ON SOME DIFFERENTIAL EQUATIONS ARISING IN A TIME DOMAIN INVERSE SCATTERING PROBLEM FOR A DISSIPATIVE WAVE EQUATION

by

David J.N. Wall

Department of Mathematics, University of Canterbury, Christchurch, New Zealand.

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Abstract

The problem of identification of one spatially varying material property, defined within a slab, from boundary measurements is examined. This inverse problem is described by a functional differential equation. Uniqueness and existence of the solution of this inverse problem and the associated direct problem is proven. Of major importance in any inverse problem are the properties of the operator mapping the boundary measurements to the material property. It is shown that this operator is continuous and differentiable.

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David J N Wall

1. Introduction

The use of a reflection kernel to characterise the scattering of waves in an inhomogeneous media in the time domain, has been well developed over the last ten years (see for example [6] – [7]) and the references cited therein). In [2] it is shown that the use of a reflection kernel combined with invariant imbedding provides a useful and convenient method for the computational solution of a variety of inverse problems. In particular this technique leads to explicit functional equations for the mapping between the reflection kernel, measured at the interface of the inhomogeneous region, and the material functions to be identified. In this paper we shall examine the properties of this mapping when the medium is dissipative. We discuss uniqueness, existence and show continuous dependence of the inverse problem solution on the given data. To do this we follow Vogel's [9] analysis of the non-dissipative problem. We shall show that the aforementioned map is linearizable, that is the map is Fréchet differentiable, and give a specific form for the differential. That the knowledge of this differential is useful in predicting the effect of noise on a solution to a "real non-dissipative inverse problem" has already been shown in [5].

The one-dimensional spatial model equation to be investigated in this paper is

$$u_{xx} - u_{tt} + A(x)u_x + B(x)u_t = 0, \tag{1.1}$$

where the independent variable x is a travel time coordinate. As shown in [4], this equation is sufficiently general to model a variety of electromagnetic and elastic wave scattering phenomena. The coefficients are to have support on the interval $x \in [0,1]$, and are assumed to be continuous. This means in the physical problem the material parameters are continuous in $(-\infty, \infty)$ thereby implying that the slab is matched to the homogeneous exterior region.

In [4] Corones and Krueger utilise the technique of invariant imbedding to derive from (1.1) the integro-partial differential equation

$$R_x^+(x,1;t) - 2R_t^+(x,1;t) = -B(x)R^+(x,1;t) - \frac{1}{2}(A(x)+B(x))\int_0^t R^+(x,1;s)R^+(x,1;t-s)\,ds, \qquad 0 \le x \le 1, 0 \le t \le 2(1-x). \tag{1.2}$$

This is the imbedding equation describing the reflection kernel at the left-hand interface at location x with the right-hand interface held at $x = 1$. The superscript $+$ is used to signify that this kernel transforms an incident wave moving in the positive x-direction from the left-hand-side of the media into a reflected wave moving in the negative x-direction. A similar equation holds for the reflection kernel at the right-hand interface, namely $R^-(0,x,t)$, describing the reflection, at location x, of an incident wave from the right-hand media with the left-hand interface held at $x = 0$. The equation satisfied by R^- can be obtained from (1.2) if x is replaced by $1-x$, then R_x^+ is replaced by $-R_x^-$, and the A term is multiplied by -1 as A involves a derivative with respect to x. (see [6] for further details). The time limit in (1.2) enables the incident wave to just traverse the slab twice, that is one return trip.

We define the triangular region in the independent variables (t, x) for which (1.2) is applicable as $D = \{(x,t) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq t \leq 2(1-x)\}$, see Figure 1. In the problems under consideration in this paper conditions on adjacent sides of the domain triangle D are specified as follows. When the problem is one of direct scattering the material functions $A(x)$, $B(x)$, $x \in [0,1]$, are known and hence

$$R^+(x,1;0) = -\frac{1}{4}(A(x) - B(x)), \quad 0 \leq x \leq 1, \tag{1.3}$$

on the side of D where $t = 0$, and it is required to calculate the reflection kernel

$$R^+(0,1;t), \quad 0 \leq t \leq 2, \tag{1.4}$$

on the side of D where $x = 0$. For the inverse problems considered in §2 and §3 these conditions are reversed, in that (1.4) is given, and after calculation of R^+ from (1.2) either A or B can be found from (1.3). A more general, and difficult inverse problem, is the reconstruction of both of the material functions A and B, it is then necessary to utilise both the equations for R^+ and R^-, this is considered in [2] - [8] and will be considered via the approach taken here in a later paper.

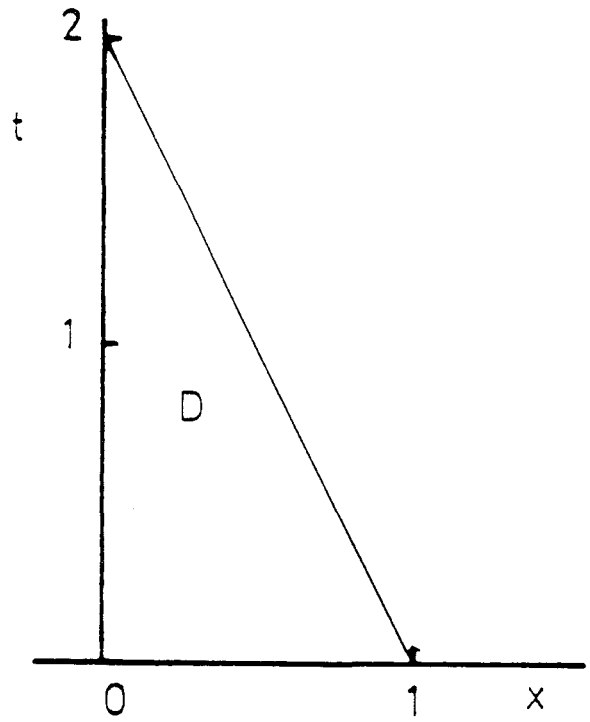

Figure 1. Illustrating the domain D for which equation (1.2) is applicable for a single return trip.

