\|f^*\|_{L^q(\mathbb{R}^N)}\|a_4\|_{L^p(\mathbb{R}^N)} + bb_1\|f^*\|_{L^q(\mathbb{R}^N)}$

$w^T(f_2, \epsilon)\psi(\|u\|)_{L^p(\mathbb{R}^N)}.$

$\mid (Fu)(x+h) - (Fu)(x) \mid$

$\leq \mid f(x+h)-f(x)\mid + \mid g_1(x+h,u(x+h))-g_1(x,u(x))\mid$

$+\mid (Gu)(x+h)-(Gu)(x)\mid$

$\leq \mid f(x+h)-f(x)\mid + \mid g_1(x+h,u(x+h))-g_1(x+h,u(x))\mid$

$+\mid g_1(x+h,u(x))-g(x,u(x))\mid + \mid (Gu)(x+h)-(Gu)(x)\mid$

$\leq \mid f(x+h)-f(x)\mid + \mid a_1(x+h)-a_1(x)\mid + l\mid u(x+h)-u(x)\mid$

$+\mid (Gu)(x+h)-(Gu)(x)\mid$

$\|\tau_h Fu - Fu\|_{L^p} \leq (\int_{B^T} \mid f(x+h)-f(x)\mid^p dx)^{\frac{1}{p}} + l(\int_{B^T} |u(x+h)-u(x)|^p dx)^{\frac{1}{p}}$

$+$

$(\int_{B^T} \mid a_1(x+h)-a_1(x)\mid^p dx)^{\frac{1}{p}} + \|\tau_h Gu - Gu\|_{L^p(B^T))}$

$\leq \|\tau_h f - f\|_{L^p(B^T)} + l\|\tau_h u - u\|_{L_p(B^T)} + |\tau_h a_1 - a_1\|_{L^p(B^T)}$

$+\|\tau_h G - G\|_{L^p(B^T)},$

$$\begin{aligned} w^T(Fx,\epsilon) &\leq w^T(f,\epsilon)+lw^T(u,\epsilon)+w^T(a_1,\epsilon)+w^T(a_3,\epsilon)+lb_2w^T(u,\epsilon)\\ &+ w^T(a_2,\epsilon)+b_1w^T(f_2,\epsilon)\|f^*\|_{L_q(\mathbb{R}^N)}\|a_4\|_{L_p(\mathbb{R}^N)}\\ &+ bb_1\|f^*\|_{L_q(\mathbb{R}^N)}\end{aligned}$$

$w^T(f_2,\epsilon)\psi(\|u\|)_{L_p(\mathbb{R}^N)}.$

Thus, we obtain

$$\begin{aligned} w^T(FX,\epsilon) &\leq w^T(f,\epsilon)+lw^T(X,\epsilon)+w^T(a_1,\epsilon)+w^T(a_3,\epsilon)+lb_2w^T(u,\epsilon)\\ &+ w^T(a_2,\epsilon)+b_1w^T(f_2,\epsilon)\|f^*\|_{L_q(\mathbb{R}^N)}\|a_4\|_{L_p(\mathbb{R}^N)}\\ &+ bb_1\|f^*\|_{L_q(\mathbb{R}^N)}\end{aligned}$$

$w^T(f_2,\epsilon)\psi(r_0).$

Also, we have $w^T(f_2,\epsilon)$ $,w^T(f,\epsilon),$ *and* $w^T(a_i,\epsilon)\to 0$ *as* $\epsilon\to\infty$ *where* $i=1,2,3$

then, we obtain

$$w(FX) \leq l(b_2+1)w(X), \qquad \textit{where } l(b_2+1)\leq 1. \tag{-13}$$

Next, let us fix an arbitrary number $T > 0$, then taking into account our assumptions,

for an arbitrary function $u \in X$. We have

$(\int_{\mathbb{R}^N}$

$\backslash B^T |(Fu)(x)|^p dx)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^N\backslash B^T} |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N\backslash B^T} |g_1(x,u(x))|^p dx\right)^{\frac{1}{p}}$

$+\left(\int_{\mathbb{R}^N\backslash B^T} \left|h_1\left(x, g_2(x,u(x)), \int_{\mathbb{R}^N} h_2(x,y,(Qu)(y))dy\right)\right|^p dx\right)^{\frac{1}{p}}$

$\leq \left(\int_{\mathbb{R}^N\backslash B^T} |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N\backslash B^T} |g_1(x,u(x)) - g_1(x,0)|^p dx\right)^{\frac{1}{p}}$

$+ \left(\int_{\mathbb{R}^N\backslash B^T} |g_1(x,0)|^p dx\right)^{\frac{1}{p}}$

$+ \left(\int_{\mathbb{R}^N\backslash B^T} |a_3(x) + b_2 \mid g_2(x,u(x)) \mid + b_1 \int_{\mathbb{R}^N} \mid h_2(x,y,(Qu)(y))dy \mid |^p dx\right)^{\frac{1}{p}}$