In the sequel we will suppress the superscript on the R term and consider only the R^+ equation. The analysis for the R^- equation follows by use of the appropriate sign changes. §2 provides the formulation of the two inverse problems considered in this paper and also gives preliminary results needed in later sections. In §3 special case characterisation results for the inverse problem are shown and the existence and uniqueness of the solution to the inverse problems posed in §2.1 and §2.2 is proven. This is done in Theorem 3.1 for the case when the data is known exactly. Theorem 3.2 is the major result of this paper, and it states that the map between the known quantities — the initial reflection kernel and one of the material functions — to the unknown material function is continuous and differentiable. It also shows that the linearisation of this map is a bounded operator. In §4 we consider the existence and uniqueness of solutions to the direct problem by similar procedures to that used in §2 and §3; this result is then used to give global existence to the inverse problem solution of §2.

2. The Equation Formulation of the Inverse Scattering Problem for one Material Function.

In this section we shall examine the inverse problem that occurs when one of the material related functions A or B, is assumed to be known and the other one is to be determined. This is of interest in practice (see [3]) and constitutes a simpler problem to the one which occurs when both A and B are to be identified. This latter case will be considered in another paper.

2.1: The material function A is known, with B to be determined.

This will correspond, in the electromagnetic problem modeled by equation (1.1), to requiring the identification of the conductivity with a priori knowledge of the material permittivity (see [3]).

On substitution of (1.3) into (1.2), to eliminate the unknown B, we obtain

$$R_x - 2R_t = -(4R(x,0) + A(x))R(x,t) - \frac{1}{2}(2A(x) + 4R(x,0)) \int_0^t R(x,t-s)R(x,s)\,ds, \qquad 0 \le x \le 1, 0 \le t \le 2(1-x). \tag{2.1}$$

In (2.1) the second argument of R, which denotes the position of the right-hand boundary point, has been suppressed as it will be in the sequel(see (1.3)). Notice that (2.1) is not a standard Volterra integro-differential equation because the unknown R in the integrand of the right-hand-side also depends upon t. Observe that the partial derivatives on the left-hand-side may be converted into a directional derivative along the characteristic direction by a suitable change of independent variable. To this end, following Vogel [9] we consider the change of independent variable $\tau = x + t/2$ in order to convert the partial derivatives on the left-hand-side of (2.1) into a directional derivative. This will transform the region D into $0 \le x \le 1$, $x \le \tau \le 1$,(see Figure 2), and on redefining the dependent variable as $u(x;\tau) = R(x, 2(\tau - x)) = R(x,t)$ we find

$$\frac{du}{dx}(x;\tau) = -(4u(x;x) + A(x))u(x;\tau) - 2(2u(x;x) + A(x)) \int_x^\tau u(x;\tau + x - s)u(x;s)\,ds, \qquad 0 \le x \le 1, x \le \tau \le 1, \tag{2.2}$$

with initial conditions

$$u(0;\tau) = R(0,2\tau) = R(0,t), \quad 0 \le \tau \le 1. \tag{2.3}$$

The inverse scattering problem can now be stated as, given $u(0;\tau)$ for $0 \le \tau \le 1$, find $u(x;x)$, for $0 \le x \le 1$, from the solution of the Volterra B-space integro-differential equation (2.2). In equation (2.2) it is seen that knowledge of u at state τ is not sufficient, it requires knowledge of u at all states between x and τ. Thus (2.2) describes a system with memory in the τ variable, but not in the x variable. The function B is then obtained from

$$B(x) = 4u(x;x) + A(x), \quad 0 \le x \le 1. \tag{2.4}$$

It is seen that (2.2) can also be considered a Banach space (B-space) valued ordinary differential equation, and we can consider $u : [0 \le x \le 1] \mapsto T$ as a mapping into the B-space T. The B-space T is the space of continuous functions $C([0,1])$ with the topology defined by the usual supremum norm $||u||_T = \sup_{\tau\in[0,1]}\{u(x;\tau)\}$, for fixed $x \in [0,1]$. This means that for fixed x the functions $u(x) = u(x;\tau)$ form the points of the B-space T. We set U to be the space of continuous functions $u : [0 \le x \le 1] \mapsto T$, that is $U = C([0 \le x \le 1], T)$, and choose the norm for U as

$$||u||_U = \sup_{0\le x\le 1} ||u||_T, \tag{2.5}$$

then $(U, \|\cdot\|_U)$ is a B-space. We shall also assume that A belongs to the parameter subspace P, $P \subset C([0,1])$ which is a B-space with the appropriate supremum norm.

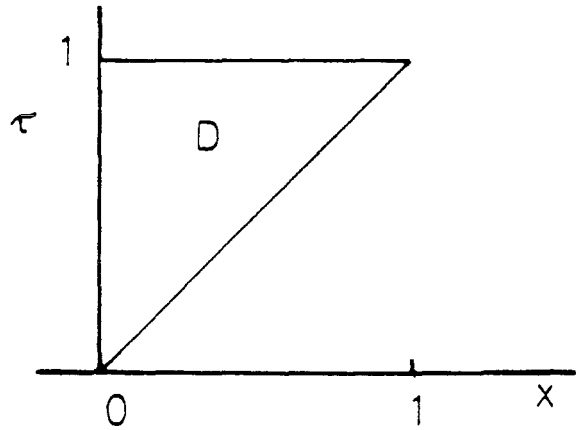

Figure 2. Transformed region D.

We now rewrite (2.2) in the standard form

$$\frac{du}{dx}(x;\tau) = F(x, u(x))(\tau), \quad 0 \le x \le 1, \tag{2.6}$$

with initial conditions

$$u(0) = u_0 = u(0;\tau), \quad 0 \le \tau \le 1, \tag{2.7}$$

where $u_0 \in T$, and the mapping functional on the right-hand-side of (2.6), $F : [0,1] \times T \mapsto T$ is to be described by

$$F(x,u)(\tau) = \begin{cases} -(4u(x;x) + A(x))u(x;\tau) - 2(2u(x;x) + A(x)) \int_x^\tau u(x;\tau + x - s)u(x;s)\,ds, & x \le \tau \le 1; \\ -(4u(x;x) + A(x))u(x;\tau) & 0 \le \tau \le x. \end{cases} \tag{2.8}$$

Note that F will be continuous at $\tau = x$.

It follows that the evolution of $u(x)$ as x increases is described by the non-linear ordinary differential equation (2.6), and where we have extended the definition of F in (2.8), in order to simplify the statement of the mathematical properties of (2.2).

The mapping between the interface reflection kernel $R(0, 2\tau)$ and the unknown material function B is given explicitly by (2.4) and (2.6) through (2.8), and it is the mathematical properties of this mapping that we wish to consider in this paper. Examination of (2.4) shows that it suffices to examine the non-linear map $G : u_0(\tau) \mapsto u(\tau;\tau)$, where $u_0 \in T$, $u(\tau;\tau) \in U$, so that $G : T \mapsto U$. Notice we have chosen to use τ as the independent variable here, rather than x as in (2.4), for convenience in describing the map G. To examine the continuity and differentiability of the functional map G we must first examine the mapping properties of F.

LEMMA 2.1. *With F defined as in (2.8) and with $A \in P$ then the mapping F has the following properties.*

(i) $F : [0,1] \times T \times P \mapsto T$.

(ii) For each $u \in T$, $F(x,u)$ is continuous with respect to x.

(iii) For each $x \in [0,1]$ and $\tau \in [0,1]$, F is Fréchet partial differentiable with respect to u, and with $F_u = \frac{\partial F}{\partial u}(x,u) : T \mapsto T$ this derivative is defined by the differential

$$(F_u(x,u)v)(\tau) = \begin{cases} -(4u(x;x) + A(x))v(x;x) - 4v(x;x)\int_x^\tau u(x;\tau + x - s)u(x;s)\,ds \\ \quad -4u(x;\tau)v(x;x) - 4(2u(x;x) + A(x))\int_x^\tau u(x;\tau + x - s)v(x;s)\,ds, & x \le \tau \le 1; \\ -(4u(x;x) + A(x))v(x;x) - 4u(x;\tau)v(x;x), & 0 \le \tau \le x, \end{cases} \tag{2.9}$$

where $v \in T$.

(iv) For $x \in [0,\gamma]$, $0 \le \gamma \le 1$ and $\tau \in [0,1]$, F is Lipschitz continuous with respect to u in the ball $B_M = \{u \in T : \|u\|_T \le M\}$, with Lipschitz constant equal to $(1+4M)\|A\|_P + 12M + 8M^2$.

(v) For each $x \in [0,1]$ and $\tau \in [0,1]$, F is continuous with respect to A.

(vi) For each $x \in [0,1]$ and $\tau \in [0,1]$, F is Fréchet partial differentiable with respect to A, and with $F_A = \frac{dF}{dA}(x,u) : P \mapsto T$ this derivative is defined by the differential

$$(F_A(x,u)(\tau))v(x) = \begin{cases} -(u(x;\tau) - 2\int_x^\tau u(x;\tau + x - s)u(x;s)\,ds)v(x), & x \le \tau \le 1; \\ -u(x;\tau)v(x), & 0 \le \tau \le x, \end{cases} \tag{2.10}$$

where $v \in P$.

(vii) For each $x \in [0,1]$ and $\tau \in [0,1]$, F_u, and F_A are jointly continuous so that F is Fréchet differentiable.

(viii) $|F(x,u)| \le (4\|u\|_T + \|A\|_P)\|u\|_T + 2(2\|u\|_T + \|A\|_P)\|u\|_T^2(\tau - x)_+$, *where* $(\tau - x)_+ = 0$, *if* $x > \tau$.

Proof: This is standard, see Appendix A. ∎

REMARK 2.1. *Concerning part (iv), until more is known about the solution of (2.6) and (2.7) nothing more can be said about γ other than it is dependent on M (see §3).*

2.2: The material function B is known, with A to be determined.

This will correspond, in the electromagnetic problem modeled by (1.1), to an inverse problem of identification of the permittivity with a priori knowledge of the material conductivity (see [2]).

On substitution of (1.3) into (1.2), to eliminate the unknown A, we obtain

$$R_x - 2R_t = -B(x)R(x,t) - \frac{1}{2}(2B(x) - 4R(x,0))\int_0^t R(x,t-s)R(x,s)\,ds, \qquad 0 \le x \le 1, 0 \le t \le 2(1-x). \tag{2.11}$$

Again notice that (2.11) is not a standard Volterra integro-differential equation because the unknown in the integrand of the right-hand-side depends upon t. Following §2.1, we utilise the same change of independent variable and the same dependent variable notation to find

$$\frac{du}{dx}(x;\tau) = -B(x)u(x;\tau) - 2(2B(x) - 2u(x;x)) \int_x^\tau u(x;\tau + x - s)u(x;s)\,ds, \qquad 0 \le x \le 1, x \le \tau \le 1, \tag{2.12}$$

with initial conditions

$$u(0;\tau) = R(0,2\tau) = R(0,t), \quad 0 \le \tau \le 1. \tag{2.13}$$

The inverse scattering problem can now be stated as — in a similar manner to §2.1 — given $u(0;\tau)$, for $0 \le \tau \le 1$, find $u(x;x)$, for $0 \le x \le 1$, from the solution of the Volterra B-space integro-differential equation (2.12). The function A is then obtained from

$$A(x) = B(x) - 4u(x;x), \quad 0 \le x \le 1. \tag{2.14}$$

We again follow §2.1, and define $u(x) = u(x;\tau)$ and consider (2.12) to be the B-space valued ordinary differential equation (2.6) with initial conditions (2.13), and where the mapping function $F : [0,1] \times T \mapsto T$ is now described by

$$F(x,u)(\tau) = \begin{cases} -B(x)u(x;\tau) - 4(B(x) - u(x;x)) \int_x^\tau u(x;\tau + x - s)u(x;s)\,ds, & x \le \tau \le 1; \\ -B(x)u(x;\tau) & 0 \le \tau \le x. \end{cases} \tag{2.15}$$

Note F is continuous at $\tau = x$.