$\leq \left(\int_{\mathbb{R}^N\backslash B^T} |f(x)|^p dx\right)^{\frac{1}{p}} + l\left(\int_{\mathbb{R}^N\backslash B^T} |u(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N\backslash B^T} |g_1(x,0)|^p dx\right)^{\frac{1}{p}}$

$+\left(\int_{\mathbb{R}^N\backslash B^T} |a_3(x)|^p dx\right)^{\frac{1}{p}} + b_2 l\left(\int_{\mathbb{R}^N\backslash B^T} |u(x)|^p dx\right)^{\frac{1}{p}} + b_2\left(\int_{\mathbb{R}^N\backslash B^T} |g_2(x,0)|^p dx\right)^{\frac{1}{p}}$

$+$

$b_1\left(\int_{\mathbb{R}^N\backslash B^T} |(\int_{\mathbb{R}^N} |k(x,y)| \times [a_4(y) + b|(Qu)(y)|]dy)|^p dx\right)^{\frac{1}{p}}$

$\leq \left(\int_{\mathbb{R}^N\backslash B^T} |f(x)|^p dx\right)^{\frac{1}{p}} + l\left(\int_{\mathbb{R}^N\backslash B^T} |u(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N\backslash B^T} |g_1(x,0)|^p dx\right)^{\frac{1}{p}}$

$+\left(\int_{\mathbb{R}^N\backslash B^T} |a_3(x)|^p dx\right)^{\frac{1}{p}} + b_2 l\left(\int_{\mathbb{R}^N\backslash B^T} |u(x)|^p dx\right)^{\frac{1}{p}}$

$+ b_2\left(\int_{\mathbb{R}^N\backslash B^T} |g_2(x,0)|^p dx\right)^{\frac{1}{p}} + b_1\left(\int_{\mathbb{R}^N\backslash B^T} (\int_{\mathbb{R}^N} |k(x,y)|^q |a_4(y)|^q dy\right)^{\frac{p}{q}} dx)^{\frac{1}{p}}$

$$+ \quad bb_1(\int_{\mathbb{R}^N \setminus B^T}(\int_{\mathbb{R}^N}|k(x,y)|^q|(Qu)(y)|^q dy)^{\frac{p}{q}}dx)^{\frac{1}{p}}$$

$\leq \|f\|_{L^p(\mathbb{R}^N \setminus B^T)} + l \parallel u \parallel_{L^p(\mathbb{R}_N \setminus B^T)} + \parallel g_1(.,0) \parallel_{L^p(\mathbb{R}_N \setminus B^T)}$

$+ \|a_3\|_{L^p(\mathbb{R}^N \setminus B^T)} + b_2 l \|u\|_{L^p(\mathbb{R}^N \setminus B^T)} + b_2\|g_2(.,0)\|_{L^p(\mathbb{R}^N \setminus B^T)}$

$+b_1$

$\parallel f^* \parallel_{L^q(\mathbb{R}^N)} . \parallel f_1 \parallel_{L^p(\mathbb{R}^N \setminus B^T)} .(\parallel a_4 \parallel_{L^p(\mathbb{R}^N \setminus B^T)} + b\psi(\|u\|)_{L^p(\mathbb{R}^N)})$.

Also we have $\|f\|_{L^p(\mathbb{R}^N \setminus B^T)}$, $\parallel g_i(.,0) \parallel_{L^p(\mathbb{R}^N \setminus B^T)}$,
$\parallel f_1 \parallel_{L^p(\mathbb{R}^N \setminus B^T)}, \|a_3\|_{L^p(\mathbb{R}^N \setminus B^T)} \to 0$

as $T \to \infty$ *where* $i = 1,2$
and hence we obtain that

$$d(FX) \leq l(b_2 + 1)d(X). \tag{-16}$$

Consequentially we infer from equation -13, -16

$$w_0(FX) \leq l(b_2 + 1)w_0(X),$$

so, the operator F satisfies all conditions of Darbo fixed point theorem, which enables us to deduce that F has at least one solution in$L^p(\mathbb{R}^N)$ *. Thus the proof is finished.*

Next, we will need the following theorem that help us in a coming example.

Theorem 3.2 *[4]*

Let $\Omega \subseteq \mathbb{R}^N$ *be a measure space and suppose* $k : \Omega \times \Omega \to \mathbb{R}$ *is a measurable function for which there is constant* $C > 0$ *such that*

$$\int_I |k(x,y)|dx \leq C \qquad a.e.\ y \in \Omega$$

and

$$\int_I |k(x,y)|dy \leq C \qquad a.e.\ x \in \Omega.$$

If $K : L^p(\Omega)$

$\to L^p(\Omega)$ *is defined by*

$$(Kf)(x) = \int_\Omega f(y)\ dy,$$

then K *is a bounded and continuous operator and* $\|K\|_1 \leq C$.