The evolution of $u(x)$ as x increases is described by the equation (2.12), and in terms of the original imbedding equations the initial data $u_0(\tau) = R(0,2\tau)$, $0 \le \tau \le 1$, is mapped to $u(\tau,\tau) = -\frac{1}{4}(A(\tau) - B(\tau))$, $0 \le \tau \le 1$, so that

$$A(\tau) = B(\tau) - 4u(\tau,\tau), \quad 0 \le \tau \le 1. \tag{2.16}$$

The mapping between the measurement and the unknown material function A is given explicitly by (2.16), (2.6) and (2.13) through (2.15). The mapping properties of the functional F are described next.

LEMMA 2.2. *With F defined as in* (2.15) *and with $B \in P$ then the mapping F has the following properties.*

(i) $F : [0,1] \times T \times P \mapsto T$.

(ii) For each $u \in T$, $F(x,u)$ is continuous with respect to x.

(iii) For each $x \in [0,1]$ and $\tau \in [0,1]$, F is Fréchet partial differentiable with respect to u, and with $F_u = \frac{\partial F}{\partial u}(x,u) : T \mapsto T$ this derivative is defined by the differential

$$(F_u(x,u)v)(\tau) = \begin{cases} -B(x)v(x;\tau) + 4v(x;x) \int_x^\tau u(x;\tau + x - s)u(x;s)\,ds \\ \quad -8(B(x) - u(x;x)) \int_x^\tau u(x;\tau + x - s)v(x;s)\,ds, & x \le \tau \le 1; \\ -B(x)v(x;\tau), & 0 \le \tau \le x, \end{cases} \tag{2.17}$$

where $v \in T$.

(iv) *For each* $x \in [0,\gamma]$, $0 \le \gamma \le 1$ *and* $\tau \in [0,1]$, F *is Lipschitz continuous with respect to* u *in the ball* $B_M = \{u \in T : \|u\|_T \le M\}$, *with Lipschitz constant equal to* $(1+8M)\|B\|_P + 12M^2$.

(v) *For each* $x \in [0,1]$ *and* $\tau \in [0,1]$, F *is continuous with respect to* A.

(vi) *For each* $x \in [0,1]$ *and* $\tau \in [0,1]$, F *is Fréchet partial differentiable with respect to* B, *and with* $F_B = \frac{\partial F}{\partial B}(x,u) : P \mapsto T$ *this derivative is defined by the differential*

$$(F_B(x,u)(\tau))v(x) = \begin{cases} -(u(x;\tau) - 4\int_x^\tau u(x;\tau+x-s)u(x;s)\,ds)v(x), & x \le \tau \le 1; \\ -u(x;\tau)v(x), & 0 \le \tau \le x, \end{cases} \tag{2.18}$$

where $v \in P$.

(vii) *For each* $x \in [0,1]$ *and* $\tau \in [0,1]$, F_u *and* F_B *are jointly continuous, so that* F *is Fréchet differentiable.*

(viii) $|F(x,u)| \le \|B\|_P\|u\|_T + 4(\|B\|_P + \|u\|_T)\|u\|_T^2(\tau - x)_+$, *where* $(\tau - x)_+ = 0$, *if* $x > \tau$.

Proof: This is standard and very similar to the proof of Lemma 2.1 which is sketched in the Appendix A.∎

Remark 2.1 is pertinent to this Lemma also. Again as in §2.1 the mathematical properties of the map $G : u_0(\tau) \to u(\tau,\tau)$, where $u_0 \in T, u(\tau,\tau) \in U$ so $G : T \to U$, is of major interest for this inverse problem.

3. Properties of the Inverse Scattering Map

We can now state the continuity and differentiability results for the map $G : T \mapsto U$ for both the problems considered in §2. The results are based upon the classical Picard–Lindelöf theorem for B-spaced valued ordinary differential equation initial value problems. See for example Zeidler [11, §§1.6, 3.3 , 3.5, 4.11] for a recent discussion of this theorem.

We are considering the problem

$$\begin{aligned} \frac{du}{dx}(x;\tau) =& F(x,u(x),p(x))(\tau), \quad 0 \le x \le 1, \\ u(0) =& u_0, \quad 0 \le \tau \le 1, \end{aligned} \tag{3.1}$$

where $u, u_o \in T$, and F is as given in either (2.8) or (2.15) depending on whether we are considering the problem of §2.1 or §2.2, respectively. Notice we have included a third variable in the dependency list of F in (3.1), this variable p is to be taken as either A or B depending on whether §2.1 or §2.2 is being considered, and $p \in P$.

Recall that with F continuous with respect to x equation (3.1) is equivalent to the integral equation

$$u(x) = u_0 + \int_0^x F(s,u(s),p(s))\,ds, \tag{3.2}$$

and the Banach fixed-point theorem applied to (3.2) as utilised in the Picard–Lindelöf theorem guarantees the existence of a unique local solution to (3.1) when F is locally Lipschitz continuous with respect to its second variable.

LEMMA 3.1. *With $u_0 \in T$, and F possessing the properties of either Lemma 2.1 or Lemma 2.2*

(i) The initial value problem (3.1) *has exactly one continuously differentiable solution $u \in B_M$ either for all $x \in [0,1]$, or else on a subinterval $[0,b]$, $0 < b \leq 1$ which is maximal with respect to extension of solution.*

(ii) The solution u depends continuously upon the initial data u_0 and the parameter p.

REMARK 3.1. *The standard Picard–Lindelöf theorem gives local existence and the standard continuation argument (see for example* [1, page 25]*) gives the maximal extension. The continuous dependence of u on the initial data result is a consequence of the Gronwall lemma, and so its modulus of continuity depends on the Lipschitz constant of F.*

In order to obtain global existence and uniqueness for solutions of (2.6) we need a comparison result. Suitable scalar comparison equations are provided via Lemma 2.1 (viii) and Lemma 2.2(viii), however even these equations appear to be intractable by analytic quadratures. However in the special case for which the a priori function A or B are known to be identically zero we find.

LEMMA 3.2.

(i) Problem §2.1: When $A \equiv 0$, if $0 < u_0(\tau) < \alpha$, then (3.1) *has a unique solution u which exists on the entire interval $0 \leq x \leq 1$. Here α is the positive zero of*

$$\frac{1}{\alpha} - 1 - \ln(1 + \frac{1}{\alpha}) = 0,$$

and $\alpha \approx 0.45$.

(ii) Problem §2.2: When $B \equiv 0$, if $|u_0(\tau)| < (2\tau)^{-1}$ then (3.1) *has a unique solution u which exists on the entire interval $0 \leq x \leq 1$.*

Proof:

(i) From Lemma 2.1(viii) a suitable comparison equation is $\frac{dy}{dx} = 4y^2(y+1)$, $y = y_0$ and this has solution

$$x = \frac{1}{y_0} - \frac{1}{y} + \ln(\frac{y_0(y+1)}{y(y_0+1)}),$$

which exists on $[0,1]$ only if $0 < y_0(\tau) < \alpha$.

(ii) From Lemma 2.2(viii) a suitable comparison equation is $\frac{dy}{dx} = 4y^3(\tau - x)$, $y = y_0$, and this has solution $y = [y_0^2/(1 - 8y_0^2(\tau x - x^2/2))]^{1/2}$, which exists on $[0,1]$ only if $|y_0(\tau)| < (2\tau)^{-1/2}$, $\tau \in [0,1]$. ∎

REMARK 3.2. *This last lemma provides a characterisation result for the inverse problem for the case of A or B identically zero, and part (ii) is a slightly stronger result than Vogel's* [9].

We shall now quote an existence and uniqueness theorem which is applicable to the inverse problems of §2 when the reflection kernel is both known exactly and to arise from such a problem. The proof of this theorem is deferred to §4.

THEOREM 3.1. *There is a unique solution to the inverse problems of §2.1 and §2.2 with* exact *initial data* $R(0,t)$, $0 \leq t \leq 2$.

We shall now examine the linearisation of the inverse scattering maps discussed earlier. In order to emphasise the dependence of the solution of (3.1) on p and on the initial values u_0 we will write $u(x;u_0;p)(\tau)$ instead of $u(x)(\tau)$ in the next lemma. It is convenient to define the partial derivatives of u with respect to p and u_0 as

$$v(x) = u_p(x, u_0, p), \text{ and } w(x) = u_{u_0}(x, u_0, p).$$

LEMMA 3.3. *With* u_0, p *and* F *possessing the properties of either Lemma 2.1 or Lemma 2.3 the mapping* $(u_0, p) \mapsto u(x;u;p)$ *is Fréchet differentiable, and with the partial derivatives* $w : T \mapsto T$ *and* $v : P \mapsto T$ *given as the solution of*

$$\begin{aligned} \frac{dv}{dx}(x) &= F_u(x, u(x), p(x))v(x) + F_p(x, u(x), p(x)), \quad v(0) = 0, \\ \frac{dw}{dx}(x) &= F_u(x, u(x), p(x))w(x), \quad w(0) = I. \end{aligned} \tag{3.3}$$

The implicit function theorem is utilised to prove this last Lemma (see [11, §4.11]).

REMARK 3.3. *The differentials corresponding to the derivatives* v *and* w *for any* $v_0 \in P$ *or* $w_0 \in T$ *are given by* vv_0 *and* ww_0, *respectively. The differential of the mapping then being given by*

$$dG(u_0, p) = G'(u_0, p)(w_0, v_0) = vv_0 + ww_0.$$

We note that the map considered for mathematical convenience throughout most of this section is the mapping of the initial value (2.3) or (2.13) and the known material function to $u(\tau;\tau)$, $\tau \in [0,1]$, and it is denoted by G, whereas the physical map, $\tilde{G}$, of interest provides the mapping $\tilde{G} : T \times P \mapsto P$. That the inverse scattering problems of §2 depend continuously on the given data can now be stated.

THEOREM 3.2. *If* $R(0,t)$, $0 \leq t \leq 2$ *is continuous in* t *then the mappings (i) from §2.1,* $\tilde{G} : (R(0,t), B(x)) \mapsto A(x)$, *and (ii) from §2.2* $\tilde{G} : (R(0,t), A(x)) \mapsto B(x)$, *are continuous and differentiable with respect to* R, *and* B *or* A *respectively. The form of the Fréchet partial derivatives of the map* $\tilde{G} : T \times P \mapsto P$ *are*

$$\begin{aligned} \tilde{G}_R &= \pm\tilde{I} + \tilde{K}(x), \\ \tilde{G}_p &= (I + \tilde{H})L(x), \end{aligned} \tag{3.4}$$

where the negative value in the expression for G_R *is taken for the problem of §2.1 and positive for problem §2.2 and where* $\tilde{I}$ *is the rescaling map* $(\tilde{I}f)(x) = f(2x)$, $0 \leq x \leq 1$, $\tilde{K}$ *and* $\tilde{H}$ *are compact mappings and* L *is a continuous map.*

Proof: The continuity and differentiability of the mapping $\tilde{G}$ follows directly from (2.4), (2.16), Lemma 3.1 and Lemma 3.2. The proof of the form of the Fréchet derivatives of the map $\tilde{G}$ follows Vogel [9] (see Appendix B for a sketch of the proof for the more general case considered here). ∎

REMARK 3.4. *The importance of the compactness of* $\tilde{K}$ *and* $\tilde{H}$ *is that these operators are bounded and hence the Fréchet derivatives of the mapping are bounded operators. The quantitative bound for a specific inverse problem must be found via computational techniques.*

4. The Direct Scattering Problem

We shall now consider the local and global existence, uniqueness and regularity of solutions to the direct scattering problem associated with (1.1) and (1.2). These properties will be used in the proof of Theorem 3.1, to show a unique solution to the inverse problems of §2 exists, but they are obviously useful in their own right as the solution of this direct problem is of current interest. The functional differential equation by which we analyse the direct problem also provides a convenient computational scheme by which to solve the problem [10].