Example: consider the integral equation

$(y_2 \frac{}{1+y_1^2+2e^{-|u(x)|}u(x))dx,}$

where

$$x = (x_1, x_2) \in \mathbb{R}^2,$$

and $\| x \|$ is the Euclidean norm. We study the solvability of this integral equation in the space $L^p(\mathbb{R}^2)$ for $p, q > 2$.
Let$f(x) = e^{-x^2}$, $g_1(x, u(x)) = \frac{\sin u}{\|x\|+4}$,
$h_2(x, y, (Qu)(y)) = \frac{e^{-(|x_1|+|y_1|)}}{(|x_2|+3)^2(|y_2|+2)^2}(\frac{y_2}{1+y_1^2} + 2e^{-|u(x)|}u(x))$,
$a(x, y) = e^{-x^2} + \frac{\sin u}{\|x\|+4}$ with $b_1 = \frac{1}{8}$, $a_3(x) = e^{-x^2}$where $a_3 \in L^p(\mathbb{R}^2)$ such that $b_2 = 1$, $g_2(x, u(x)) = \frac{\sin u}{\|x\|+4}$.
Hence the norm

$$\| f \|_{L^p(\mathbb{R}^2)} = (\frac{\pi}{p})^{\frac{1}{p}}.$$

Next the functions $g_i(x, u(x)), i = 1, 2$ satisfy the assumption(ii) with $a_i(x) = \frac{1}{\|x\|+4}$, $l = \frac{1}{4}$, indeed

$$\begin{aligned}
|g_i(x, u) - g_i(y, v)| &= |\frac{\sin u}{\| x \| +4} - \frac{\sin v}{\| y \| +4}| \\
&\leq | \frac{1}{\| x \| +4} - \frac{1}{\| y \| +4} || \sin u | + \frac{1}{\| y \| +4} | \sin u - \sin v | \\
&\leq |\frac{1}{\| x \| +4} - \frac{1}{\| y \| +4}| + \frac{1}{4} | u - v | \\
&= | a_i(x) - a_i(y) | + l | u - v |
\end{aligned}$$

where $a_i(x) \in L_p(\mathbb{R}^2)$ with norm

$$\| a_i \|_{L^p(\mathbb{R}^2)} = \left(\frac{4\pi(2)^{1-p}}{(p-1)(p-2)} \right)^{\frac{1}{p}},$$

where $a_4 = \frac{y_2}{1+y_1^2}$, with $\|a_4\|_{L^p(\mathbb{R}^2)} = 0$, also

$$k(x, y) = \frac{e^{-(|x_1|+|y_1|)}}{(| x_2 | +3)^2(| y_2 | +2)^2},$$

$f^*(x) = \frac{e^{-|x_1|}}{(|x_2|+3)^2}$, $f_1(x) = f_2(x) = \frac{e^{-(|x_1|)}}{(|x_2|+2)^2}$ we see that $f_1, f_2 \in L_{p(\mathbb{R}^2)}$, $f^* \in L_{q(\mathbb{R}^2)}$. Also we have

$$\int_{\mathbb{R}^2} | k(x, y) | dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(|x_1|+|y_1|)}}{(| x_2 | +3)^2(| y_2 | +2)^2} dx_1 dx_2 \leq \frac{1}{3},$$

$$\int_{\mathbb{R}^2} | k(x, y) | dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(|x_1|+|y_1|)}}{(| x_2 | +3)^2(| y_2 | +2)^2} dy_1 dy_2 \leq \frac{2}{9},$$

and thus from the theorem $\| K \|_1 \leq \frac{1}{3}$ furthermore $b = 2$, $Q(u)(x) = e^{-|u(x)|}u(x)$)satisfies the assumption with$\psi(t) = t$. Finally, the inequality from assumption (vi) has the form

$$\begin{aligned} & \| \quad f \|_{L_p(\mathbb{R}^2)} + lr_0 + \| g_1(x,0) \|_{L_p(\mathbb{R}^2)} + \| a_3 \|_{L_p(\mathbb{R}^2)} + b_2 l r_0 \\ + & \quad b_2 \| g_2(x,0) \|_{L_p(\mathbb{R}^2)} + b_1 \| K \|_1 \| a_4 \|_{L_p(\mathbb{R}^2)} + bb_1 \| K \|_1 \psi(r_0) \end{aligned}$$

$\leq r_0$,

$$2(\frac{\pi}{p})^{\frac{1}{p}} + \frac{1}{2}r_0 + (\frac{1}{4})(\frac{1}{3})r_0 \leq r_0.$$

Thus, for the number $r_0 = (\frac{24}{5})$

$\times(\frac{\pi}{p})^{\frac{1}{p}}$. Hence all the assumptions are satisfied and so, Eq.(3.4) has at least one solution in $L^p(\mathbb{R}^2)$.

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Recommendation: Based on above report, manuscript is

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Data Availability (excluding Review articles)

Applicable.

Supplementary Materials

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Funding Statement

The research was self-sposored by the author.

Acknowledgments

The authors would like to express their thanks to the referee for his helpful comments and suggestions, which greatly improved the presentation of this paper.