We shall consider the change of independent variables $y = x + t/2$ in order to convert the partial derivatives on the left-hand-side of (1.2) into a directional derivative. This will transform the region D into $0 \le y \le 1$, $0 \le t \le 2y$, see Figure 3, and on redefining the dependent variable as $u(t;y) = -2R(y - t/2, t) = -2R(x,t)$ we find

$$\frac{du}{dt} = -\frac{1}{2}B(y-t/2)u(t;y) - \frac{1}{8}(A(y-t/2)+B(y-t/2))\int_0^t u(s;y)u(t-s;y)\,ds, \quad 0 \le t \le 2y, 0 \le y \le 1, \quad (4.1)$$

with initial conditions

$$u(0;y) = -\frac{1}{2}(A(y) - B(y)) = -2R(y,0), \quad 0 \le y \le 1. \quad (4.2)$$

The direct scattering problem can now be stated as given $u(0;y)$, for $0 \le y \le 1$, find $u(t;t/2)$, for $0 \le t \le 2$, (or equivalently $u(2y;y)$, for $0 \le y \le 1$) from the solution of the Volterra functional differential equation (4.1).

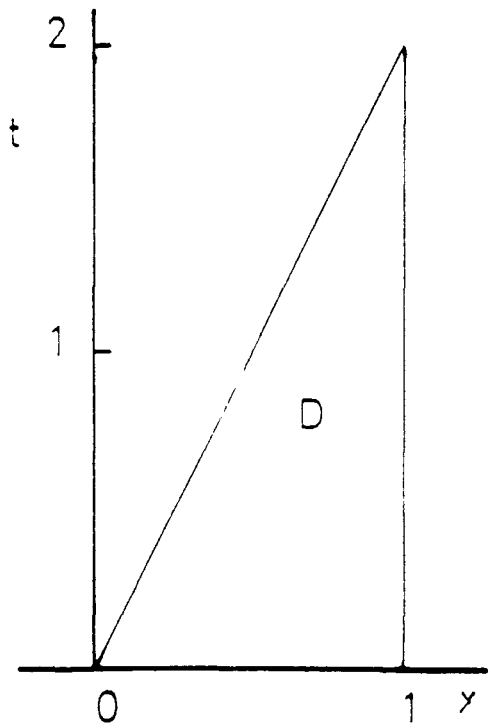

Figure 3. Domain of definition of (4.1).

In order to consider the theoretical aspects of the solution of (4.1) we find it convenient as in §2 and §3 to consider equations (4.1), (4.2) through B-space ordinary differential equation theory. We therefore rewrite (4.1) in the standard form

$$\frac{du}{dt}(t;y) = F(t, u(t))(y), \quad 0 \le t \le 2, \quad (4.3)$$

with initial conditions

$$u(0) = u_0 = u(0; y), \quad 0 \le y \le 1, \tag{4.4}$$

where $u_0, u \in Y$ and Y is the B-space of continuous functions $C([0,1])$ with norm

$$||u||_Y = \sup_{y \in [0,1]} \{u(t; y)\},$$

for fixed $t \in [0, 2]$. Again as in §2 this will mean for fixed t the function $u(t) = u(t; y) = -2R(x, t)$ will form points of the B-space Y. The mapping function on the right-hand-side of (4.3) $F : [0, 1] \times Y \mapsto Y$ is described by

$$F(t, u)(y) = \begin{cases} -\frac{1}{2}B(y - t/2)u(t; y) - \frac{1}{8}(A(y - t/2) + B(y - t/2)) \int_0^t u(s; y)u(t - s; y)\, ds, & 0 \le t \le 2y; \\ 0 & 2y \le t \le 2. \end{cases} \tag{4.5}$$

Notice F is continuous at $t = 2y$, with the definition $A(x) = B(x) = 0$, $x < 0$, and in equation (4.3) the knowledge of u at state t is not sufficient, but it requires knowledge of u at all states between 0 and t. Thus (4.3) describes a system with memory in the t variable, but not in the y variable. This later property will mean (4.1) is particularly efficient for computation (see [10]).

We define U_α to be the space of continuous functions $U_\alpha = C([0 \le t \le \alpha], Y)$, $0 \le \alpha \le 2$, with an appropriate norm modeled on (2.5). As in §2 to proceed further we need the regularity properties of F.

LEMMA 4.1. *With F defined as in* (4.5) *and with $A, B \in P$ then the mapping F has the following properties.*

(i) $F : [0, 2] \times Y \times P \times P \mapsto Y$.

(ii) For each $u \in Y$, $F(t, u)$ is continuous with respect to t.

(iii) For each $t \in [0, 2]$ and $y \in [0, 1]$, F is Fréchet partial differentiable with respect to u and with $F_u = \frac{\partial F}{\partial u}(t, u) : T \mapsto T$ this derivative is defined by the differential

$$(F_u(t, u)v)(y) = \begin{cases} -\frac{1}{2}B(y - t/2)v(t; y) - \frac{1}{4}(A(y - t/2) \\ \quad +B(y - t/2)) \int_0^t u(t - s; y)v(s; y)\, ds, & 0 \le t \le 2y; \\ 0, & 2y \le t \le 2. \end{cases} \tag{4.6}$$

(iv) For each $t \in [0, \alpha]$, $0 < \alpha \le 2$ and $y \in [0, 1]$, F is Lipschitz continuous with respect to u in the ball $B_M = \{u \in U_\alpha : ||u||_{U_\alpha} \le M\}$, with Lipschitz constant equal to $\frac{1}{2}||B||_P + \frac{1}{4}(||A||_P + ||B||_P)M$.

(v) For each $t \in [0, 2]$ and $y \in [0, 1]$, F is continuous with respect to A and B.

(vi) For each $t \in [0, 2]$ and $y \in [0, 1]$, F is Fréchet partial differentiable with respect to A or B.

(vii) $|F(x, u)| \le \frac{1}{2}||B||_P||u||_Y(t) + \frac{1}{8}(||A + B||_P) \int_0^t ||u||_Y(s)||u||_Y(t - s)\, ds$, where the notation $||u||_Y(s)$ is used to explicitly illustrate that the scalar quantity $||u||_Y$ is a function of the ordinate s.

Proof: Again this follows a similar lines to Lemma 2.1. ∎

The Picard-Lindelöf theorem now guarantees existence of a unique local solution to (4.3) and Lemma 3.1 can be taken to apply with appropriate changes, viz $x \in [0,1]$ replaced by $t \in [0,2]$, $0 \leq b \leq 2$ and the parameter p is now taken to represent A and B. Because of the local nature of this result (due to the Lipschitz result in Lemma 4.1 part (iv)) we resort to a comparison equation to assert the existence of a unique solution to (4.3) on the whole interval $0 \leq t \leq 2$.

THEOREM 4.1. *With $u_0 \in Y$, $A, B \in P$, and F possessing the properties of Lemma 4.1 the direct scattering problem* (4.3) *has exactly one continuously differentiable solution $u \in U$ for all $t \in [0,2]$ which depends continuously on the initial data* (4.2) *and the parameters A and B.*

Proof: Part (vii) of Lemma 4.1 shows that a suitable scalar comparison equation for (4.3) is

$$\frac{dw}{dt} = \alpha w + \beta \int_0^t w(s)w(t-s)\,ds, \quad w(0) = w_0, \tag{4.7}$$

with $\alpha = \frac{1}{2}||B||_P$, $\beta = \frac{1}{8}||A+B||_P$ and $w_0 = ||u||_Y$. Equation (4.7) has a solution

$$w(t) = \exp(\alpha t) 2\sqrt{\beta w_0} \mathrm{I}_1(2t\sqrt{\beta w_0})/t, \tag{4.8}$$

(see Appendix C), where I_1 is the modified Bessel function of the first kind. The local existence theory for (4.3) assures it has a unique solution in $C([0,b])$, $0 < b \leq 1$, and examination of (4.8) shows that this solution is continuous in the region $t \in [0,2]$. It therefore follows that (4.3) has a unique solution on $[0,2]$.

The Gronwall Lemma and the equivalent of Lemma 3.2 provides the continuity of the solution of (4.3) on the initial data and parameters, because of the results of parts (iv) and (vi) of Lemma 4.1. ∎

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1: The fact that a solution to (3.1) exists when the reflection kernel is exact follows directly from Theorem 4.1. In particular the continuity of the solution with respect to the material functions A and B is required; but this existence implies immediately an a priori estimate for F in (3.1), that is $||F||_Y < k$ for all $x \in [0,1]$, $\tau \in [0,1]$, which yields the result (see [11, §3.3]). ∎

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Appendix A

Proof of Lemma 2.1: We shall sketch the major steps.

(i) We must show $F(x,u)(\tau)$ is continuous in τ for fixed $x \in [0,1]$, and this follows from

$$\begin{aligned}|F(x,u)(t) - F(x,u)(\tau)| \leq\ & |4u(x;x) + A(x)|\ |u(x;t) - u(x;\tau)| + 2|2u(x;x) + A(x)|\times \\ & (\max_{0\leq w\leq x-\tau} |u(x;\tau+w) - u(x;t+w)|\ \|u\|_T\ |t-x| + \|u\|_T|\tau - t|) \\ & \to 0 \text{ as } |t-\tau| \to 0,\end{aligned}$$

provided $u \in T$ and A bounded.

(ii) To prove continuity of F with respect to x and for $u \in T$ let $0 \leq x, y \leq 1$ and $\tau > y$, then for fixed u

$$\begin{aligned}|F(x,u)(\tau) - F(y,u)(\tau)| \leq\ & 4|u(y;y)u(y;\tau) - u(x;x)u(x;\tau)| + |A(y)u(y;\tau) - A(x)u(x;\tau)| \\ & + 2(2|u(y;y) - u(x;x)| + |A(y) - A(x)|)\{\int_x^\tau |u(x;\tau+x-s)u(x;s)|\,ds \\ & + 2(2|u(y;y)| + |A(y)|)\{\int_y^\tau |u(y;\tau+y-s)u(y;s) - u(x;\tau+x-s)u(x;s)|\,ds \\ & - \int_x^y |u(x;\tau+x-s)u(x;s)|\,ds\} \\ \leq\ & 4|u(y;y)u(y;\tau) - u(x;x)u(x;\tau)| + |A(y)u(y;\tau) - A(x)u(x;\tau)| \\ & + 2(2|u(y;y) - u(x;x)| + |A(y) - A(x)|)\|u\|_T^2|\tau - x| \\ & + 2(2\|u\|_T + \|A\|_P)\times \\ & \{\max_{0\leq w\leq \tau-y} |u(y;\tau+w)u(y;\tau-w) - u(x;\tau+w)u(x;\tau-w)||x-y| \\ & + \|u\|_T^2|y-x|\} \\ & \to 0 \text{ as } |x-y| \to 0,\end{aligned}$$

provided $u \in T$ and $A \in P$. When $\tau \leq y$ the continuity follows more simply.

(iii) To prove that (2.17) is indeed the Fréchet differential of (2.8) we must show that the operator $F_u(x,u)$ is linear (obvious), bounded that is $F_u \in \mathcal{L}(T,T)$ (and this follows immediately because of the initial assumptions on u and A), and

$$\|F(x,u+v) - F(x,u) - F_u(x,u)v\|_T = o(\|v\|_T),$$

for $u, v \in T$. This last requirement can be easily shown by first forming the difference $F(x,u+v) - F(x,u)$ and then subtracting (2.17) from the resultant. This leads to

$$\begin{aligned}F(x,u+v) - F(x,u) - F_u(x,u)v =\ & -4v(x;x)v(x;\tau) - 8v(x;x)\int_x^\tau u(x;\tau+x-s)v(x;s)\,ds \\ & - 2(2u(x;x) + A(x) + 2v(x;x))\int_x^\tau v(x;\tau+x-s)v(x;s)\,ds,\end{aligned}$$

and so

$$\begin{aligned}|F(x,u+v) - F(x,u) - F_u(x,u)v| \leq\ & 4\|v\|_T^2 + 8\|v\|_T^2\|u\|_T|\tau - x| \\ & + 2(|2u(x;x) + A(x) + 2v(x;x))|\|v\|_T^2|\tau - x| \\ =\ & o(\|v\|_T).\end{aligned}$$

Note the commutativity of the convolution operator has been invoked to obtain (2.7)

(iv) To find the Lipschitz constant for the function F we observe that the mean-value theorem provides a simple way to do this because of the result (iii). For $u, v \in T$, and with $||u||_T \leq M$ and $||v||_T \leq M$ then

$$||F(x,u) - F(x,v)||_T \leq ||u - v||_T \sup_{0 \leq \alpha \leq 1} ||F_u(x, \alpha u + (1-\alpha)v)||_T,$$

thus it remains to bound the Fréchet derivative of F.

Now for any $w \in T$

$$\begin{aligned}|(F_u(x,w)v)(\tau)| \leq & (|4w(x;x)| + |A(x)|)|v(x;\tau)| + 4|w(x;\tau)v(x;x)| + 4|v(x;x| \int_x^\tau ||w||_T^2 \, ds \\ & + 4(2|w(x;x)| + |A(x)|) \int_x^\tau ||w||_T ||v||_T \, ds \\ \leq & 12||w||_T||v||_T + 8||w||_T^2||v||_T + ||A||_P(1 + 4||w||_T)||v||_T.\end{aligned}$$

Thus on using the definition of the operator norm

$$||F_u(x,w)(\tau)||_T \leq 12||w||_T + 8||w||_T^2 + ||A||_P(1 + 4||w||_T),$$

and as $||\alpha u + (1-\alpha)v|| \leq \alpha||u|| + (1-\alpha)||v|| \leq M$, with $||u||_T \leq M$ and $||v||_T \leq M$ it follows

$$||F_u(x, \alpha u + (1-\alpha)v)||_T \leq 12M + 8M^2 + ||A||_P(1 + 4M).$$

Observe that in the generalised mean-value theorem we have restricted attention to a convex set by virtue of the ball B_M being convex.

(v) The continuity of F with respect to A follows immediately as F is a linear function of A.

(vi) The Fréchet derivative of F with respect to A can be easily found formally via linearisation and its proof follows the lines of part (iii).

(vii) This follows the proof of parts (ii) and (v).

(viii) From (2.8) we easily find the inequality

$$|F| \leq (4|u(x;x)| + |A(x)|)|u(x;\tau)| + 2(2|u(x;x) + |A(x)|) \int_x^\tau |u(x;\tau + x - s)u(x;s)| \, ds,$$

and as $|u(x;s)| \leq ||u||_T$ for all $s \in [0,1]$ with x fixed

$$|F| \leq (4||u||_T + ||A||_P)||u||_T + 2(2||u||_T + ||A||_P)||u||_T^2|\tau - x|.$$

Appendix B

Proof of Theorem 3.2: Note we shall outline the proof for G_p since the proof for G_R is almost identical to Vogel's [9] (with a slight change in notation). The first equation (3.3) can be written as the integral equation

$$vv_0(x) = \int_0^x F_p(x,u,p)v_0\,dx + \int_0^x F_u(x,u,p)vv_0\,dx, \quad 0 \le x \le 1. \tag{B.1}$$

By defining the second integral operator in the right-hand-side of (B.1) as L_2, where $L_2 : C([0,x],P) \mapsto U$ we see

$$\|L_2\|_U^n \le \|F_u(x,u,p)\|_U^n/(n!).$$

Use of Lemma 2.1 or 2.2 parts (iv) shows that the series $\sum_{n=1}^{\infty} L_2^n$ converges in the uniform operator topology to an operator $H \in \mathcal{L}(U,U)$. Let us denote the extension of $z \in T$ to U by Ez and the pointwise restriction at x of $z \in U$ by $P_x z = z(x)$ so that $E : T \mapsto U$ and $P_x : U \mapsto T$. We define the first integral operator on the right-hand-side of (B.1) to be L_1, it can then be shown that L_1 is a continuous operator on the same spaces as L_2. It follows from (B.1) that

$$vv_0 = (I - L_2)^{-1}L_2v_0 = (I + H)L_2v_0,$$

so that

$$G_p = P_xIL_2 + P_xHL_2,$$

where the composition P_xI is the identity on T, and it can be shown $P_xH : U_x \mapsto U_x$ is compact via the Arzelà-Ascoli theorem. ∎

Appendix C

The analytic solution to (4.7) can be found by use of the Laplace transform. We denote the Laplace transform of a function $w(t)$ by $\widehat{w}(s)$ or $\mathcal{L}(w)(s)$, where s is the Laplace transform independent variable. Then (4.7) becomes on taking its transform

$$\beta\widehat{w}^2 + (\alpha - s)\widehat{w} + w_0 = 0$$

On taking the negative branch of the square root function, one of the roots of this equation yields

$$\widehat{w}(s) = (2\beta)^{-1}((s-\alpha) - \sqrt{(s-\alpha)^2 - 4\beta w_0}\ .$$

The use of the inverse Laplace transformation, the shift theorem, and the transform pair

$$\mathcal{L}(\frac{a}{t}\mathrm{I}_1(at)) = (s - \sqrt{s^2 - a^2}),$$

then yields the solution (4.8).